Unstructured Adaptive Grid and Grid-Free Methods for Edge Plasmas Fluid Simulations

O.V.Batishcheva, A.A.Batishcheva, A.S.Kholodov
S.I.Krasheninnikov, D.J.Sigmar

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Plasma Science and Fusion Center
Massachusetts Institute of Technology
Cambridge MA 02139 USA

\(^a\) also at: Lodestar Research Corporation
Boulder CO 80301 USA, and Keldysh Institute for
Applied Mathematics, Moscow 125047 Russia
\(^b\) Moscow Institute of Physics and Technology
Dolgoprudny 141700 Russia
\(^c\) also at: Kurchatov Institute Research Center
Moscow 123182 Russia

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Abstract

Fluid modeling of edge plasma is usually performed using a finite-difference scheme on a fixed structured grid. However, both experimental measurement and numerical simulation show presence of front-like regions characterized by sharp variations of main plasma parameters like temperature and radiation power. This is caused in part by (i) strong non-linearities in the fluid equation coefficients due to abrupt changes of various plasma reaction rates as a function of temperature and by (ii) high anisotropy of the plasma transport along and across magnetic field lines. Manual mesh adoption is usually applied to allow better resolution of the regions with sharp gradients. However, such an approach is very time consuming and limited. To overcome this problem we propose to use adaptive unstructured meshes constructed with a new quasi one-dimensional adaption algorithm. This approach is fast and conservative because we use a new finite-volumes scheme. The price for adaptation is high, because numerical algorithms became much more complicated. To avoid unwanted complexity we suggest an alternative use of a grid-free method, which requires no connectivity of arbitrarily placed vortices. To benchmark the methods and codes in two dimensions we find analytical and semi-analytical solutions of the non-linear diffusion - radiation equation, which may have sharp fronts, unconnected boundaries, and bifurcated solutions. We use these solutions to study the efficiency of the proposed numerical algorithms.
I. Introduction

Numerical simulation of the edge plasma is an extremely challenging computational problem. The primary reasons for this are coming from a wide range of both time scales and space scales, which are intrinsic to edge plasmas. Another big source of difficulties is driven by the strong non-linearities of the fluid equations due to the abrupt thresholds in the atomic physics rate constants and the huge anisotropy of plasma transport coefficients along and across the magnetic field. The predictive simulation of tokamak scrape-off layer plasmas in the promising detached regimes requires the resolution of sharp fronts, associated with essential radiation, ionization, and recombination processes. These fronts may have complicated spatial structure and are characterized by short millimeter scale lengths, provided machine dimension are of the order of meters. The location of these fronts is \textit{a priori} unknown. However, location and parameters of these fronts are crucial for the accurate modeling of edge plasma required for the design of future fusion reactors and for the interpretation of experimental results from research tokamaks. We note, that fine spatial structure requires enhanced resolution in both angular and radial directions. Due to this reason, the usual structured meshes are very limited. One cannot refine such a mesh locally by adding a single node, because at least one row or one column must be added at a time. To keep the simulations on a reasonable time frame we have to keep the number of nodes limited. Thus, the edge plasma modeling is ideally suited to the unstructured adaptive grid algorithms. Of course, we have to keep the separation of grid vertices inversely proportional to the local plasma gradients. Another interesting feature of the SOL plasma is that it may allow several stable solutions for the same input conditions [Wising,1996]. Several attempts have been made previously to develop static unstructured meshes and algorithms for fluid codes [Zanino,1996], and to simulate SOL plasma in realistic conditions [Marchand,1998].

To illustrate the difficulty of the fluid modeling of edge plasmas we have focus on a single two-dimensional non-stationary non-linear heat conduction - radiation equation for the plasma temperature, $T$, with strongly anisotropic non-linear heat conductivities, and
with a non-linear radiation term, $R$,

$$\frac{\partial T}{\partial t} - \nabla_x K_x \nabla_x T - \nabla_y K_y \nabla_y T = -R(T, x, y). \tag{1}$$

