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# Endomorphisms of the shift dynamical system, discrete derivatives, and applications

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ABSTRACT

All continuous endomorphisms $f_\infty$ of the shift dynamical system $S$ on the 2-adic integers $\mathbb{Z}_2$ are induced by some $f : \mathcal{B}_n \to \{0, 1\}$, where $n$ is a positive integer, $\mathcal{B}_n$ is the set of $n$-blocks over $\{0, 1\}$, and $f_\infty(x) = y_0y_1y_2\ldots$ where for all $i \in \mathbb{N}$, $y_i = f(x_ix_{i+1}\ldots x_{i+n-1})$. Define $D : \mathbb{Z}_2 \to \mathbb{Z}_2$ to be the endomorphism of $S$ induced by the map $\{(00, 0), (01, 1), (10, 1), (11, 0)\}$ and $V : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $V(x) = -1 - x$. We prove that $D$, $V \circ D$, $S$, and $V \circ S$ are conjugate to $S$ and are the only continuous endomorphisms of $S$ whose parity vector function is solenoidal. We investigate the properties of $D$ as a dynamical system, and use $D$ to construct a conjugacy from the $3x + 1$ function $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ to a parity-neutral dynamical system. We also construct a conjugacy $R$ from $D$ to $T$. We apply these results to establish that, in order to prove the $3x + 1$ conjecture, it suffices to show that for any $m \in \mathbb{Z}^+$, there exists some $n \in \mathbb{N}$ such that $R^{-1}(m)$ has binary representation of the form $\overline{x_0x_1\ldots x_{2^n-1}}$ or $x_0\overline{x_1x_2\ldots x_{2^n}}$.



## 1. Introduction

A discrete dynamical system is a function from a set or metric space to itself [5]. Given two dynamical systems $f : X \to X$ and $g : Y \to Y$, a function $h : X \to Y$ is a **morphism** from $f$ to $g$ if $h \circ f = g \circ h$. A morphism from a dynamical system to itself is called an **endomorphism**. A bijective morphism is called a **conjugacy**, and a bijective endomorphism is called an **autoconjugacy**. Note that conjugacies on metric spaces are not assumed to be continuous.

Let $\mathbb{Z}_2$ be the ring of 2-adic integers. Each element of $\mathbb{Z}_2$ is a formal sum $\sum_{i=0}^{\infty} 2^i x_i$ where $x_i \in \{0, 1\}$ for all $i \in \mathbb{N}$. The binary representation of $x = \sum_{i=0}^{\infty} 2^i x_i$ is the infinite sequence of zeroes and ones $x_0x_1x_2\ldots$. (Throughout this paper $x_{i-1}$ will denote the $i$th digit of the binary representation of a 2-adic integer $x$.) Note that $\mathbb{Z} \subseteq \mathbb{Z}_2$. For example, $13 = 1\cdot 2^0 + 0\cdot 2^1 + 1\cdot 2^2 + 1\cdot 2^3$, so the 2-adic binary representation of 13 is $1011\overline{0}$, where the overbar represents repeating digits as in decimal notation. The binary representation of $-1$ is $\overline{1}$, since $\overline{1} + 1 = 1\overline{1} + 1\overline{0} = \overline{0} = 0$.

By interpreting $\mathbb{Z}_2$ as the set of all binary sequences, there is a natural topology on $\mathbb{Z}_2$, namely the product topology induced by the discrete topology on $\{0, 1\}$. This topology is also induced by the metric $\delta$ on $\mathbb{Z}_2$ defined by $\delta(x, y) = 2^{-k}$ where $k$ is the smallest natural number such that $x_k \neq y_k$.

The shift dynamical system, $S : \mathbb{Z}_2 \to \mathbb{Z}_2$, is a well-known map, continuous with respect to the 2-adic topology, defined by $S(x_0x_1x_2\ldots) = x_1x_2x_3\ldots$. This map can be extended to the shift map $\sigma$ on binary bi-infinite sequences $\ldots x_{-2}x_{-1}x_0x_1x_2\ldots$ by defining $\sigma(x) = y$ where $y_i = x_{i+1}$ for all integers $i$.

In [3], Hedlund classified all continuous endomorphisms of the shift dynamical system $\sigma$ on bi-infinite sequence space ($\{0, 1\}^{\mathbb{Z}}$ with the product topology). Lind and Marcus [5] also stated this result, referring to the continuous endomorphisms of $\sigma$ as *sliding block codes*.

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In Section 2, we will show that the continuous endomorphisms of $S$ on $\mathbb{Z}_2$ can be classified as follows. For each $n \in \mathbb{Z}^+$, let $\mathcal{B}_n$ be the set of all binary sequences (or *blocks*) of length $n$. Then every continuous endomorphism of $S$ is induced by a function $f : \mathcal{B}_n \to \{0, 1\}$ for some $n$. The endomorphism induced by such an $f$ is the map $f_\infty : \mathbb{Z}_2 \to \mathbb{Z}_2$ defined by $f_\infty(x) = y_0y_1y_2 \ldots$ where $y_i = f(x_ix_{i+1} \ldots x_{i+n-1})$ for all $i \in \mathbb{N}$. These results are analogous to those already obtained for $\sigma$ on $\{0, 1\}^{\mathbb{Z}}$.

These endomorphisms have applications to the famous $3x+1$ conjecture. This conjecture states that the $T$-orbit $\{T^i(x)\}_{i=0}^{\infty}$ of any positive integer $x$ contains 1, where $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ is defined by

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ (3x+1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

In [4], Lagarias proved that there exists a continuous conjugacy $\Phi$ from $S$ to $T$, whose inverse is also continuous. Since conjugacies preserve dynamics (fixed points, cycles, divergent orbits, etc.), the dynamics of $S$ are the same as those of $T$. Furthermore, we can combine these results to classify all continuous endomorphisms of $T$. A map $H$ is a continuous endomorphism of $T$ if and only if $H = \Phi \circ f_\infty \circ \Phi^{-1}$ for some continuous endomorphism $f_\infty$ of $S$.

Hedlund also showed that exactly two of the continuous endomorphisms of $\sigma$ are autoconjugacies. It can be shown that this is true for $\mathbb{Z}_2$ as well (cf. [3,6]). The two continuous autoconjugacies of $S$ are the bit complement map $V = f_\infty$ where $f$ is the map sending the block 0 to 1 and the block 1 to 0, and the identity map $\mathcal{I} = \mathbf{1}_{\mathbb{Z}_2}$ (induced by the map sending 0 to 0 and 1 to 1). Monks and Yazinski [6] investigated the corresponding autoconjugacies of $T$, namely $\Omega = \Phi \circ V \circ \Phi^{-1}$ and the identity map, respectively.

Continuing the line of research of Monks and Yazinski, it is natural to investigate the continuous endomorphisms of $S$ which are not autoconjugacies. Note that each of these maps, in addition to being an endomorphism of $S$, is a dynamical system in its own right. As such, it is natural to ask which of these dynamical systems are conjugate to $S$ (and hence to $T$).

Let $f : B_2 \to \{0, 1\}$ be defined by $f(00) = f(11) = 0$ and $f(01) = f(10) = 1$, and define the discrete derivative $D : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $D = f_\infty$. In Section 5, we find that $D$ is in fact conjugate to $T$. Furthermore, the dynamical systems $D$, $S$, and their "duals" (formed by interchanging the symbols 0 and 1) are the only endomorphisms of the shift dynamical system having a certain property (see Section 3, Theorem 3.3). In Section 4, we thoroughly investigate the dynamics of $D : \mathbb{Z}_2 \to \mathbb{Z}_2$, and apply these results to the $3x + 1$ conjecture in Section 5.

## 2. Continuous endomorphisms of the shift map

We begin by classifying all continuous endomorphisms of the shift dynamical system $S : \mathbb{Z}_2 \to \mathbb{Z}_2$. As in the classification of the continuous endomorphisms of the shift map on bi-infinite sequence space, each such endomorphism is characterized by a "block code" as follows.

**Definition 1.** Let $\mathcal{B}_n$ denote the set of all length-$n$ sequences $x_0x_1 \ldots x_{n-1}$ where each $x_i \in \{0, 1\}$. For any function $f : \mathcal{B}_n \to \{0, 1\}$, we define $f_\infty : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $f_\infty(x) = y$ where $y_i = f(x_ix_{i+1} \ldots x_{i+n-1})$.

