18.175 Theory of Probability
Fall 2008

Section 27

Laws of the Iterated Logarithm.

For convenience of notations let us denote $\ell(t) = \log\log t$.

Theorem 65 *(LIL) Let W_t be the Brownian motion and $u(t) = \sqrt{2t\ell(t)}$. Then*

$$\limsup_{t\to\infty} \frac{W_t}{u(t)} = 1.$$

Let us briefly describe the main idea that gives origin to the function $u(t)$. For $a > 1$, consider a geometric sequence $t = a^k$ and take a look at the probabilities of the following events

$$\begin{aligned}\mathbb{P}\Big(W_{a^k} \geq Lu(a^k)\Big) = \mathbb{P}\Big(\frac{W_{a^k}}{\sqrt{a^k}} \geq \frac{Lu(a^k)}{\sqrt{a^k}}\Big) &\sim \frac{1}{\sqrt{2\pi}} \frac{1}{L\sqrt{2\ell(a^k)}} \exp\Big(-\frac{1}{2}\,\frac{L^2 2a^k \ell(a^k)}{a^k}\Big) \\ &\sim \frac{1}{\sqrt{2\pi}} \frac{1}{L\sqrt{2\ell(a^k)}} \Big(\frac{1}{k\log a}\Big)^{L^2}. \qquad (27.0.1)\end{aligned}$$

This series will converge or diverge depending on whether $L > 1$ or $L < 1$. Even though these events are not independent in some sense they are "almost independent" and the Borel-Cantelli lemma would imply that the upper limit of W_{a^k} behaves like $u(a^k)$. Some technical work will complete this main idea. Let us start with the following.

Lemma 49 *For any $\varepsilon > 0$,*

$$\limsup_{s\to\infty} \ \sup\Big\{\frac{|W_t - W_s|}{u(s)} \ : \ s \leq t \leq (1+\varepsilon)s\Big\} \leq 4\sqrt{\varepsilon} \quad a.s.$$

Proof. Let $\varepsilon, \alpha > 0$, $t_k = (1+\varepsilon)^k$ and $M_k = \alpha u(t_k)$. By symmetry, (25.0.1) and the Gaussian tail estimate in Lemma 46

$$\begin{aligned}\mathbb{P}\Big(\sup_{t_k \leq t \leq t_{k+1}} |W_t - W_{t_k}| \geq M_k\Big) &\leq 2\mathbb{P}\Big(\sup_{0\leq t\leq t_{k+1}-t_k} W_t \geq M_k\Big) \\ &= 4\mathcal{N}(0, t_{k+1} - t_k)(M_k, \infty) \leq 4\exp\Big(-\frac{1}{2}\frac{M_k^2}{(t_{k+1}-t_k)}\Big) \\ &\leq 4\exp\Big(-\frac{\alpha^2 2t_k\ell(t_k)}{2\varepsilon t_k}\Big) = 4\Big(\frac{1}{k\log(1+\varepsilon)}\Big)^{\frac{\alpha^2}{\varepsilon}}.\end{aligned}$$

If $\alpha^2 > \varepsilon$, the sum of these probabilities converges and by the Borel-Cantelli lemma, for large enough k,

$$\sup_{t_k\leq t\leq t_{k+1}} |W_t - W_{t_k}| \leq \alpha u(t_k).$$

It is easy to see that for small enough ε, $u(t_{k+1})/u(t_k) < 1+\varepsilon \le 2$. If k is such that $t_k \le s \le t_{k+1}$ then, clearly, $t_k \le s \le t \le t_{k+2}$ and, therefore, for large enough k,

$$|W_t - W_s| \le 2\alpha u(t_k) + \alpha u(t_{k+1}) \le (2\alpha + \alpha(1+\varepsilon))u(s) \le 4\alpha u(s).$$

Letting $\alpha \to \sqrt{\varepsilon}$ over some sequence finishes the proof.

□

Proof of Theorem 65. For $L = 1+\gamma > 1$, (27.0.1) and the Borel-Cantelli lemma imply that

$$W_{t_k} \le (1+\gamma)u(t_k)$$

for large enough k. If $t_k = (1+\varepsilon)^k$ then Lemma 49 implies that with probability one for large enough t (if $t_k \le t < t_{k+1}$)

$$\frac{W_t}{u(t)} = \frac{W_{t_k}}{u(t_k)}\frac{u(t_k)}{u(t)} + \frac{W_t - W_{t_k}}{u(t_k)}\frac{u(t_k)}{u(t)} \le (1+\gamma) + 4\sqrt{\varepsilon}.$$