Here, $x$ and $y$ are directions perpendicular and parallel to the magnetic field, respectively. Though this equation has no analytical solution for a carbon-like radiation function $R(T)$ [which is shown in Fig.1, along with a non-linear $K_y \propto T^{2.5}$ ($K_x$ was fixed)]

$$R(T) = R_0 \, c(T) \times \begin{cases} \exp \left\{ -w \right\} \frac{(T/T_0)^3}{T^{2.5}(Ta-T)}, & T < T_0 \\ (2 - \exp \left\{ -w \right\}) \frac{T_0^3}{T^{2.5}}, & T \geq T_0, \end{cases} \tag{2}$$

where $w = \frac{2a - T}{g}$, $c(T) = 1 + a \exp \left\{ -w^2 \right\}$, $R_0$, $T_0$, $a$, $b$ and $g$ are constants. It was shown that this Eq.(1) with $R$ given by Eq.(2) can exhibit a bifurcation of its solutions [Krasheninnikov,1997]. It was also determined by the modeling that around the bifurcation threshold, the numerical solution of Eqs.(1-2) is extremely sensitive to the spatial resolution. Figs.2 b and a represent fully converged solutions of Eq.(1) for a particular set of parameters on an structured non-uniform grid. The only difference between these solutions is that one was obtained on a fine mesh with $10^5$ nodes, and the other on a 10 times finer mesh with total above $10^6$ nodes. All other numerical parameters and boundary conditions were the same. Such behavior is quite typical, as will be shown later for the one-dimensional heat conduction equation. However, in reality, $R$ may have several maxima. Thus, there could be few regions with sensitive bifurcated solutions.

From the numerical point of view, to reliably prove our methods and code accuracy we have to have test problems with exact analytical or easy to find numerical solutions of Eq.(1), which in addition exhibit some of the listed SOL plasma properties.

The structure of the paper is the following. In the Section II we describe the test problems for two-dimensional non-linear diffusion-radiation equation, and discuss the numerical results obtained for each of them using our adaptive approach. Main details of the new adaptive grid method are discussed in Section III. The grid free method is introduced in Section IV. The results of modeling of the two test problems with the grid-free method are presented at the end of Section IV. Main results are summarized in the Conclusions.
II. Model Analytical Problems.

We construct test problems to exhibit main features of the SOL plasma in tokamaks. The latter may include (i) few orders of magnitude variation of temperature along magnetic field, (ii) presence of several non-connected boundaries, and, possibly, (iii) allowing bifurcated solutions. Furthermore, we want the solutions to contain controllable sharp gradients (fronts) for benchmarking purposes. Below we are specifying three such problems, for which analytical or semi-analytical solutions of the nonlinear Eq.(1) had been found.

A. Two unconnected boundaries.

Axisymmetrical models of the tokamak scrape-off layer plasma usually have two unconnected boundaries. First is an internal boundary, which corresponds to a core plasma, with high temperature, $T_0^C$. The second boundary is external, which corresponds to a cold edge, located at the open magnetic lines, with the plasma at a much lower temperature $T_E^0 \ll T_0^C$.

It’s possible in some cases (e.g for $\beta = \alpha$, see below) to find an analytical solution of Eq.(1), which mimics this feature. This solution is characterized by sharp, but still controllable gradients. The nine-parameter \{\sigma, \alpha, a, b, c, x_c, y_c, T_0^C, T_E^0\} family of the positive, smooth stationary analytical solutions of Eq.(1) is given as

$$T_b(x, y) = \begin{cases} \left\{(\alpha + 1) T_C \left(\exp\{-\delta\} + T_E\right)\right\}^{\frac{1}{\alpha+1}}, & r > c > a \\ \left\{(\alpha + 1) T_C \left(2 - \exp\{-\delta\} + T_E\right)\right\}^{\frac{1}{\alpha+1}}, & r \leq c < b \end{cases}$$

where $T_0^C$ and $T_E^0$ are defined as follows:

$$\exp\{-\delta\} = \begin{cases} \frac{T_0^C}{(\alpha+1)T_C} - T_E, & r > c \\ T_E + 2 - \frac{T_0^C}{(\alpha+1)T_C}, & r \leq c. \end{cases}$$

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in a ring $0 < a \leq r \equiv \sqrt{(x - x_c)^2 + (y - y_c)^2} \leq b$ with non-linear heat conduction coefficients taken in the usual fixed power form as $K_x = T^\alpha$ and $K_y = T^\beta$, and with non-linear term on the right hand side

$$R(T) = \pm 2T_0^{0.5}(4\delta - \frac{\Delta}{\Delta + c} - 3)\exp\{-\delta\},$$

where $\Delta = \sigma \delta^{0.25}$, and $\delta(T)$ are defined as follows:

$$\exp\{-\delta\} = \begin{cases} \frac{T_0^{\alpha+1}}{(\alpha+1)T_C} - T_E, & r > c \\ T_E + 2 - \frac{T_0^{\alpha+1}}{(\alpha+1)T_C}, & r \leq c. \end{cases}$$
The imposed boundary conditions preserve constant \( T_C^0 \) and \( T_E^0 \) as

\[
T_C^0 = T(r = a) = \{2(\alpha + 1)(T_E^0 + 2T_C^0)\}^{\frac{1}{\alpha + 1}} \\
\{(\alpha + 1)T_E^0T_C^0\}^{\frac{1}{\alpha + 1}} = T(r = b) \equiv T_E^0
\]  

