18.175 Theory of Probability
Fall 2008

Section 16

Convergence on metric spaces. Portmanteau Theorem. Lipschitz Functions.

Let (S, d) be a metric space and $\mathcal{B}$ - a Borel σ-algebra generated by open sets. Let us recall that $\mathbb{P}_n \to \mathbb{P}$ weakly on $\mathcal{B}$ if

$$\int f d\mathbb{P}_n \to \int f d\mathbb{P}$$

for all $f \in C_b(S)$ - real-valued bounded continuous functions on S.

For a set $A \subseteq S$, we denote by $\bar{A}$ the closure of A, $\text{int}A$ - interior of A and $\partial A = \bar{A} \setminus \text{int}A$ - boundary of A. A is called a *continuity set* of $\mathbb{P}$ if $\mathbb{P}(\partial A) = 0$.

Theorem 36 *(Portmanteau theorem) The following are equivalent.*

1. $\mathbb{P}_n \to \mathbb{P}$ *weakly.*
2. *For any open set* $U \subseteq S$, $\liminf_{n\to\infty} \mathbb{P}_n(U) \geq \mathbb{P}(U)$.
3. *For any closed set* $F \subseteq S$, $\limsup_{n\to\infty} \mathbb{P}_n(F) \leq \mathbb{P}(F)$.
4. *For any continuity set* A *of* $\mathbb{P}$, $\lim_{n\to\infty} \mathbb{P}_n(A) = \mathbb{P}(A)$.

Proof.

1$\Longrightarrow$2. Let U be an open set and $F = U^c$. Consider a sequence of functions in $C_b(S)$

$$f_m(s) = \min(1, md(s, F))$$

such that $f_m(s) \uparrow \text{I}_U(s)$. (This is not necessarily true if U is not open.) Since $\mathbb{P}_n \to \mathbb{P}$,

$$\mathbb{P}_n(U) \geq \int f_m d\mathbb{P}_n \to \int f_m d\mathbb{P} \quad \text{as } n \to \infty \text{ and } \quad \liminf_{n\to\infty} \mathbb{P}_n(U) \geq \int f_m d\mathbb{P}.$$

Letting $m \to \infty$, by monotone convergence theorem.

$$\liminf_{n\to\infty} \mathbb{P}_n(U) \geq \int \text{I}_U d\mathbb{P} = \mathbb{P}(U).$$

2$\Longleftrightarrow$3. By taking complements.

2, 3$\Longrightarrow$4. Since int$A$ is open and $\bar{A}$ is closed and int$A \subseteq \bar{A}$, by 2 and 3,

$$\mathbb{P}(\text{int}A) \le \liminf_{n\to\infty} \mathbb{P}_n(\text{int}A) \le \limsup_{n\to\infty} \mathbb{P}_n(\bar{A}) \le \mathbb{P}(\bar{A}).$$

If $\mathbb{P}(\partial A) = 0$ then $\mathbb{P}(\bar{A}) = \mathbb{P}(\text{int}A) = \mathbb{P}(A)$ and, therefore, $\lim \mathbb{P}_n(A) = \mathbb{P}(A)$.

4$\Longrightarrow$1. Consider $f \in C_b(S)$ and let $F_y = \{s \in S : f(s) = y\}$ be a level set of f. There exist at most countably many y such that $\mathbb{P}(F_y) > 0$. Therefore, for any $\varepsilon > 0$ we can find a sequence $a_1 \le \ldots \le a_N$ such that

$$\max(a_{k+1} - a_k) \le \varepsilon, \ \ \mathbb{P}(F_{a_k}) = 0 \quad \text{for all } k$$

and the range of f is inside the interval (a_1, a_N). Let

$$B_k = \{s \in S : a_k \le f(s) < a_{k+1}\} \quad \text{and} \quad f_\varepsilon(s) = \sum a_k \mathrm{I}(s \in B_k).$$

Since f is continuous, $\partial B_k \subseteq F_{a_k} \cup F_{a_{k+1}}$ and $\mathbb{P}(\partial B_k) = 0$. By 4,

$$\int f_\varepsilon d\mathbb{P}_n = \sum_k a_k \mathbb{P}_n(B_k) \to \sum_k a_k \mathbb{P}(B_k) = \int f_\varepsilon d\mathbb{P}.$$

Since, by construction, $|f_\varepsilon(s) - f(s)| \le \varepsilon$, letting $\varepsilon \to 0$ proves that $\int f d\mathbb{P}_n \to \int f d\mathbb{P}$.