Letting $\varepsilon, \gamma \to 0$ over some sequences proves that with probability one

$$\limsup_{t\to\infty} \frac{W_t}{u(t)} \le 1.$$

To prove that upper limit is equal to one we will use the Borel-Cantelli lemma for independent increments $W_{a^k} - W_{a^{k-1}}$ for large values of the parameter $a > 1$. If $0 < \gamma < 1$ then, similarly to (27.0.1),

$$\mathbb{P}\Big(W_{a^k} - W_{a^{k-1}} \ge (1-\gamma)u(a^k - a^{k-1})\Big) \sim \frac{1}{\sqrt{2\pi}}\frac{1}{(1-\gamma)\sqrt{2\ell(a^k - a^{k-1})}}\Big(\frac{1}{\log(a^k - a^{k-1})}\Big)^{(1-\gamma)^2}.$$

The series diverges and, since these events are independent, they occur infinitely often with probability one. We already proved (by (27.0.1)) that for $\varepsilon > 0$ for large enough k, $W_{a^k}/u(a^k) \le 1+\varepsilon$ and, therefore, by symmetry $W_{a^k}/u(a^k) \ge -(1+\varepsilon)$. This gives

$$\begin{aligned}
\frac{W_{a^k}}{u(a^k)} &\ge (1-\gamma)\frac{u(a^k - a^{k-1})}{u(a^k)} + \frac{W_{a^{k-1}}}{u(a^k)} \\
&\ge (1-\gamma)\frac{u(a^k - a^{k-1})}{u(a^k)} - (1+\varepsilon)\frac{u(a^{k-1})}{u(a^k)} \\
&= (1-\gamma)\sqrt{\frac{(a^k - a^{k-1})\ell(a^k - a^{k-1})}{a^k\ell(a^k)}} - (1+\varepsilon)\sqrt{\frac{a^{k-1}\ell(a^{k-1})}{a^k\ell(a^k)}}
\end{aligned}$$

and

$$\limsup_{t\to\infty}\frac{W_t}{u(t)} \ge \limsup_{k\to\infty}\frac{W_{a^k}}{u(a^k)} \ge (1-\gamma)\sqrt{\Big(1-\frac{1}{a}\Big)} - (1+\varepsilon)\sqrt{\frac{1}{a}}.$$

Letting $\gamma \to 0$ and $a \to \infty$ over some sequences proves that the upper limit is equal to one.

□

The LIL for Brownian motion will imply the LIL for sums of independent random variables via Skorohod's imbedding.

Theorem 66 *Suppose that $Y_1, \ldots, Y_n$ are i.i.d. and $\mathbb{E}Y_i = 0, \mathbb{E}Y_i^2 = 1$. If $S_n = Y_1 + \ldots + Y_n$ then*

$$\limsup_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}} = 1 \ \textit{a.s.}$$

Proof. Let us define a stopping time $\tau(1)$ such that $W_{\tau(1)} \stackrel{\mathcal{L}}{=} Y_1$. By Markov property, the increment of the process after stopping time is independent of the process before stopping time and has the law of the Brownian motion. Therefore, we can define $\tau(2)$ such that $W_{\tau(1)+\tau(2)} - W_{\tau(1)} \stackrel{\mathcal{L}}{=} Y_2$ and, by independence,

$W_{\tau(1)+\tau(2)} \stackrel{\mathcal{L}}{=} Y_1 + Y_2$ and $\tau(1), \tau(2)$ are i.i.d. By induction, we can define i.i.d. $\tau(1), \ldots, \tau(n)$ such that $S_n \stackrel{\mathcal{L}}{=} W_{T(n)}$ where $T(n) = \tau(1) + \ldots + \tau(n)$. We have

$$\frac{S_n}{u(n)} \stackrel{\mathcal{L}}{=} \frac{W_{T(n)}}{u(n)} = \frac{W_n}{u(n)} + \frac{W_{T(n)} - W_n}{u(n)}.$$

By the LIL for the Brownian motion,

$$\limsup_{n\to\infty} \frac{W_n}{u(n)} = 1.$$

By the strong law of large numbers, $T(n)/n \to \mathbb{E}\tau(1) = \mathbb{E}Y_1^2 = 1$ a.s. For any $\varepsilon > 0$, Lemma 49 implies that for large n

$$\frac{|W_{T(n)} - W_n|}{u(n)} \le 4\sqrt{\varepsilon}$$

and letting $\varepsilon \to 0$ finishes the proof.

□

LIL for Brownian motion also implies a *local LIL*:

$$\limsup_{t\to 0} \frac{W_t}{\sqrt{2t\ell(1/t)}} = 1.$$

It is easy to check that if W_t is a Brownian motion then $tW_{1/t}$ is also the Brownian motion and the result follows by a change of variable $t \to 1/t$. To check that $tW_{1/t}$ is a Brownian motion notice that for $t < s$,

$$\mathbb{E}tW_{1/t}\Big(sW_{1/s} - tW_{1/t}\Big) = st\frac{1}{s} - t^2\frac{1}{t} = t - t = 0$$

and

$$\mathbb{E}\Big(tW_{1/t} - sW_{1/s}\Big)^2 = t + s - 2t = s - t.$$

□