(6)

The adaptation of an adjustable grid to a particular stationary solution of the Eq.\(^{(1)}\) with the key parameters \( \sigma = 0.02, \ b = 0.5, \ \alpha = 0.15, \ T_C^0 = 100, \ T_E^0 = 1, \ x_c = y_c = 0.5 \) is shown in Fig.3 and Fig.4. In Fig.3b we draw the initial unstructured triangular mesh. It has about uniform resolution in both \( x \) and \( y \) directions throughout the domain. The total number of triangular elements, \( NT \), and the total number of vertices, \( NV \), are about 5K and 2.5K, respectively. A numerical solution of the Eq.\(^{(1)}\), \( T_{n,1} \), obtained with this initial mesh, is shown in Fig.3a. The relative accuracy, \( \varepsilon = \max_{i=1,N} \left| 1 - T_{n,1}(x_i, y_i) \right| \), is \( \approx 1 \), where \( x_i, y_i \) are the coordinates of the \( i \)th vertex. After the solution was obtained, the grid was allowed to accommodate to the calculated gradients in both directions. Next, a new stationary numerical solution, \( T_{n,2} \), was calculated, and the iterative procedure was repeated a few times. The adaptive mesh obtained after 10 iterations is shown in the Fig.4b. The total number of nodes is about 5K. We note, that the ratio of the number of triangles (\( NE \approx 10K \)) to the number of nodes is 1.97. This indicates, that the adapted mesh is close to an ideal one, which asymptotically, as \( N \to \infty \), has a \( NE/NV \) ratio of 2. The "adapted" numerical solution, \( T_{n,10} \), is shown in Fig.4a, and is characterized with a much better relative error, \( \varepsilon \), of the order of \( 10^{-3} \), because of a much finer mesh.

Next, an analytical non-stationary solution of Eq.\(^{(1)}\) (where \( c \) may vary with time) was used to study the efficiency of the grid adaption algorithm. Fig.5 shows the consecutive patterns of mesh automatic adaptation to a moving temperature gradient front with few parameters modified (\( \sigma = 0.05 \) and \( \alpha = 0.08 \)), which was shifting axisymmetically towards the axis. The total number of nodes in this run was around 6K and remained almost constant, while a 3 digit agreement of the numerical to the analytical solution was preserved. We note, that the solution given by Eq.\(^{(5)}\) with Dirichlet boundary conditions may be used as a test not only in a circular region, but in an arbitrary shaped domain, with any number of unconnected boundaries.
B. V-shaped fronts.

As it was previously mentioned, the radiation fronts might have an arbitrary shape. In the divertor they are usually V-shaped. For $\beta \neq \alpha$ we have found a non-linear analytical solution of Eq.(1), which demonstrates exactly this feature in a rectangular $x \times y \in [0, 1] \times [0, L_y]$ domain

$$T_b(x, y) = \begin{cases} 0.5e^{-\sigma(y_f(x) - y)^p}, & y < y_f(x) \\ 1 - 0.5e^{-\sigma(y - y_f(x))^p}, & y \geq y_f(x), \end{cases}$$

with a controllable "parabolic" front $y_f(x)$

$$y_f = y_v + r \left| x - x_v \right|^s.$$  \hspace{1cm} (8)

The solution depends on six free parameters $\{\sigma, p, x_v, y_v, r, s\}$. The specially constructed radiation sink term $R$ reads as

$$R(T, x, y) = \begin{cases} T^{\alpha+1}(\alpha a^2 + b) + T^{\beta+1}(\beta c^2 + d), & y < y_f(x) \\ T^{\alpha-1}[\alpha(T - 1)^2 a^2 + T(T - 1)b] + \\ T^{\beta-1}[\beta(T - 1)^2 c^2 + T(T - 1)d], & y \geq y_f(x). \end{cases}$$

