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Construction of bosonic symmetry-protected-trivial states and their topological invariants via $G \times SO(\infty)$ nonlinear σ models

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It has been shown that the bosonic symmetry-protected-trivial (SPT) phases with pure gauge anomalous boundary can all be realized via nonlinear σ models (NLσMs) of the symmetry group G with various topological terms. Those SPT phases (called the pure SPT phases) can be classified by group cohomology $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. But there are also SPT phases with mixed gauge-gravity anomalous boundary (which will be called the mixed SPT phases). Some of the mixed SPT states were also referred as the beyond-group-cohomology SPT states. In this paper, we show that those beyond-group-cohomology SPT states are actually within another type of group cohomology classification. More precisely, we show that both the pure and the mixed SPT phases can be realized by $G \times SO(\infty)$ NLσMs with various topological terms. Through the group cohomology $\mathcal{H}^d[G \times SO(\infty),\mathbb{R}/\mathbb{Z}]$, we find that the set of our constructed SPT phases in d-dimensional space-time are described by $E^d(G) \rtimes \oplus_{k=1}^{d-1}\mathcal{H}^k(G,\text{iTO}_L^{d-k}) \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ where G may contain time reversal. Here iTO_L^d is the set of the topologically ordered phases in d-dimensional space-time that have no topological excitations, and one has $\text{iTO}_L^1 = \text{iTO}_L^2 = \text{iTO}_L^4 = \text{iTO}_L^6 = 0$, $\text{iTO}_L^3 = \mathbb{Z}$, $\text{iTO}_L^5 = \mathbb{Z}_2$, $\text{iTO}_L^7 = 2\mathbb{Z}$. For $G = U(1) \rtimes Z_2^T$ (charge conservation and time-reversal symmetry), we find that the mixed SPT phases beyond $\mathcal{H}^d[U(1) \rtimes Z_2^T,\mathbb{R}/\mathbb{Z}]$ are described by $\mathbb{Z}_2$ in $3+1$D, $\mathbb{Z}$ in $4+1$D, $3\mathbb{Z}_2$ in $5+1$D, and $4\mathbb{Z}_2$ in $6+1$D. Our construction also gives us the topological invariants that fully characterize the corresponding SPT and iTO phases. Through several examples, we show how the universal physical properties of SPT phases can be obtained from those topological invariants.

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I. INTRODUCTION AND RESULTS

A. Gapped quantum liquid without topological excitations

In 2009, in a study of the Haldane phase [1] of spin-1 chain using space-time tensor network [2], it was found that, from the entanglement point of view, the Haldane state is really a trivial product state. So the nontrivialness of Haldane phase must be contained in the way how symmetry and short-range entanglement [3] get intertwined. This led to the notion of *symmetry-protected-trivial (SPT) order* (also known as *symmetry-protected-topological order*). Shortly after, the concept of SPT order allowed us to classify [4–6] all $1+1$D gapped phases for interacting bosons/spins and fermions [7–11]. This result is quickly generalized to higher dimensions where a large class of SPT phases is constructed using group cohomology theory [12–14].

Such a higher-dimension construction is based on G nonlinear σ model (NLσM)[13–15]

$$\mathcal{L} = \frac{1}{\lambda}(\partial g)^2 + i\,L_{\text{top}}^d(g^{-1}\partial g), \quad g(x) \in G, \tag{1}$$

with topological term L_{top}^d in $\lambda \to \infty$ limit. Since the topological term L_{top}^d is classified by the elements in group cohomology class $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ [13–15], this allows us to show that such kind of SPT states are classified by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. (See Appendix A for an introduction of group cohomology.) Later, it was realized that there also exist time-reversal-protected SPT states that are beyond the $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ description [16–18].

We like to point out that there are many other ways to construct SPT states, which include Chern-Simons theories [19,20], NLσMs of symmetric space [16,21–25], projective construction [26–28], domain wall decoration [29], string-net [18], layered construction [17], higher gauge theories [30–32], etc.

SPT states are *gapped quantum liquids* [3,33], characterized by having no *topological excitations* [34,35], and having no *topological order* [36–39]. E_8 bosonic quantum Hall state [19,40] described by the E_8 K-matrix [41–47] is also a gapped quantum liquid with no topological excitations, but it has a nontrivial topological order. We will refer such kind of topologically ordered states as invertible topologically ordered (iTO) states [35,48] (see Table I). Bosonic SPT and iTO states are the simplest kind of gapped quantum liquids. In this paper, we will try to develop a systematic theory for those phases. The main result is Eq. (33) which generalizes the $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ description of the SPT phases, so that the new description also includes the time-reversal-protected SPT phases beyond the $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ description. This result is derived in Sec. V. Applying Eq. (33) to simple symmetry groups, we obtain Table II for the SPT phases produced by NLσMs.

B. Probing SPT phases and topological invariants

The above is about the construction of SPT states. But how to probe and measure different SPT orders in the ground state of a generic system? The SPT states have no topological order. Thus, their fixed-point partition function $Z_{\text{fixed}}(M^d)$ on a closed space-time manifold M^d is trivial $Z_{\text{fixed}}(M^d) = 1$ [35], and cannot be used to probe different SPT orders. However, if we add the G-symmetry twists [49,50] to the space-time by gauging the onsite symmetry G [51–53], we may get a nontrivial fixed-point partition function $Z_{\text{fixed}}(M^d,A) \in U(1)$*which is a pure* $U(1)$ *phase* [35] that depends on A. Here, A is the background nondynamical gauge field that describes the symmetry twist. The fixed-point partition

TABLE I. The L-type bosonic iTO phases realized by the $SO(\infty)$ NLσMs in d-dimensional space-time form an Abelian group σiTO_L^d. (The meaning of "L-type" is defined in Sec. I D, and one can ignore such a qualifier in the first reading.) More general L-type bosonic iTO phases realized by the NLσMs, Chern-Simons theories, etc., form a bigger Abelian group iTO_L^d. The generating topological invariants $W_{\text{top}}^d(\Gamma)$ are also listed.

Dim.	σiTO_L^d	W_{top}^d	iTO_L^d	W_{top}^d
$d=0+1$	0		0	
$d=1+1$	0		0	
$d=2+1$	$\mathbb{Z}$	ω_3	$\mathbb{Z}$	$\frac{1}{3}\omega_3$
$d=3+1$	0		0	
$d=4+1$	$\mathbb{Z}_2$	$\frac{1}{2}\text{w}_2\text{w}_3$	$\mathbb{Z}_2$	$\frac{1}{2}\text{w}_2\text{w}_3$
$d=5+1$	0		0	
$d=6+1$	$2\mathbb{Z}$	$\omega_7^{p_1^2}$, $\omega_7^{p_2}$	$2\mathbb{Z}$	$\frac{\omega_7^{p_1^2}-2\omega_7^{p_2}}{5}$, $\frac{-2\omega_7^{p_1^2}+5\omega_7^{p_2}}{9}$

TABLE III. (Color online) The L-type $U(1)$ SPT phases.

$d=$	$\text{LSPT}_{U(1)}^d$	Generators W_{top}^d
$0+1$	$\mathbb{Z}$	a
$1+1$	0	
$2+1$	$\mathbb{Z}$	ac_1
$3+1$	0	
$4+1$	$\mathbb{Z}\oplus\mathbb{Z}$	ac_1^2, $\frac{1}{3}ap_1$
$5+1$	0	
$6+1$	$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$	ac_1^3, $\frac{1}{3}ac_1p_1$, $\frac{1}{2}\text{w}_2\text{w}_3c_1$

function $Z_{\text{fixed}}(M^d,A)$ is robust against any smooth change of the local Lagrangian $\mathcal{L}$ that preserve the symmetry, and is a topological invariant. Such type of topological invariants should completely describe the SPT states that have no topological order. In this paper, we will express such universal fixed-point partition function in terms of topological invariant W_{top}^d (which is a d-form, or more precisely, a d-cocycle):

$$Z_{\text{fixed}}(M^d,A) = e^{i\int_{M^d} 2\pi W_{\text{top}}^d(A,\Gamma)}, \tag{2}$$

where Γ is the connection on M^d. We will use W_{top}^d to characterize the SPT phases.

Even without the symmetry, the fixed-point partition function $Z_{\text{fixed}}(M^d)$ can still be a pure $U(1)$ phase that depends on the topologies of space-time. In this case, the fixed-point partition function describes an iTO state [35]. Thus, we can also use W_{top}^d to characterize the iTO phases. We believe that the function $e^{i\int_{M^d} 2\pi W_{\text{top}}^d(A,\Gamma)}$ that maps various closed space-time manifolds M^d with various G-symmetry twist A to the $U(1)$ value, completely characterizes the iTO phases and the SPT phases [49]. So in this paper, we will often use W_{top}^d to label/describe iTO and SPT phases.

We like to point out that the topological invariant $W_{\text{top}}^d(A)$ is given by a cocycle ω_d in $\mathcal{H}^d[G\times SO(\infty),\mathbb{R}/\mathbb{Z}]$. Equation (58) tells us how to calculate $e^{i\int_{M^d} 2\pi W_{\text{top}}^d(A)}$, from ω_d, the space-time manifold M^d, and the symmetry twist A. So $e^{i\int_{M^d} 2\pi W_{\text{top}}^d(A)}$ is well defined.

C. Simple SPT phases and their physical properties

In Tables III–VIII, we list the generators $W_{\text{top}}^d(A,\Gamma)$ of those topological invariants for simple SPT phases. The $U(1)$-symmetry twist on the space-time M^d is described by a vector potential one-form A and the Z_n-symmetry twist is described by a vector potential one-form A_{Z_n} with vanishing curl that satisfies $\oint A_{Z_n}=0 \bmod 2\pi/n$. However, in the tables, we use the normalized one-form $a\equiv A/2\pi$ and $a_1\equiv nA_{Z_n}/2\pi$. Also in the table, $c_1=da$ is the first Chern-Class, w_i is the Stiefel-Whitney classes, and p_1 the first Pontryagin classes for the tangent bundle of M^d. The results in black are for the pure SPT phases [which are defined as the SPT phases described by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$]. The results in blue are for the mixed SPT phases described by $\oplus_{k=1}^{d-1}H^k(BG,\text{iTO}_L^{d-k})$. The results in red are for the extra mixed SPT phases described by $E^d(G)$ [see Eq. (33)].

Those topological invariants fully characterize the corresponding topological phases. All the universal physical properties [16,49,51,54–57] of the topological phases can be derived from those topological invariants. This is the approach used in [49]. In the following, we will discuss some of the simple cases as examples. We find that the topological invariants allow us to "see" and obtain many universal physical properties easily.

TABLE II. (Color online) The L-type bosonic SPT phases realized by the $G\times SO(\infty)$ NLσMs, which are described by $E^d(G)\rtimes\oplus_{k=1}^{d-1}H^k(BG,\text{iTO}_L^{d-k})\oplus\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. The results in black are the pure SPT phases described by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ first discovered in [13]. The pure SPT states have boundaries that carry only pure "gauge" anomaly. The results in blue are the mixed SPT phases described by $\oplus_{k=1}^{d-1}H^k(BG,\text{iTO}_L^{d-k})$. The results in red are the extra mixed SPT phases described by $E^d(G)$. The mixed SPT states have boundaries that carry mixed gauge-gravity anomaly.

Symmetry	$0+1$D	$1+1$D	$2+1$D	$3+1$D	$4+1$D	$5+1$D	$6+1$D
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0	$\mathbb{Z}_n\oplus\mathbb{Z}_n$	$\mathbb{Z}_{\langle n,2\rangle}$	$\mathbb{Z}_n\oplus\mathbb{Z}_n\oplus\mathbb{Z}_{\langle n,2\rangle}$
Z_2^T	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	0	$\mathbb{Z}_2\oplus 2\mathbb{Z}_2$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$
$U(1)\rtimes Z_2=O_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2\oplus\mathbb{Z}_2$	$2\mathbb{Z}_2\oplus 2\mathbb{Z}_2$	$\mathbb{Z}\oplus 2\mathbb{Z}_2\oplus\mathbb{Z}\oplus 3\mathbb{Z}_2$
$U(1)\times Z_2^T$	0	$2\mathbb{Z}_2$	0	$3\mathbb{Z}_2\oplus\mathbb{Z}_2$	0	$4\mathbb{Z}_2\oplus 3\mathbb{Z}_2$	$2\mathbb{Z}_2\oplus\mathbb{Z}_2$
$U(1)\rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}$	$2\mathbb{Z}_2\oplus 2\mathbb{Z}_2$	$2\mathbb{Z}_2\oplus 3\mathbb{Z}_2\oplus\mathbb{Z}_2$

TABLE IV. (Color online) The L-type Z_2 SPT phases.

$d =$	$\text{LSPT}^d_{Z_2}$	Generators W^d_{top}
$0+1$	$\mathbb{Z}_2$	$\frac{1}{2}a_1$
$1+1$	0	
$2+1$	$\mathbb{Z}_2$	$\frac{1}{2}a_1^3$
$3+1$	0	
$4+1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\frac{1}{2}a_1^5$, $\frac{1}{2}a_1 p_1$
$5+1$	$\mathbb{Z}_2$	$\frac{1}{2}a_1 \text{w}_2\text{w}_3$
$6+1$	$\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\frac{1}{2}a_1^7$, $\frac{1}{2}a_1^3 p_1$, $\frac{1}{2}a_1^2 \text{w}_2\text{w}_3$

1. $U(1)$ SPT states in Table III

The $0+1$D $U(1)$ SPT phases are classified by $k \in \mathbb{Z}$ with a gauge topological invariant

$$W^1_{\text{top}}(A) = k\frac{A}{2\pi}. \tag{3}$$

It describes a $U(1)$ symmetric ground state with charge k. The $\mathbb{Z}$ class of $2+1$D $U(1)$ SPT phases is generated by $W^3_{\text{top}} = ac_1$, or

$$W^3_{\text{top}}(A) = \frac{A\,dA}{(2\pi)^2}, \tag{4}$$

where $A\,dA$ is the wedge product of one-form A and two-form dA: $A\,dA = A \wedge dA$. Those SPT states have even-integer Hall conductances $\sigma_{xy} = \frac{\text{even}}{2\pi}$ [15,19,20,58].

The above are the pure $U(1)$ SPT states whose boundary has only pure $U(1)$ anomalies. The $\mathbb{Z}$ class of $4+1$D $U(1)$ SPT phases introduced in Ref. [59] are mixed SPT states. The generating state is described by (see Appendix I)

$$W^5_{\text{top}}(A,\Gamma) = \frac{1}{3}\frac{Ap_1}{2\pi} = -\frac{1}{3}\beta(A/2\pi)\omega_3 = -\frac{1}{3}\frac{dA}{2\pi}\omega_3, \tag{5}$$

where ω_3 is a gravitational Chern-Simons three-form: $d\omega_3 = p_1$. Also, β is the natural map $\beta : \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \to \mathcal{H}^{d+1}(G,\mathbb{Z})$ that maps $a \in \mathcal{H}^1[U(1),\mathbb{R}/\mathbb{Z}]$ to $\beta(a) = c_1 \in \mathcal{H}^2[U(1),\mathbb{Z}]$. One of the physical properties of such a state is its dimension reduction: we put the state on space-time $M^5 = M^2 \times M^3$ and put 2π $U(1)$ flux through M^2. In the large M^3 limit, the effective theory on M^3 is described by effective Lagrangian $W^3_{\text{top}}(\Gamma) = -\frac{1}{3}\omega_3$, which is a E_8 quantum Hall state with chiral central charge $c = 8$. If M^3 has a boundary, the boundary will carry the gapless chiral edge state of E_8 quantum Hall state. Note that the boundary of M^3 can be viewed as the core of a $U(1)$ monopole (which forms a loop in four spatial dimensions). So the core of a $U(1)$ monopole will carry the gapless chiral edge state of E_8 quantum Hall state.

