18.175 Theory of Probability
Fall 2008

Section 18

Convergence and uniform tightness.

In this section, we will make several connections between convergence of measures and uniform tightness on general metric spaces, which are similar to the results in the Euclidean setting. First, we will show that, in some sense, uniform tightness is necessary for convergence of laws.

Theorem 43 *If $\mathbb{P}_n \to \mathbb{P}_0$ on S and each $\mathbb{P}_n$ is tight for $n \geq 0$, then $(\mathbb{P}_n)_{n\geq 0}$ is uniformly tight.*

Proof. Since $\mathbb{P}_n \to \mathbb{P}_0$ and $\mathbb{P}_0$ is tight, by Theorem 41, the Levy-Prohorov metric $\rho(\mathbb{P}_n, \mathbb{P}_0) \to 0$. Given $\varepsilon > 0$, let us take a compact K such that $\mathbb{P}_0(K) > 1 - \varepsilon$. By definition of ρ,

$$1 - \varepsilon < \mathbb{P}_0(K) \leq \mathbb{P}_n\Big(K^{\rho(\mathbb{P}_n,\mathbb{P}_0)+\frac{1}{n}}\Big) + \rho(\mathbb{P}_n, \mathbb{P}_0) + \frac{1}{n}$$

and, therefore,

$$a(n) = \inf\Big\{\delta > 0 : \mathbb{P}_n(K^\delta) > 1 - \varepsilon\Big\} \to 0.$$

By regularity of measure $\mathbb{P}_n$, any measurable set A can be approximated by its closed subset F. Since $\mathbb{P}_n$ is tight, we can choose a compact of measure close to one, and intersecting it with the closed subset F, we can approximate any set A by its compact subset. Therefore, there exists a compact $K_n \subseteq K^{2a(n)}$ such that $\mathbb{P}_n(K_n) > 1 - \varepsilon$. Let

$$L = K \bigcup (\cup_{n\geq 1} K_n).$$

Then $\mathbb{P}_n(L) \geq \mathbb{P}_n(K_n) > 1 - \varepsilon$. It remains to show that L is compact. Consider a sequence (x_n) on L. There are two possibilities. First, if there exists an infinite subsequence $(x_{n(k)})$ that belongs to one of the compacts K_j then it has a converging subsubsequence in K_j and as a result in L. If not, then there exists a subsequence $(x_{n(k)})$ such that $x_{n(k)} \in K_{m(k)}$ and $m(k) \to \infty$ as $k \to \infty$. Since

$$K_{m(k)} \subseteq K^{2a(m(k))}$$

there exists $y_k \in K$ such that

$$d(x_{n(k)}, y_k) \leq 2a(m(k)).$$

Since K is compact, the sequence $y_k \in K$ has a converging subsequence $y_{k(r)} \to y \in K$ which implies that $d(x_{n(k(r))}, y) \to 0$, i.e. $x_{n(k(r))} \to y \in L$. Therefore, L is compact.

□

We already know from the Selection Theorem in Section 8 that any uniformly tight sequence of laws on any metric space has a converging subsequence. Under additional assumptions on (S, d) we can complement the Selection Theorem and make some connections to the metrics defined in the previous section.

Theorem 44 *Let (S, d) be a complete separable metric space and A be a subset of probability laws on S. Then the following are equivalent.*

1. *A is uniformly tight.*

2. *For any sequence $\mathbb{P}_n \in A$ there exists a converging subsequence $\mathbb{P}_{n(k)} \to \mathbb{P}$ where $\mathbb{P}$ is a law on S.*

3. *A has the compact closure on the space of probability laws equipped with the Levy-Prohorov or bounded Lipschitz metrics ρ or β.*

4. *A is totally bounded with respect to ρ or β.*

Remark. Implications $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4$ hold without completeness assumption and the only implication where completeness will be used is $4 \Longrightarrow 1$.

Proof. $1 \Longrightarrow 2$. Any sequence $\mathbb{P}_n \in A$ is uniformly tight and, by selection theorem, there exists a converging subsequence.