The coefficients $c, d, a,$ and $b$ are defined as

$$c(x, y) = \sigma p \left( y_f - y \right)^{p-1},$$

$$d(x, y) = \sigma p \left( y_f - y \right)^{p-2} \left[ \sigma p \left( y_f - y \right)^p + p - 1 \right],$$

$$a(x, y) = -\sigma p \left( y_f - y \right)^{p-1} \delta,$$

$$b(x, y) = \sigma p \left( y_f - y \right)^{p-2} \left[ \sigma p \left( y_f - y \right)^p - p + 1 \right] \delta^2 - (y_f - y) \Delta,$$

by means of $\Delta$ and $\delta$:

$$\delta = rs(x - x_v)^{s-1},$$

$$\Delta = rs(s - 1)(x - x_v)^{s-2},$$

where we have additionally assumed that $p \geq 2$ and $s \geq 2$.  

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Two examples of adaptive mesh application to the second test problem are shown in Figs. 6 and 7. Fig. 6 illustrate the mesh adaption for a fixed parameters $\sigma = 0.05$, $p = s = 2$, $x_v = 0.4$, $y_v = 30$, $r = 200$, $L_y = 100$. We present here the initial mesh, and the grid after two and six iterative adjustments to the stationary solution. We have additionally demanded for each of the triangle elements to have one side parallel to the magnetic field, which is strictly vertical in this case. However, from Fig. 6 one can see, that the constraint of the alignment of the elements with the magnetic field direction is in conflict with smooth variation of the mesh sizes.

The grid adaptation saturates when the temperature difference between any two adjacent cells becomes smaller than some prescribed value. Figs. 7a-d show that final mesh refinement depends on the solution sharpness, which is determined by a variable parameter $\sigma$, provided the rest of the parameters were fixed. Figs. 7a and 7c represent the numerical solutions, while Fig. 7b and 7d show the "final" meshes for $\sigma = 0.03$ and for $\sigma = 0.12$, respectively. Initial grid was the same. One can see that the number of the nodes has increased four times, while the gradient became ten times larger. Thus, we can conclude, that the unstructured mesh allows us to have 25 times less nodes to achieve the same accuracy, compared to the common non-uniform structured rectangular grid.
C. Bifurcated Solutions.

In this part we introduce bifurcated solutions of the steady-state one-dimensional nonlinear reduction of the heat conduction-radiation equation. The equation trivially follows from Eq. (1) by assuming that \( \frac{\partial}{\partial t} = \frac{\partial}{\partial x} \equiv 0. \)

\[
\frac{\partial}{\partial y} K_y(T) \frac{\partial T}{\partial y} = R(T). 
\]  

(16)

This equation may be reduced to a full integral by (i) multiplying by \( q_y = K_y \frac{\partial T}{\partial y} \), and by (ii) switching to a new independent variable \( T \) instead of \( x \), resulting in

\[
q_y^2(y) = q_0^2 - \int_{T(y)}^{T(L_y)} (2K_y(T')) R(T')^{0.5} dT',
\]

(17)

where \( q_0 \) is the incoming heat flux at the \( y = L_y \) end of the interval, which serves as a first boundary conditions. The second boundary condition at \( y = 0 \) is "responsible" for the bifurcation, because it was chosen to be dependent on the solution as

\[
q_y = \gamma T, \quad y = 0
\]

(18)

If the energy transmission factor \( \gamma \) is small, we may have two different solutions of the equation, provided the incoming heat flux rises to \( q_0 \approx \int_0^\infty (2K_y(T') R(T')^{0.5} dT' \). The first solution corresponds to a hot plasma profile, with no radiated power. Thus, the heat flux is almost constant, \( q_y \approx q_0 \), as well as the temperature, \( T \approx q_0 / \gamma \). The second solution corresponds to a cold branch with almost all heat energy radiated, and, thus, low temperature, \( T_D \), at \( x = 0 \), compared to the temperature, \( T_m \), at \( x = L_y \). If the radiation function, \( R(T) \), is localized in \( T \), then there could be an interval of \( q_0 \) values, where two solutions co-exist.