□

Lipschitz functions. For a function $f : S \to \mathbb{R}$, let us define a Lipschitz semi-norm by

$$||f||_{\mathrm{L}} = \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Clearly, $||f||_{\mathrm{L}} = 0$ iff f is constant so $||f||_{\mathrm{L}}$ is not a norm. Let us define a *bounded Lipschitz* norm by

$$||f||_{\mathrm{BL}} = ||f||_{\mathrm{L}} + ||f||_\infty,$$

where $||f||_\infty = \sup_{s\in S} |f(s)|$. Let

$$BL(S,d) = \Big\{ f : S \to \mathbb{R} \, : \, ||f||_{\mathrm{BL}} < \infty \Big\}$$

be a set of all bounded Lipschitz functions.

Lemma 32 *If $f, g \in BL(S,d)$ then $fg \in BL(S,d)$ and $||fg||_{\mathrm{BL}} \le ||f||_{\mathrm{BL}} ||g||_{\mathrm{BL}}$.*

Proof. First of all, $||fg||_\infty \le ||f||_\infty ||g||_\infty$. We can write,

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\le |f(x)(g(x) - g(y))| + |g(y)(f(x) - f(y))| \\ &\le ||f||_\infty ||g||_{\mathrm{L}} d(x,y) + ||g||_\infty ||f||_{\mathrm{L}} d(x,y) \end{aligned}$$

and, therefore,

$$||fg||_{\mathrm{BL}} \le ||f||_\infty ||g||_\infty + ||f||_\infty ||g||_{\mathrm{L}} + ||g||_\infty ||f||_{\mathrm{L}} \le ||f||_{\mathrm{BL}} ||g||_{\mathrm{BL}}.$$

□

Let us recall the notations $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$ and let $* = \wedge$ or $\vee$.

Lemma 33 *The following hold.*

1. $||f_1 * \cdots * f_k||_{\mathrm{L}} \le \max_{1\le i\le k} ||f_i||_{\mathrm{L}}$.
2. $||f_1 * \cdots * f_k||_{\mathrm{BL}} \le 2\max_{1\le i\le k} ||f_i||_{\mathrm{BL}}$.

Proof. *Proof of 1.* It is enough to consider $k = 2$. For specificity, take $* = \vee$. Given $x, y \in S$, suppose that

$$f_1 \vee f_2(x) \geq f_1 \vee f_2(y) = f_1(y).$$

Then

$$\begin{aligned}|f_1 \vee f_2(y) - f_1 \vee f_2(x)| = f_1 \vee f_2(x) - f_1 \vee f_2(y) &\leq \begin{cases} f_1(x) - f_1(y), & \text{if } f_1(x) \geq f_2(x) \\ f_2(x) - f_2(y), & \text{otherwise} \end{cases} \\ &\leq ||f_1||_{\mathrm{L}} \vee ||f_2||_{\mathrm{L}} d(x, y).\end{aligned}$$

This finishes the proof of 1.

Proof of 2. First of all, obviously,

$$||f_1 * \cdots * f_k||_\infty \leq \max_{1 \leq i \leq k} ||f_i||_\infty.$$

Therefore, using 1,

$$||f_1 * \cdots * f_k||_{\mathrm{BL}} \leq \max_i ||f_i||_\infty + \max_i ||f_i||_{\mathrm{L}} \leq 2 \max_i ||f_i||_{\mathrm{BL}}.$$

□

Theorem 37 *(Extension theorem) Given a set $A \subseteq S$ and a bounded Lipschitz function $f \in BL(A, d)$ on A, there exists an extension $h \in BL(S, d)$ such that*

$$f = h \quad \textit{on} \quad A \quad \textit{and} \quad ||h||_{\mathrm{BL}} = ||f||_{\mathrm{BL}}.$$

