18.175 Theory of Probability
Fall 2008

18.175. Pset 1.

1. Let $\{X_i\}_{i\geq 1}$ be i.i.d. If $S_n/n \to 0$ a.s., show that $E|X_1| < \infty$ and, therefore, $EX_1 = 0$.

2. If $u : [0,1]^k \to \mathbb{R}$ is continuous then show that

$$\sum_{0\leq j_1,\ldots,j_k\leq n} u\Bigl(\frac{j_1}{n},\ldots,\frac{j_k}{n}\Bigr)\prod_{i\leq k}\binom{n}{j_i}x_i^{j_i}(1-x_i)^{n-j_i} \to u(x_1,\ldots,x_k)$$

uniformly on $[0,1]^k$.

3. $\mu_n = 1/(n+1)$ for $n \geq 0$ is a sequence of moments of a distribution on $[0,1]$. Find

$$p_k^{(n)} = \binom{n}{k}(-1)^{n-k}\Delta^{n-k}\mu_k,\;\; k = 0,\ldots,n$$

where Δ is the difference operator.

4. Let $\{X_i\}_{i\geq 1}$ be independent with $EX_i = 0$ and $EX_i^2 < \infty$. Let $b_i \leq b_{i+1}$ and $b_i \to \infty$. If $\sum_{i\geq 1} EX_i^2/b_i^2 < \infty$ show that $S_n/b_n \to 0$ a.s.

Remark. Notice that in the proof of SLLN we showed that

$$\sum_{i\geq 1}\frac{1}{i^2}EX_i^2 I(X_i \leq i) < \infty,$$

so this gives a new proof of SLLN.

5. Let $\{X_i\}_{i\geq 1}$ be i.i.d. with $EX_1 = 0$ and $EX_1^2 < \infty$. Prove that

$$\frac{S_n}{n^{1/2}(\log n)^{1/2+\delta}} \to 0 \text{ a.s.}$$

for any $\delta > 0$.

6. Let $\{X_i\}_{i\geq 1}$ be i.i.d. and $EX_1 > 0$. Given $a > 0$, show that $E\tau < \infty$ for $\tau = \inf\{k \geq 1 : S_k > a\}$.

7. A function

$$\mathrm{ch}(\lambda) = \frac{e^{\lambda} + e^{-\lambda}}{2}$$

is a moment generating function of a Rademacher r.v. Is $\mathrm{ch}^m(\lambda)$ a m.g.f. if $0 < m < 1$?