To illustrate this feature, we consider the following simple model problem in the interval \( y \in [0, 1] \). We take the source term in the form

\[
R(T) = C \frac{T}{T_0} \exp \left\{ - \left( \frac{T - T_0}{\sigma} \right)^2 \right\}.
\]

(19)

Results for the arbitrary set of coefficients \( [C = 50, T_0 = 15, \sigma = 4, \gamma = 0.03, \text{ and } K_y = 10^{-4} T^{2.5}] \) are presented in Figs. 8-10. In the Fig. 8a the key parameters of the cold
solution, which exists for a limited interval of $q_0 \in [0, q_B]$, are presented. The "hot" solution, on the contrary, exists at the unlimited interval $[q_A, \infty]$, as can be seen in Fig.8b. Thus, there is a broad interval $q_0 \in [q_A, q_B] \approx [0.7928.., 8.455..]$ where the bifurcated solutions occur.

Both cold and hot coexisting solutions are shown in Figs. 9a and 10a near low and upper bifurcation thresholds, respectively. The profiles of the heat flux are shown in the Figs.9b and 10b, correspondingly.

Next, these one-dimensional solutions were used to benchmark the two-dimensional code. The sharp variation of the two-dimensional solution in a box geometry is shown in Fig.11, where the spatial distribution of a radiation sink, $R(x, y)$, is presented. As can be seen, the location of its maximum moves fast with small variation of $q_0$. 
III. Adaptive Grid Method.

We believe our approach is novel, and exhibits a composite nature. Overall it may be called "finite-elements and finite-volumes hybrid", because it combines features of both methods. In our method, the unknowns, \( \{T_i\} \), are defined at the moving grid vertices locations, \( \{x_i, y_i\}, i = 1, NV_n \). The total number of vertices, \( NV_n \), may vary from the previous iteration, \( n \), to the next iteration of the mesh adaption. The triangular grid is formed using these vertices, as shown in Fig. 12. The number of the triangular elements, \( \{E_k\}, k = 1, NE_n \) with individual surface areas \( \{S_k\} \), is roughly twice bigger than the number of vertices, \( NE_n \approx 2NV_n \), at any iteration.

The scheme is conservative. We achieve this important feature without introducing any new grids, and without constructing Dirichlet cells, as is usually done. Instead, we are proposing the following new finite volume approach for the triangular mesh:

(i) all quantities are strictly cell averaged, and the number of cells is equal to the number of nodes;
(ii) ith cell, \( C_i \), is a composition of all the triangles (one or more), which have the ith node as one of its corners:

\[
C_i = \bigcup_{m=1}^{M_i} E_{km}, \quad (x_i, y_i) \in E_{km}. \tag{20}
\]

Thus, the same triangular element \( E_k \) will be used in the forming of one, two or three different cell elements. Despite this fact, the conservation law may be now written in the generalized form as

\[
\sum_{i=1}^{NE} \sum_{j=1}^{J_i} T_{ij} S_{ij} w_{ij} = \sum_i j_i^1 - \sum_i R_i \tag{21}
\]

where \( j_i^1 \) are the heat fluxes through the external boundaries, \( R_i \) are the integrated radiation losses, and \( w_{ij} = 1/3, 2/3, \) or 1 is an individual weight of a particular triangular element.

To make a final step towards a fully conservative scheme, we have to set all fluxes through adjacent cell surfaces to be the same. This is done by using a continuous 4-point interpolation which includes two "exchanging" cells centers (\( i \) and \( j \) in Fig.12), and two vertices, \( s_1 \) and
s_2$, which form the surface:

\[ f_{i-j}(x, y) = T_i + c_{x}^{j-i}(x-x_i) + c_{y}^{j-i}(y-y_i), \]

This interpolation function can be always found for our mesh, and it is always symmetrical, provided $f_{i-j}(x_i, y_i) = T_i$ for the other three corners of the quadrilateral. Now, to calculate, for example, the parallel heat flux through the $i-j$ surface, we need to calculate the integral:

\[ q_{x}^{i-j} = \int_{x_1}^{x_2} K_y \nabla_y f_{i-j} dx, \]

which may be performed numerically. However, we take this integral analytically, which requires some linear algebra manipulations. In the non-linear case, we use one step linearization which works fine, because the grid is adjusted to the local gradient in both $x$ and $y$ directions. One can see that if the mesh is aligned with the magnetic field, then the parallel flux does not contaminate the perpendicular flux due to the vanishing $dx$.