TABLE V. (Color online) The L-type Z_2^T SPT phases.

$d =$	$\text{LSPT}^d_{Z_2^T}$	Generators W^d_{top}
$0+1$	0	
$1+1$	$\mathbb{Z}_2$	$\frac{1}{2}\text{w}_1^2$
$2+1$	0	
$3+1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\frac{1}{2}\text{w}_1^4$, $\frac{1}{2}p_1$
$4+1$	0	
$5+1$	$\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\frac{1}{2}\text{w}_1^6$, $\frac{1}{2}\text{w}_1^2 p_1$ $\frac{1}{2}\text{w}_1\text{w}_2\text{w}_3$
$6+1$	$\mathbb{Z}_2$	$\frac{1}{2}\text{w}_1^2\text{w}_2\text{w}_3$

TABLE VI. (Color online) The L-type $U(1) \times Z_2^T$ SPT phases.

$d =$	$\text{LSPT}^d_{U(1)\times Z_2^T}$	Generators W^d_{top}
$0+1$	0	$\frac{1}{2}\text{w}_1$
$1+1$	$2\mathbb{Z}_2$	$\frac{1}{2}\text{w}_1^2$, $\frac{1}{2}c_1$
$2+1$	0	
$3+1$	$3\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\frac{1}{2}c_1^2$, $\frac{1}{2}\text{w}_1^2 c_1$, $\frac{1}{2}\text{w}_1^4$, $\frac{1}{2}p_1$
$4+1$	0	
$5+1$	$4\mathbb{Z}_2$	$\frac{1}{2}c_1^3$, $\frac{1}{2}\text{w}_1^2 c_1^2$, $\frac{1}{2}\text{w}_1^4 c_1$, $\frac{1}{2}\text{w}_1^6$
	$3\mathbb{Z}_2$	$\frac{1}{2}c_1 p_1$, $\frac{1}{2}\text{w}_1^2 p_1$, $\frac{1}{2}\text{w}_1\text{w}_2\text{w}_3$
$6+1$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\frac{1}{2}c_1\text{w}_2\text{w}_3$, $\frac{1}{2}\text{w}_1^2\text{w}_2\text{w}_3$, $\frac{1}{2}\text{w}_1 c_1 p_1$

Since the monopole loop in 4D space can be viewed as a boundary of $U(1)$ vortex sheet in 4D space, the above physical probe also leads to a mechanism of the $U(1)$ SPT states: we start with a $U(1)$ symmetry breaking state. We then proliferate the $U(1)$ vortex sheets to restore the $U(1)$ to obtain a trivial $U(1)$ symmetric state. However, if we bind the E_8 state to the vortex sheets, proliferate the new $U(1)$ vortex sheets will produce a nontrivial $U(1)$ SPT state. In general, a probe of SPT state will often lead to a mechanism of the SPT state.

If the mixed $U(1)$ SPT state is realized by a continuum field theory, then we can have another topological invariant: we can put the state on a spatial manifold of topology $\mathbb{C}P^2$ or $T^4 = (S^1)^4$. Since $\int_{\mathbb{C}P^2}\frac{1}{3}p_1 - \int_{T^4}\frac{1}{3}p_1 = 1$, we find that the ground state on $\mathbb{C}P^2$ and on T^4 will carry different $U(1)$ charges (differ by one unit). We like to stress that the above result is a field theory result, which requires the lattice model to have a long correlation length much bigger than the lattice constant.

2. Z_2 SPT states in Table IV

The $2+1$D Z_2 SPT state described by

$$W^3_{\text{top}}(A_{Z_2}) = \tfrac{1}{2}a_1^3 \tag{6}$$

is the first discovered SPT state beyond $1+1$D [12]. Here, $a_1 = A_{Z_2}/2\pi$ is the Z_2 connection that describes the Z_2-symmetry twist on space-time. However, a_1^3 is not the wedge product of three one-forms: $a_1^3 \neq a_1 \wedge a_1 \wedge a_1$. a_1^3 is the cup product $a_1^3 \equiv a_1 \cup a_1 \cup a_1$, after we view a_1 as a 1-cocycle

TABLE VII. (Color online) The L-type $U(1) \rtimes Z_2^T$ SPT phases.

$d =$	$\text{LSPT}^d_{U(1)\rtimes Z_2^T}$	Generators W^d_{top}
$0+1$	$\mathbb{Z}$	a
$1+1$	$\mathbb{Z}_2$	$\frac{1}{2}\text{w}_1^2$
$2+1$	$\mathbb{Z}_2$	$\frac{1}{2}\text{w}_1 c_1$
$3+1$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\frac{1}{2}c_1^2$, $\frac{1}{2}\text{w}_1^4$, $\frac{1}{2}p_1$
$4+1$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$	ac_1^2, $\frac{1}{2}\text{w}_1^3 c_1$, $\frac{1}{3}ap_1$
$5+1$	$2\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\frac{1}{2}\text{w}_1^2 c_1^2$, $\frac{1}{2}\text{w}_1^6$, $\frac{1}{2}\text{w}_1^2 p_1$, $\frac{1}{2}\text{w}_1\text{w}_2\text{w}_3$
$6+1$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\frac{1}{2}\text{w}_1 c_1^3$, $\frac{1}{2}\text{w}_1^5 c_1$, $\frac{1}{2}\text{w}_1 c_1 p_1$
	$3\mathbb{Z}_2$	$\frac{1}{2}\text{w}_1 c_1 p_1$, $\frac{1}{2}c_1\text{w}_2\text{w}_3$, $\frac{1}{2}\text{w}_1^2\text{w}_2\text{w}_3$

TABLE VIII. (Color online) The L-type O_2 SPT phases.

$d=$	$\text{LSPT}^d_{O_2}$	Generators W^d_{top}
$0+1$	$\mathbb{Z}_2$	$\frac{1}{2}a_1$
$1+1$	$\mathbb{Z}_2$	$\frac{1}{2}c_1$
$2+1$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$ac_1,\ \frac{1}{2}a_1^3$
$3+1$	$\mathbb{Z}_2$	$\frac{1}{2}a_1^2c_1$
$4+1$	$2\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\frac{1}{2}a_1^5,\ \frac{1}{2}a_1c_1^2,\ \frac{1}{2}a_1p_1$
$5+1$	$2\mathbb{Z}_2\oplus 2\mathbb{Z}_2$	$\frac{1}{2}c_1^3,\ \frac{1}{2}a_1^4c_1,\ \frac{1}{2}c_1p_1,\ \frac{1}{2}a_1\mathrm{w}_2\mathrm{w}_3$
$6+1$	$\mathbb{Z}\oplus 2\mathbb{Z}_2$	$ac_1^3,\ \frac{1}{2}a_1^7,\ \frac{1}{2}a_1^3c_1^2$
	$\mathbb{Z}\oplus 3\mathbb{Z}_2$	$\frac{1}{3}c_1^2\omega_3,\ \frac{1}{2}a_1^3p_1,\ \frac{1}{2}a_1^2\mathrm{w}_2\mathrm{w}_3,\ \frac{1}{2}c_1\mathrm{w}_2\mathrm{w}_3,$

in $H^1(M^3,\mathbb{Z}_2)$. The cup product of cocycles generalizes the wedge product of differential forms.

But, how to compute the action amplitude $e^{i\int_{M^3}2\pi W^3_{\text{top}}(A_{Z_2})} = e^{i\pi\int_{M^3}a_1^3}$ that involves cup products? One can use the defining relation (58) to compute $e^{i\int_{M^3}2\pi W^3_{\text{top}}(A_{Z_2})}$. First, we note that the cocycle $a_1\in\mathcal{H}^1(Z_2,\mathbb{Z}_2)$ is given by

$$a_1(1)=1,\quad a_1(-1)=-1, \tag{7}$$

where $\{1,-1\}$ form the group Z_2. The cocylce for the cup product a_1^3 is simply given by

$$a_1^3(g_0,g_1,g_2)=a_1(g_0)a_1(g_1)a_1(g_2), \tag{8}$$

which is a cocycle in $\mathcal{H}^3(Z_2,\mathbb{Z}_2)$. Then, $\omega_3(g_0,g_1,g_2)=\frac{1}{2}a_1^3(g_0,g_1,g_2)$ is a cocycle in $\mathcal{H}^3(Z_2,\mathbb{R}/\mathbb{Z})$, that describes our Z_2 SPT state. This allows us to use Eq. (58) to compute $e^{i\int_{M^3}2\pi W^3_{\text{top}}(A_{Z_2})}$.

However, there are simpler ways to compute $e^{i\int_{M^3}2\pi W^3_{\text{top}}(A_{Z_2})}$. According to the Poincaré duality, an i-cocycle x_i in a d-dimensional manifold M^d is dual to a $(d-i)$-cycle [i.e., a $(d-i)$-dimensional closed sub-manifold] X^{d-i} in M^d. In our case, a_1 is dual to a 2D closed surface N^2 in M^3, and the 2D closed surface is the surface across which we perform the Z_2-symmetry twist. We will denote the Poincaré dual of x_i as $[x_i]^*=X^{d-i}$. Under the Poincaré duality, the cup product has a geometric meaning: Let X^{d-i} be the dual of x_i and Y^{d-j} be the dual of y_j. Then the cup product of $x_i\cup y_i$ is a $(i+j)$-cocycle z_{i+j}, whose dual is a $(d-i-j)$-cycle Z^{d-i-j}. We find that Z^{d-i-j} is simply the intersection of X^{d-i} and Y^{d-j}: $Z^{d-i-j}=X^{d-i}\cap Y^{d-j}$. In other words,

$$x_i\cup y_i = z_{i+j}\quad\longleftrightarrow\quad [x_i]^*\cap[y_i]^*=[z_{i+j}]^*. \tag{9}$$

So to calculate $\int_{M^3}a_1^3$, we need to choose three different 2D surfaces N_1^2, N_2^2, N_3^2 that describe the equivalent Z_2-symmetry twists a_1. Then

$$\int_{M^3}a_1^3 = \text{number of the points in } N_1^2\cap N_2^2\cap N_3^2 \text{ mod } 2. \tag{10}$$

There is another way to calculate $\int_{M^3}a_1^3$. Let N^2 be a 2D surface in space-time M^3 that describes the Z_2-symmetry twist a_1. We choose the space-time M^3 to have a form $M^2\times S^1$ where S^1 is the time direction. At each time slice, the surface

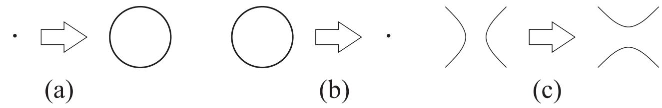

FIG. 1. (a) A loop creation. (b) A loop annihilation. (c) A line reconnection.

of symmetry twist N^2 becomes loops in the space M^2. Then (see Fig. 1)

$$\int_{M^3}a_1^3 = \text{number of loop creation/annihilation} + \text{number of line reconnection} \quad \text{mod } 2, \tag{11}$$

as we go around the time loop S^1. (Such a result leads to the picture in Ref. [51].)

To show the relation between Eqs. (10) and (11), we split each point on N^2 into three points 1, 2, 3 (see Fig. 2), which split N^2 into three nearby 2D surfaces N_1^2, N_2^2, and N_3^2. Then from Fig. 3, we can see the relation between Eqs. (10) and (11).

Equation (11) is consistent with the result in Ref. [50] where we considered a space-time $T^2\times I$, where $I=[0,1]$ is a 1D line segment for time $t\in[0,1]=I$. Then we added a Z_2-symmetry twist on a torus T^2 at $t=0$ [see Fig. 4(a)]. Next, we evolved such a Z_2 twist at $t=0$ to the one described by Fig. 4(c) at $t=1$, via the process Fig. 4(a) → Fig. 4(b)→ Fig. 4(c). Last, we clued the tori at $t=0$ and at $t=1$ together to form a closed space-time, *after we do a double Dehn twist on one of the tori*. Reference [50] showed that the value of the topological invariant on such a space-time with such a Z_2 twist is nontrivial: $\int_{M^3}a_1^3=1$ mod 2, through an explicit calculation. In this paper, we see that the nontrivial value comes from the fact that there is one line reconnection in the process Fig. 4(a) → Fig. 4(b) → Fig. 4(c).

Using the result (11), we can show that the end of the Z_2-symmetry twist line (which is called the monodromy defect [49]) must carry a fractional spin $\frac{1}{4}$ mod 1 and a semion fractional statistics [51].

Let us use $|\,\rangle_{\text{def}}$ to represent the many-body wave function with a monodromy defect. We first consider the spin of such a defect to see if the spin is fractionalized or not [60,61]. Under a 360° rotation, the monodromy defect (the end of Z_2-twist line) is changed to $|\,\rangle_{\text{def}}$. Since $|\,\rangle_{\text{def}}$ and $|\,\rangle_{\text{def}}$ are always different even after we deform and reconnect the Z_2-twist lines, $|\,\rangle_{\text{def}}$ is not an eigenstate of 360° rotation and does not carry a definite spin.

To construct the eigenstates of 360° rotation, let us make another 360° rotation to $|\,\rangle_{\text{def}}$. To do that, we first use the line

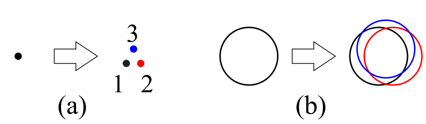

FIG. 2. (Color online) (a) A point is split into three points. (b) A surface N^2 is split into three surfaces N_1^2, N_2^2, N_3^2.

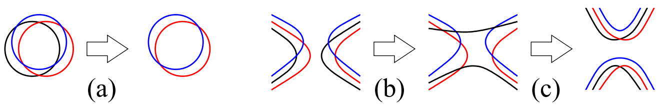

FIG. 3. (Color online) Loop annihilation: (a) as we shrink the black circle to a point, the black line sweeps across the intersection of the red and blue lines once. This means that N_1^2, N_2^2, N_3^2 intersect once in the loop annihilation/creation process. Line reconnection: as we deform the black lines in process (b), the black lines sweep across the intersection of red and blue lines once. But in process (c), no line sweeps across the intersection of the other two lines. This means that N_1^2, N_2^2, N_3^2 intersect once in the line reconnection process.