$2 \Longrightarrow 3$. Since (S, d) is separable, by Theorem 41, $\mathbb{P}_n \to \mathbb{P}$ if and only if $\rho(\mathbb{P}_n, \mathbb{P})$ or $\beta(\mathbb{P}_n, \mathbb{P}) \to 0$. Every sequence in the closure $\bar{A}$ can be approximated by a sequence in A. That sequence has a converging subsequence that, obviously, converges to an element in $\bar{A}$ which means that the closure of A is compact.

$3 \Longrightarrow 4$. Compact sets are totally bounded and, therefore, if the closure $\bar{A}$ is compact, the set A is totally bounded.

$4 \Longrightarrow 1$. Since $\rho \leq 2\sqrt{\beta}$, we will only deal with ρ. For any $\varepsilon > 0$, there exists a finite subset $B \subseteq A$ such that $A \subseteq B^{\varepsilon}$. Since (S, d) is complete and separable, by Ulam's theorem, for each $\mathbb{P} \in B$ there exists a compact $K_{\mathbb{P}}$ such that $\mathbb{P}(K_{\mathbb{P}}) > 1 - \varepsilon$. Therefore,

$$K_B = \bigcup_{\mathbb{P} \in B} K_{\mathbb{P}} \text{ is a compact and } \mathbb{P}(K_B) > 1 - \varepsilon \text{ for all } \mathbb{P} \in B.$$

For any $\varepsilon > 0$, let F be a finite set such that $K_B \subseteq F^{\varepsilon}$ (here we will denote by F^{ε} the closed ε-neighborhood of F). Since $A \subseteq B^{\varepsilon}$, for any $\mathbb{Q} \in A$ there exists $\mathbb{P} \in B$ such that $\rho(\mathbb{Q}, \mathbb{P}) < \varepsilon$ and, therefore,

$$1 - \varepsilon \leq \mathbb{P}(K_B) \leq \mathbb{P}(F^{\varepsilon}) \leq \mathbb{Q}(F^{2\varepsilon}) + \varepsilon.$$

Thus, $1 - 2\varepsilon \leq \mathbb{Q}(F^{2\varepsilon})$ for all $\mathbb{Q} \in A$. Given $\delta > 0$, take $\varepsilon_m = \delta / 2^{m+1}$ and find F_m as above, i.e.

$$1 - \frac{\delta}{2^m} \leq \mathbb{Q}\Big(F_m^{\delta/2^m}\Big).$$

Then $\mathbb{Q}\Big(\bigcap_{m \geq 1} F_m^{\delta/2^m}\Big) \geq 1 - \sum_{m \geq 1} \frac{\delta}{2^m} = 1 - \delta$. Finally, $L = \bigcap_{m \geq 1} F_m^{\delta/2^m}$ is compact because it is closed and totally bounded by construction, and S is complete.

□

Corollary 5 *(Prohorov) The set of laws on a complete separable metric space is complete with respect to metrics ρ or β.*

Proof. If a sequence of laws is Cauchy w.r.t. ρ or β then it is totally bounded and by previous theorem it has a converging subsequence. Obviously, Cauchy sequence will converge to the same limit.

□

Finally, let us state as a result the idea which appeared in Lemma 19 in Section 9.

Lemma 36 *Suppose that $(\mathbb{P}_n)$ is uniformly tight on a metric space (S, d). Suppose that all converging subsequences $(\mathbb{P}_{n(k)})$ converge to the same limit, i.e. if $\mathbb{P}_{n(k)} \to \mathbb{P}_0$ then $\mathbb{P}_0$ is independent of $(n(k))$. Then $\mathbb{P}_n \to \mathbb{P}_0$.*

Proof. Any subsequence $(\mathbb{P}_{n(k)})$ is uniformly tight and, by the selection theorem, it has a converging subsubsequence $(\mathbb{P}_{n(k(r))})$ which has to converge to $\mathbb{P}_0$. Lemma 13 in Section 8 finishes the proof.

□

This will be very useful when proving convergence of laws on metric spaces, such as $C([0,1])$, for example. If we can prove that $(\mathbb{P}_n)$ is uniformly tight and, assuming that a subsequence converges, can identify the unique limit, then the sequence $\mathbb{P}_n$ must converge to the same limit.