As was mentioned, the unknowns are cell-averaged. However, the "natural" approximation of the integral of a sink $R$ over the cell as a sum of the partial integrals over sub-triangles as $\sum \int_{E_k} R dx dy \approx R(T_i) \sum S_k$ is insufficient due to the high sensitivity of the numerical solution, $T_i$ on the $R(T)$. We integrate the sink in the following way:

(i) two coefficients, $a_x^k$ and $b_y^k$, for a 3-point bi-linear interpolation function

\[ f_{3}^{k}(x, y) = T_{k_1} + a_x^k(x-x_{k_1}) + b_y^k(y-y_{k_1}) \]

are obtained for each of the triangular elements (which can be done always);

(ii) the integrals $\int_{E_k} R(f_{3}^{k}(x, y)) dx dy$ are calculated numerically by a) subdivision into smaller triangles, or (in some cases) b) analytically by the local linearization of the radiation function, provided $R(T)$ is differentiated numerically.

The set of implicit algebraic equations is solved either by using direct solvers for sparse matrices, or by an iterative relaxation method. A typical run with 10K element takes several minutes on a workstation. However, for the meshes with 100K nodes, the CPU time may easily exceed a few hours. To speed-up calculations, we are envisioning application of modern multigrid methods [Knoll,1998].
The format of the unstructured mesh in the code is the following: list of vertices, list of the vertices connectivity, list of elements connectivity, plus some information on the alignment with the magnetic field. The format uses another important constraint, i.e. that the number of neighboring elements is limited to three. This demand may be in conflict with the demand of the alignment of the mesh with the field. One can see in the Fig.11, that the triangle at the beginning of a new magnetic surface is not aligned. We plan to remove the fixed number of neighbors constraint in the future.

As was mentioned previously, the grid adaption in a 2D domain is performed using the following quasi-1D procedure. The region is covered initially by a set of lines aligned with the magnetic field. These lines may be clearly seen in Figs.3b and 6, where the "field" is respectively radial and vertical. Vertices are distributed along each of the lines using a one-dimensional algorithm described in the work [Batishchev,1998]. The algorithm distributes the prescribed number of nodes with the spacings inversely proportional to the gradients, calculated along the line. The adaption to the cross-field gradients is performed in accordance with the following procedure: if the perpendicular gradient becomes locally high, then a sub-line, which is also parallel to the magnetic field, is generated. The new line breaks if the gradient drops back. The quasi-1D procedure is then applied to each new sub-line and the mesh is regenerated every time from the very beginning. The number of levels of sub-division is unlimited, but it rarely exceeded 10, which corresponds to a 1000-fold refinement of the initial grid.

The proposed approach to build two-dimensional unstructured adaptive grids is very efficient because it is effectively reduced to a set of one-dimensional problems. Mesh refinement time is negligible, compared to the solution time of the implicit system. As a side benefit, it automatically gives us triangular elements with one side parallel to the field. This property was found to be vital to treat the large anisotropy of the cross- and along-field diffusivity coefficients.
IV. Grid-free approach.

We have to admit that the price for adaptivity is high, because the grids become unstructured, time-varying and a challenge to manage. Fortunately there are grid-free methods, which effectively deal with a "cloud of vertices" with no mutual connectivity required. Local stencils are determined on-the-fly from the stability and accuracy conditions (see below). We study applicability of one such method [Kholodov, 1991] to the edge plasma problems. This approach constructs the difference scheme using a positive approximation for any second-order non-linear elliptic equation. The method can deal with arbitrary shape domain, bounded by any number of unconnected boundaries. The approach is based on the properties of the difference schemes in the space of undetermined coefficients for functions defined on a cloud of arbitrarily located vertices. It is characterized by the following features: (i) ideal for complex geometry, (ii) explicit, (iii) unconditionally stable, (iv) high order accurate.

For illustration, let us consider the second order stationary quasi-linear elliptic equation.

$$
\begin{align*}
V_{xx} + e_{12}V_{xy} + e_{22}V_{yy} + e_1 V_x + e_2 V_y &= F(x, y, V) \\
e_{22} &> e_{12}^2/4, \quad e_{ml} = e_{m}(x, y, V), \quad e_l = e_l(x, y, V), \quad m, l = 1, 2,
\end{align*}
$$

with the Dirichlet boundary conditions:

$$
V_r = V^P(g),
$$

where $g$ is the distance along the boundary. Let us first introduce boundary nodes $r_i = (x_i, y_i), i = 1, T$, and specify a grid function $V_i$, as given by Eq.(27). Next, let us place (arbitrary) $K$ additional nodes inside the simulation domain: $r_k = (x_k, y_k), k = 1, K$, and assign desired values, $V_k$, to each of them.