**Theorem 2.1.** *A map $F : \mathbb{Z}_2 \to \mathbb{Z}_2$ is a continuous endomorphism of the shift map $S$ if and only if there is a positive integer $n$ such that $F = f_\infty$ for some $f : \mathcal{B}_n \to \{0, 1\}$.*

**Proof.** First note that $\mathbb{Z}_2$ is homeomorphic to the (middle thirds) Cantor set. (See [2].) The Cantor set is a closed and bounded subset of $\mathbb{R}$, so it is compact by the Heine–Borel theorem. Hence, $\mathbb{Z}_2$ is a compact metric space.

Let $n$ be a positive integer, and let $f : \mathcal{B}_n \to \{0, 1\}$ be arbitrary. We show $f_\infty$ is a continuous endomorphism of $S$.

To show $f_\infty$ is continuous, we show that the inverse image of every open ball is open. Let $B(x, \epsilon)$ be an arbitrary open ball in the metric space $\mathbb{Z}_2$. Let $k$ be the smallest nonnegative integer such that $2^{-k} < \epsilon$. Then $B(x, \epsilon)$ is the set of all 2-adic integers $y$ such that the first $k$ digits of $y$ are the same as the first $k$ digits of $x$.

Let $a \in f_\infty^{-1}(B(x, \epsilon))$ be arbitrary, and let $b \in B(a, 2^{-(k+n-2)})$. Note that the first $k + n - 1$ digits of $b$ are $a_0 \ldots a_{k+n-2}$. Then for any nonnegative integer $m \le k - 1$, we have $(f_\infty(b))_m = f(b_mb_{m+1} \ldots b_{m+n-1}) = f(a_ma_{m+1} \ldots a_{m+n-1}) = x_m$. Hence the first $k$ digits of $f_\infty(b)$ are the same as those of $x$, so it follows that $f_\infty(b) \in B(x, \epsilon)$. Since $b$ was arbitrary, it follows that any member of $B(a, 2^{-(k+n-2)})$ maps to an element of $B(x, \epsilon)$ under $f_\infty$. Hence, $B(a, 2^{-(k+n-2)}) \subset f_\infty^{-1}(B(x, \epsilon))$. Since $a$ was arbitrary, it follows that $f_\infty^{-1}(B(x, \epsilon))$ is open, as desired.

To show $f_\infty$ is an endomorphism of $S$, let $x \in \mathbb{Z}_2$ be arbitrary. Then for any positive integer $i$,

$$\begin{aligned}(f_\infty(S(x)))_i &= f(S(x)_iS(x)_{i+1} \ldots S(x)_{i+n-1}) \\ &= f(x_{i+1}x_{i+2} \ldots x_{i+n}) \\ &= (f_\infty(x))_{i+1} \\ &= (S(f_\infty(x)))_i.\end{aligned}$$

Hence, $f_\infty$ is a continuous endomorphism of $S$.

It now remains to show that such maps are the only continuous endomorphisms of $S$. Let $F : \mathbb{Z}_2 \to \mathbb{Z}_2$ be a continuous endomorphism of $S$. Since $\mathbb{Z}_2$ is a compact metric space and $F$ is continuous, it follows by the Heine–Cantor theorem that $F$

is uniformly continuous. Hence, choosing $\epsilon = 1$, there is a positive real number $\delta > 0$ such that any two elements $x$ and $y$ of $\mathbb{Z}_2$ which are separated by at most $\delta$ have the property that the distance between $F(x)$ and $F(y)$ is less than $\epsilon = 1$, i.e. they match in the first digit.

Let $n$ be the smallest positive integer such that $2^{-n} < \delta$. Then any two elements $x$ and $y$ having $x_0 \dots x_{n-1} = y_0 \dots y_{n-1}$ satisfy $(F(x))_0 = (F(y))_0$. We can now define the map $f : \mathcal{B}_n \to \{0, 1\}$ by $f(a_0a_1 \dots a_{n-1}) = (F(a_0a_1 \dots a_{n-1}000\dots))_0$. We show that $F = f_\infty$.

Since $F$ is an endomorphism of $S$, we have $F \circ S = S \circ F$. We have that $F(x)_0 = f(x_0x_1 \dots x_{n-1}) = f_\infty(x)_0$ for any $x$. We use this as the base case to show by induction that $F(x)_i = f_\infty(x)_i$ for any nonnegative integer $i$ and $x \in \mathbb{Z}_2$. Let $i$ be a positive integer and assume $F(x)_{i-1} = f_\infty(x)_{i-1}$ for any $x \in \mathbb{Z}_2$. Then since $f_\infty$ commutes with $S$ by the above argument, we have

$$\begin{aligned}(F(x))_i &= (S(F(x)))_{i-1}\\ &= (F(S(x)))_{i-1}\\ &= (f_\infty(S(x)))_{i-1}\\ &= (S(f_\infty(x)))_{i-1}\\ &= (f_\infty(x))_i.\end{aligned}$$

This completes the induction. ■

## 3. Conjugacies to the shift dynamical system

For any $x, y \in \mathbb{Z}_2$, we write $x \underset{n}{\equiv} y$ if $x$ is congruent to $y$ mod $2^n$, i.e. if the binary representations of $x$ and $y$ match in the first $n$ digits. We extend this notation to include finite sequences, for example, $1\overline{011} \underset{2}{\equiv} 100$. Lagarias defined $\Phi^{-1}$ by $\Phi^{-1}(x) = a_0a_1a_2 \dots$ where $a_i \underset{1}{\equiv} T^i(x)$. We call $\Phi^{-1}$ the *T-parity vector function* and generalize this definition as follows.

**Definition 2.** Let $F : \mathbb{Z}_2 \to \mathbb{Z}_2$. The $F$ **-parity vector function** is the map $P_F : \mathbb{Z}_2 \to \mathbb{Z}_2$ given by $P_F(x) = a_0a_1a_2 \dots$ where $a_i \in \{0, 1\}$ and $a_i \underset{1}{\equiv} F^i(x)$ for all $i \in \mathbb{N}$.

It is easily shown that the parity vector function $P_F$ of every dynamical system $F : \mathbb{Z}_2 \to \mathbb{Z}_2$ is a morphism from $F$ to $S$. To see this, let $x \in \mathbb{Z}_2$ and let $a = P_F(x)$. Then $S(P_f(x)) = a_1a_2a_3 \dots$ by the definition of $S$. By the definition of $P_F$, $P_F(F(x)) = b_0b_1b_2 \dots$ where $b_i \underset{1}{\equiv} F^i(F(x))$. Thus $b_i \underset{1}{\equiv} F^{i+1}(x) \underset{1}{\equiv} a_{i+1}$ for all $i \in \mathbb{N}$, so $P_F(F(x)) = S(P_F(x))$. Therefore $P_F \circ F = S \circ P_F$.

Note that $F$ is not assumed to be continuous in the definition above. In the case that $F$ is continuous with respect to the 2-adic topology, the composition of continuous functions $F^i$ is also continuous for each $i$. Thus, if $F$ is continuous then its parity vector function $P_F$ is continuous as well.

Since every parity vector function is a morphism, it is natural to ask which of these are bijections and therefore conjugacies. The following theorem classifies all functions that are conjugate to $S$ by their parity vector functions.

**Theorem 3.1.** *Let $F : \mathbb{Z}_2 \to \mathbb{Z}_2$, not necessarily continuous. Then $P_F$ is a conjugacy from $F$ to $S$ if and only if $F = P^{-1} \circ S \circ P$ for some parity-preserving bijection $P : \mathbb{Z}_2 \to \mathbb{Z}_2$ (and in this situation $P_F = P$).*

**Proof.** Assume $P_F$ is a conjugacy from $F$ to $S$. Then $F = P_F^{-1} \circ S \circ P_F$ by the definition of conjugacy. By definition, $P_F$ is parity-preserving, since $P_F(x) \underset{1}{\equiv} x$.

Now assume that there exists a parity-preserving bijection $P : \mathbb{Z}_2 \to \mathbb{Z}_2$ such that $F = P^{-1} \circ S \circ P$. It follows by induction on $n$ that $F^n = P^{-1} \circ S^n \circ P$ for all $n \in \mathbb{Z}^+$.

Let $x \in \mathbb{Z}_2$. Then for all $n \in \mathbb{Z}^+$, $F^n(x) \underset{1}{\equiv} P^{-1}(S^n(P(x))) \underset{1}{\equiv} S^n(P(x))$ since $P$ is parity-preserving. Let $a = P(x)$. Then $S^n(P(x)) \underset{1}{\equiv} a_n$, so $F^n(x) \underset{1}{\equiv} a_n$ for all $n$, and thus $P(x) = P_F(x)$. Since $x$ was arbitrary, $P = P_F$. Also, we know $P$ is a conjugacy from $F$ to $S$, so $P_F$ is a conjugacy from $F$ to $S$ as well. ■

Lagarias [4] showed that $\Phi^{-1} = P_T$ is bijective by showing it has a property later named in [1]. Bernstein and Lagarias called a function $h : \mathbb{Z}_2 \to \mathbb{Z}_2$ **solenoidal** if for all $k \in \mathbb{Z}^+$, $x \underset{k}{\equiv} y \Leftrightarrow h(x) \underset{k}{\equiv} h(y)$. Such a map induces a permutation of $\mathbb{Z}_2/2^k\mathbb{Z}_2$ for all $k \in \mathbb{Z}^+$.

