18.175 Theory of Probability
Fall 2008

Section 22

Stochastic Processes. Brownian Motion.

We have developed a general theory of convergence of laws on (separable) metric spaces and in the following two sections we will look at some specific examples of convergence on the spaces of continuous functions $(C[0,1], \|\cdot\|_\infty)$ and $(C(\mathbb{R}_+), d)$, where d is a metric metrizing uniform convergence on compacts. These examples will describe a certain central limit theorem type results on these spaces and in this section we will define the corresponding limiting Gaussian laws, namely, the Brownian motion and Brownian bridge. We will start with basic definitions and basic regularity results in the presence of continuity. Given a set T and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a *stochastic process* is a function

$$X_t(\omega) = X(t,\omega) : T \times \Omega \to \mathbb{R}$$

such that for each $t \in T$, $X_t : \Omega \to \mathbb{R}$ is a random variable, i.e. a measurable function. In other words, a stochastic process is a collection of random variables X_t indexed by a set T. A stochastic process is often defined by specifying finite dimensional (f.d.) distributions $\mathbb{P}_F = \mathcal{L}(\{X_t\}_{t\in F})$ for all finite subsets $F \subseteq T$. Kolmogorov's theorem then guarantees the existence of a probability space on which the process is defined, under the natural consistency condition

$$F_1 \subseteq F_2 \Rightarrow \mathbb{P}_{F_1} = \mathbb{P}_{F_2}\Big|_{\mathbb{R}^{F_1}}.$$

One can also think of a process as a function on Ω with values in $\mathbb{R}^T = \{f : T \to \mathbb{R}\}$, because for a fixed $\omega \in \Omega$, $X_t(\omega) \in \mathbb{R}^T$ is a (random) function of t. In Kolmogorov's theorem, given a family of consistent f.d. distributions, a process was defined on the probability space $(\mathbb{R}^T, \mathcal{B}_T)$, where $\mathcal{B}_T$ is the cylindrical σ-algebra generated by the algebra of cylinders $B \times \mathbb{R}^{T\setminus F}$ for Borel sets B in $\mathbb{R}^F$ and all finite F. When T is uncountable, some very natural sets such as

$$\Big\{\sup_{t\in T} X_t > 1\Big\} = \bigcup_{t\in T}\{X_t > 1\}$$

might be not measurable on $\mathcal{B}_T$. However, in our examples we will deal with continuous processes that possess additional regularity properties.

Definition. If (T,d) is a metric space then a process X_t is called *sample continuous* if for all $\omega \in \Omega$, $X_t(\omega) \in C(T,d)$ - the space of continuous function on (T,d). The process X_t is called *continuous in probability* if $X_t \to X_{t_0}$ in probability whenever $t \to t_0$.

Example. Let $T = [0,1]$, $(\Omega, \mathbb{P}) = ([0,1], \lambda)$ where λ is the Lebesgue measure. Let $X_t(\omega) = \mathrm{I}(t = \omega)$ and $X'_t(\omega) = 0$. F.d. distributions of these processes are the same because for any fixed $t \in [0,1]$,

$$\mathbb{P}(X_t = 0) = \mathbb{P}(X'_t = 0) = 1.$$

However, $\mathbb{P}(X_t \text{ is continuous}) = 0$ but for X'_t this probability is 1.

□

Definition. Let (T, d) be a metric space. The process X_t is *measurable* if

$$X_t(\omega) : T \times \Omega \to \mathbb{R}$$

is jointly measurable on the product space $(T, \mathcal{B}) \times (\Omega, \mathcal{F})$, where $\mathcal{B}$ is the Borel σ-algebra on T.

Lemma 44 *If (T, d) is a separable metric space and X_t is sample continuous then X_t is measurable.*

Proof. Let $(S_j)_{j\geq 1}$ be a measurable partition of T such that $\mathrm{diam}(S_j) \leq \frac{1}{n}$. For each non-empty S_j, let us take a point $t_j \in S_j$ and define

$$X_t^n(\omega) = X_{t_j}(\omega) \quad \text{for} \quad t \in S_j.$$

$X_t^n(\omega)$ is, obviously, measurable on $T \times \Omega$ because for any Borel set A on $\mathbb{R}$,

$$\Big\{(\omega, t) \,:\, X_t^n(\omega) \in A\Big\} = \bigcup_{j\geq 1} \Big\{\omega \,:\, X_{t_j}(\omega) \in A\Big\} \times S_j.$$

$X_t(w)$ is sample continuous and, therefore, $X_t^n(\omega) \to X_t(\omega)$ for all (ω, t). Hence, X is also measurable.

