18.175 Theory of Probability
Fall 2008

Section 25

Markov property of Brownian motion. Reflection principles.

We showed that the empirical process converges to Brownian bridge on $(C([0,1]), \|\cdot\|_\infty)$. As a result, the distribution of a continuous function of the process will also converge, for example,

$$\sup_{0\le t\le 1} |X_t^n| \to \sup_{0\le t\le 1} |B_t|$$

weakly. We will compute the distribution of this supremum in Theorem 60 below but first we will start with a simpler example to illustrate the so called strong Markov property of the Brownian motion.

Given a process W_t on $(C[0,\infty), d)$, let $\mathcal{F}_t = \sigma(W_s; s \le t)$. A random variable τ is called a stopping time if

$$\{\tau \le t\} \in \mathcal{F}_t \quad \text{for all} \quad t \ge 0.$$

For example, a hitting time $\tau_c = \inf\{t > 0, W_t = c\}, c > 0$, is a stopping time because, by sample continuity,

$$\{\tau_c \le t\} = \bigcap_{q<c} \bigcup_{r<t} \{W_r > q\}$$

where the intersection and union are over rational numbers q, r. If W_t is the Brownian motion then strong Markov property of W_t states, informally, that the increment process $W_{\tau+t} - W_\tau$ after the stopping time is independent of the σ-algebra $\mathcal{F}_\tau$ generated by W_t up to the stopping time τ and, moreover, $W_{\tau+t} - W_\tau$ has the same distribution as W_t. This property is very similar to the property of stopping times for sums of i.i.d. random variables, in Section 7. However, to avoid subtle measure theoretic considerations, we will simply approximate arbitrary stopping times by dyadic stopping times for which Markov property can be used more directly, by summing over all possible values. If τ is a stopping time then

$$\tau_n = \frac{\lfloor 2^n \tau \rfloor + 1}{2^n}$$

is also a stopping time. Indeed, if

$$\frac{k}{2^n} \le \tau < \frac{k+1}{2^n} \quad \text{then} \quad \tau_n = \frac{k+1}{2^n}$$

and, therefore, for any $t \ge 0$, if $\frac{l}{2^n} \le t < \frac{l+1}{2^n}$ then

$$\{\tau_n \le t\} = \left\{\tau < \frac{l}{2^n}\right\} = \bigcup_{q<l/2^n} \{\tau \le q\} \in \mathcal{F}_t.$$

By construction, $\tau_n \downarrow \tau$ and, by continuity, $W_{\tau_n} \to W_\tau$ a.s. Let us demonstrate how to use Markov property for these dyadic approximations in the computation of the following probability,

$$\mathbb{P}(\sup_{t\leq b} W_t \geq c) = \mathbb{P}(\tau_c \leq b)$$

for $c > 0$. For dyadic approximation τ_n of τ_c, we can write

$$\begin{aligned}\mathbb{P}(\tau_n \leq b, W_b - W_{\tau_n} \geq 0) &= \sum_{k\geq 0} \mathbb{P}\Big(\overbrace{\tau_n = k/2^n \leq b}^{in\ \mathcal{F}_{k/2^n}}, \overbrace{W_b - W_{k/2^n}}^{indep.\ of\ \mathcal{F}_{k/2^n}} \geq 0\Big) \\ &= \frac{1}{2}\sum_{k\geq 0} \mathbb{P}\Big(\tau_n = \frac{k}{2^n} \leq b\Big) = \frac{1}{2}\mathbb{P}(\tau_n \leq b).\end{aligned}$$

Letting $n \to \infty$ and applying the portmanteau theorem,

$$\mathbb{P}(\tau_c \leq b, W_b - W_{\tau_c} \geq 0) = \frac{1}{2}\mathbb{P}(\tau_c \leq b),$$

since both sets $\{\tau_c = b\}$ and $\{W_b - W_{\tau_c} = 0\}$ are the sets of continuity because

$$\{\tau_c = b\} \subseteq \{W_b = c\} \quad \text{and} \quad \{W_b - W_{\tau_c} = 0\} \subseteq \{W_b = c\}$$

and $\mathbb{P}(W_b = c) = 0$. Finally, this implies that

$$\mathbb{P}(W_b \geq c) = \mathbb{P}(\tau_c \leq b, W_b - W_{\tau_c} \geq 0) = \frac{1}{2}\mathbb{P}(\tau_c \leq b) = \frac{1}{2}\mathbb{P}\Big(\sup_{t\leq b} W_t \geq c\Big)$$

and, therefore,

$$\mathbb{P}\Big(\sup_{t\leq b} W_t \geq c\Big) = \mathbb{P}(\tau_c \leq b) = 2\mathcal{N}(0,b)(c,\infty) = 2\int_{c/\sqrt{b}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx. \tag{25.0.1}$$

The p.d.f. of τ_c satisfies

$$f_{\tau_c}(b) = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2b}} \cdot \frac{c}{b^{3/2}} = \mathcal{O}(b^{-3/2})$$

as $b \to +\infty$, which means that $\mathbb{E}\tau_c = \infty$.