Bernstein and Lagarias [1] also showed that any solenoidal map $h : \mathbb{Z}_2 \to \mathbb{Z}_2$ is an isometry (bijective and continuous with continuous inverse). Since $P_F$ is a morphism from $F$ to $S$, we obtain the following corollary.

**Corollary 3.2.** *Let $F : \mathbb{Z}_2 \to \mathbb{Z}_2$. If $P_F$ is solenoidal, then $F$ is continuous and $P_F$ is a conjugacy from $F$ to $S$.*

Hence, we can prove that a function is conjugate to the shift map by showing that its parity vector function is solenoidal. In particular, it is of interest to determine which continuous endomorphisms of $S$ have a solenoidal parity vector function. In order to classify these, we define a specific endomorphism $D$ as follows.

**Definition 3.** Let $f : B_2 \to \{0, 1\}$ be the map $\{(00, 0), (01, 1), (10, 1), (11, 0)\}$. We define the **discrete derivative** $D : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $D = f_\infty$.

Note that $D(x)$ is obtained by replacing each subsequence $x_i x_{i+1}$ of the 2-adic binary representation of $x$ with

$$x'_i = |x_i - x_{i+1}|,$$

so $D$ resembles a discrete derivative, explaining our nomenclature. (The natural extension of this map to bi-infinite sequences has been discussed in [5], pp. 4, 16.)

Let $V : \mathbb{Z}_2 \to \mathbb{Z}_2$ be the map $V(x) = -1 - x$. Note that $V(x)$ is obtained by interchanging the symbols 0 and 1 in the binary representation of $x$. The "dual" $V \circ D$ of $D$ is induced by $\{(00, 1), (01, 0), (10, 0), (11, 1)\}$ and is essentially the same as $D$ if one were to interchange the symbols 0 and 1. For simplicity of notation we let $\mathcal{P} = P_D$.

**Theorem 3.3.** *The functions $D$, $V \circ D$, $S$, and $V \circ S$ are the only continuous endomorphisms of $S$ with solenoidal parity vector functions.*

Combining this theorem with Corollary 3.2, we obtain the following result.

**Corollary 3.4.** *$D$ is conjugate to $S$ by its parity vector function $\mathcal{P}$.*

Before we present the proof of Theorem 3.3 we first prove two technical lemmas.

**Definition 4.** For every positive integer $n \geq 2$, define $d : B_n \to B_{n-1}$ by $d(x_0x_1 \dots x_{n-1}) = y_0y_1 \dots y_{n-2}$ where $y_i = |x_i - x_{i+1}|$ for $0 \leq i \leq n-2$.

Note that $d$ is essentially $D$ defined on finite sequences.

**Lemma 3.5.** *Let $x \in \mathbb{Z}_2$, $n \in \mathbb{Z}^+$, and $y = D^n(x)$. For all $i \in \mathbb{N}$, $y_i = d^n(x_i x_{i+1} \dots x_{i+n})$.*

**Proof.** We proceed by induction on $n$. For the base case, $n = 1$, we see that for all $i$, $y_i = |x_i - x_{i+1}| = d(x_i x_{i+1})$ by the definition of $D$ and $d$.

Assume the assertion is true for $n$, and let $i \in \mathbb{N}$. Then $d^{n+1}(x_i x_{i+1} \dots x_{i+n+1}) = d^n(d(x_i x_{i+1} \dots x_{i+n+1})) = d^n(z_i z_{i+1} \dots z_{i+n})$ where $z_j = \left|x_j - x_{j+1}\right|$ for all $j$. Note that $D(x) = z_0 z_1 z_2 \dots$ by the definition of $D$. Let $y = D^n(D(x))$. By the inductive hypothesis, we have $y_i = d^n(z_i z_{i+1} \dots z_{i+n})$. Thus $D^{n+1}(x) = D^n(D(x)) = y$ and $d^{n+1}(x_i x_{i+1} \dots x_{i+n+1}) = d^n(z_i z_{i+1} \dots z_{i+n}) = y_i$, so our induction is complete. ■

**Lemma 3.6.** *Let $n \in \mathbb{Z}^+$, $x_0x_1 \dots x_{n-1}x_n \in B_{n+1}$, and $v = 1 - x_n$. Then $d^n(x_0x_1 \dots x_{n-1}x_n) \neq d^n(x_0x_1 \dots x_{n-1}v)$.*

**Proof.** Again, we show this by induction on $n$. The base case, $n = 1$, is clearly true since $d(01) \neq d(00)$ and $d(11) \neq d(10)$.

Let $n \in \mathbb{Z}^+$ and assume the assertion is true for $n$. Let $x_0x_1 \dots x_n x_{n+1} \in B_{n+2}$ and define $v = 1 - x_{n+1}$. Let $d(x_0x_1 \dots x_n x_{n+1}) = y_0y_1 \dots y_{n-1}y_n$. Then $d(x_0x_1 \dots x_n v) = y_0y_1 \dots y_{n-1}w$ where $w = d(x_n v)$. We know $w = d(x_n v) \neq d(x_n x_{n+1}) = y_n$, and since $w, y_n \in \{0, 1\}$, we conclude that $w = 1 - y_n$. By the inductive hypothesis, we have

$$\begin{aligned} d^{n+1}(x_0x_1 \dots x_n x_{n+1}) &= d^n(d(x_0x_1 \dots x_n x_{n+1})) \\ &= d^n(y_0y_1 \dots y_{n-1}y_n) \\ &\neq d^n(y_0y_1 \dots y_{n-1}w) \\ &= d^n(d(x_0x_1 \dots x_n v)) \\ &= d^{n+1}(x_0x_1 \dots x_n v) \end{aligned}$$

and the induction is complete. ■

We are now ready to prove Theorem 3.3.

**Proof.** We first show that $\mathcal{P}$ is solenoidal. Let $k \in \mathbb{Z}^+$ and $x \in \mathbb{Z}_2$. For all $i \leq k - 1$, we have by Lemma 3.5 that $D^i(x) \underset{1}{\equiv} d^i(x_0x_1 \dots x_i)$. Thus the finite sequence $a_0 \dots a_{k-1}$ where $a_i \underset{1}{\equiv} D^i(x)$ is entirely determined by the first $k$ digits of $x$, i.e. $x \underset{k}{\equiv} y \Rightarrow \mathcal{P}(x) \underset{k}{\equiv} \mathcal{P}(y)$.

Let $x, y \in \mathbb{Z}_2$ be such that $\mathcal{P}(x) \underset{k}{\equiv} \mathcal{P}(y)$ and let $a_0 \dots a_{k-1}$ be the first $k$ digits of $\mathcal{P}(x)$ and $\mathcal{P}(y)$. We will show that $x \underset{k}{\equiv} y$. Assume to the contrary that $x \underset{k}{\not\equiv} y$. Then $x_0x_1 \dots x_{k-1} \neq y_0y_1 \dots y_{k-1}$. Let $j$ be the smallest nonnegative integer such that $x_j \neq y_j$ (note that $j < k$), so that $y_0y_1 \dots y_{j-1} = x_0x_1 \dots x_{j-1}$ and $y_j = 1 - x_j$. Then by Lemma 3.5, we have $a_j \underset{1}{\equiv} D^j(x) \underset{1}{\equiv} d^j(x_0x_1 \dots x_j)$ and $a_j \underset{1}{\equiv} D^j(x) \underset{1}{\equiv} d^j(y_0y_1 \dots y_j) = d^j(x_0x_1 \dots x_{j-1}y_j)$. But by Lemma 3.6, $d^j(x_0x_1 \dots x_j) \neq d^j(x_0x_1 \dots x_{j-1}y_j)$, so $a_j \neq a_j$, a contradiction. We conclude that $x \underset{k}{\equiv} y$, and hence $\mathcal{P}$ is solenoidal.

Observe that $V \circ D$ is induced by $\{(00, 1), (01, 0), (10, 0), (11, 1)\}$, which is exactly the same map as that which induces $D$ except with 0 and 1 interchanged. With this in mind, we see that since $\mathcal{P}$ is solenoidal, $P_{V \circ D}$ must be solenoidal as well.