$\square$

If (T, d) is a compact metric space and X_t is a sample continuous process indexed by T then we can think of X_t as an element of the metric space of continuous functions $(C(T, d), ||\cdot||_\infty)$, rather then simply an element of $\mathbb{R}^T$. We can define measurable events on this space in two different ways. On one hand, we have the natural Borel σ-algebra $\mathcal{B}$ on $C(T)$ generated by the open (or closed) balls

$$B_g(\varepsilon) = \{f \in C(T) : ||f - g||_\infty < \varepsilon\}.$$

On the other hand, if we think of $C(T)$ as a subspace of $\mathbb{R}^T$, we can consider a σ-algebra

$$S_T = \Big\{B \cap C(T) : B \in \mathcal{B}_T\Big\}$$

which is the intersection of the cylindrical σ-algebra $\mathcal{B}_T$ with $C(T)$. It turns out that these two definitions coincide. An important implication of this is that the law of any sample continuous process X_t on (T, d) is completely determined by its finite dimensional distributions.

Lemma 45 *If (T, d) is a separable metric space then $\mathcal{B} = S_T$.*

Proof. Let us first show that $S_T \subseteq \mathcal{B}$. Any element of the cylindrical algebra that generates the cylindrical σ-algebra $\mathcal{B}_T$ is given by

$$B \times \mathbb{R}^{T\setminus F} \quad \text{for a finite } F \subseteq T \text{ and for some Borel set } \; B \subseteq \mathbb{R}^F.$$

Then

$$\big(B \times \mathbb{R}^{T\setminus F}\big) \bigcap C(T) = \Big\{x \in C(T) \,:\, (x_t)_{t\in F} \in B\Big\} = \big\{\pi_F(x) \in B\big\}$$

where $\pi_F : C(T) \to \mathbb{R}^F$ is the finite dimensional projection such that $\pi_F(x) = (x_t)_{t\in F}$. Projection π_F is, obviously, continuous in the $||\cdot||_\infty$ norm and, therefore, measurable on the Borel σ-algebra $\mathcal{B}$ generated by open sets in the $\|\cdot\|_\infty$ norm. This implies that $\big\{\pi_F(x) \in B\big\} \in \mathcal{B}$ and, thus, $S_T \subseteq \mathcal{B}$. Let us now show that $\mathcal{B} \subseteq S_T$. Let T' be a countable dense subset of T. Then, by continuity, any closed ε-ball in $C(T)$ can be written as

$$\big\{f \in C(T) : ||f - g||_\infty \leq \varepsilon\big\} = \bigcap_{t\in T'} \big\{f \in C(T) \,:\, |f(t) - g(t)| \leq \varepsilon\big\} \in S_T$$

and this finished the proof.

$\square$

In the remainder of the section we will define two specific sample continuous stochastic processes.

Brownian motion. Brownian motion is a sample continuous process X_t on $T=\mathbb{R}_+$ such that (a) the distribution of X_t is centered Gaussian for each $t\geq 0$; (b) $X_0=0$ and $\mathbb{E}X_1^2=1$; (c) if $t<s$ then X_t and X_s-X_t are independent and $\mathcal{L}(X_s-X_t)=\mathcal{L}(X_{s-t})$. If we denote $\sigma^2(t)=\mathrm{Var}(X_t)$ then these properties imply

$$\sigma^2(nt)=n\sigma^2(t),\ \ \sigma^2\Bigl(\frac{t}{m}\Bigr)=\frac{1}{m}\sigma^2(t)\ \ \text{ and }\ \ \sigma^2(qt)=q\sigma^2(t)$$

for all rational q. Since $\sigma^2(1)=1$, $\sigma^2(q)=q$ for all rational q and, by sample continuity, $\sigma^2(t)=t$ for all $t\geq 0$. Therefore, for $s<t$,

$$\mathbb{E}X_sX_t=\mathbb{E}X_s(X_s+(X_t-X_s))=s=\min(t,s).$$

As a result, we can give an equivalent definition.

Definition. Brownian motion is a sample continuous centered Gaussian process X_t for $t\in[0,\infty)$ with the covariance $\mathrm{Cov}(X_t,X_s)=\min(t,s)$.

Without the requirement of sample continuity, the existence of such process follows from Kolmogorov's theorem since all finite dimensional distributions are consistent by construction. However, we still need to prove that there exists a sample continuous version of the process. We start with a simple estimate.