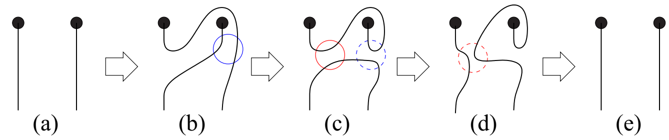

FIG. 5. (Color online) Deformation of the Z_2-twist lines and two reconnection moves, plus an exchange of two defects and a 360° rotation of one of the defects, change the configuration (a) back to itself. Note that from (a) to (b) we exchange the two defects, and from (d) to (e) we rotate one of the defects by 360°. The combination of those moves does not generate any phase since the number of the reconnection move is even.

reconnection move in Fig. 1(c) to change $|\circlearrowleft\rangle_{\text{def}} \to -|\circlearrowleft\rangle_{\text{def}}$. A 360° rotation on $|\circlearrowleft\rangle_{\text{def}}$ gives us $|\,|\,\rangle_{\text{def}}$.

We see that a 360° rotation changes $(|\,|\,\rangle_{\text{def}}, |\circlearrowleft\rangle_{\text{def}})$ to $(|\circlearrowleft\rangle_{\text{def}}, -|\circlearrowleft\rangle_{\text{def}})$. We find that $|\,|\,\rangle_{\text{def}} + i|\circlearrowleft\rangle_{\text{def}}$ is the eigenstate of the 360° rotation with eigenvalue $-i$, and $|\,|\,\rangle_{\text{def}} - i|\circlearrowleft\rangle_{\text{def}}$ is the other eigenstate of the 360° rotation with eigenvalue i. So the defect $|\,|\,\rangle_{\text{def}} + i|\circlearrowleft\rangle_{\text{def}}$ has a spin $-\frac{1}{4}$, and the defect $|\,|\,\rangle_{\text{def}} - i|\circlearrowleft\rangle_{\text{def}}$ has a spin $\frac{1}{4}$.

If one believes in the spin-statistics theorem, one may guess that the defects $|\,|\,\rangle_{\text{def}} + i|\circlearrowleft\rangle_{\text{def}}$ and $|\,|\,\rangle_{\text{def}} - i|\circlearrowleft\rangle_{\text{def}}$ are semions. This guess is indeed correct. From Fig. 5, we see that we can use deformation of Z_2-twist lines and two reconnection moves to generate an exchange of the two defects and a 360° rotation of one of the defects. Such operations allow us to show that Figs. 5(a) and 5(e) have the same amplitude, which means that an exchange of two defects followed by a 360° rotation of one of the defects does not generate any phase. This is nothing but the spin-statistics theorem.

The above understanding of geometric meaning of the topological invariant $\frac{1}{2}a_1^3$ in terms of Z_2-twist domain wall also leads to a mechanism of the Z_2 SPT state. Consider a quantum Ising model on 2D triangle lattice

$$H = -J \sum_{\langle ij\rangle} \sigma_i^z \sigma_j^z - g \sum_i \sigma_i^x, \tag{12}$$

where $\sigma^{x,y,z}$ are the Pauli matrices and $\langle ij\rangle$ are nearest neighbors. Such a model can be described by the path integral

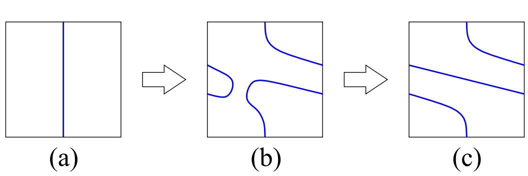

FIG. 4. (Color online) (a) A Z_2-symmetry twist on a torus. (c) The Z_2-symmetry twist obtained from (a) by double Dehn twist. ($\to$ $\to$ c) contains a line reconnection.

of the domain walls between $\sigma^z = 1$ and -1 in space-time. However, all domain walls in space-time have an amplitude of $+1$.

In order to have the nontrivial Z_2 SPT state, we need to modify the domain wall amplitudes in the path integral to allow them to have values ± 1. The ± 1 is assigned based on the following rules: as time evolves, a domain wall loop creation/annihilation will contribute to a -1 to the domain wall amplitude. A domain wall line reconnection will also contribute to a -1 to the domain wall amplitude. Those additional -1's can be implemented through local Hamiltonian. We simply need to modify the $-\sum_i \sigma_i^x$ term which creates the fluctuations of the domain walls:

$$H = -J \sum_{\langle ij\rangle} \sigma_i^z \sigma_j^z - g \sum_i \sigma_i^x \left(- e^{i\frac{\pi}{4}\sum_{\mu=1}^{6}(1-\sigma_{i,\mu}^z \sigma_{i,\mu+1}^z)} \right), \tag{13}$$

where $\sum_{\mu=1}^{6}(1-\sigma_{i,\mu}^z\sigma_{i,\mu+1}^z)$ is the sum over all six spins neighboring the site i. (In fact, we can set $J=0$.) The factor $-e^{i\frac{\pi}{4}\sum_{\mu=1}^{6}(1-\sigma_{i,\mu}^z\sigma_{i,\mu+1}^z)}$ contributes to a -1 when the spin flip generated by σ^x creates/annihilates a small loop of domain walls or causes a reconnection of the domain walls. The factor $-e^{i\frac{\pi}{4}\sum_{\mu=1}^{6}(1-\sigma_{i,\mu}^z\sigma_{i,\mu+1}^z)}$ contributes to a $+1$ when the spin flip only deforms the shape of the domain walls. This is the Hamiltonian obtained in Ref. [51].

Now let us switch to the $4+1$D Z_2 SPT described by (see Appendix I)

$$W_{\text{top}}^5(A) = \tfrac{1}{2}a_1 p_1 = \beta(a_1)\omega_3 = a_1^2\omega_3, \tag{14}$$

which is a new mixed SPT phase first discovered in this paper. Here β is the natural map $\beta : \mathcal{H}^d(G,\mathbb{Z}_2) \to \mathcal{H}^{d+1}(G,\mathbb{Z})$ that maps $a_1 \in \mathcal{H}^1(Z_2,\mathbb{Z}_2)$ to $\beta(a_1) = Sq^1(a_1) = a_1^2 \in \mathcal{H}^2(Z_2,\mathbb{Z})$ [see Appendix E and also Eq. (5)]. We note that $\int_M \frac{2}{3}p_1 = 0$ mod 2. Hence we can rewrite $p_1 = \frac{1}{3}p_1 + \frac{2}{3}p_1 = \frac{1}{3}p_1$ if we concern about mod 2 numbers. The above topological invariant can be rewritten as

$$W_{\text{top}}^5(A) = \tfrac{1}{2}a_1\tfrac{1}{3}p_1 = a_1^2\tfrac{1}{3}\omega_3. \tag{15}$$

One of the physical properties of such a Z_2 SPT state is its dimension reduction: we put the state on space-time $M^5 = S^2 \times M^3$ and choose the Z_2 twist a_1 to create two *identical* monodromy defects on S^2 (see Fig. 6). [The physics of two *identical* monodromy defects was discussed in detail in Ref. [49] and here we follow a similar approach. Also we may embed Z_2 into $U(1)$ and view the Z_2 monodromy defect

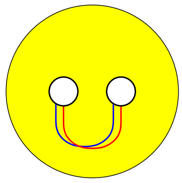

FIG. 6. (Color online) Two identical Z_2 monodromy defects on S^2. The boundary across which we do the Z_2 twist is split into the red and blue curves. Note that the splitting is identical at the two monodromy defects. The red and blue lines cross once, indicating that $\int_{S^2} a_1^2 = 1$.

as the $U(1)$ π flux.] For such a design of S^2 and a_1, we have $\int_{S^2} a_1^2 = 1 \bmod 2$ (see Fig. 6). We then take the large M^3 limit, and examine the induced effective theory on M^3. The induced effective Lagrangian must have a form $\mathcal{L} = 2\pi k \frac{1}{3}\omega_3$ with $k = 1 \bmod 2$, which describes a topologically ordered state with chiral central charge $8k$. If M^3 has a boundary, the boundary will carry the gapless chiral edge state of chiral central charge $8k$.

We like to remark that adding two Z_2 monodromy defects to S^2 is not a small perturbation. Inducing an E_8 bosonic quantum Hall state on M^3 by a large perturbation on S^2 does not imply the parent state on $S^2 \times M^3$ to be nontrivial. Even when the parent state is trivial, a large perturbation on S^2 can still induce an E_8 state on M^3. However, what we have shown is that *two identical* Z_2 monodromy defects on S^2 induce an *odd* number of E_8 states on M^3. This can happen only when the parent state on $S^2 \times M^3$ is nontrivial.

We may choose another dimension reduction by putting the state on space-time $M^5 = S^1 \times M^4$ and adding a Z_2 twist by threading a Z_2-flux line through the S^1. We then take the large M^4 limit. The effective theory on M^4 will be described by effective Lagrangian $\mathcal{L}_{\text{eff}} = \pi \frac{1}{3} p_1$. When M^4 has a boundary, $\partial M^4 \neq \emptyset$, the system on the $M^3 = \partial M^4$ must have chiral central charge $c = 4 \bmod 8$. In other words, if the four-dimensional space has a three-dimensional boundary $S^1 \times M^2$ and if we thread a Z_2-flux line through the S^1, then the state on M^2 will have a gravitational response described by a gravitational Chern-Simons effective Lagrangian $\mathcal{L}_{\text{eff}} = k\pi \frac{1}{3}\omega_3$, with $k = 1 \bmod 2$. Such a state on M^2 is either gapless or has a nontrivial topological order, regardless if the symmetry is broken on the boundary or not.

Let us assume that the Z_2 SPT state has a gapped symmetry breaking boundary. The above result implies that if we have a symmetry breaking domain wall on S^1, then the induced boundary state on M^2 must be topologically ordered with a chiral central charge $c = 4 \bmod 8$. (The mod 8 comes from the possibility that the modified local Hamiltonian at the domain wall may add several copies of E_8 bosonic quantum Hall states.) We see that a Z_2 symmetry breaking domain wall on the boundary carries a $2+1$D topologically ordered state with a chiral central charge $c = 4 \bmod 8$.

3. Z_2^T, $U(1) \times Z_2^T$, and $U(1) \rtimes Z_2^T$ SPT states in Tables V–VII

Tables V–VII list the so-called realizable topological invariants, which can be produced via our NLσM construction. The potential topological invariants (which may or may not be realizable) for those symmetries have been calculated in Ref. [62] using cobordism approach and in [48] using spectrum approach. For the topological invariants that generate the $\mathbb{Z}_2$ classes, our realizable topological invariants agree with the potential topological invariants obtained in Ref. [62]. For the topological invariants that generate the $\mathbb{Z}$ classes, our realizable topological invariants only form a subset of the potential topological invariant obtained in Refs. [48,62].

In $1+1$D, all those time-reversal-protected SPT phases contain one described by

$$W^2_{\text{top}}(A,\Gamma) = \tfrac{1}{2}\mathrm{w}_1^2. \tag{16}$$

Here, we would like to remark that time-reversal symmetry and space-time mirror reflection symmetry should be regarded as the same symmetry [62,63]. If a system has no time-reversal symmetry, then we can only use orientable space-time to probe it. Putting a system with no time-reversal symmetry on a nonorientable space-time is like adding a boundary to the system. If a system has a time-reversal symmetry, then we can use nonorientable space-time to probe it, and in this case, the Z_2^T twist is described by $a_1 = \mathrm{w}_1$. Since $\mathrm{w}_1 \neq 0$ only on nonorientable manifolds, the Z_2^T twist is nontrivial only on nonorientable manifolds. So we should use a nonorientable space-time to probe the above time-reversal-protected SPT phase. In fact, the above topological invariant can be detected on $\mathbb{R}P^2$: $\int_{\mathbb{R}P^2} \mathrm{w}_1^2 = 1 \bmod 2$ (see Fig. 7).

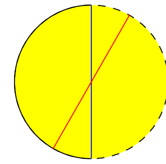

FIG. 7. (Color online) The shaded disk represents a two-dimensional manifold $\mathbb{R}P^2$, where the opposite points on the boundary are identified ($\hat{\boldsymbol{r}} \sim -\hat{\boldsymbol{r}}$). The blue and red lines are two noncontractible loops in $\mathbb{R}P^2$. Consider a Z_2 twist a_1 described by $[a_1]^* =$ a contractible loop. Then the blue and red lines represent the same Z_2 twist a_1. For such a Z_2 twist, we find that $\int_{\mathbb{R}P^2} a_1^2 = 1$ since the blue and red lines cross once. The above Z_2 twist is also the orientation reversing twist. So $a_1 = \mathrm{w}_1$ and we have $\int_{\mathbb{R}P^2} \mathrm{w}_1^2 = 1$.

In the following we will explain how the above topological invariant ensures the degenerate ground states at the boundaries of 1D space. We first consider the partition of a single boundary point over a time loop S^1 [see Fig. 8(a)]. Such a

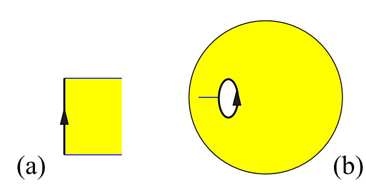

FIG. 8. (Color online) (a) The path integral of a single boundary point of 1D space over the time loop S^1. The shaded area represents the $1+1$D space-time. The two ends of the thick line are identified to form a loop S^1. The two blue lines are also identified. (b) The loop S^1 is extended to a sphere with a hole. The identified blue line is also shown.

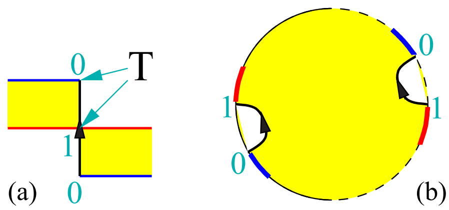

FIG. 9. (Color online) (a) The path integral of a single boundary point over the time loop S^1 with two time-reversal transformations at points 0 and 1. The shaded area represents the $1+1$D space-time. The two ends of the thick line are identified to form a loop S^1. The two blue lines are also identified after a horizontal reflection. The two red lines on the two sides of the thick line are identified as well after a horizontal reflection. (b) The shaded disk represents a two-dimensional manifold $\mathbb{R}P^2$, where the opposite points on the boundary are identified ($\hat{\boldsymbol{r}} \sim -\hat{\boldsymbol{r}}$). The loop S^1 in (a) is extended to the $\mathbb{R}P^2$ with a hole in (b). The two red lines and the two blue lines in (a) are also shown in (b).

partition function on S^1 is defined by first extending S^1 into a sphere with a hole S^2_{hole} [see Fig. 8(b)], and then we use the $1+1$D partition function defined on S^2_{hole} (from the path integral of $e^{i\int_{S^2_{\text{hole}}} W^2_{\text{top}}(A,\Gamma)}$) to define the partition function on S^1. We find that such a partition function on S^1 is trivial $Z=1$.