For $P_S$, let $x \in \mathbb{Z}_2$. By the definition of $S$, for all $k \in \mathbb{N}$, $S^k(x) \underset{1}{\equiv} x_k$. Thus $P_S(x) = x_0x_1x_2 \ldots = x$ and therefore $P_S = \mathcal{I}$. Since $\mathcal{I}$ is clearly solenoidal, $P_S$ is as well.

Let $v_i = 1 - x_i$ for all $i \in \mathbb{N}$. Note that the "dual" shift map $V \circ S$ is induced by the function $\{(00, 1), (01, 0), (10, 1), (11, 0)\}$, so $V \circ S(x) = v_1v_2v_3 \ldots$. Similarly, $(V \circ S)^2(x) = x_2x_3x_4 \ldots$. Continuing this pattern, it follows by induction that

$$(V \circ S)^n(x) = \begin{cases} x_nx_{n+1}x_{n+2}\ldots & \text{if } n \text{ is even} \\ v_nv_{n+1}v_{n+2}\ldots & \text{if } n \text{ is odd.} \end{cases}$$

Taking the $V \circ S$-orbit of $x$ mod 2, we obtain $P_{V\circ S}(x) = x_0v_1x_2v_3x_4v_5 \ldots$. This implies that the first $k$ digits of $P_{V\circ S}(x)$ are entirely determined by the first $k$ digits of $x$ and vice versa, and thus $P_{V\circ S}$ is solenoidal.

We now know that the parity vector functions of $D$, $V \circ D$, $S$, and $V \circ S$ are solenoidal. To show that these are the only ones, we first eliminate all endomorphisms induced by a map $f : B_1 \to \{0, 1\}$. Clearly $P_V$ and $P_{\mathcal{I}}$ are not solenoidal, since $P_V(x)$ is either $\overline{10}$ or $\overline{01}$ for all $x$ by the definition of $V$, and $P_{\mathcal{I}}(x)$ is either $\overline{1}$ or $\overline{0}$ for all $x$ by the definition of $\mathcal{I}$. The trivial maps induced by $\{(0, 0), (1, 0)\}$ and $\{(0, 1), (1, 1)\}$ map everything to $\overline{0}$ and $\overline{1}$ respectively, and thus their parity vector functions are not solenoidal.

We now examine endomorphisms induced by $f : B_2 \to \{0, 1\}$. There are sixteen such maps, four of which are equivalent to the endomorphisms induced by a map $f : B_1 \to \{0, 1\}$. For example, if $s = \{(00, 0), (01, 0), (10, 1), (11, 1)\}$, then $f_\infty = \mathcal{I}$ since the second digit is irrelevant. Another four are $D$, $V \circ D$, $S$, and $V \circ S$. The remaining eight maps are induced by a function which sends three of 00, 01, 10, 11 to 0 and the other to 1 or vice versa. Consider as an illustrative case $s = \{(00, 1), (01, 1), (10, 1), (11, 0)\}$. In this case, $f_\infty$ never maps an even 2-adic integer to an even 2-adic integer, since whether $x_0x_1$ is 00 or 01, $f_\infty(x)$ begins with 1. Thus $P_{f_\infty}(x)$ cannot have 00 as its first two digits, and it is not solenoidal. The other seven cases are similar.

Finally, we show by induction that for any $n \geq 1$ and any $f : B_n \to \{0, 1\}$, either $f_\infty \in \{D, V \circ D, S, V \circ S\}$ or $P_{f_\infty}$ is not solenoidal. The base cases $n = 1$ and $n = 2$ are done above.

Let $n \geq 2$, assume the assertion is true for $n$, and let $f : B_{n+1} \to \{0, 1\}$. We consider two cases.

*Case* 1: Suppose that for all $b = b_0b_1 \ldots b_n$ and $c = c_0c_1 \ldots c_n \in B_{n+1}$, $s(b) = s(c)$ whenever $b \underset{n}{\equiv} c$. Then $f_\infty = t_\infty$ where $t : B_n \to \{0, 1\}$ is defined by $t(b_0b_1 \ldots b_{n-1}) = s(b_0b_1 \ldots b_{n-1}0) = s(b_0b_1 \ldots b_{n-1}1)$. By the inductive hypothesis, either $t_\infty$ is a member of $\{D, V \circ D, S, V \circ S\}$ or $P_{t_\infty}$ is not solenoidal, and we are done.

*Case* 2: Suppose that for some $b_0b_1 \ldots b_{n-1} \in B_n$, the digits $s(b_0b_1 \ldots b_{n-1}0)$ and $s(b_0b_1 \ldots b_{n-1}1)$ are distinct. Let $x, y \in \mathbb{Z}_2$ be such that $x \underset{n+1}{\equiv} b_0b_1 \ldots b_{n-1}0$ and $y \underset{n+1}{\equiv} b_0b_1 \ldots b_{n-1}1$. Then $f_\infty(x) \underset{1}{\not\equiv} f_\infty(y)$, and thus $P_{f_\infty}(x) \underset{2}{\not\equiv} P_{f_\infty}(y)$. Also, since $n \geq 2$, we have $x \underset{2}{\equiv} y$. Hence, $P_{f_\infty}$ does not induce a permutation on $\mathbb{Z}_2/2^2\mathbb{Z}_2$, so $P_{f_\infty}$ is not solenoidal.

This completes the induction, and we conclude that $D$, $V \circ D$, $S$, and $V \circ S$ are the only endomorphisms of $S$ with solenoidal parity vector functions. ∎

## 4. Dynamics of *D*

Let us consider the implications of Theorem 3.3 and Corollary 3.4. The map $D$, although defined as a specific endomorphism of $S$, is actually conjugate to $S$ when viewed as a dynamical system on its own. In addition, $D$ is special in that only $D$, $S$ itself, and their duals $V \circ D$ and $V \circ S$ have solenoidal parity vector functions. This provides incentive to further investigate the dynamical system $D : \mathbb{Z}_2 \to \mathbb{Z}_2$.

To begin our investigation of the dynamics of $D$, we observe some properties of the function itself.

**Lemma 4.1.** *Let $x \in \mathbb{Z}_2$ and $y = D(x)$. Then for any $i \in \mathbb{N}$, $y_i = |x_i - x_{i+1}|$, $x_{i+1} = |x_i - y_i|$, and $x_i = |x_{i+1} - y_i|$.*

**Proof.** Let $i \in \mathbb{N}$. There are four cases to consider: $x_ix_{i+1} = 00, 01, 10,$ or $11$.

*Case* 1: Suppose $x_ix_{i+1} = 01$. By the definition of $D$, $y_i = |x_i - x_{i+1}| = 1$. Also, $x_{i+1} = 1 = |0 - 1| = |x_i - y_i|$ and $x_i = 0 = |1 - 1| = |x_{i+1} - y_i|$.

The remaining three cases are similar. ∎

The symmetry of $D$ revealed by Lemma 4.1 implies a surprising and beautiful symmetry of the function $\mathcal{P}$, the $D$-parity vector function.

**Theorem 4.2.** *$\mathcal{P}^2 = \mathcal{I}$. Equivalently, $\mathcal{P} = \mathcal{P}^{-1}$.*

**Proof.** Let $x \in \mathbb{Z}_2$, and let $A$ be the infinite matrix defined as follows. For all $i, j \in \mathbb{N}$, $A[i, j]$ is $a_j$ where $a = D^i(x)$, i.e. the $i + 1$st row of $A$ consists of the digits of $D^i(x)$. Note that the leftmost column of $A$ (with $j = 0$) consists of the digits of $\mathcal{P}(x)$.

By Lemma 4.1, we see that for all $i, j \in \mathbb{N}$, $A[i, j + 1] = |A[i, j] - A[i + 1, j]|$. Let $j \in \mathbb{N}$. Define $d_i = A[i, j]$ and $e_i = A[i, j + 1]$for all $i$. Then for all $i \in \mathbb{N}$, $e_i = |d_i - d_{i+1}|$, so by the definition of $D$, $D(d_0d_1d_2 \ldots) = e_0e_1e_2 \ldots$. Thus the 2-adic integer formed by the entries of the $j + 1$st column in $A$ is $D$ of the 2-adic integer formed by the $j$th column for any $j$. This implies that for all $j \in \mathbb{N}$, the digits of $D^j(\mathcal{P}(x))$ are the entries of the $j + 1$st column of $A$, so $D^j(\mathcal{P}(x)) \underset{1}{\equiv} A[0, j] = x_j$. By the definition of $\mathcal{P}$, $\mathcal{P}(\mathcal{P}(x)) = x_0x_1x_2 \ldots = x$. We conclude that $\mathcal{P}^2 = \mathcal{I}$. ∎

Theorem 4.2 shows, remarkably, that the $D$-parity vector of the $D$-parity vector of a 2-adic integer is itself. In other words, $\mathcal{P}$ is an involution.