□

Reflection principles. If x_t is the Brownian motion then $y_t = x_t - tx_1$ is the Brownian bridge for $t \in [0,1]$. The next lemma shows that we can think of the Brownian bridge as the Brownian motion conditioned to be equal to zero at time $t = 1$ (pinned down Brownian motion).

Lemma 48 *Conditional distribution of x_t given $|x_1| < \varepsilon$ converges to the law of y_t,*

$$\mathcal{L}\big(x_t \big| |x_1| < \varepsilon\big) \to \mathcal{L}(y_t)$$

as $\varepsilon \downarrow 0$.

Proof. Notice that $y_t = x_t - tx_1$ is independent of x_1 because their covariance

$$\mathbb{E}y_t x_1 = \mathbb{E}x_t x_1 - t\mathbb{E}x_1^2 = t - t = 0.$$

Therefore, the Brownian motion can be written as a sum $x_t = y_t + tx_1$ of the Brownian bridge and independent process tx_1. Therefore, if we define a random variable η_ε with distribution $\mathcal{L}(\eta_\varepsilon) = \mathcal{L}\big(x_1 \big| |x_1| < \varepsilon\big)$ independent of y_t then

$$\mathcal{L}\big(x_t \big| |x_1| < \varepsilon\big) = \mathcal{L}(y_t + t\eta_\varepsilon) \to \mathcal{L}(y_t).$$

as $\varepsilon \downarrow 0$.

□

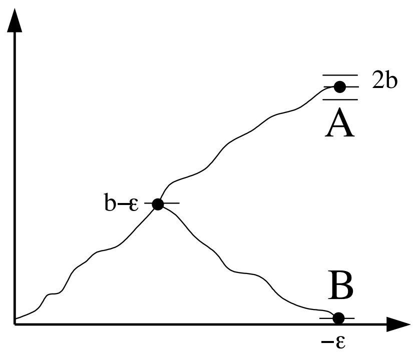

Figure 25.1: Reflecting the Brownian motion.

Theorem 59 *If y_t is the Brownian bridge then for all $b > 0$,*

$$\mathbb{P}\Big(\sup_{t\in[0,1]} y_t \geq b\Big) = e^{-2b^2}.$$

Proof. Since $y_t = x_t - tx_1$ and x_1 are independent, we can write

$$\mathbb{P}(\exists t : y_t = b) = \frac{\mathbb{P}(\exists t : x_t - tx_1 = b, |x_1| < \varepsilon)}{\mathbb{P}(|x_1| < \varepsilon)} = \frac{\mathbb{P}(\exists t : x_t = b + tx_1, |x_1| < \varepsilon)}{\mathbb{P}(|x_1| < \varepsilon)}.$$

We can estimate the numerator from below and above by

$$\mathbb{P}\big(\exists t : x_t > b + \varepsilon, |x_1| < \varepsilon\big) \leq \mathbb{P}\big(\exists t : x_t = b + tx_1, |x_1| < \varepsilon\big) \leq \mathbb{P}\big(\exists t : x_t \geq b - \varepsilon, |x_1| < \varepsilon\big).$$

Let us first analyze the upper bound. If we define a hitting time $\tau = \inf\{t : x_t = b - \varepsilon\}$ then $x_\tau = b - \varepsilon$ and

$$\mathbb{P}\big(\exists t : x_t \geq b - \varepsilon, |x_1| < \varepsilon\big) = \mathbb{P}\big(\tau \leq 1, |x_1| < \varepsilon\big) = \mathbb{P}\big(\tau \leq 1, x_1 - x_\tau \in (-b, -b + 2\varepsilon)\big).$$