Now, we like to consider the partition of a single boundary point over a time loop S^1, but now with two time-reversal transformations inserted [see Fig. 9(a)], where the time reversal is implemented as mirror reflection in the transverse direction. Next, we extend Fig. 9(a) into a $\mathbb{R}P^2$ with a hole $\mathbb{R}P^2_{\text{hole}}$ [see Fig. 9(b)]. Since (after taking the small hole limit) $\int_{\mathbb{R}P^2} W^2_{\text{top}}(A,\Gamma) = \frac{1}{2}\int_{\mathbb{R}P^2} \mathrm{w}_1^2 = \frac{1}{2}$ mod 1, we find that the partition function on S^1 with two time-reversal transformations is nontrivial $Z=-1$. This implies that $T^2=-1$ when acting on the states on a single boundary point. The states on a single boundary must form Kramers doublets, and degenerate.

From the Tables V–VII, we also see that most generators of $3+1$D time-reversal SPT states are pure SPT states described by $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$. All mixed time-reversal SPT states are generated by a single generator

$$W^4_{\text{top}}(A,\Gamma) = \tfrac{1}{2}p_1, \tag{17}$$

which is a mixed Z_2^T SPT state [16]. In other words, all mixed time-reversal SPT states can be obtained from the pure SPT states by stacking with one copy of the above mixed Z_2^T SPT state.

4. $U(1) \rtimes Z_2 = O_2$ SPT states in Table VIII

The $1+1$D O_2 SPT state is characterized by the following topological invariant:

$$W^2_{\text{top}}(A) = \frac{1}{2}\frac{dA}{2\pi}. \tag{18}$$

Let us explain how such a topological invariant ensures the degenerate ground states at the boundaries of 1D space. Let us consider a $1+1$D space-time S^2_{hole} which is S^2 with a small hole [see Fig. 8(b)]. The partition function for S^2_{hole} can be viewed as the effective theory for the boundary $S^1 = \partial S^2_{\text{hole}}$, which is the partition function for a single boundary point of 1D space over the time loop S^1 [see Fig. 8(a)]. Since the partition function on S^2_{hole} changes sign as we add 2π $U(1)$ flux to S^2_{hole}, this means that a 2π $U(1)$ rotation acting on the states on a single boundary point will change the sign of the states. So the states on a single boundary point must form a projective representation of O_2 where the 2π $U(1)$ rotation is represented by -1. Such a projective representation is always even dimensional, and the states on a single boundary point must have an even degeneracy.

From the $2+1$D topological invariants, we see that the $2+1$D O_2 SPT is actually the $2+1$D $U(1)$ SPT state (by ignoring Z_2) and the $2+1$D Z_2 SPT state (by ignoring $U(1)$).

In $3+1$D, we have a pure $3+1$D O_2 SPT state described by

$$W^4_{\text{top}}(A) = \tfrac{1}{2}a_1^2 c_1, \tag{19}$$

which is a SPT state for quantum spin systems.

To construct a physical probe for the above $U(1) \rtimes Z_2$ SPT state, we first note that the topological invariant (19) is invariant under time reversal (mod 2π). So the corresponding $U(1) \rtimes Z_2$ SPT state is compatible with time-reversal symmetry. If we assume the $U(1) \rtimes Z_2$ SPT state also has the time-reversal symmetry, then we can design the following probe for the $U(1) \rtimes Z_2$ SPT state. We choose the $3+1$D space-time to be $S^2 \times M^2$, and put 2π $U(1)$ flux through S^2, where S^2 is actually a lattice. But such 2π flux is in a form of two *identical* thin π fluxes, with each π flux going through a single unit cell in S^2. Such a configuration has $\int_{S^2} c_1 = 1$ mod 2, and at the same time, does not break the $U(1) \rtimes Z_2$ symmetry.

In the large M^2 limit, the dimension-reduced theory on M^2 is described by a topological invariant $W^2_{\text{top}} = \frac{1}{2}a_1^2$. However, due to an identity $a_1^2 = \mathrm{w}_1 a_1$ in two-dimensional space, $\int_{M^2} a_1^2 = \int_{M^2} \mathrm{w}_1 a_1 = 0$ mod 2, if M^2 is orientable (since $\mathrm{w}_1 = 0$ iff the manifold is orientable). The topological invariant $W^2_{\text{top}} = \frac{1}{2}a_1^2$ can be detected only on nonorientable M^2. This is where we need the time-reversal symmetry: in the presence of time-reversal symmetry, we can use nonorientable M^2 to probe the topological invariant.

Let Z_2^t be the symmetry group generated by the combined Z_2 transformation and time-reversal Z_2^T transformation. Let a_1^t be the Z_2^t twist. Then we have $a_1^t = a_1 = \mathrm{w}_1$. Thus the topological invariant can be rewritten as $W^2_{\text{top}} = \frac{1}{2}\mathrm{w}_1^2$, which describes a $1+1$D SPT state protected by time-reversal symmetry Z_2^t.

We like to remark that threading two thin π-flux lines through S^2 is not a small perturbation. Inducing a Z_2^t SPT state on M^2 by a large perturbation on S^2 does not imply the parent state on $S^2 \times M^2$ to be nontrivial. Even when the parent state is trivial, a large perturbation on S^2 can still induce a Z_2^t SPT state on M^2. However, what we have shown is that threading *two identical* thin π-flux lines through S^2 induces *one* Z_2^t SPT state on M^2. This can happen only when the parent state on $S^2 \times M^2$ is nontrivial.

D. Realizable and potential topological invariants

After discussing the physical consequences of various topological invariants, let us turn to study the topological invariants themselves. It turns out that the topological invariants for iTO states satisfy many self-consistent conditions. Solving those conditions allows us to obtain self-consistent topological invariants, which will be called *potential* gauge-gravity topological invariants. References [48,59,62–64] studied the topological invariants from this angle and only the potential gauge-gravity topological invariants are studied. For example, when there is no symmetry, the following type of potential gauge-gravity topological invariants were found: (1) The $2+1$D potential gravitational topological invariants are described by $\mathbb{Z}$ [35,40,65,66], which are generated by

$$W^3_{\text{top}}(\Gamma) = \tfrac{1}{3}\omega_3(\Gamma), \tag{20}$$

where $\omega_3(\Gamma)$ is the gravitational Chern-Simons term that is defined via $d\omega_3 = p_1$, with p_i the ith Pontryagin class. In Ref. [48], it was suggested that the $2+1$D potential gravitational topological invariants are generated by

$$W^3_{\text{top}}(\Gamma) = \tfrac{1}{6}\omega_3(\Gamma). \tag{21}$$

(2) The $4+1$D potential gravitational topological invariants are described by $\mathbb{Z}_2$ [35,62,63], which are generated by

$$W^5_{\text{top}}(\Gamma) = \tfrac{1}{2}\mathrm{w}_2\mathrm{w}_3, \tag{22}$$

where w_i is the ith Stiefel-Whitney class. (3) The $6+1$D potential gravitational topological invariants are described by $2\mathbb{Z}$ [35], which are generated by

$$\begin{aligned} W^7_{\text{top}}(\Gamma) &= \frac{\omega_7^{p_1^2} - 2\omega_7^{p_2}}{5}, \\ W^7_{\text{top}}(\Gamma) &= \frac{-2\omega_7^{p_1^2} + 5\omega_7^{p_2}}{9}, \end{aligned} \tag{23}$$

where the gravitational Chern-Simons terms are defined by $d\omega_7^{p_1^2} = p_1^2$ and $d\omega_7^{p_2} = p_2$.

The potential topological invariants in Eqs. (20), (22), and (23) have a close relation to the orientated d-dimensional cobordism group Ω_d^{SO} [62–64], which are Abelian groups generated by the Stiefel-Whitney classes w_i and the Pontryagin classes p_i. For example, $\Omega_4^{SO} = \mathbb{Z}$ is generated by the Pontryagin class $\frac{1}{3}p_1$ and $\Omega_8^{SO} = 2\mathbb{Z}$ by $\frac{\omega_7^{p_1^2}-2\omega_7^{p_2}}{5}$ and $\frac{-2\omega_7^{p_1^2}+5\omega_7^{p_2}}{9}$. Also $\Omega_5^{SO} = \mathbb{Z}_2$ is generated by Stiefel-Whitney class $\mathrm{w}_2\mathrm{w}_3$. In this case, the set of potential gravitational topological invariants in d-dimensional space-time (denoted as PiTO_L^d) are exactly those Stiefel-Whitney classes and the Pontryagin classes that describe the cobordism group Ω_d^{SO}:

$$\begin{aligned} \text{Tor}(\text{PiTO}_L^d) &= \text{Tor}(\Omega_d^{SO}), \\ \text{Free}(\text{PiTO}_L^d) &= \text{Free}(\Omega_{d+1}^{SO}). \end{aligned} \tag{24}$$

Note that PiTO_L^d and Ω_d^{SO} are discrete Abelian groups. "Tor" and "Free" are the torsion part and the free part of the discrete Abelian groups.

However, we do not know if those potential gauge-gravity topological invariants can all be realized or produced by local bosonic systems. In this paper, we will study this issue. However, to address this issue, we need to first clarify the meaning of "realizable by local bosonic systems."

We note that there are two types of local bosonic systems: L type and H type [35]. L-type local bosonic systems are systems described by local bosonic Lagrangians. L-type systems have well-defined partition functions for space-time that can be any manifolds. H-type local bosonic systems are systems described by local bosonic Hamiltonians. H-type systems have well-defined partition functions only for any space-time that are mapping tori. (A mapping torus is a fiber bundle over S^1.) An L-type system always corresponds to an H-type system. However, an H-type system may not correspond to an L-type system. For example, SPT phases described by group cohomology and the NLσMs are L-type topological phases (and they are also H-type topological phases). The E_8 bosonic quantum Hall state is defined as an H-type topological phase. However, it is not clear if it is an L-type topological phase or not. In the following, we will argue that any quantum Hall state is also an L-type topological phase (i.e., realizable by space-time path integral, that is well defined on any space-time manifold).

In this paper, we will only consider L-type bosonic quantum systems. We will study which potential gauge-gravity topological invariants are realizable by L-type local bosonic systems. We will use $SO(\infty)\times G$ NLσMs (1) to try to realize those potential gauge-gravity topological invariants. After adding the G-symmetry twist and choosing a curved space-time M^d, the "gauged" $SO(\infty)\times G$ NLσMs (1) become [67–69]

$$\begin{aligned} &\mathcal{L} = \frac{1}{\lambda}[(\partial + iA + i\Gamma)g]^2 + i\,L^d_{\text{top}}[g^{-1}(\partial + iA + i\Gamma)g], \\ &g(x) \in G\times SO, \quad SO \equiv SO(\infty), \end{aligned} \tag{25}$$

where the space-time connection Γ couples to $SO(\infty)$ and the "gauge" connection A couples to G. The induced gauge-gravity topological term $L^d_{\text{top}}[g^{-1}(\partial + iA + i\Gamma)g]$ is classified by group cohomology $\mathcal{H}^d[G\times SO,\mathbb{R}/\mathbb{Z}]$. After we integrate out the matter fields g, the above gauged NLσM will produce a partition function that gives rise to a *realizable gauge-gravity topological invariant* $W^d_{\text{top}}(A,\Gamma)$ via

$$Z(M^d,A) = e^{i\int_{M^d} 2\pi W^d_{\text{top}}(A,\Gamma)}. \tag{26}$$

{See [70] for a study of gauged topological terms described by $[\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})]$ for continuous groups.}

The set of potential gauge-gravity topological terms contains the set of realizable gauge-gravity topological terms. More precisely, the two sets are related by a map

$$\{L^d_{\text{top}}[g^{-1}(\partial + iA + i\Gamma)g]\} \to \{W^d_{\text{top}}(A,\Gamma)\}. \tag{27}$$

However, the map may not be one-to-one and may not be surjective.

For example, when there is no symmetry, we find that the following types of realizable gauge-gravity topological invariants were generated by the above NLσM (see Table I):

(1) Those $2+1$D realizable gravitational topological invariants are described by $\mathbb{Z}$, which are generated by

$$W^3_{\text{top}}(\Gamma) = \omega_3(\Gamma). \tag{28}$$

The corresponding generating topological state has a chiral central charge $c = 24$ at the edge. So, the stacking of three E_8 bosonic quantum Hall states can be realized by a well-defined L-type local bosonic system. It is not clear if a single E_8 bosonic quantum Hall state can be realized by an L-type local bosonic system or not. However, we know that a single E_8

bosonic quantum Hall state can be realized by an H-type local bosonic system.

(2) Those $4+1$D realizable gravitational topological invariants are described by $\mathbb{Z}_2$, which are generated by

$$W^5_{\text{top}}(\Gamma) = \tfrac{1}{2}\text{w}_2\text{w}_3. \tag{29}$$

[Note that $\mathcal{H}^5(SO,\mathbb{R}/\mathbb{Z})$ is also $\mathbb{Z}_2$ in this case.] In fact, we will show that all the potential gauge-gravity topological invariants that generate a finite group are realizable by the $SO(\infty)$ NLσMs, which are L-type local bosonic systems.

(3) $\mathcal{H}^6(SO,\mathbb{R}/\mathbb{Z}) = 2\mathbb{Z}_2$, and there are four different types of $SO(\infty)$ NLσMs (with four different topological terms). However, the four different topological terms in the NLσMs all reduce to the same trivial gravitational topological invariant $W^6_{\text{top}}(\Gamma)$ after we integrate out the matter field g, suggesting that all the four NLσMs give rise to the same topological order.

(4) Those $6+1$D realizable gravitational topological invariants are described by $2\mathbb{Z}$, which are generated by

$$W^7_{\text{top}}(\Gamma) = \omega_7^{p_1^2}, \qquad W^7_{\text{top}}(\Gamma) = \omega_7^{p_2}. \tag{30}$$

We see that only part of the potential gravitational topological invariants are realizable by the $SO(\infty)$ NLσMs.

However, it is possible that $SO(\infty)$ NLσMs do not realize all possible L-type iTOs. In the following, we will argue that the $2+1$D E_8 bosonic quantum Hall state is an L-type iTO. $SO(\infty)$ NLσM cannot realize the E_8 state since it has a central charge $c = 8$ and a topological invariant $\frac{1}{3}\omega_3$.

In fact, we will argue that any quantum Hall state is an L-type topologically ordered state. Certainly, by definition, any quantum Hall state, being realizable by some interacting Hamiltonians, is an H-type topologically ordered state. The issue is if we can have a path-integral description that can be defined on any closed space-time manifold. At first, it seems that such a path-integral description does not exist and a quantum Hall state cannot be an L-type topological order. This is because quantum Hall state is defined with respect to a nonzero background magnetic field, a closed two-form field ($B = dA$) in $2+1$D space-time. This seems to imply that a path-integral description of quantum Hall state exists only on space-time that admits everywhere nonzero closed two-form field.