It is well-known that any function $h : X \to Y$ induces an equivalence relation $\approx$ on $X$ defined by $x \approx y$ if and only if $h(x) = h(y)$. This equivalence relation in turn induces a quotient set $Q_h$ of equivalence classes mod $\approx$. Consider the quotient set $Q_D$ induced by $D$. Due to the symmetry of $D$ shown in Lemma 4.1, we have the following:

**Theorem 4.3.** $Q_D = \{\{x, V(x)\} \mid x \in \mathbb{Z}_2\}$.

**Proof.** Let $x, y \in \mathbb{Z}_2$ and $v = V(x)$. Assume $y = D(x)$. By Lemma 4.1,

$$x_{i+1} = |x_i - y_i| \tag{4.1}$$

for all $i \geq 0$. If $x_0 = 0$, Eq. (4.1) is a recursion for the sequence $x_0, x_1, x_2, \ldots$ and thus there is exactly one even $x$ such that $D(x) = y$. Similarly, there is exactly one odd $x$ such that $D(x) = y$. Therefore, each class in the quotient set induced by $D$ has two elements, one even and one odd. By the definition of $V$, $v_i = 1 - x_i$ for all $i$. Thus for all $i$, $|v_i - v_{i+1}| = |(1 - x_i) - (1 - x_{i+1})| = |x_i - x_{i+1}| = y_i$ and so $D(V(x)) = y = D(x)$. We conclude that each equivalence class mod $\approx$ consists of two elements, $x$ and $V(x)$. ∎

*4.1. Periodic points*

It is desirable to classify the fixed points and periodic points of any dynamical system. There are exactly two fixed points of $S$, namely $\overline{0}$ and $\overline{1}$. Since $D$ is conjugate to $S$ there are exactly two fixed points of $D$, namely $\overline{0}$ and $1\overline{0}$. To classify the remaining periodic points of $D$, we introduce some new notation.

**Definition 5.** Let $x$ be a 2-adic integer with an eventually repeating binary representation $x_0x_1 \ldots x_{t-1}\overline{x_t x_{t+1} \ldots x_{t+m-1}}$. Then $x$ is in **reduced form** if and only if $x_{t-1} \neq x_{t+m-1}$ and $m$ is the least integer such that $x$ can be expressed in this form. For any $x$ having reduced form $x_0x_1 \ldots x_{t-1}\overline{x_t x_{t+1} \ldots x_{t+m-1}}$, we define the $S$-**period length** $\|x\| = m$ and the $S$-**preperiod length** $\underline{x} = t$.

Note that $x$ is cyclic for $S$ if and only if $\underline{x} = 0$.

**Definition 6.** An eventually repeating 2-adic integer that has reduced form

$$x_0x_1 \ldots x_{t-1}\overline{x_t x_{t+1} \ldots x_{t+m-1}}$$

is **half-flipped** if and only if $m$ is even and for all $i \geq t$, $x_i = 1 - x_{i+m/2}$.

For instance, the 2-adic integers $\overline{1100}$ and $010\overline{110100}$ are half-flipped.

In order to avoid confusion between 2-adic integers which are periodic (or eventually periodic) points of $D$ and those having repeating (or eventually repeating) binary representation, we will refer to the former as $D$ **-periodic** (or **eventually** $D$-**periodic**) and the latter as **repeating** or **eventually repeating**. Note that $x$ has an eventually periodic $S$-orbit if and only if $x$ is eventually repeating. It is much less obvious which 2-adic integers have an eventually periodic $D$-orbit, so we prove several lemmas about $D$-orbits to answer this question.

**Lemma 4.4.** *Let $x$ be an eventually repeating 2-adic integer. Then*

$$\|D(x)\| = \begin{cases} \|x\| & \text{if } x \text{ is not half-flipped} \\ \frac{1}{2}\|x\| & \text{if } x \text{ is half-flipped}. \end{cases}$$

**Proof.** Let $m = \|x\|$ and $t = \underline{x}$, with $x = x_0x_1 \ldots x_{t-1}\overline{x_t x_{t+1} \ldots x_{t+m-1}}$ in reduced form. Let $x' = S^t(x) = \overline{x_t x_{t+1} \ldots x_{t+m-1}}$, so that for all $i \in \mathbb{N}$, $x'_i = x'_{m+i}$, i.e. $\|x'\| = \|x\| = m$. Note that since $D$ is an endomorphism of $S$, $S^t(D(x)) = D(S^t(x)) = D(x')$, so $\|D(x)\| = \|S^t(D(x))\| = \|D(x')\|$. We proceed to find $\|D(x')\|$.

Let $y = D(x')$ and $n = \|D(x')\|$. For all $i \in \mathbb{N}$, $y_{m+i} = |x'_{m+i} - x'_{i+m+1}| = |x'_i - x'_{i+1}| = y_i$. Thus $n$ divides $m$.

If $x'$ is half-flipped, then for all $i$, $x'_i = 1 - x_{i+m/2}$, and $y_{i+m/2} = |x'_{i+m/2} - x'_{i+m/2+1}| = |1 - x'_i - (1 - x'_{i+1})| = |x'_i - x'_{i+1}| = y_i$. Therefore

$$x' \text{ is half-flipped} \Rightarrow n \leq \frac{m}{2}. \tag{4.2}$$

Consider the case $x'_0 = 0$. We have two cases: either $x'_{n-1} = y_{n-1}$ or $x'_{n-1} \neq y_{n-1}$.

*Case* 1: Suppose $x'_{n-1} = y_{n-1}$. Then by Lemma 4.1, $x'_n = |x'_{n-1} - y_{n-1}| = 0 = x'_0$. This being our base case, we show by induction that for all $i \in \mathbb{N}$, $x'_{n+i} = x'_i$. Let $j \in \mathbb{N}$ and assume $x'_{n+j} = x'_j$. Then $x'_{n+j+1} = |x'_{n+j} - y_{n+j}| = |x'_j - y_j| = x'_{j+1}$, completing the induction. We now have $m \mid n$ and $n \mid m$, so $n = m$. Thus $\|D(x)\| = \|x\|$. It follows from (4.2) that $x$ is not half-flipped, and the theorem holds in this case.

*Case* 2: Suppose $x'_{n-1} \neq y_{n-1}$. Then by Lemma 4.1, $x'_n = \left|x'_{n-1} - y_{n-1}\right| = 1 = 1 - x'_0$. This being our base case, we show by induction that for all $i \in \mathbb{N}$, $x'_{n+i} = 1 - x'_i$. Let $j \in \mathbb{N}$ and assume $x'_{n+j} = 1 - x'_j$. Then $x'_{n+j+1} = \left|x'_{n+j} - y_{n+j}\right| = \left|1 - x'_j - y_j\right| \neq \left|x_j - y_j\right| = x'_{j+1}$, and therefore $x'_{n+j+1} = 1 - x'_{j+1}$, completing the induction. This implies that $m \neq n$, and since $n \mid m$, we conclude that $n \leq \frac{1}{2}m$. Also, for all $i \in \mathbb{N}$, $x'_{2n+i} = 1 - x'_{n+i} = 1 - (1 - x'_i) = x'_i$. Therefore $m \leq 2n$. Since $n \leq \frac{1}{2}m$ and $\frac{1}{2}m \leq n$, we have $n = \frac{1}{2}m$. Thus $\left\|D(x')\right\| = \frac{1}{2}\|x\|$. Finally, making the substitution $n = \frac{1}{2}m$ we have that for all $i \in \mathbb{N}$, $x'_{m/2+i} = 1 - x'_i$, so $x$ is half-flipped as well.

Hence the theorem holds for $x'_0 = 0$. The proof for the case $x'_0 = 1$ is analogous. ■

**Lemma 4.5.** *Let $x$ be an eventually repeating 2-adic integer. Then for all $k \in \mathbb{N}$, $\underline{D^k(x)} = \underline{x}$.*

**Proof.** Let $y = D(x)$, $m = \|x\|$, and $t = \underline{x}$, so that

$$x = x_0x_1 \dots x_{t-1}\overline{x_tx_{t+1} \dots x_{t+m-1}}$$

in reduced form. Then $D(x) = y_0y_1 \dots y_{t-1}\overline{y_ty_{t+1} \dots y_{t+m-1}}$, but not necessarily in reduced form. We consider two cases: either $x$ is half-flipped or $x$ is not half-flipped.