Proof. Let us first find an extension such that $||h||_{\mathrm{L}} = ||f||_{\mathrm{L}}$. We will start by extending f to one point $x \in S \setminus A$. The value $y = h(x)$ must satisfy

$$|y - f(s)| \leq \|f\|_{\mathrm{L}} d(x, s) \quad \text{for all} \quad s \in A$$

or, equivalently,

$$\inf_{s \in A} (f(s) + ||f||_{\mathrm{L}} d(x, s)) \geq y \geq \sup_{s \in A} (f(s) - ||f||_{\mathrm{L}} d(x, s)).$$

Such y exists iff for all $s_1, s_2 \in A$,

$$f(s_1) + ||f||_{\mathrm{L}} d(x, s_1) \geq f(s_2) - ||f||_{\mathrm{L}} d(x, s_2).$$

This inequality is satisfied because by triangle inequality

$$f(s_2) - f(s_1) \leq ||f||_{\mathrm{L}} d(s_1, s_2) \leq ||f||_{\mathrm{L}} (d(s_1, x) + d(s_2, x)).$$

It remains to apply Zorn's lemma to show that f can be extended to the entire S. Define order by inclusion:

$$f_1 \prec f_2 \text{ if } f_1 \text{ is defined on } A_1,\ f_2 \text{ - on } A_2,\ A_1 \subseteq A_2,\ f_1 = f_2 \text{ on } A_1 \text{ and } \|f_1\|_{\mathrm{L}} = \|f_2\|_{\mathrm{L}}.$$

For any chain $\{f_\alpha\}, f = \bigcup f_\alpha \succ f_\alpha$. By Zorn's lemma there exists a maximal element h. It is defined on the entire S because, otherwise, we could extend to one more point. To extend preserving BL norm take

$$h' = (h \wedge ||f||_\infty) \vee (-||f||_\infty).$$

By part 1 of previous lemma, it is easy to see that $||h'||_{\mathrm{BL}} = ||f||_{\mathrm{BL}}$.

□

Stone-Weierstrass Theorem.

A set $A \subseteq S$ is *totally bounded* if for any $\varepsilon > 0$ there exists a finite ε-cover of A, i.e. a set of points $a_1, \ldots, a_N$ such that

$$A \subseteq \bigcup_{i \leq N} B(a_i, \varepsilon),$$

where $B(a, \varepsilon) = \{y \in S : d(a, y) \leq \varepsilon\}$ is a ball of radius ε centered at a. Let us recall the following theorem from analysis.

Theorem 38 *(Arzela-Ascoli) Let (S, d) be a compact metric space and let $(C(S), d_\infty)$ be the space of continuous real-valued functions on S with uniform convergence metric*

$$d_\infty(f, g) = \sup_{x \in S} |f(x) - g(x)|.$$

A subset $\mathcal{F} \subseteq C(S)$ is totally bounded in d_∞ metric iff $\mathcal{F}$ is equicontinuous and uniformly bounded.

Remark. Equicontinuous means that for any $\varepsilon >$ there exists $\delta > 0$ such that if $d(x, y) \leq \delta$ then for all $f \in \mathcal{F}$, $|f(x) - f(y)| \leq \varepsilon$.

Theorem 39 *(Stone-Weierstrass) Let (S, d) be a compact metric space and $\mathcal{F} \subseteq C(S)$ is such that*

1. *$\mathcal{F}$ is algebra, i.e. for all $f, g \in \mathcal{F}, c \in \mathbb{R}$, we have $cf + g \in \mathcal{F}, fg \in \mathcal{F}$.*
2. *$\mathcal{F}$ separates points, i.e. if $x \neq y \in S$ then there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.*
3. *$\mathcal{F}$ contains constants.*

Then $\mathcal{F}$ is dense in $C(S)$.