However, as stressed in Refs. [71,72], a quantum Hall state of filling fraction $\nu = p/q$ always contains an n-cluster structure. Also, the closed two-form field $B = dA$ in $2+1$D space-time may contain "magnetic monopoles." If those "magnetic monopoles" are quantized as multiples of nq, they will correspond to changing magnetic field by nq flux quanta each time. Changing magnetic field by nq flux quanta and changing particle number by pn-clusters is like adding a product state to a gapped quantum liquid discussed in Ref. [33], which represents a "smooth" change of the quantum Hall state. Since everywhere nonzero closed two-form field $B = dA$ with "magnetic monopoles" can be defined on any $2+1$D space-time, we can have a path-integral description of any quantum Hall state, such that the path integral is well defined on any space-time manifold. We conclude that quantum Hall states, such as the E_8 state, are L-type topologically ordered states. Therefore, the gravitational topological invariant

$$W^3_{\text{top}}(\Gamma) = \tfrac{1}{3}\omega_3 \tag{31}$$

is realizable by a $2+1$D L-type iTO, i.e., an E_8 state (see Table I).

E. A construction of L-type realizable pure and mixed SPT phases

Now, let us include symmetry and discuss SPT phases (i.e., L-type topological phases with short range entanglement). We like to point out that some SPT states are characterized by boundary effective theory with anomalous symmetry [69,73,74], which is commonly referred as gauge anomaly (or 't Hooft anomaly). Those SPT states are classified by group cohomology $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ of the symmetry group G. We also know that the boundaries of topologically ordered states [36–39] realize and (almost[1]) classify all pure gravitational anomalies [35]. So one may wonder, the boundary of what kind of order realizes mixed gauge-gravity anomalies? The answer is SPT order. This is because the mixed gauge-gravity anomalies are present only if we have the symmetry. Such SPT order is also beyond the $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ description since the mixed gauge-gravity anomalies are beyond the pure gauge anomalies. We will refer to this new class of SPT states as *mixed SPT states* and refer to the SPT states with only the pure gauge anomalies as *pure SPT states*. We would like to mention that the gauge anomalies and mixed gauge-gravity anomalies have played a key role in the classification of free-electron topological insulators/superconductors [75,76].

The main result of this paper is a classification of both pure and mixed SPT states realized by the NLσMs:

$$\begin{aligned}\sigma\text{LSPT}^d_G &= \frac{\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})}{\Lambda^d(G)}\\ &= E^d(G) \rtimes \left[\oplus_{k=1}^{d-1} H^k\big(BG,\sigma\text{iTO}_L^{d-k}\big) \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})\right],\end{aligned} \tag{32}$$

where σLSPT^d_G is the Abelian group formed by the L-type G SPT phases in d-dimensional space-time produced by the NLσMs, and σiTO^d_L is the Abelian group formed by the L-type iTO phases in d-dimensional space-time *produced by the* NLσMs. Also $\Lambda^d(G)$ is a subgroup of $\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$.

Replacing σiTO^d_L by iTO^d_L, the Abelian group formed by the L-type iTO phases in d-dimensional space-time, we obtain more general SPT states described by LSPT^d_G:

$$\text{LSPT}^d_G = E^d(G) \rtimes \left[\oplus_{k=1}^{d-1} H^k\big(BG,\text{iTO}_L^{d-k}\big) \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})\right]. \tag{33}$$

If G contains time-reversal transformation, it will have a nontrivial action $\mathbb{R}/\mathbb{Z} \to -\mathbb{R}/\mathbb{Z}$ and $\text{iTO}^{d-k}_L \to -\text{iTO}^{d-k}_L$.

[1] For example, the pure $2+1$D gravitational anomalies described by *unquantized* thermal Hall conductivity are not classified by topologically ordered states.

Also, BG is the classifying space of G and $H^k(BG,\mathbb{M})$ is the topological cohomology class on BG.

Note that stacking two topological phases $\mathcal{C}_1$ and $\mathcal{C}_2$ together will produce another topological phase $\mathcal{C}_3$. We denote such a stacking operation as $\mathcal{C}_1 \boxplus \mathcal{C}_2 = \mathcal{C}_3$. Under $\boxplus$, the topological phases form a commutative monoid [35]. In general, a topological phase $\mathcal{C}$ may not have an inverse, i.e., we can not find another topological phase $\mathcal{C}'$ such that $\mathcal{C} \boxplus \mathcal{C}' = 0$ is a trivial product state. This is why topological phases form a commutative monoid, instead of an Abelian group. However, a subset of topological phases can have the inverse and form an Abelian group. Those topological phases are called invertible [35,48]. One can show that a topological phase is invertible iff it has no topological excitations [35,48]. Therefore, all SPT phases are invertible. Some topological orders are also invertible, which are called invertible topological orders (iTO). SPT phases and iTO phases form Abelian groups under the stacking $\boxplus$ operation. So for SPT states and iTO states, we can replace $\boxplus$ by $+$:

$$\mathcal{C}_1 \boxplus \mathcal{C}_2 = \mathcal{C}_1 + \mathcal{C}_2. \tag{34}$$

So LSPT^d_G and iTO^d_L can be viewed as modules over the ring $\mathbb{Z}$, and they can appear as the coefficients in group cohomology.

The result (32) can be understood in two ways. It means that the SPT states constructed from NLσMs are all described by $\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, but in a many-to-one fashion; i.e., $\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ contain a subgroup $\Lambda^d(G)$ that different elements in $\Lambda^d(G)$ correspond to the same SPT state. It also means that the constructed SPT states are described by $\oplus_{k=1}^{d-1} H^k(BG,\sigma\text{iTO}_L^{d-k}) \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, but in a one-to-many fashion; i.e., each element of $\oplus_{k=1}^{d-1} H^k(BG,\sigma\text{iTO}_L^{d-k}) \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ corresponds to several SPT states that form a group $E^d(G)$. The group $\Lambda^d(G)$ and $E^d(G)$ can be calculated but we do not have a simple expression for them (see Sec. V).

In Eq. (33), LSPT^d_G includes both pure and mixed SPT states. The group cohomology class $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ describes the pure SPT phases, and the group cohomology class $E^d(G) \rtimes \oplus_{k=1}^{d-1} H^k(BG,\text{iTO}_L^{d-k})$ describes the mixed SPT phases. We would like to mention that an expression of the form Eq. (33) was first proposed in [49] in a study of topological invariants of SPT states. We see that our NLσMs construction can produce mixed SPT phases with and without time-reversal symmetry. We have used Eq. (33) to compute the SPT phases for some simple symmetry groups (see Table II).

The formal group cohomology methods employed for obtaining the result (33) directly shed light on the physics of these phases. The SPT states described by $\mathcal{H}^1(G,\text{iTO}_L^{d-1})$ in Eq. (33) can be constructed using the decorated domain walls proposed in [29]. Other SPT states described $H^k(BG,\text{iTO}_L^{d-k})$ can be obtained by a generalization of the decorated domain wall construction [59,77,78], which will be called the nested construction [79]. The formal methods also lead to physical/numerical probes for these phases [16,49,51,54–57]. In addition, these methods are easy to generalize to fermionic systems [79,80], and provide answers for the physically important situation of continuous symmetries (like charge conservation).

We also studied the potential SPT phases (i.e., might not realizable) for a non-onsite symmetry, the mirror reflection symmetry Z_2^M. The Abelian group formed by those SPT phases is denoted as $\text{PSPT}^d_{Z_2^M}$. Following Refs. [62–64], we find that $\text{PSPT}^d_{Z_2^M}$ is given by a quotient of the unoriented cobordism groups Ω_d^O:

$$\text{PSPT}^d_{Z_2^M} = \Omega_d^O/\bar{\Omega}_d^{SO}, \tag{35}$$

where $\bar{\Omega}_d^{SO}$ is the orientation invariant subgroup of Ω_d^{SO} (i.e., the manifold M^d and its orientation reversal $-M^d$ belong to the same oriented cobordism class). It is interesting to see

$$\text{PSPT}^d_{Z_2^M} = \text{LSPT}^d_{Z_2^T} \tag{36}$$

(see Table V).

We want to remark that, in this paper, the time-reversal transformation is defined as the complex conjugation transformation (see Sec. II B), without the $t \to -t$ transformation. The mirror reflection corresponds to the $t \to -t$ transformation. The time-reversal symmetry used in Refs. [62–64,81] is actually the mirror reflection symmetry Z_2^M in this paper. The two ways to implement time-reversal symmetry should lead to the same result as demonstrated by Eq. (36), despite the involved mathematics, the cobordism approach, and NLσM approach are very different.

F. Discrete gauge anomalies, discrete mixed gauge-gravity anomalies, and invertible discrete gravitational anomalies

First, let us explain the meaning of *discrete anomalies*. All the commonly known anomalies are discrete in the sense that different anomalies form a discrete set. However, there are continuous gauge/gravitational anomalies labeled by one or more continuous parameters [35,69]. In this section, we only consider discrete anomalies.

Since the boundaries of SPT states realize all pure gauge anomalies, as a result, group cohomology $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ systematically describes all the perturbative and global gauge anomalies [69,73]. For topological orders, we found that they can be systematically described by tensor category theory [3,35,82–86] and tensor network [87–89], and those theories also systematically describe all the perturbative and global gravitational anomalies [35].

More precisely, the discrete pure bosonic gauge anomalies in d-dimensional space-time are described by $\mathcal{H}^{d+1}(G,\mathbb{R}/\mathbb{Z})$. The discrete invertible pure bosonic gravitational anomalies in d-dimensional space-time are described $\text{iTO}_L^{d+1} \simeq (\Omega_{SO}^{d+2}) \oplus (\Omega_{SO}^{d+1})$. The discrete mixed bosonic gauge-gravity anomalies are described by $E^d(G) \rtimes \oplus_{k=1}^{d} H^k(BG,\text{iTO}_L^{d-k+1})$.

In Table I, we list the generators of the topological invariants $W^d_{\text{top}}(\Gamma)$. Those topological invariants describe various bosonic invertible gravitational anomalies in one lower dimension. For example, $W^3_{\text{top}}(\Gamma) = \frac{1}{3}\omega_3$ describes the well-known perturbative gravitational anomaly in $1+1$D chiral boson theories. The topological invariant $W^4_{\text{top}}(\Gamma) = \frac{1}{2}\mathrm{w}_2\mathrm{w}_3$ implies a new type of bosonic global gravitational anomaly in $4+1$D bosonic theories. In Tables III, IV, VI, VII, V, and VIII, we list the generators of the topological invariants $W^d_{\text{top}}(A,\Gamma)/2\pi$ for some simple groups. Those topological invariants describe various bosonic anomalies for those groups at one lower dimension. For example, $W^3_{\text{top}}(\Gamma) = \frac{1}{(2\pi)^2} A\, dA$ describes the well-known perturbative $U(1)$ gauge anomaly in $1+1$D chiral boson theories. The topological invariant $W^4_{\text{top}}(A_{O_2}) = \frac{1}{2} a_1^2 c_1$ implies a new type of bosonic global O_2 gauge anomaly in $2+1$D bosonic theories. In fact, all the non-$\mathbb{Z}$-type topological invariants in the tables give rise to a new type of bosonic global gauge/gravity/mixed anomalies in one lower dimension.

Note that the invertible anomalies are the usual anomalies people talked about. They can be canceled by other anomalies. The anomalies, defined by the absence of well-defined realization in the same dimension, can be noninvertible (i.e., cannot be canceled by any other anomalies) [35]. All pure gauge and mixed gauge-gravity anomalies are invertible, but most gravitational anomalies are not invertible [35].

G. Relations between the H-type and the L-type topological phases

We have introduced the concept of potential SPT phases PSPT^d_G (which may or may not be realizable), H-type SPT phases HSPT^d_G (which are realizable by H-type local quantum systems), and L-type SPT phases LSPT^d_G (which are realizable by L-type local quantum systems). Those SPT phases are related

$$\begin{aligned} \text{LSPT}^d_G &\subset \text{PSPT}^d_G, \\ \text{HSPT}^d_G &\subset \text{PSPT}^d_G, \\ \text{LSPT}^d_G &\to \text{HSPT}^d_G. \end{aligned} \tag{37}$$

where $\subset$ represents subgroup and $\to$ is a group homomorphism. Similarly, we also introduced the concept of potential iTO phases iTO^d_P (which may or may not be realizable), H-type iTO phases iTO^d_H (which are realizable by H-type local quantum systems), and L-type iTO phases iTO^d_L (which are realizable by L-type local quantum systems). Those iTO phases are related

$$\begin{aligned} \text{iTO}^d_L &\subset \text{iTO}^d_P, \\ \text{iTO}^d_H &\subset \text{iTO}^d_P, \\ \text{iTO}^d_L &\to \text{iTO}^d_H. \end{aligned} \tag{38}$$

In condensed matter physics, we are interested in iTO^d_H and HSPT^d_G. (A study on the H-type topological phases can be found in Refs. [35,48].) But in this paper, we will mainly discuss iTO^d_L and LSPT^d_G. The SPT states constructed in Refs. [12–14] belong to LSPT^d_G (and they also belong to HSPT^d_G). The SPT states constructed in Refs. [16–18,31] belong to HSPT^d_G. In Refs. [48,59,62–64] only the potential SPT states PSPT^d_G are studied.

H. Organization of this paper

In Sec. II, we review the NLσM construction of the pure SPT states. In Sec. III, we generalize the NLσM construction to cover the mixed SPT states and iTO states. In Sec. IV, a construction L-type iTO order is discussed. Using such a construction (Sec. V), we proposed a construction of the pure and the mixed SPT states of the L-type. In Sec. VI, we discussed the L-type SPT states protected by the mirror reflection symmetry.

II. GROUP COHOMOLOGY AND THE L-TYPE PURE SPT STATES

An L-type pure SPT state in d-dimensional space-time M^d can be realized by a NLσM with the symmetry group G as the target space

$$Z = \int D[g]\, e^{-\int_{M^d}[\frac{1}{\lambda}[\partial g(x)]^2 + i\, L^d_{\text{top}}(g^{-1}\partial g)]}, \quad g(x) \in G \tag{39}$$

in the large λ limit. Here we treat the space-time as a (random) lattice which can be viewed as a d-dimensional complex. The space-time complex has vertices, edges, triangles, tetrahedrons, etc. The field $g(x)$ live on the vertices and $\partial g(x)$ live on the edges. So $\int_{M^d}$ is in fact a sum over the vertices, edges, and other simplices of the lattice. ∂ is the lattice difference between vertices connected by edges. The above action S actually defines a lattice theory [13,14].

Under renormalization group transformations, λ flows to infinity. So the fixed point action contains only the topological term. In this section, we will describe such a fixed-point theory on a space-time lattice [51,68,90]. The space-time lattice is a triangulation of the space-time. So we will start by describing such a triangulation.

A. Discretize space-time

Let M^d_{tri} be a triangulation of the d-dimensional space-time. We will call the triangulation M^d_{tri} as a space-time complex, and a cell in the complex as a simplex. In order to define a generic lattice theory on the space-time complex M^d_{tri}, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure [13,14,91]. A branching structure is a choice of orientation of each edge in the d-dimensional complex so that there is no oriented loop on any triangle (see Fig. 10).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming edges, and the second vertex is the vertex with only one incoming edge, etc. So the simplex in Fig. 10(a) has the following vertex ordering: 0,1,2,3.