For dyadic approximation as above

$$\begin{aligned}
\mathbb{P}(\tau_n \leq 1, x_1 - x_{\tau_n} \in (-b, -b + 2\varepsilon)) &= \sum_{k\geq 0} \mathbb{P}\Big(\tau_n = \frac{k}{2^n} \leq 1, x_1 - x_{k/2^n} \in (-b, -b + 2\varepsilon)\Big) \\
&= \sum_{k\geq 0} \mathbb{P}\Big(\tau_n = \frac{k}{2^n} \leq 1\Big)\mathbb{P}\Big(x_1 - x_{k/2^n} \in (-b, -b + 2\varepsilon)\Big) \\
&= \sum_{k\geq 0} \mathbb{P}\Big(\tau_n = \frac{k}{2^n} \leq 1\Big)\mathbb{P}\Big(x_1 - x_{k/2^n} \in (b - 2\varepsilon, b)\Big) \\
&= \mathbb{P}(\tau_n \leq 1, x_1 - x_{\tau_n} \in (b - 2\varepsilon, b))
\end{aligned}$$

where in the third line we used the fact that the distribution of $x_1 - x_{k/2^n}$ is symmetric around zero and, thus, we "reflected" the Brownian motion after stopping time τ as in figure 25.1. Therefore, in the limit $n \to \infty$ we get

$$\mathbb{P}\big(\exists t : x_t \geq b - \varepsilon, |x_1| < \varepsilon\big) = \mathbb{P}\big(\tau \leq 1, x_1 - x_\tau \in (b - 2\varepsilon, b)\big) = \mathbb{P}\big(x_1 \in (2b - 3\varepsilon, 2b - \varepsilon)\big)$$

because the fact that $x_1 \in (2b - 3\varepsilon, 2b - \varepsilon)$ automatically implies that $\tau \leq 1$ for $b > 0$ and ε small enough. Finally, this proves that

$$\mathbb{P}(\exists t : x_t = b) \leq \frac{\mathbb{P}(x_1 \in (2b - 3\varepsilon, 2b - \varepsilon))}{\mathbb{P}(x_1 \in (-\varepsilon, \varepsilon))} \to e^{-2b^2}$$

as $\varepsilon \to 0$. The lower bound can be analyzed similarly.

□

Theorem 60 *(Kolmogorov-Smirnov) If y_t is the Brownian bridge then for all $b > 0$,*

$$\mathbb{P}\Big(\sup_{0\le t\le 1}|y_t| \ge b\Big) = 2\sum_{n\ge 1}(-1)^{n-1}e^{-2n^2b^2}.$$

Proof. For $n \ge 1$, consider an event

$$A_n = \big\{\exists t_1 < \dots < t_n \le 1 : y_{t_j} = (-1)^{j-1}b\big\}$$

and let τ_b and τ_{-b} be the hitting times of b and $-b$. By symmetry of the distribution of the process y_t,

$$\mathbb{P}\Big(\sup_{0\le t\le 1}|y_t| \ge b\Big) = \mathbb{P}\big(\tau_b \text{ or } \tau_{-b} \le 1\big) = 2\mathbb{P}(A_1, \tau_b < \tau_{-b}).$$

Again, by symmetry,

$$\mathbb{P}(A_n, \tau_b < \tau_{-b}) = \mathbb{P}(A_n) - \mathbb{P}(A_n, \tau_{-b} < \tau_b) = \mathbb{P}(A_n) - \mathbb{P}(A_{n+1}, \tau_b < \tau_{-b}).$$

By induction,

$$\mathbb{P}(A_1, \tau_b < \tau_{-b}) = \mathbb{P}(A_1) - \mathbb{P}(A_2) + \ldots + (-1)^{n-1}\mathbb{P}(A_n, \tau_b < \tau_{-b}).$$

As in Theorem 59, reflecting the Brownian motion each time we hit b or $-b$, one can show that

$$\mathbb{P}(A_n) = \lim_{\varepsilon\to 0}\frac{\mathbb{P}\big(x_1 \in (2nb-\varepsilon, 2nb+\varepsilon)\big)}{\mathbb{P}\big(x_1 \in (-\varepsilon,\varepsilon)\big)} = e^{-\frac{1}{2}(2nb)^2} = e^{-2n^2b^2}$$

and this finishes the proof.

□

Given $a, b > 0$, let us compute the probability that a Brownian bridge crosses one of the levels $-a$ or b.