Lemma 46 *If $f(c)=\mathcal{N}(0,1)(c,\infty)$ is the tail probability of the standard normal distribution then*

$$f(c)\leq e^{-\frac{c^2}{2}}\ \ \textit{for all}\ \ c>0.$$

Proof. We have

$$f(c)=\frac{1}{\sqrt{2\pi}}\int_c^\infty e^{-\frac{x^2}{2}}dx\leq\frac{1}{\sqrt{2\pi}}\int_c^\infty\frac{x}{c}e^{-\frac{x^2}{2}}dx=\frac{1}{\sqrt{2\pi}}\frac{1}{c}e^{-\frac{c^2}{2}}.$$

If $c>1/\sqrt{2\pi}$ then $f(c)\leq\exp(-c^2/2)$. If $c\leq 1/\sqrt{2\pi}$ then a simpler estimate gives the result

$$f(c)\leq f(0)=\frac{1}{2}\leq\exp\Bigl(-\frac{1}{2}\Bigl(\frac{1}{\sqrt{2\pi}}\Bigr)^2\Bigr)\leq e^{-\frac{c^2}{2}}.$$

□

Theorem 54 *There exists a continuous version of the Brownian motion.*

Proof. It is obviously enough to define X_t on the interval $[0,1]$. Given a process X_t that has f.d. distributions of the Brownian process but is not necessarily continuous, let us define for $n\geq 1$,

$$V_k=X_{\frac{k+1}{2^n}}-X_{\frac{k}{2^n}}\ \ \text{for}\ \ k=0,\ldots,2^n-1.$$

The variance $\mathrm{Var}(V_k)=1/2^n$ and, by the above lemma,

$$\mathbb{P}\Bigl(\max_k|V_k|\geq\frac{1}{n^2}\Bigr)\leq 2^n\mathbb{P}\Bigl(|V_1|\geq\frac{1}{n^2}\Bigr)\leq 2^{n+1}\exp\Bigl(-\frac{2^{n-1}}{n^4}\Bigr).$$

The right hand side is summable over $n\geq 1$ and, by the Borel-Cantelli lemma,

$$\mathbb{P}\Bigl(\Bigl\{\max_k|V_k|\geq\frac{1}{n^2}\Bigr\}\text{ i.o.}\Bigr)=0. \tag{22.0.1}$$

Given $t\in[0,1]$ and its dyadic decomposition $t=\sum_{j=1}^\infty\frac{t_j}{2^j}$ for $t_j\in\{0,1\}$, let us define $t(n)=\sum_{j=1}^n\frac{t_j}{2^j}$ so that

$$X_{t(n)}-X_{t(n-1)}\in\{0\}\cup\{V_k:k=0,\ldots,2^n-1\}.$$

Then, the sequence

$$X_{t(n)}=0+\sum_{1\leq j\leq n}(X_{t(j)}-X_{t(j-1)})$$

converges almost surely to some limit Z_t because by (22.0.1) with probability one

$$|X_{t(n)} - X_{t(n-1)}| \le n^{-2}$$

for large enough (random) $n \ge n_0(\omega)$. By construction, $Z_t = X_t$ on the dense subset of all dyadic $t \in [0,1]$. If we can prove that Z_t is sample continuous then all f.d. distributions of Z_t and X_t will coincide, which means that Z_t is a continuous version of the Brownian motion. Take any $t, s \in [0,1]$ such that $|t-s| \le 2^{-n}$. If $t(n) = \frac{k}{2^n}$ and $s(n) = \frac{m}{2^n}$, then $|k-m| \in \{0,1\}$. As a result, $|X_{t(n)} - X_{s(n)}|$ is either equal to 0 or one of the increments $|V_k|$ and, by (22.0.1), $|X_{t(n)} - X_{s(n)}| \le n^{-2}$ for large enough n. Finally,

$$\begin{aligned}
|Z_t - Z_s| &\le |Z_t - X_{t(n)}| + |X_{t(n)} - X_{s(n)}| + |X_{s(n)} - Z_s| \\
&\le \sum_{l \ge n} \frac{1}{l^2} + \frac{1}{n^2} + \sum_{l \ge n} \frac{1}{l^2} \le \frac{c}{n}
\end{aligned}$$

which proves the continuity of Z_t. On the event in (22.0.1) of probability zero we set $Z_t = 0$.

□

Definition. A sample continuous centered Gaussian process B_t for $t \in [0,1]$ is called a *Brownian bridge* if

$$\mathbb{E} B_t B_s = s(1-t) \quad \text{for} \quad s < t.$$

Such process exists because if X_t is a Brownian motion then $B_t = X_t - tX_1$ is a Brownian bridge, since for $s < t$,

$$\mathbb{E} B_s B_t = \mathbb{E}(X_t - tX_1)(X_s - sX_1) = s - st - ts + st = s(1-t).$$

Notice that $B_0 = B_1 = 0$.

□