The branching structure also gives the simplex (and its subsimplices) an orientation denoted by $s_{ij\ldots k} = 1,*$. Figure 10 illustrates two 3-simplices with opposite orientations $s_{0123} = 1$ and $s_{0123} = *$. The red arrows indicate the orientations of the 2-simplices which are the subsimplices of the 3-simplices.

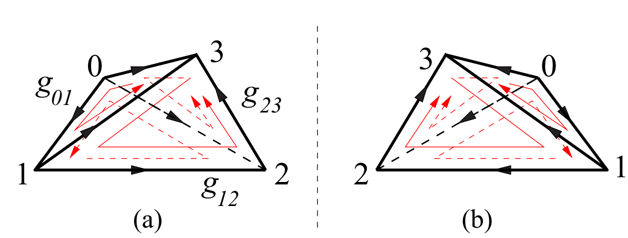

FIG. 10. (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

The black arrows on the edges indicate the orientations of the 1-simplices.

B. G NLσM on a space-time lattice

In our lattice NLσM, the degrees of freedom live on the vertices of the space-time complex, which are described by $g_i \in G$ where i labels the vertices.

The action amplitude $e^{-S_{\text{cell}}}$ for a d cell $(ij\dots k)$ is a complex function of g_i: $A_{ij\dots k}(\{g_i\})$. The total action amplitude e^{-S} for a configuration (or a path) is given by

$$e^{-S} = \prod_{(ij\dots k)} [A_{ij\dots k}(\{g_i\})]^{s_{ij\dots k}}, \tag{40}$$

where $\prod_{(ij\dots k)}$ is the product over all the d cells $(ij\dots k)$. Note that the contribution from a d cell $(ij\dots k)$ is $A_{ij\dots k}(\{g_i\})$ or $A^*_{ij\dots k}(\{g_i\})$ depending on the orientation $s_{ij\dots k}$ of the cell. Our lattice G NLσM is defined by following imaginary-time path integral (or partition function)

$$Z_{\text{gauge}} = \sum_{\{g_i\}} \prod_{(ij\dots k)} [A_{ij\dots k}(\{g_i\})]^{s_{ij\dots k}}, \tag{41}$$

where the action amplitude $A_{ij\dots k}(\{g_i\})$ is invariant or covariant under the G-symmetry transformation $g_i \to g_i' = gg_i$, $g \in G$:

$$A_{ij\dots k}(\{gg_i\}) = A^{S(g)}_{ij\dots k}(\{g_i\}). \tag{42}$$

Note that here we allow G to contain time-reversal symmetry. In H-type theory (i.e., in Hamiltonian quantum theory) the time-reversal transformation is implemented by complex conjugation without reversing the time $t \to -t$ (there is no time to reverse in Hamiltonian quantum theory). Generalizing that to L-type theory, we will also implement time-reversal transformation by complex conjugation without reversing the time $t \to -t$. This is the implementation used in [13,14]. $S(g)$ in Eq. (42) describes the effect of complex conjugation. $S(g) = 1$ if g contains no time-reversal transformation and $S(g) = *$ if g contains a time-reversal transformation.

The fixed-point theory contains only the pure topological term. Such a pure topological term can be constructed from a group cocycle $\nu_d \in \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. Note that a group cocycle $\nu_d(g_0,g_1,\dots,g_d)$, $g_i \in G$ is a map from G^{d+1} to $\mathbb{R}/\mathbb{Z}$ (see Appendix A). We can express the action amplitude $A_{ij\dots k}(\{g_i\})$ that corresponds to a pure topological term as [13,14]

$$A_{01\dots d}(\{g_i\}) = e^{2\pi\, i\, \nu_d(g_0,g_1,\dots,g_d)}. \tag{43}$$

Due to the symmetry condition (A3), the action amplitude $A_{ij\dots k}(\{g_i\})$ is invariant/covariant under the G-symmetry transformation. Due to the cocycle condition (A5), the total action amplitude on a closed space-time M^d is always equal to 1:

$$e^{i\int_{M^d} L^d_{\text{top}}(g^{-1}\partial g)} = \prod_{(ij\dots k)} [A_{ij\dots k}(\{h_{ij}\},\{g_i\})]^{s_{ij\dots k}} = 1. \tag{44}$$

Also, two cocycles different by a coboundary [see Eq. (A6)] can be smoothly deformed into each other without affecting the condition (44). In other words, the connected components of the fixed-point theories that satisfy the condition (44) are described by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. This way, we show that the fixed points of the G NLσMs are classified by the elements of $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$.

We like to remark that for continuous group, the cocycle $\nu_d(g_0,g_1,\dots,g_d)$ do not need to be continuous function of g_i. It can be a measurable function.

C. Adding the G-symmetry twist

The above bosonic system may be in different SPT phases for different choices of the topological term [i.e., for different choices of group cocycles $\nu_d \in \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$]. But how can we be sure that the system is indeed in different SPT phases? One way to address such a question is to find measurable topological invariants, and show that different cocycles give rise to different values for the topological invariants.

In this section, we will assume that the symmetry group does not contain time reversal. In this case, the universal topological invariants for SPT state can be constructed systematically by twisting (or "gauging") the onsite symmetry [51,67,68,92] and study the gauged bosonic model

$$Z(A) = \int D[g]\, e^{\int_{M^d}[\frac{1}{\lambda}[(\partial - i\,A)g]^2 + i\, L^d_{\text{top}}[g^{-1}(\partial - i\,A)g]]}. \tag{45}$$

Note that the gauge field A just represents space-time-dependent coupling constants, which is not dynamical (i.e., we do not integrate out the gauge field A in the path integral). Since the SPT state is gapped for large λ, in large space-time limit, the partition function has a form

$$Z(A) = e^{-\epsilon_0 V_{\text{space-time}}}\, e^{i\int_{M^d} 2\pi W^d_{\text{top}}(A)}, \tag{46}$$

where ϵ_0 is the ground state energy density and $V_{\text{space-time}}$ is the volume of the space-time manifold M^d. The term $\int_{M^d} 2\pi W^d_{\text{top}}(A)$ represents the volume independent term in the partition function and is conjectured to be universal (i.e., independent of any small local change of the Lagrangian that preserves the symmetry) [35]. Such a term is called *the realizable gauge topological term (or topological invariant)*, which is referred as the SPT invariant in Refs. [49,50]. The SPT invariants are the topological invariants that are believed to be able to characterize and distinguish any SPT phases.

The topological invariant is gauge invariant, i.e., for any closed space-time manifold M^d:

$$\int_{M^d} W^d_{\text{top}}(A^g) - \int_{M^d} W^d_{\text{top}}(A) = 0 \bmod 1,$$
$$A^g = g^{-1}Ag + i\, g^{-1}dg, \tag{47}$$

where we have treated A as the gauge field one-form. Also, as a topological invariant, $W^d_{\text{top}}(A)$ does not depend on the metrics of the space-time. For example, $W^d_{\text{top}}(A)$ can be a Chern-Simons term $\frac{2k}{4\pi}\text{Tr}A\,dA$, $k \in \mathbb{Z}$ in 2 + 1D or a θ term $\frac{\theta}{(2\pi)^2}\, dA\,dA$ in 3 + 1D. The presence of nontrivial topological invariant $Z_{\text{fixed}}(A) = e^{i\int_{M^d} 2\pi W^d_{\text{top}}(A)}$ indicates the presence of nontrivial SPT phase.

In the above, we described the symmetry twist in the continuous field theory. On lattice, the symmetry twist can be achieved by introducing $h_{ij} \in G$ for each edge ij in the space-time complex M^d_{tri}. The twisted theory (i.e., the "gauged" theory) is described by the total action amplitude e^{-S}:

$$e^{-S} = \prod_{(ij\dots k)} [\tilde{A}_{ij\dots k}(\{h_{ij}\},\{g_i\})]^{s_{ij\dots k}}. \tag{48}$$

The imaginary-time path integral (or partition function) is given by

$$Z(\{h_{ij}\}) = \sum_{\{g_i\}} \prod_{(ij\ldots k)} [\tilde{A}_{ij\ldots k}(\{h_{ij}\},\{g_i\})]^{s_{ij\ldots k}}. \quad (49)$$

We see that only g_i are dynamical. h_{ij} are nondynamical background probe fields. The above action amplitude $\prod_{(ij\ldots k)}[\tilde{A}_{ij\ldots k}(\{h_{ij}\},\{g_i\})]^{s_{ij\ldots k}}$ on closed space-time complex ($\partial M^d = \emptyset$) should be invariant under the "gauge" transformation

$$h_{ij} \to g'_{ij} = h_i h_{ij} h_j^{-1}, \;\; g_i \to g'_i = h_i g_i, \;\; h_i \in G \quad (50)$$

and covariant under the global symmetry transformation

$$h_{ij} \to h'_{ij} = g h_{ij} g^{-1}, \; g_i \to g'_i = g g_i, \;\; g \in G \quad (51)$$

$$\tilde{A}_{ij\ldots k}(\{h_{ij}\},\{g_i\}) = \tilde{A}^{S(g)}_{ij\ldots k}(\{h'_{ij}\},\{g'_i\}). \quad (52)$$

The gauged action amplitudes $\tilde{A}_{ij\ldots k}(\{h_{ij}\},\{g_i\})$ are obtained from the ungauged action amplitudes $A_{ij\ldots k}(\{g_i\})$ in the following way (where we assume G is discrete):

$$\tilde{A}_{01\ldots d}(\{h_{ij}\},\{g_i\}) = 0 \text{ if } h_{ij}h_{jk} \neq h_{ik},$$
$$\tilde{A}_{01\ldots d}(\{h_{ij}\},\{g_i\}) = A_{01\cdots d}(h_0g_0,h_1g_1,\ldots,h_dg_d), \quad (53)$$

where h_i are given by

$$h_0 = 1, \quad h_1 = h_0h_{01}, \quad h_2 = h_1h_{12}, \quad h_3 = h_2h_{23}, \ldots . \quad (54)$$

At a fixed point, the twisted action amplitude $A_{ij\ldots k}(\{h_{ij}\},\{g_i\})$ is given by

$$\tilde{A}_{01\ldots d}(\{h_{ij}\},\{g_i\}) = e^{2\pi\, i\, \nu_d(h_0g_0,h_1g_1,\ldots,h_dg_d)}$$
$$= e^{2\pi\, i\, \omega_d(g_0^{-1}h_{01}g_1,\ldots,g_{d-1}^{-1}h_{d-1,d}g_d)} \text{ if } h_{ij}h_{jk} = h_{ik},$$

where ω_d is the inhomogeneous cocycle corresponding to ν_d:

$$\omega_d(h_{01},h_{12},\ldots,h_{d-1,d}) = \nu_d(h_0,h_1,\ldots,h_d). \quad (55)$$

By rewriting the partition function as [see Eq. (55)]

$$Z(\{h_{ij}\}) = \sum_{\{g_i\}} \prod_{(ij\ldots k)} [A_{ij\ldots k}(\{g_i^{-1}h_{ij}g_j\},\{1\})]^{s_{ij\ldots k}} \quad (56)$$

we find that the partition function is explicitly gauge invariant and symmetric.

The topological invariant $W^d_{\text{top}}(A)$ is given by the fixed-point partition function for the twisted theory

$$e^{i\int_{M^d} 2\pi W^d_{\text{top}}(A)} = Z_{\text{fixed}}(\{h_{ij}\}) = Z_{\text{fixed}}(A). \quad (57)$$

The twisted fixed-point partition function $Z_{\text{fixed}}(\{h_{ij}\})$ or $Z_{\text{fixed}}(A)$ is nontrivial and depends on the symmetry twist h_{ij} (or gauge connection A). We see that different realizable topological invariants $W^d_{\text{top}}(A)$ are classified and given explicitly by the elements of group cohomology $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$:

$$e^{i\int_{M^d} 2\pi W^d_{\text{top}}(A)} = \prod_{(ij\ldots k)} [e^{2\pi\, i\, \omega_d(\{h_{ij}\})}]^{s_{ij\ldots k}}, \quad (58)$$

where $\omega_d(h_1,\ldots,h_d)$ is an inhomogeneous cocycle in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, and $\{h_{ij}\}$ on the edges complex M^d_{tri} define the symmetry twist A in space-time M^d. Equation (58) tells us how to calculate $e^{i\int_{M^d} 2\pi W^d_{\text{top}}(A)}$, given cocycle ω_d, the space-time manifold M^d_{tri}, and the symmetry twist $A = \{h_{ij}\}$.

We can also see this within the field theory. The realizable gauge topological invariant $W^d_{\text{top}}(A)$ and the NLσM topological term $L^d_{\text{top}}[g^{-1}(\partial - i\,A)g]$ are directly related:

$$L^d_{\text{top}}[g^{-1}(\partial - i\,A)g] = 2\pi W^d_{\text{top}}(A). \quad (59)$$

Since the NLσM topological terms $L^d_{\text{top}}[g^{-1}(\partial - i\,A)g]$ are classified by the group cohomology $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ of the symmetry group G. The realizable gauge topological invariants $W^d_{\text{top}}(A)$ are also classified by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$.

The gauge topological term (or topological invariant) $W^d_{\text{top}}(A)$ can be defined for both continuous and discrete symmetry groups G. In general, it is a generalization of the Chern-Simons term [67,92,93]. It describes the response of the quantum ground state. We hope that the ground states in different quantum phases will produce different responses, which correspond to different classes of gauge topological terms, that cannot be smoothly deformed into each other. So we can use such a term to study and classify pure SPT phases.

We would like to point out that there are two kinds of topological invariants. The topological invariants corresponding to $[\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})]$ are called *locally null topological invariants*. They have the following defining properties:

(i) $\int_{M^d} W^d_{\text{top}}(A)$ are well defined for any symmetry twists A.

(ii) $\int_{M^d} W^d_{\text{top}}(A)$ does not depend on any small smooth change of the symmetry twist:

$$\int_{M^d} W^d_{\text{top}}(A + \delta A) = \int_{M^d} W^d_{\text{top}}(A). \quad (60)$$

The topological invariants corresponding to $[\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})]$ are called *Chern-Simons topological invariants*. The Chern-Simons topological invariants are only well defined for some symmetry twists A. In general, only the difference

$$\int_{\tilde{M}^d} W^d_{\text{top}}(\tilde{A}) - \int_{M^d} W^d_{\text{top}}(A) \quad (61)$$

is well defined, provided that there exists a $(d+1)$-dimensional manifold N^{d+1} such that $\partial N^{d+1} = \tilde{M}^d \cup (-M^d)$ and the gauge connections A on M^d and $\tilde{A}$ on $\tilde{M}^d$ can be extended to N^{d+1} (see Appendix B).

Now, two questions naturally arise: (1) How to write the most general topological invariants $W^d_{\text{top}}(A)$ (i.e., the most general topological invariants) which are self-consistent? We will call such topological invariants as *potential topological invariants*. (2) Can we show that every potential topological invariant can be induced by some symmetric local bosonic model, after we gauge the onsite symmetry?

In Appendix B, we will address these two questions. We find that the potential gauge topological invariants $W^d_{\text{top}}(A)$ are described by $H^{d+1}(BG,\mathbb{Z})$, which are all realizable since $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) = H^{d+1}(BG,\mathbb{Z})$.