Theorem 61 *(Two-sided boundary) If $a, b > 0$ then*

$$\mathbb{P}(\exists t : y_t = -a \text{ or } b) = \sum_{n\ge 0}\Big(e^{-2(na+(n+1)b)^2} + e^{-2((n+1)a+nb)^2}\Big) - \sum_{n\ge 1}2e^{-2n^2(a+b)^2}. \qquad (25.0.2)$$

Proof. We have

$$\mathbb{P}(\exists t : y_t = -a \text{ or } b) = \mathbb{P}(\exists t : y_t = -a, \tau_{-a} < \tau_b) + \mathbb{P}(\exists t : y_t = b, \tau_b < \tau_{-a}).$$

If we introduce the events

$$B_n = \big\{\exists t_1 < \ldots < t_n : y_{t_1} = b, y_{t_2} = -a, \ldots\big\}$$

and

$$A_n = \big\{\exists t_1 < \ldots < t_n : y_{t_1} = -a, y_{t_2} = b, \ldots\big\}$$

then, as in the previous theorem,

$$\mathbb{P}(B_n, \tau_b < \tau_{-a}) = \mathbb{P}(B_n) - \mathbb{P}(B_n, \tau_{-a} < \tau_b) = \mathbb{P}(B_n) - \mathbb{P}(A_{n+1}, \tau_{-a} < \tau_b)$$

and, similarly,

$$\mathbb{P}(A_n, \tau_{-a} < \tau_b) = \mathbb{P}(A_n) - \mathbb{P}(B_{n+1}, \tau_b < \tau_{-a}).$$

By induction,

$$\mathbb{P}(\exists t : y_t = -a \text{ or } b) = \sum_{n=1}^{\infty}(-1)^{n-1}(\mathbb{P}(A_n) + \mathbb{P}(B_n)).$$

Probabilities of the events A_n and B_n can be computed using the reflection principle as above,

$$\mathbb{P}(A_{2n}) = \mathbb{P}(B_{2n}) = e^{-2n^2(a+b)^2},\ \ \mathbb{P}(B_{2n+1}) = e^{-2(na+(n+1)b)^2},\ \ \mathbb{P}(A_{2n+1}) = e^{-2((n+1)a+nb)^2}$$

and this finishes the proof.

□

If $X = -\inf y_t$ and $Y = \sup y_t$ then *the spread* of the process y_t is $\xi = X + Y$.

Theorem 62 *(Distribution of the spread) For any $t > 0$,*

$$\mathbb{P}\big(\xi \leq t\big) = 1 - \sum_{n\geq 1}(8n^2t^2 - 2)e^{-2n^2t^2}.$$

Proof. First of all, (25.0.2) gives the joint c.d.f. of (X, Y) because

$$F(a,b) = \mathbb{P}(X < a, Y < b) = \mathbb{P}(-a < \inf y_t, \sup y_t < b) = 1 - \mathbb{P}(\exists t : y_t = -a \text{ or } b).$$

If $f(a,b) = \partial^2 F/\partial a \partial b$ is the joint p.d.f. of (X, Y) then the c.d.f of the spread $X + Y$ is

$$\mathbb{P}\big(Y + X \leq t\big) = \int_0^t \int_0^{t-a} f(a,b)\, db\, da.$$

The inner integral is

$$\int_0^{t-a} f(a,b)db = \frac{\partial F}{\partial a}(a, t-a) - \frac{\partial F}{\partial a}(a, 0).$$

Since

$$\begin{aligned}
\frac{\partial F}{\partial a}(a,b) &= \sum_{n\geq 0} 4n\big(na + (n+1)b\big)\, e^{-2(na+(n+1)b)^2} \\
&+ \sum_{n\geq 0} 4(n+1)\big((n+1)a + nb\big)\, e^{-2((n+1)a+nb)^2} \\
&- \sum_{n\geq 1} 8n^2(a+b)\, e^{-2n^2(a+b)^2},
\end{aligned}$$

plugging in the values $b = t - a$ and $b = 0$ gives

$$\int_0^{t-a} f(a,b)db = \sum_{n\geq 0} 4n((n+1)t - a)e^{-2((n+1)t-a)^2} + \sum_{n\geq 0} 4(n+1)(nt + a)e^{-2(nt+a)^2} - \sum_{n\geq 1} 8n^2te^{-2n^2t^2}.$$

Integrating over $a \in [0, t]$,

$$\begin{aligned}
\mathbb{P}\big(Y + X \leq t\big) &= \sum_{n\geq 0}(2n+1)\Big(e^{-2n^2t^2} - e^{-2(n+1)^2t^2}\Big) - \sum_{n\geq 1} 8n^2t^2e^{-2n^2t^2} \\
&= 1 + 2\sum_{n\geq 1} e^{-2n^2t^2} - \sum_{n\geq 1} 8n^2t^2e^{-2n^2t^2},
\end{aligned}$$

and this finishes the proof.

□