*Case* 1: Suppose $x$ is not half-flipped. By Lemma 4.4, $\|D(x)\| = m$. Also, by the definition of $t$, $x_{t-1} \neq x_{t+m-1}$. Thus

$$y_{t-1} = |x_{t-1} - x_t| \neq |x_{t+m-1} - x_{t+m}| = y_{t+m-1}$$

so $y_0y_1 \dots y_{t-1}\overline{y_ty_{t+1} \dots y_{t+m-1}}$ is in reduced form. We conclude that $\underline{D(x)} = t = \underline{x}$.

*Case* 2: Suppose $x$ is half-flipped. By Lemma 4.4, $\|D(x)\| = \frac{1}{2}m$. It follows that $D(x) = y_0y_1 \dots y_{t-1}\overline{y_ty_{t+1} \dots y_{t+m/2-1}}$. By the definition of half-flipped and $t$, $x_{t+m/2-1} = 1 - x_{t+m-1} = x_{t-1}$ and $x_{t+m/2} = 1 - x_t$. Therefore

$$\begin{aligned} y_{t-1} &= |x_{t-1} - x_t| \\ &= \left|x_{t+m/2-1} - (1 - x_{t+m/2})\right| \\ &\neq \left|x_{t+m/2-1} - x_{t+m/2}\right| \\ &= y_{t+m/2-1} \end{aligned}$$

so $y_0y_1 \dots y_{t-1}\overline{y_ty_{t+1} \dots y_{t+m/2-1}}$ is in reduced form. We conclude that $\underline{D(x)} = t = \underline{x}$.

Therefore, $\underline{D(x)} = \underline{x}$ for all $x \in \mathbb{Z}_2$. It follows by induction that for all $k \in \mathbb{N}$, $\underline{D^k(x)} = \underline{x}$. ■

We are now ready to classify all 2-adic integers which are eventually $D$-periodic.

**Theorem 4.6.** *Let $x \in \mathbb{Z}_2$. Then $x$ is eventually $D$-periodic if and only if it is eventually $S$-periodic, i.e. its 2-adic binary representation is eventually repeating.*

**Proof.** Assume that the 2-adic binary representation of $x$ is eventually repeating (so that $x$ is eventually $S$-periodic), with $x = x_0x_1 \dots x_{t-1}\overline{x_tx_{t+1} \dots x_{t+m-1}}$ where $t = \underline{x}$ and $m = \|x\|$. Let $a$ be the greatest odd divisor of $m$, with $m = a \cdot 2^b$. Lemma 4.4 implies that for any $k, n \in \mathbb{N}$ with $k < n$, $\left\|D^k(x)\right\| = a \cdot 2^{b'}$ and $\|D^n(x)\| = a \cdot 2^{b''}$ for some $b', b'' \in \mathbb{N}$ with $b \geq b' \geq b''$. Hence the sequence $\left\{\log_2\left(\frac{1}{a}\left\|D^k(x)\right\|\right)\right\}_{k=0}^{\infty}$ is a non-increasing sequence of nonnegative integers, and thus is eventually constant. Let $\beta$ be the minimum value of $\log_2\left(\frac{1}{a}\left\|D^k(x)\right\|\right)$ over all $k$, so that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\|D^n(x)\| = a \cdot 2^\beta$. Define $c = a \cdot 2^\beta$. For all $n \geq N$, there are at most $2^c$ possibilities for the repeating digits of $D^n(x)$, and by Lemma 4.5, there are at most $2^{\underline{x}}$ possibilities for the first $\underline{x}$ digits of $D^n(x)$. Thus there are at most $2^c \cdot 2^{\underline{x}} = 2^{c+\underline{x}}$ possibilities for the values of $D^n(x)$ for all $n \geq N$. By the pigeonhole principle, two of $D^N(x), D^{N+1}(x), \dots, D^{N+2^{c+\underline{x}}}(x)$ are equal, and thus the $D$-orbit of $x$ is eventually periodic. So if $x$ is eventually repeating then $x$ is eventually $D$-periodic.

Now assume that the 2-adic representation of $x$ is not eventually repeating, and assume to the contrary that $x$ is eventually $D$-periodic. Then $\mathcal{P}(x)$ is eventually repeating. So the $D$-orbit of $\mathcal{P}(x)$ is eventually periodic, and thus $\mathcal{P}(\mathcal{P}(x))$ is eventually repeating as well. But Theorem 4.2 implies $\mathcal{P}(\mathcal{P}(x)) = x$, and $x$ is not eventually repeating by assumption. This contradiction completes the proof. ■

Note that Theorem 4.6 is not a consequence of $D$ being conjugate to $S$, for $D = \mathcal{P}S\mathcal{P}^{-1} = \mathcal{P}S\mathcal{P}$ implies that $x$ is eventually periodic for $D$ if and only if $\mathcal{P}(x)$ is eventually periodic for $S$.

In the proof of Theorem 4.6, we found that the $S$-period length of elements in the $D$-orbit of $x$ is either divided by 2 or remains constant with each iteration, until the orbit becomes periodic and the $S$-period length $\|x\|$ stabilizes. However, the value of $\|x\|$ at which it stabilizes may be even. For example, $x = \overline{100111}$ has the periodic $D$-orbit $\overline{100111}, \overline{101000}, \overline{111001}, \overline{001010}, \overline{011110}, \overline{100010}, \dots$.

### 4.2. Eventually fixed points

We now classify those 2-adic integers whose $D$-orbit contains a fixed point (0 or 1).

**Lemma 4.7.** *Let $n \in \mathbb{N}$ and $a = a_0a_1a_2 \ldots a_{2^n-1} \in B_{2^n}$. Then*

$$d^{2^n-1}(a) = \left(\sum_{i=0}^{2^n-1} a_i\right) \bmod 2$$

$$\text{i.e. } d^{2^n-1}(a) = \begin{cases} 0 & \text{if } a \text{ contains an even number of 1's among its digits} \\ 1 & \text{otherwise.} \end{cases}$$

**Proof.** We proceed by induction on $n$. The base case, $n = 0$, is trivial since $d^{2^0-1}(1) = d^0(1) = 1$ and $d^{2^0-1}(0) = d^0(0) = 0$.

Let $n \in \mathbb{N}$ and assume the assertion is true for $n$. Let $a_0a_1a_2 \ldots a_{2^{n+1}-1} \in B_{2^{n+1}}$, and let $b = a_0a_1a_2 \ldots a_{2^n-1}$ and $c = a_{2^n}a_{2^n+1} \ldots a_{2^{n+1}-1}$ be the first and second halves of $a$. We now consider two cases.

*Case* 1: Suppose $\sum_{i=0}^{2^{n+1}-1} a_i \underset{1}{\equiv} 0$, i.e. $a$ has an even number of 1's among its digits. We have

$$\left(\sum_{i=0}^{2^n-1} a_i\right) + \left(\sum_{i=2^n}^{2^{n+1}-1} a_i\right) = \sum_{i=0}^{2^{n+1}-1} a_i \underset{1}{\equiv} 0$$

and therefore $\sum_{i=0}^{2^n-1} a_i \underset{1}{\equiv} \sum_{i=2^n}^{2^{n+1}-1} a_i$. By the inductive hypothesis, $d^{2^n-1}(b) = d^{2^n-1}(c)$. Let $z_0z_1 \ldots z_{2^n}$ be the digits of $d^{2^n-1}(a)$. Note that $z_0 = d^{2^n-1}(b)$ and $z_{2^n} = d^{2^n-1}(c)$, so $z_0 = z_{2^n}$. Now, consider all subsequences of $z_0z_1 \ldots z_{2^n}$ of length 2. Such a subsequence $z_iz_{i+1}$ is a **switch** if $z_i \neq z_{i+1}$. Clearly, the first and last digit will match if and only if there are an even number of switches, so in this case there are an even number of switches in $z_0z_1 \ldots z_{2^n}$. Since each 1 in $d(z_0z_1 \ldots z_{2^n})$ corresponds to a switch in $z_0z_1 \ldots z_{2^n}$, there are an even number of 1's among the digits of $d(z_0z_1 \ldots z_{2^n})$. By the definition of $d$, $d(z_0z_1 \ldots z_{2^n}) \in B_{2^n}$. Using the inductive hypothesis a second time, we have

$$d^{2^{n+1}-1}(a) = d^{2^n-1}(d(d^{2^n-1}(a))) = d^{2^n-1}(d(z_0z_1 \ldots z_{2^n})) = 0$$

and the induction is complete.