D. $G \times G'$ pure SPT states

In this section, we will study $G \times G'$ pure SPT states described by group cohomology $\mathcal{H}^d(G \times G',\mathbb{R}/\mathbb{Z})$. This result will be useful for later discussions. First, we can use the following version of Künneth formula [49,69]:

$$\mathcal{H}^d(G \times G',\mathbb{R}/\mathbb{Z}) \simeq \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^d(G',\mathbb{R}/\mathbb{Z}) \oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(G',\mathbb{R}/\mathbb{Z})] \quad (62)$$

to compute $\mathcal{H}^d(G \times G',\mathbb{R}/\mathbb{Z})$. In addition, the above Künneth formula can help us to construct topological invariants to probe the $G \times G'$ SPT order [49].

For example, a G SPT order in d-dimensional space-time can be probed by a map W^d_{top}, that maps a closed space-time M^d with a G-symmetry twist A to a number in $\mathbb{R}/\mathbb{Z}$:

$$\int_{M^d} W^d_{\text{top}}(A) \in \mathbb{R}/\mathbb{Z}. \quad (63)$$

Such a map is nothing but the topological invariant that we discussed before. At the same time, the topological invariant can also be viewed as a cocycle in $H^d[BG,\mathbb{R}/\mathbb{Z}]$ since it is a map for the G bundles (i.e., the G-symmetry twists) on M^d to $\mathbb{R}/\mathbb{Z}$, and the G bundles on M^d is classified by the embedding of M^d into the classifying space. Different SPT states will lead to different maps. We believe that the map W^d_{top} fully characterizes the G SPT states described by $\mathcal{H}^d[G,\mathbb{R}/\mathbb{Z}]$ (see Appendix B) [49,50].

Similarly, for the $G \times G'$ pure SPT states described by $H^k[BG,\mathcal{H}^{d-k}(G',\mathbb{R}/\mathbb{Z})]$, they can also be probed by a map $W^d_{\mathcal{H}^k}$, that maps a closed space-time M^k with a G-symmetry twist A_G on M^k to an element in $H^{d-k}(G',\mathbb{R}/\mathbb{Z})$. This is simply a dimension reduction: we consider a space-time of the form $M^k \times M^{d-k}$, add a G-symmetry twist A_G on M^k, and then take a large M^{d-k} limit. The system can be viewed as a $(d-k)$-dimensional G' SPT state on M^{d-k}, which is described by an element in $H^{d-k}(G',\mathbb{R}/\mathbb{Z})$. Such a dimension reduction can be formally written as

$$\int_{M^k} W^d_{\mathcal{H}^k}(A_G) \in \mathcal{H}^{d-k}(G',\mathbb{R}/\mathbb{Z}), \quad (64)$$

which has the same structure as Eq. (63). The map $W^d_{\mathcal{H}^k}$ can be viewed as a cocycle in $H^k[BG,\mathcal{H}^{d-k}(G',\mathbb{R}/\mathbb{Z})]$. Such a map fully characterizes the $G \times G'$ pure SPT states described by $H^k[BG,\mathcal{H}^{d-k}(G',\mathbb{R}/\mathbb{Z})]$.

The dimension reduction discussed above reveals the physical meaning of the Künneth formula. We will use such a physical picture to obtain the key result of this paper.

III. CONSTRUCTING PURE AND MIXED SPT STATES, AS WELL AS iTO STATES

A. SPT states, gauge anomalies, and mixed gauge-gravity anomalies

So far, we have reviewed the group cohomology approach to pure SPT states. It was pointed out in Ref. [69] that (a) the SPT orders [described by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$] and pure gauge anomalies in one lower dimension are directly related and (b) the topological orders and gravitational anomalies in one lower dimension are directly related. This suggests that the SPT orders beyond $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ [16–18,31,59,62,63] and mixed gauge-gravity anomalies are closely related [59]. This line of thinking gives us a deeper understanding of generic SPT states. In this section, we are going to construct local bosonic models that systematically realize iTOs, pure SPT orders (associated with pure gauge anomaly), and mixed SPT orders (associated with mixed gauge-gravity anomaly).

B. Realizable L-type SPT and iTO phases

One of the key properties of SPT states is that they do not contain any nontrivial topological excitations [12–14]. In [35] it was conjectured that a gapped quantum liquid state has no nontrivial topological excitations iff its fixed-point partition function is a pure $U(1)$ phase.

However, when we study the pure SPT orders described by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ using G NLσMs, we only add the symmetry twists, which are associated with the G bundles on the space-time, to induce the nontrivial $U(1)$-phase-valued partition function. This is why we only get pure gauge anomalies in such an approach. To get the gravitational anomalies and the mixed gauge-gravity anomalies, we must include the space-time twist, described by the nontrivial tangent bundle of the space-time as well. The tangent bundle is a $SO_d \equiv SO(d)$ bundle. Thus, to include the gravitational anomalies and the mixed gauge-gravity anomalies, as well as the pure gauge anomalies, we simply need to consider a $SO_d \times G$ NLσM with topological term $L^d_{\text{top}}(g^{-1}\partial g)$ where $g(x) \in SO_d \times G$. We can gauge the G symmetry to probe the SPT states and the pure gauge anomalies as before. We can also choose nonflat space-time to probe the SPT states (and the gravitational anomalies), that corresponds to couple the SO_d part of the NLσM to the connection of the tangent bundle of the space-time. We will see that using $G \times SO_d$ NLσMs, we can obtain a topological invariant $W^d_{\text{top}}(A,\Gamma)$ that contains both the gauge G connection A and the gravitational SO_d connection Γ. Such kind of bosonic NLσM is capable of producing the pure SPT states that are associated with pure gauge anomalies, as well as the mixed SPT states that are associated with mixed gauge-gravity anomalies. It can also produce iTO states, if we choose a trivial symmetry group G.

Here we would like to remark that we can also use a $G \times SO_n$ NLσM with $n > d$ to produce the SPT states and iTO states. The stability consideration suggests that we should take $n = \infty$. So we will use $G \times SO$ NLσM to study the new topological states, where $SO \equiv SO_\infty$.

Repeating the discussion in Sec. II B, we find that the realizable gauge-gravity topological invariants $W^d_{\text{top}}(A,\Gamma)$ in the $G \times SO$ NLσM can be constructed from each element in the group cohomology class $\mathcal{H}^d(G \times SO,\mathbb{R}/\mathbb{Z})$. However, because of the restrictive relation between the gravitational connection Γ and the topology of the space-time (see Appendix C), the correspondence is not one-to-one: different elements in $\mathcal{H}^d(G \times SO,\mathbb{R}/\mathbb{Z})$ may produce the same realizable gauge-gravity topological invariant $W^d_{\text{top}}(A,\Gamma)$ after integrating out the matter field g. The reason is the following. For two topological invariants $W^d_{\text{top}}(A,\Gamma)$ and $\tilde W^d_{\text{top}}(A,\Gamma)$ obtained from two cocycles ν_d and $\tilde\nu_d$ in $\mathcal{H}^d(G \times SO,\mathbb{R}/\mathbb{Z})$, it is possible that

$$e^{i\int_{M^d} 2\pi W^d_{\text{top}}(A,\Gamma)} = e^{i\int_{M^d} 2\pi \tilde W^d_{\text{top}}(A,\Gamma)} \quad (65)$$

on any closed space-time M^d. In this case, we should view $W^d_{\text{top}}(A,\Gamma)$ and $\tilde W^d_{\text{top}}(A,\Gamma)$ as the same topological invariant. (Note that the above two topological invariants can be distinguished if the SO connection Γ is not restricted to be the connection of the tangent bundle of the space-time M^d.) Thus, $\mathcal{H}^d(G\times SO,\mathbb{R}/\mathbb{Z})$ contains a subgroup $\Lambda^d(G)$ such that the realizable gauge-gravity topological invariants $W^d_{\text{top}}(A,\Gamma)$ have a one-to-one correspondence with the elements in $\mathcal{H}^d(G\times SO,\mathbb{R}/\mathbb{Z})/\Lambda^d(G)$. Those different NL$\sigma$Ms, that produce differently the realizable gauge-gravity topological invariants $W^d_{\text{top}}(A,\Gamma)$, realize different L-type topological phases with no topological excitations.

In Appendix C, we will discuss potential gauge-gravity topological invariants $W^d_{\text{top}}(A,\Gamma)$. We find that the locally null potential gauge-gravity topological invariants $W^d_{\text{top}}(A,\Gamma)$ are described by a subgroup of $\mathcal{H}^d(G\times SO,\mathbb{R}/\mathbb{Z})$, which are all realizable. We also find that Chern-Simons potential gauge-gravity topological invariants $W^d_{\text{top}}(A,\Gamma)$ are described by a subgroup of $H^{d+1}[B(SO\times G),\mathbb{Z}(\frac{1}{n})]$. NL$\sigma$Ms can only realize those that are also in $H^{d+1}[B(G\times SO),\mathbb{Z}]$.

IV. iTO STATES

Using Pontryagin class and Stiefel-Whitney class, one can show that different L-type potential iTO phases (i.e., may not be realizable) are described by $\mathbb{Z}$ in three dimensions [40], $\mathbb{Z}_2$ in four dimensions [35,62], and $2\mathbb{Z}$ in seven dimensions, where the dimensions d are the space-time dimensions. In this section, we will reexamine those results using the approaches discussed above, and try to understand which L-type potential iTO can be realized by SO NLσMs. We will show that the above potential topologically ordered phases described by Stiefel-Whitney class are always realizable, while only a subset of those described by Pontryagin classes are realizable by SO NLσMs. The result is summarized in Table I.

A. Classification of SO NLσMs

Since we do not have any symmetry, the realizable gauge-gravity topological invariants produced by the NLσMs are covered by $\mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z}) = H^{d+1}(BSO,\mathbb{Z})$, $d>1$. In Appendix D, we calculated the ring $H^*(BSO,\mathbb{Z})$. In low dimensions, we have

$$\begin{aligned}
&H^0(BSO,\mathbb{Z})=\mathbb{Z},\\
&H^1(BSO,\mathbb{Z})=0,\\
&H^2(BSO,\mathbb{Z})=0,\\
&H^3(BSO,\mathbb{Z})=\mathbb{Z}_2,\ \text{basis } \beta(\mathrm{w}_2),\\
&H^4(BSO,\mathbb{Z})=\mathbb{Z},\ \text{basis } p_1,\\
&H^5(BSO,\mathbb{Z})=\mathbb{Z}_2,\ \text{basis } \beta(\mathrm{w}_4),\\
&H^6(BSO,\mathbb{Z})=\mathbb{Z}_2,\ \text{basis } \beta(\mathrm{w}_2)\beta(\mathrm{w}_2),\\
&H^7(BSO,\mathbb{Z})=2\mathbb{Z}_2,\ \text{basis } \beta(\mathrm{w}_6),\mathrm{w}_2^2\beta(\mathrm{w}_2),\\
&H^8(BSO,\mathbb{Z})=2\mathbb{Z}\oplus\mathbb{Z}_2,\ \text{basis } p_1^2,p_2,\beta(\mathrm{w}_2)\beta(\mathrm{w}_4).
\end{aligned}\tag{66}$$

We note that due to the relation $\mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z}) = H^{d+1}(BSO,\mathbb{Z})$, the d-dimensional gauge-gravity topological invariants $W^d_{\text{top}}(A,\Gamma)$ (with values in $\mathbb{R}/\mathbb{Z}$) are promoted to $(d+1)$-dimensional topological invariants $K^{d+1}(A,\Gamma)$ (with values in $\mathbb{Z}$). In the above, we also listed the basis of those topological invariants, so that a generic topological invariant $K^{d+1}(A,\Gamma)$ is a superposition of those bases. In the following, we list $\mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z})$ and the basis of their topological invariants $W^d_{\text{top}}(A,\Gamma)$:

$$\begin{aligned}
&\mathcal{H}^0(SO,\mathbb{R}/\mathbb{Z})=0,\\
&\mathcal{H}^1(SO,\mathbb{R}/\mathbb{Z})=0,\\
&\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2,\ \text{basis } \tfrac{1}{2}\mathrm{w}_2,\\
&\mathcal{H}^3(SO,\mathbb{R}/\mathbb{Z})=\mathbb{Z},\ \text{basis } \omega_3,\\
&\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2,\ \text{basis } \tfrac{1}{2}\mathrm{w}_4,\\
&\mathcal{H}^5(SO,\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2,\ \text{basis } \tfrac{1}{2}\mathrm{w}_2(\mathrm{w}_1\mathrm{w}_2+\mathrm{w}_3),\\
&\mathcal{H}^6(SO,\mathbb{R}/\mathbb{Z})=2\mathbb{Z}_2,\ \text{basis } \tfrac{1}{2}\mathrm{w}_6,\tfrac{1}{2}\mathrm{w}_2^3,\\
&\mathcal{H}^7(SO,\mathbb{R}/\mathbb{Z})=2\mathbb{Z}\oplus\mathbb{Z}_2,\ \text{basis } \omega_7^{p_1^2},\omega_7^{p_2},\tfrac{1}{2}(\mathrm{w}_1\mathrm{w}_2+\mathrm{w}_3)\mathrm{w}_4.
\end{aligned}\tag{67}$$

The above basis gives rise to the basis in Eq. (66) through the natural map $\tilde\beta:\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})\to\mathcal{H}^{d+1}(G,\mathbb{Z})$ (see Appendix E).

We see that $\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2$, which implies that a realizable gauge-gravity topological invariant exists in 1 + 1D, provided that we probe the SO NLσM by an arbitrary SO bundle on an oriented 1 + 1D space-time manifold M^2:

$$\int_{M^2} W^2_{\text{top}}(\Gamma_{SO}) = \int_{M^2}\frac{m}{2}\mathrm{w}_2^{SO},\qquad m=0,1\tag{68}$$

where Γ_{SO} is the connection of the SO bundle on M^2 and w_i^{SO} are the Stiefel-Whitney classes for the SO bundle. However, the SO bundle on M^2 is restricted: it must be the tangent bundle of M^2. So we actually have

$$\int_{M^2} W^2_{\text{top}}(\Gamma) = \int_{M^2}\frac{m}{2}\mathrm{w}_2,\qquad m=0,1\tag{69}$$

where Γ is the connection of the tangent bundle on M^2 and w_i are the Stiefel-Whitney classes for the tangent bundle. The Stiefel-Whitney classes for the tangent bundle have some special relations. In fact, we have (1) a manifold is orientable iff $\mathrm{w}_1=0$; (2) a manifold admits a spin structure iff $\mathrm{w}_2=0$. Since all closed orientable two-dimensional manifolds are spin, thus both w_1 and w_2 vanish for tangent bundles of M^2. The realizable gauge-gravity topological invariant cannot be probed by any oriented space-time M^2. Thus, the above realizable gauge-gravity topological invariants described by $\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})$ collapse to zero in 1 + 1D. There is no iTO in 1 + 1D [or in other words, $\sigma\text{iTO}^2_L = H^2(SO,\mathbb{R}/\mathbb{Z})/\Lambda^2 = 0$].