Now we assume, that the stencil is not fixed, and for each inner point $r_k$ let us consider a linear difference expression

$$
V_k = \sum_{j \geq j_k}^{j,j \geq 10} a_{kj} V_j + \beta_0 F_k + \beta_1 F_{xk} + \beta_2 F_{yk}
$$

(28)
which involves at least ten vertices. They do not necessarily have to have consecutive subscripts. The boundary nodes may be included as well.

Next, we expand difference expression Eq.(28) in a Taylor series in the vicinity of the pivot point \( r_k \), and demand the second order accuracy of the approximation. This will require the satisfaction of the following conditions:

\[
\sum_j a_{kj} = 1 \tag{29}
\]

\[
\sum_j a_{kj} X_j (1 - \frac{1}{2} X_j (e_1 - X_j (e_1^2 + F_v - e_{1x})/3 + Y_j e_{1y})) = 0 \tag{30}
\]

\[
\sum_j a_{kj} Y_j \left( 1 - \frac{1}{2} X_j^2 (e_1 - X_j (e_1^2 + X_j (e_2 F_v - e_1 e_2 + e_{2x})/3 + Y_j (F_v - e_{2y}))) \right) = 0 \tag{31}
\]

\[
\sum_j a_{kj} X_j (Y_j - \frac{1}{2} X_j (e_1 + X_j (e_2 F_v - e_1 e_2 + e_{2x})/3 + Y_j (F_v - e_{2y}))) = 0 \tag{32}
\]

\[
\sum_j a_{kj} X_j (Y_j^2 - X_j^2 (e_2 + X_j (e_1^2 + e_1 e_2 - e_{2x})/3 + Y_j (e_2 - e_{2y}))) = 0 \tag{33}
\]

\[
\sum_j a_{kj} X_j (3Y_j (Y_j - e_{12} X_j) - x_j^2 (e_{22} - e_{12}^2)) = 0 \tag{34}
\]

\[
\sum_j a_{kj} (Y_j^3 - e_{22} X_j^3 (3Y_j - e_{12} X_j)) = 0 \tag{35}
\]

\[
2\beta_0 = \sum_j a_{kj} X_j^2 (1 - e_{11} X_j/3)/2 \tag{36}
\]

\[
6\beta_1 = - \sum_j a_{kj} X_j^3 \tag{37}
\]

\[
2\beta_2 = - \sum_j a_{kj} X_j^2 Y_j \tag{38}
\]

where we used the notations \( X_j = x_j - x_k \) and \( Y_j = y_j - y_k \). Usually, the neighbors for each point can be chosen so that

\[
a_{kj} \geq 0, \quad j = j_1, j_2, J \geq 10. \tag{39}
\]

This condition guarantees that the finite difference operator will be monotonic and positively defined, which is known to be crucial for the stability. The search for the neighbors is performed in each of the 20 sectors (see Fig.13), to satisfy the conditions Eqs.(29-38). In the non-linear case the iterations, with possible refreshment of the neighbors, are required.
cannot proof that system Eqs.(29-38) may be always be solvable. However, our experience shows that we were always able to do so, even in the non-linear cases.

A first attempt has been made to apply this method to scrape-off layer plasma related model problems. We illustrate the method’s efficiency by solving two problems with diffusion dominance.

*Model problem I.*

Quasilinear heat conduction-radiation equation in a box geometry

\[ V_{xx} + C_y(x, y, V)V_{yy} = S(x, y, V), \quad X \times Y \in L_x \times L_y \]  

with the following boundary conditions: at \( y = L_y \) the incoming heat flux, \( q_0(x) \), is fixed, while at the other boundaries \( V \) is set to a small positive value. The contours of the calculated \( V \) are shown in the Fig.14. We note that the arrangement of vertices was regular, and their total number was about 1K.

*Model problem II.*

Heat conduction equation in a domain with two non-connected boundaries. For illustration we picked the Alcator C-Mod like geometry. The boundary conditions were just two fixed temperatures: hot core, \( T_C=100 \), and cold edge, \( T_E=1 \). The simulation domain is covered by a cloud of arbitrarily placed nodes (about 3K in this run). Their spatial distribution is shown in Fig.15. Fig.16 shows the numerical solution which was obtained using this cloud of vertices and the grid-free method.
V. Conclusions.