Corollary 3 *If (S, d) is a compact space then $BL(S, d)$ is dense in $C(S)$.*

Proof. For $\mathcal{F} = BL(S, d)$ in the Stone-Weierstrass theorem, 3 is obvious, 1 follows from Lemma 32 and 2 follows from the extension Theorem 37, since a function defined on two points $x \neq y$ such that $f(x) \neq f(y)$ can be extended to the entire S.

□

Proof of Theorem 39. Consider bounded $f \in \mathcal{F}$, i.e. $|f(x)| \leq M$. A function $x \to |x|$ defined on the interval $[-M, M]$ can be uniformly approximated by polynomials of x by the Weierstrass theorem on the real line or, for example, using Bernstein's polynomials. Therefore, $|f(x)|$ can be uniformly approximated by polynomials of $f(x)$, and by properties 1 and 3, by functions in $\mathcal{F}$. Therefore, if $\bar{\mathcal{F}}$ is the closure of $\mathcal{F}$ in d_∞ norm then for any $f \in \bar{\mathcal{F}}$ its absolute value $|f| \in \bar{\mathcal{F}}$. Therefore, for any $f, g \in \bar{\mathcal{F}}$ we have

$$\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g| \in \bar{\mathcal{F}}, \ \max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \in \bar{\mathcal{F}}. \tag{16.0.1}$$

Given any points $x \neq y$ and $c, d \in \mathbb{R}$ one can always find $f \in \mathcal{F}$ such that $f(x) = c$ and $f(y) = d$. Indeed, by property 2 we can find $g \in \mathcal{F}$ such that $g(x) \neq g(y)$ and, as a result, a system of equations

$$ag(x) + b = c, \ \ ag(y) + b = d$$

has a solution a, b. Then the function $f = ag + b$ satisfies the above and it is in $\mathcal{F}$ by 1.

Take $h \in C(S)$ and fix x. For any y let $f_y \in \mathcal{F}$ be such that

$$f_y(x) = h(x), \ \ f_y(y) = h(y).$$

By continuity of f_y, for any $y \in S$ there exists an open neighborhood U_y of y such that

$$f_y(s) \geq h(s) - \varepsilon \text{ for } s \in U_y.$$

Since (U_y) is an open cover of the compact S, there exists a finite subcover $U_{y_1}, \ldots, U_{y_N}$. Let us define a function

$$f^x(s) = \max(f_{y_1}(s), \ldots, f_{y_N}(s)) \in \bar{\mathcal{F}} \text{ by (16.0.1)}.$$

By construction, it has the following properties:

$$f^x(x) = h(x), \ \ f^x(s) \geq h(s) - \varepsilon \quad \text{for all} \quad s \in S.$$

Again, by continuity of $f^x(s)$ there exists an open neighborhood U_x of x such that

$$f^x(s) \le h(s) + \varepsilon \quad \text{for} \quad s \in U_x.$$

Take a finite subcover $U_{x_1}, \ldots, U_{x_M}$ and define

$$h'(s) = \min\bigl(f^{x_1}(s), \ldots, f^{x_M}(s)\bigr) \in \bar{\mathcal{F}} \text{ by (16.0.1).}$$

By construction, $h'(s) \le h(s) + \varepsilon$ and $h'(s) \ge h(s) - \varepsilon$ for all $s \in S$ which means that $d_\infty(h', h) \le \varepsilon$. Since $h' \in \bar{\mathcal{F}}$, this proves that $\bar{\mathcal{F}}$ is dense in $C(S)$.

□

Corollary 4 *If (S, d) is a compact space then $C(S)$ is separable in d_∞.*

Remark. Recall that this fact was used in the proof of the Selection Theorem, which was proved for general metric spaces.

Proof. By the above theorem, $BL(S, d)$ is dense in $C(S)$. For any integer $n \ge 1$, the set $\{f : ||f||_{\mathrm{BL}} \le n\}$ is uniformly bounded and equicontinuous. By the Arzela-Ascoli theorem, it is totally bounded and, therefore, separable which can be seen by taking finite $1/m$-covers for all $m \ge 1$. The union

$$\bigcup \{||f||_{\mathrm{BL}} \le n\} = BL(S, d)$$

is therefore separable in $C(S)$ which is, as a result, also separable.

□