*Case* 2: Suppose $\sum_{i=0}^{2^{n+1}-1} a_i \underset{1}{\equiv} 1$. By an argument similar to that of Case 1, we have $d^{2^{n+1}-1}(a) = 1$ and the induction is complete. ■

**Theorem 4.8.** *Let $n \in \mathbb{N}$ and $a = a_0a_1a_2 \ldots a_{2^n} \in B_{2^n+1}$. Then $d^{2^n}(a) = d(a_0a_{2^n})$.*

**Proof.** Suppose $d(a)$ has an even number of 1's among its digits. As in the proof of Lemma 4.7, we know there are an even number of switches in $a$, so $a_0 = a_{2^n}$. But by Lemma 4.7, $d^{2^n-1}(d(a)) = 0 = d(a_0a_{2^n})$. Similarly, if $d(a)$ has an odd number of 1's among its digits then $a_0 \neq a_{2^n}$, and $d^{2^n-1}(d(a)) = 1 = d(a_0a_{2^n})$. Thus in all cases $d^{2^n}(a) = d(a_0a_{2^n})$. ■

Theorem 4.8 gives us an easy method for computing large iterations of $D$ without computing each individual iteration. For example, if we wish to compute $D^8(x_0x_1x_2 \ldots)$, we merely compute $d(x_0x_8)$, $d(x_1x_9)$, etc., which yields the digits of $D^8(x_0x_1x_2 \ldots)$ in one step rather than eight. This technique is also of use in the proof of the following theorem, which classifies the 2-adic integers whose $D$-orbit is eventually fixed.

**Theorem 4.9.** *The $D$-orbit of $x$ is eventually fixed if and only if the reduced form of $x$ is either $\overline{x_0x_1 \ldots x_{2^n-1}}$ (in which case it eventually maps to 0) or $x_0\overline{x_1x_2 \ldots x_{2^n}}$ (which eventually maps to 1) for some $n \in \mathbb{N}$.*

**Proof.** We first show that the $D$-orbit of $x$ contains 0 if and only if the reduced form of $x$ is $\overline{x_0x_1 \ldots x_{2^n-1}}$ for some $n \in \mathbb{N}$. Assume the $D$-orbit of $x$ eventually contains 0. Since $\|0\| = 1$, we know by Lemma 4.4 that $\|x\| = 1 \cdot 2^n$ for some $n \in \mathbb{N}$.

Now, assume to the contrary that $\underline{x} \neq 0$. Then by Lemma 4.5, for all $k \in \mathbb{N}$, $\underline{D^k(x)} = \underline{x} > 0$. However, $\underline{0} = 0$, so the $D$-orbit of $x$ cannot eventually contain 0. We conclude that our assumption was false and $\underline{x} = 0$. Thus the reduced form of $x$ is $\overline{x_0x_1 \ldots x_{2^n-1}}$ for some $n \in \mathbb{N}$.

Assume $x = \overline{x_0x_1 \ldots x_{2^n-1}}$ in reduced form. Let $y = D^{2^n}(x)$. By Theorem 4.8 and Lemma 3.5, we have for all $i \in \mathbb{N}$, $y_i = d(x_ix_{i+2^n})$. Since $x_i = x_{i+2^n}$, $y_i = 0$ for all $i$ and thus $D^{2^n}(x) = \overline{0} = 0$.

We now show that the $D$-orbit of $x$ contains 1 if and only if the reduced form of $x$ is $x_0\overline{x_1x_2 \ldots x_{2^n}}$ for some $n \in \mathbb{N}$. Since $D$ is an endomorphism of $S$, we have $S(D^j(x)) = D^j(S(x))$ for all $j \in \mathbb{N}$. Assume $D^k(x) = 1$ for some $k \in \mathbb{N}$. Then $D^k(S(x)) = S(D^k(x)) = S(1) = 0$. By the above argument, $S(x) = \overline{x_1x_2 \ldots x_{2^n}}$ in reduced form for some $n \in \mathbb{N}$. By the definition of $S$, $x$ either has reduced form $\overline{x_0x_1x_2 \ldots x_{2^n-1}}$ or $x_0\overline{x_1x_2 \ldots x_{2^n}}$. By Lemma 4.5, $\underline{x} = \underline{D^k(x)} = \underline{1} = 1$, so $x = x_0\overline{x_1x_2 \ldots x_{2^n}}$ in reduced form.

Assume $x = x_0\overline{x_1x_2 \ldots x_{2^n}}$ in reduced form. By the above argument, $D^{2^n}(S(x)) = D^{2^n}(\overline{x_1x_2 \ldots x_{2^n}}) = 0$. Therefore $S(D^{2^n}(x)) = 0$ as well, so $D^{2^n}(x)$ is either 0 or 1 by the definition of $S$. By Lemma 4.5, $\underline{D^{2^n}(x)} = \underline{x} = 1$, so $D^{2^n}(x) = 1$. ■

### 4.3. The D-Orbit of an Integer

Any nonnegative integer is eventually repeating (ending in $\overline{0}$), so all nonnegative integers are eventually $D$-periodic by Theorem 4.6. Surprisingly, they all are purely periodic points of $D$ with minimum period $2^n$ for some $n \in \mathbb{N}$, as we now show.

**Theorem 4.10.** *Let $x$ be a nonnegative integer. Then $x$ is a purely periodic point of $D$ with minimum period $2^n$ being the smallest power of 2 that is at least as large as the $S$-preperiod length of $x$, i.e. $2^n \geq \underline{x}$.*

**Proof.** Let $t = \underline{x}$. By Lemma 4.5, for any $i \in \mathbb{N}$, $\underline{D^i(x)} = t$ as well. Thus for all $i \in \mathbb{N}$, $2^{t-1} \leq D^i(x) < 2^t$ by the definition of 2-adic integer.

Let $x_0x_1x_2 \ldots x_{t-1}\overline{0}$ be the 2-adic expansion of $x$, and $y_0y_1 \ldots y_{t-1}\overline{0}$ the 2-adic expansion of $D^{2^n}(x)$. Then by Theorem 4.8, we have that for all $i \in \mathbb{N}$, $y_i = d^{2^n}(x_ix_{i+1} \ldots x_{i+2^n}) = d(x_ix_{i+2^n}) = d(x_i0) = x_i$. Thus $D^{2^n}(x) = x$, and $x$ is $D$-periodic with minimum period dividing $2^n$. Note that if $x$ is 0 or 1, $2^n = 1$, so $2^n$ must be the minimum period of $x$ in both of these cases.

Assume that $x > 1$ and the minimum $D$-period of $x$ is less than $2^n$. Since it divides $2^n$ it must be $2^k$ for some $k \leq n-1$. Also, since $n$ is the smallest natural number such that $2^n \geq t$, we have $2^{n-1} < t$, and thus $2^k < t$ as well. Let $z_0z_1 \ldots z_{t-1}\overline{0}$ be the 2-adic expansion of $D^{2^k}(x)$. Since $t - 2^k - 1 \geq 0$, we have $z_{t-2^k-1} = d^{2^k}(x_{t-2^k-1}x_{t-2^k} \ldots x_{t-1}) = d(x_{t-2^k-1}x_{t-1}) = d(x_{t-2^k-1}1) \neq x_{t-2^k-1}$. Therefore $D^{2^k}(x) \neq x$, and $x$ is not $D$-periodic with minimum period $2^k$. We conclude that the assumption was incorrect and thus $2^n$ is the minimum period of $x$. ■

Negative integers have a 2-adic expansion ending in $\overline{1}$. This is because for any $x \in \mathbb{Z}_2$, $-1-x = V(x)$ by binary arithmetic, so $-x = V(x) + 1$. Therefore, if $x$ is a positive integer, $-x$ is one more than $V(x)$, which ends in $\overline{1}$. Notice that $D$ of a negative integer is a positive integer, so by Theorem 4.10, the $D$-orbit of a negative integer enters a cycle of positive integers after one iteration.

These facts are consistent with the duality of $\mathcal{P}$ seen in Theorem 4.2. Given a 2-adic integer $x$ whose reduced form is $\overline{x_0x_1 \ldots x_{2^n-1}}$ or $x_0\overline{x_1x_2 \ldots x_{2^n}}$, we have by Theorem 4.9 that $\mathcal{P}(x)$ is an integer. Also, given a 2-adic integer $x$ which is also an integer, we have by Theorem 4.10 that $\mathcal{P}(x)$ has reduced form $\overline{x_0x_1 \ldots x_{2^n-1}}$ or $x_0\overline{x_1x_2 \ldots x_{2^n}}$.

## 5. Applications to the 3*x* + 1 conjecture

Recall that the $3x + 1$ conjecture states that the $T$-orbit of any positive integer contains 1, or equivalently, eventually enters the $\overline{1, 2}$ cycle.