i A new algorithm for the adjustment of the unstructured adaptive triangular grid is proposed. It allows to build very effectively two-dimensional meshes using a quasi-one-dimensional approach. The grid is aligned with the magnetic field, while its variable resolution takes into account local plasma gradients in both directions. A new discretization scheme has been employed for the edge plasma applications. Our approach combines flexibility of finite-elements and conservative properties of finite volumes methods. The method was benchmarked on several non-linear test problems with anisotropic coefficients, sharp fronts, and bifurcated solutions. The total number of nodes required to achieve the same accuracy used with standard structured meshes was found to be almost two orders smaller.

ii A fast and accurate grid-free method was applied to the scrape-off layer plasma problems. The method is unconditionally stable, and very flexible. It can easily simulate regions with complex geometry and with non-connected boundaries. It is not conservative, but extremely efficient even for non-linear cases. We suggest that it will be used in conjunction with other implicit methods as a preconditioner in fluid codes.

iii Exact analytical and semi-analytical solutions of the non-linear diffusion-radiation equation with edge plasma related properties were obtained. They may exhibit bifurcation of the solutions, contain sharp fronts, and non-connected boundaries. The structure of these solutions is controllable via adjustable parameters. This makes them very useful for the benchmarking of various numerical methods and codes, which are being developed for non-linear plasma problems.
References:


Figure captions.

Fig.1. Carbon-like radiation, $R(T)$, and classical parallel heat conduction, $K_y$, coefficient are sharply varying functions of the plasma temperature, $T$.

Fig.2. Position of the radiative front is very sensitive to mesh refinement near the bifurcation threshold.

Fig.3. a) stationary solution of 1st test problem on a coarse initial mesh, shown in b).

Fig.4. a) steady-state solution of the same problem as in the Fig.3, but on the adapted fine mesh, shown in b).

Fig.5. Automatic mesh adoption to the moving circular front.

Fig.6. Field-aligned unstructured triangular grid adaption to a V-shaped stationary front.

Left-to-right: initial mesh, grid after 2 adjustments, and after 6 iterations.

Fig.7. Final adaptive mesh depends on the local gradients. The sharper the gradients the
more magnetic line subdivisions will be required, before algorithm saturates.

Fig.8. One-dimensional bifurcation study. Parameters of the a) cold and b) branches of the solutions. There is a broad region of $q_0$, where two solutions coexist.

Fig.9. Hot and cold solutions, $T(y)$ of the 1D diffusion-radiation equation, a), and corresponding heat flux, $q(y)$, profiles, b); $q_0$ is slightly above lower threshold, $q_A$.

Fig.10. Same as Fig.9 a and b, but for incoming heat flux, $q_0$, slightly lower the upper threshold, $q_B$.

Fig.11. Illustration of a big shift of radiation profile $R(x, t)$ versus small variation of incoming heat flux: a) $q_0 = 8.45$, and b) $q_0 = 8.455$.

Fig.12. A fragment of actual unstructured triangular adaptive mesh, showing the definition of cell volume, node dependence region, calculation of fluxes, and mesh alignment with the magnetic field, B.

Fig.13. kth vertex neighborhood is divided into 20 sectors to assure the construction of the second order monotonic difference scheme by the grid-free method.

Fig.14. Contours of $V$ as calculated for the second order quasilinear equation in the box geometry.

Fig.15. Cloud of arbitrarily placed vertices for the Alcator C-Mod geometry with two un-connected boundaries.

Fig.16. Contours of temperature profile, which was obtained by applying grid-free method to a cloud of nodes, shown in the Fig.15.
Fig. 1
Fig. 2
$\sigma = 0.02$

$N=2560$
$E=4845$

Fig. 3a
Fig. 4a

$\sigma = 0.02$

$N = 5352$

$E = 10553$
Fig. 5: The evolution of temperature front moving from $T_C^0$ to $T_E=1$. 

- Initial state: $T_C^0$ with a moving front.
- Transition: The front moves from the inner to the outer region.
- Final state: $T_E=1$ with the front completely moved out.
\[ \sigma = 0.03 \]

\[ \sigma = 0.12 \]

Fig. 7
Fig. 8
Fig. 9

(a) 

"hot" \[ T(y) \]

"cold"

(b) 

"hot" \[ q_A = 0.7928 \ldots \]

"cold" 

q(y)

Fig. 9
Fig. 10

(a) \[ T(y) \]

(b) \[ q_B = 8.455.. \]
Temperature profile

$q_0 = -8.455$

Fig. 11b
new magnetic line started

Fig. 12
Fig. 13
Fig. 14
Fig. 16

$T_C = 100$

$T_E = 1$