Corollary 3.4 states that $\mathcal{P}$ is a conjugacy from $D$ to $S$. Also, as stated in the introduction, $\Phi$ is a conjugacy from $S$ to $T$. Since the composition of conjugacies is a conjugacy, this implies that $D$, the endomorphism of $S$ resembling a discrete derivative, is conjugate to $T$, the famous $3x + 1$ function.

**Theorem 5.1.** *The map $R = \Phi \circ \mathcal{P}$ is a conjugacy from $D$ to $T$.*

Thus $T$ and $D$ have the same dynamics, and hence to solve the $3x + 1$ conjecture it suffices to have an understanding of the dynamics of $D$ and the correspondence $R$ between the orbits of $D$ and those of $T$.

Having studied the dynamics of $D$ in Section 4, we turn our attention to understanding the correspondence $R$. Since $\overline{1, 2}$ and $\overline{2, 1}$ are the unique 2-cycles of the dynamical system $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ and $\overline{3, 2}$ and $\overline{2, 3}$ are 2-cycles of $D : \mathbb{Z}_2 \to \mathbb{Z}_2$, these 2-cycles of $D$ must be unique. Thus, since $R$ preserves parity, $R(3) = 1$ and $R(2) = 2$. Similarly, $R(0) = 0$ and $R(1) = -1$ since they are fixed points of corresponding parity of the two dynamical systems.

By an argument similar to the proof of Theorem 4.9, the $D$-orbit of a 2-adic integer $x$ eventually enters the $\overline{3, 2}$ cycle (or, equivalently, the $\overline{2, 3}$ cycle) if and only if $x$ has reduced form $x_0x_1\overline{x_2x_3 \ldots x_{2^n+1}}$ for some $n \in \mathbb{N}$. However, since an element $x$ in the dynamical system $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ eventually enters the $\overline{1, 2}$ cycle if and only if the $D$-orbit of $R^{-1}(x)$ eventually enters the $\overline{3, 2}$ cycle, we have the following equivalence theorem.

**Theorem 5.2.** *The following statements are equivalent:*

(1) *The $3x + 1$ conjecture is true.*

(2) *For all positive integers $m$, $R^{-1}(m)$ has reduced form $x_0x_1\overline{x_2x_3 \ldots x_{2^n+1}}$ for some $n \in \mathbb{N}$.*

Thus it suffices to determine $R^{-1}$ on positive integers in order to solve the $3x+1$ conjecture. In particular, it would suffice to find a tractable formula for $R^{-1}(m)$ for positive integers $m$.

There is yet another way that $D$ can be of use in solving the $3x + 1$ conjecture, and that is in its role as an endomorphism of the shift map.

Recall that Monks and Yazinski [6] defined $\Omega = \Phi \circ V \circ \Phi^{-1}$, and showed that $\Omega$ is the unique nontrivial continuous autoconjugacy of $T$ and that $\Omega^2 = \mathcal{I}$. They also defined an equivalence relation $\sim$ on $\mathbb{Z}_2$ by $x \sim y \Leftrightarrow (x = y \text{ or } x = \Omega(y))$. This induces a set of equivalence classes $\mathbb{Z}_2/\sim\, = \{\{x, \Omega(x)\} \mid x \in \mathbb{Z}_2\}$, and note that each equivalence class in $\mathbb{Z}_2/\sim$ consists of two elements of opposite parity. This enables one to define a parity-neutral map $\Psi$ as follows.

**Definition 7.** The **parity-neutral** $3x + 1$ **map** $\Psi : \mathbb{Z}_2/\sim\, \to \mathbb{Z}_2/\sim$ is the map given by $\Psi(\{x, \Omega(x)\}) = \{T(x), \Omega(T(x))\}$.

Monks and Yazinski also showed that the $3x+1$ conjecture is equivalent to the claim that the $\Psi$-orbit of any $X \in \mathbb{Z}_2/\sim$ contains $\{1, 2\}$.

Making use of the endomorphism $D$, the following theorem improves upon this result.

**Theorem 5.3.** *The dynamical system* $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ *is conjugate to* $\Psi : \mathbb{Z}_2/\sim \to \mathbb{Z}_2/\sim$.

**Proof.** Define $H = \Phi \circ D \circ \Phi^{-1}$. Since $D$ is an endomorphism of $S$ and $\Phi$ is a conjugacy from $S$ to $T$, $H$ is an endomorphism of $T$. Recall that $H$ induces the quotient set $Q_H$ discussed in Section 4. We now show that $Q_H = \mathbb{Z}_2/\sim$. By Theorem 4.3, $D \circ V = D$, so

$$\begin{aligned} H \circ \Omega &= (\Phi \circ D \circ \Phi^{-1}) \circ (\Phi \circ V \circ \Phi^{-1}) \\ &= \Phi \circ D \circ V \circ \Phi^{-1} \\ &= \Phi \circ D \circ \Phi^{-1} \\ &= H. \end{aligned}$$

Thus for all $x \in \mathbb{Z}_2$, $H(x) = H(\Omega(x))$, so $\{x, \Omega(x)\}$ is a subset of the equivalence class of $x$ in $Q_H$.

To see that these are the only elements in the equivalence class of $x$, let $y \in \mathbb{Z}_2$ and assume $y \neq x$ and $H(y) = H(x)$. Then $\Phi\left(D\left(\Phi^{-1}(x)\right)\right) = \Phi(D(\Phi^{-1}(y)))$, and since $\Phi$ and $\Phi^{-1}$ are bijections, $\Phi^{-1}(x) \neq \Phi^{-1}(y)$ and $D(\Phi^{-1}(x)) = D(\Phi^{-1}(y))$. Therefore $\Phi^{-1}(x) = V(\Phi^{-1}(y))$ by Theorem 4.3. Thus $x = \Phi \circ V \circ \Phi^{-1}(y) = \Omega(y)$. Therefore, $Q_H = \mathbb{Z}_2/\sim$.

Now define $G : \mathbb{Z}_2/\sim \to \mathbb{Z}_2$ by $G(\{x, \Omega(x)\}) = H(x) = H(\Omega(x))$. By the definition of $Q_H$, $G$ is injective. Also, since $D$ is surjective and $\Phi$ and $\Phi^{-1}$ are bijective, $H$ is surjective as well, and therefore $G$ is surjective. Thus $G$ is a bijection. Finally, for any $x \in \mathbb{Z}_2$,

$$\begin{aligned} G(\Psi(\{x, \Omega(x)\})) &= G(\{T(x), T(\Omega(x))\}) \\ &= G(\{T(x), \Omega(T(x))\}) \\ &= H(T(x)) \\ &= T(H(x)) \\ &= T(G(\{x, \Omega(x)\})) \end{aligned}$$

and therefore $G \circ \Psi = T \circ G$.O So $G$ is a conjugacy from $\Psi$ to $T$. ∎

This theorem is fascinating, for it proves that the parity-neutral function $\Psi$ is conjugate to, and thus has the same dynamical structure as, the function $T$ defined piecewise on even and odd 2-adic integers.

## 6. Conclusion

We have discovered an interesting finite subset of the set of all continuous endomorphisms of $S$ in that $D$, $V \circ D$, $S$, and $V \circ S$ are the only such maps whose parity vector functions are solenoidal. In addition, each of these four maps are conjugate to $S$ when viewed as dynamical systems on $\mathbb{Z}_2$, and we have seen that the "discrete derivative" $D$ has fascinating dynamics. In particular, we have proven that $x$ is eventually $D$-periodic if and only if it is eventually repeating, and have classified all eventually fixed points (Theorem 4.9) and the $D$-orbits of integers (Theorem 4.10) as well. We have observed that $D$ exhibits remarkable symmetry in that $Q_D = \{\{x, V(x)\} \mid x \in \mathbb{Z}_2\}$ and that $\mathcal{P}$ is an involution. Given that $D$ has such rich structure, it would be of interest to study the dynamics of other continuous endomorphisms of $S$ and their applications as an area of future research.

We have also seen that the map $D$ has applications to other branches of mathematics. Using Lagarias's result that $S$ is conjugate to $T$, we have demonstrated that $D$ is conjugate to $T$ via $R$, and thus that to prove the $3x+1$ conjecture, it suffices to show that for all positive integers $m$, $R^{-1}(m)$ has reduced form $x_0x_1\overline{x_2x_3\ldots x_{2^n+1}}$ for some $n \in \mathbb{N}$. Using $D$, we have also constructed a conjugacy $G$ between $T$ and the parity-neutral function $\Psi$. Hence, our results open the door to future research on the conjugacies $R$ and $G$, motivated by the possibility of making progress on the $3x+1$ conjecture.


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