B. Relations between Stiefel-Whitney classes

We see that to understand the realizable gauge-gravity topological invariants, whether they collapse to zero or not, it is important to understand all relations that the Stiefel-Whitney classes must satisfy, when the Stiefel-Whitney classes come from a tangent bundle. To obtain such relations, let us first consider the Stiefel-Whitney classes for an arbitrary O vector bundle on a d-dimensional space.

We note that the total Stiefel-Whitney class $w = 1+\mathrm{w}_1+\mathrm{w}_2+\dots$ is related to the total Wu class $u = 1+u_1+u_2+\dots$

through the total Steenrod square:

$$w = Sq(u), \quad Sq = 1 + Sq^1 + Sq^2 + \dots . \tag{70}$$

Therefore,

$$\mathrm{w}_n = \sum_{i=0}^{n} Sq^i u_{n-i}. \tag{71}$$

The Steenrod squares have the following properties:

$$Sq^i(x_j) = 0,\ i > j,\ \ Sq^j(x_j) = x_j x_j,\ \ Sq^0 = 1, \tag{72}$$

for any $x_j \in H^j(X^d,\mathbb{Z}_2)$. Thus,

$$u_n = \mathrm{w}_n + \sum_{i=1,2i\leqslant n} Sq^i u_{n-i}. \tag{73}$$

This allows us to compute u_n iteratively, using the Wu formula

$$\begin{aligned} &Sq^i(\mathrm{w}_j) = 0,\ \ i > j, \quad Sq^i(\mathrm{w}_i) = \mathrm{w}_i\mathrm{w}_i, \\ &Sq^i(\mathrm{w}_j) = \mathrm{w}_i\mathrm{w}_j + \sum_{k=1}^{i} \frac{(j-i-1+k)!}{(j-i-1)!k!}\mathrm{w}_{i-k}\mathrm{w}_{j+k},\ \ i < j \end{aligned} \tag{74}$$

and the Steenrod relation

$$Sq^n(xy) = \sum_{i=0}^{n} Sq^i(x) Sq^{n-i}(y). \tag{75}$$

We find

$$\begin{aligned} u_0 &= 1, \\ u_1 &= \mathrm{w}_1, \\ u_2 &= \mathrm{w}_1^2 + \mathrm{w}_2, \\ u_3 &= \mathrm{w}_1\mathrm{w}_2, \\ u_4 &= \mathrm{w}_1^4 + \mathrm{w}_2^2 + \mathrm{w}_1\mathrm{w}_3 + \mathrm{w}_4, \\ u_5 &= \mathrm{w}_1^3\mathrm{w}_2 + \mathrm{w}_1\mathrm{w}_2^2 + \mathrm{w}_1^2\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_4, \\ u_6 &= \mathrm{w}_1^2\mathrm{w}_2^2 + \mathrm{w}_1^3\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_2\mathrm{w}_3 + \mathrm{w}_3^2 + \mathrm{w}_1^2\mathrm{w}_4 + \mathrm{w}_2\mathrm{w}_4, \\ u_7 &= \mathrm{w}_1^2\mathrm{w}_2\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_3^2 + \mathrm{w}_1\mathrm{w}_2\mathrm{w}_4, \\ u_8 &= \mathrm{w}_1^8 + \mathrm{w}_2^4 + \mathrm{w}_1^2\mathrm{w}_3^2 + \mathrm{w}_1^2\mathrm{w}_2\mathrm{w}_4 + \mathrm{w}_1\mathrm{w}_3\mathrm{w}_4 + \mathrm{w}_4^2 \\ &\quad + \mathrm{w}_1^3\mathrm{w}_5 + \mathrm{w}_3\mathrm{w}_5 + \mathrm{w}_1^2\mathrm{w}_6 + \mathrm{w}_2\mathrm{w}_6 + \mathrm{w}_1\mathrm{w}_7 + \mathrm{w}_8. \end{aligned} \tag{76}$$

We note that the Steenrod squares form an algebra

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \frac{(b-j-1)!}{(a-2j)!(b-a+j-1)!} Sq^{a+b-j} Sq^j, \quad 0 < a < 2b \tag{77}$$

which leads to the relation $Sq^1Sq^1 = 0$ used in the last section.

If the O vector bundle on d-dimensional space M^d happens to be the tangent bundle of M^d, then the Steenrod square and the Wu class satisfy

$$Sq^{d-j}(x_j) = u_{d-j}x_j \text{ for any } x_j \in H^j(X^d,\mathbb{Z}_2). \tag{78}$$

(1) If we choose x_j to be a combination of Stiefel-Whitney classes, the above will generate many relations between Stiefel-Whitney classes. (2) Since $Sq^i(x_j) = 0$ if $i > j$, therefore $u_i x_{d-i} = 0$ for any $x_{d-i} \in H^{d-i}(X^d,\mathbb{Z}_2)$ if $i > d - i$. Thus, for d-dimensional manifold, the Wu class $u_i = 0$ if $2i > d$. Also, $Sq^n \dots Sq^m(u_i) = 0$ if $2i > d$. This also gives us relations among Stiefel-Whitney classes. (3) Last, there is another type of relation. In $4n$ dimensions, the mod 2 reduction of Pontryagin classes $p_{i_1}p_{i_2}\dots$, $n = i_1 + i_2 + \dots$, should be regarded as zero. The reason is explained below Eq. (84). This leads to the relations for d-dimensional manifold

$$\mathrm{w}_{2i_1}^2\mathrm{w}_{2i_2}^2 \dots = 0, \quad \text{if} \quad 2i_1 + 2i_2 + \dots = d. \tag{79}$$

σiTO$_L^d$ is given by $H^{d+1}(BSO,\mathbb{Z})$ after quotient out all those relations.

C. iTO phases in low dimensions

In two-dimensional space-time $\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z}) = H^3(BSO,\mathbb{Z}) = \mathbb{Z}_2$ which is generated by $W_{\text{top}}^2 = \frac{1}{2}\mathrm{w}_2$. So σiTO$_L^2$ may be nontrivial. The relations $u_2 = u_3 = 0$ give us

$$\mathrm{w}_1^2 + \mathrm{w}_2 = 0. \tag{80}$$

Since M^2 is oriented, $\mathrm{w}_1 = 0$. We see that $\mathrm{w}_2 = 0$. W_{top}^2 vanishes, and there is no realizable gauge-gravity topological invariant in 1 + 1D. So σiTO$_L^2 = 0$.

In 2 + 1D space-time, the corresponding $\mathcal{H}^3(SO,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$ is generated by $W_{\text{top}}^3 = \omega_3$. There is no relation involving ω_3. So σiTO$_L^3 = \mathbb{Z}$. The generating topological invariant $W_{\text{top}}^3(\Gamma) = \omega_3$ describes an iTO state with chiral central charge $c = 24$.

In 3 + 1D space-time, the corresponding $\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$ is generated by the gauge-gravity topological invariant $W_{\text{top}}^4 = \frac{1}{2}\mathrm{w}_4$. The Wu classes $u_3 = u_4 = 0$ can lead to relations between the Stiefel-Whitney classes, which give us

$$\mathrm{w}_1\mathrm{w}_2 = \mathrm{w}_1^4 + \mathrm{w}_1\mathrm{w}_3 + \mathrm{w}_2^2 + \mathrm{w}_4 = 0. \tag{81}$$

Other relations can be obtained by applying the Steenrod squares to the above:

$$Sq^1(\mathrm{w}_1\mathrm{w}_3) = \mathrm{w}_1\mathrm{w}_3 = 0. \tag{82}$$

Additional relations can be obtained from Eq. (78):

$$\begin{aligned} Sq^1(\mathrm{w}_3) &= u_1\mathrm{w}_3 \to \mathrm{w}_1\mathrm{w}_3 = \mathrm{w}_1\mathrm{w}_3, \\ Sq^2(\mathrm{w}_2) &= u_2\mathrm{w}_2 \to \mathrm{w}_2^2 = \mathrm{w}_1^2\mathrm{w}_2 + \mathrm{w}_2^2. \end{aligned} \tag{83}$$

We see that $\mathrm{w}_4 = \mathrm{w}_2^2$, but nothing restricts w_2^2. Naively, this suggests that $\mathrm{w}_2^2 \in H^4(M^4,\mathbb{Z}_2)$ is a realizable gauge-gravity topological invariant in 3 + 1D:

$$W_{\text{top}}^4(\Gamma) = \tfrac{1}{2}\mathrm{w}_2^2. \tag{84}$$

However, there is a relation between Pontryagin classes and Stiefel-Whitney classes (see Appendix F):

$$\mathrm{w}_{2i}^2 = p_i \bmod 2 \tag{85}$$

on any closed oriented manifolds M^{4i} of dimension $4i$. Thus, w_2^2 is part of Pontryagin class p_1. The topological invariant $W_{\text{top}}^4(\Gamma) = \frac{1}{2}\mathrm{w}_2^2 = \frac{1}{2}p_1$ is realizable, but also smoothly connects to the trivial case via the Pontryagin class: $W_{\text{top}}^4(\Gamma) = \frac{\theta}{2\pi}p_1$, where θ can go from π to 0 smoothly. There is no realizable gauge-gravity topological invariant in 3 + 1D that cannot connect to zero. Thus, $\sigma\text{iTO}_L^4 = 0$. In general, such kind of reasoning gives rise to Eq. (79).

In 4 + 1D space-time, the corresponding $\mathcal{H}^5(SO,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$ is generated by the gauge-gravity topological invariant $W_{\text{top}}^4 = \frac{1}{2}\text{w}_2(\text{w}_1\text{w}_2 + \text{w}_3)$. The Wu classes $u_3 = u_4 = u_5 = 0$ can lead to relations between the Stiefel-Whitney classes, which give us

$$\text{w}_4 + \text{w}_2^2 = 0. \tag{86}$$

$\text{w}_4 + \text{w}_2^2 = 0$ is just a generator of the relations. Other relations can be obtained by applying the Steenrod squares

$$Sq^1(\text{w}_4 + \text{w}_2^2) = \text{w}_1\text{w}_4 + \text{w}_5 = 0. \tag{87}$$

Additional relations can be obtained from Eq. (78):

$$\begin{aligned} &Sq^1(\text{w}_4) = u_1\text{w}_4 \to \text{w}_1\text{w}_4 + \text{w}_5 = \text{w}_1\text{w}_4,\\ &Sq^2(\text{w}_3) = u_2\text{w}_3 \to \text{w}_2\text{w}_3 + \text{w}_1\text{w}_4 + \text{w}_5 = \text{w}_1^2\text{w}_3 + \text{w}_2\text{w}_3. \end{aligned} \tag{88}$$

We see that w_5 must vanish, but nothing restricts w_2w_3. So we have a realizable gauge-gravity topological invariant in 4 + 1D:

$$W_{\text{top}}^5(\Gamma) = \tfrac{1}{2}\text{w}_2\text{w}_3. \tag{89}$$

Thus, $\sigma\text{iTO}_L^5 = \mathbb{Z}_2$.

In 5 + 1D space-time, the corresponding $\mathcal{H}^6(SO,\mathbb{R}/\mathbb{Z}) = 2\mathbb{Z}_2$ is generated by the gauge-gravity topological invariant $W_{\text{top}}^4 = \frac{1}{2}\text{w}_6, \frac{1}{2}\text{w}_2^3$. The Wu classes $u_4 = u_5 = u_6 = 0$ give us

$$\text{w}_4 + \text{w}_2^2 = \text{w}_2\text{w}_4 + \text{w}_3^2 = 0. \tag{90}$$

Other relations can be obtained by applying the Steenrod squares

$$\begin{aligned} Sq^1(\text{w}_4 + \text{w}_2^2) &= \text{w}_1\text{w}_4 + \text{w}_5 = 0,\\ Sq^2(\text{w}_4 + \text{w}_2^2) &= \text{w}_1^2\text{w}_2^2 + \text{w}_3^2 + \text{w}_2\text{w}_4 + \text{w}_6 = 0,\\ Sq^1Sq^1(\text{w}_4 + \text{w}_2^2) &= 0. \end{aligned} \tag{91}$$

We see that w_6 must vanish, and $\text{w}_2\text{w}_4 = \text{w}_2^3 = \text{w}_3^2$. Additional relations can be obtained from Eq. (78):

$$\begin{aligned} &Sq^1(\text{w}_5) = u_1\text{w}_5 \to \text{w}_1\text{w}_5 = \text{w}_1\text{w}_5,\\ &Sq^2(\text{w}_4) = u_2\text{w}_4 \to \text{w}_2\text{w}_4 + \text{w}_6 = \text{w}_1^2\text{w}_4 + \text{w}_2\text{w}_4,\\ &Sq^3(\text{w}_3) = u_3\text{w}_3 \to \text{w}_3\text{w}_3 = \text{w}_1\text{w}_2\text{w}_3. \end{aligned} \tag{92}$$

We see that $\text{w}_2\text{w}_4 = \text{w}_2^3 = \text{w}_3^2 = 0$. So $\sigma\text{iTO}_L^6 = 0$.

In 6 + 1D space-time, the corresponding $\mathcal{H}^8(SO,\mathbb{R}/\mathbb{Z}) = 2\mathbb{Z} \oplus \mathbb{Z}_2$ is generated by the gauge-gravity topological invariant $W_{\text{top}}^4 = \omega_7^{p_1^2}, \omega_7^{p_2}, \frac{1}{2}(\text{w}_1\text{w}_2 + \text{w}_3)\text{w}_4$. The Wu classes $u_4 = u_5 = u_6 = u_7 = 0$ give us

$$\text{w}_4 + \text{w}_2^2 = \text{w}_2\text{w}_4 + \text{w}_3^2 = 0. \tag{93}$$

Other relations can be obtained by applying the Steenrod squares (setting $\text{w}_1 = 0$)

$$\begin{aligned} Sq^1(\text{w}_4 + \text{w}_2^2) &= \text{w}_1\text{w}_4 + \text{w}_5 = 0,\\ Sq^2(\text{w}_4 + \text{w}_2^2) &= \text{w}_1^2\text{w}_2^2 + \text{w}_3^2 + \text{w}_2\text{w}_4 + \text{w}_6 = 0,\\ Sq^1(\text{w}_2\text{w}_4 + \text{w}_3^2) &= \text{w}_3\text{w}_4 + \text{w}_2\text{w}_5 = 0. \end{aligned} \tag{94}$$

Additional relations can be obtained from Eq. (78) (setting $\text{w}_1 = 0$):

$$\begin{aligned} Sq^1(\text{w}_6) &= u_1\text{w}_6 \to \text{w}_7 = 0,\\ Sq^1(\text{w}_2^3) &= u_1\text{w}_2^3 \to \text{w}_2^2\text{w}_3 = 0,\\ Sq^2(\text{w}_2\text{w}_3) &= u_3\text{w}_2\text{w}_3 \to \text{w}_2^2\text{w}_3 + \text{w}_2\text{w}_5. \end{aligned} \tag{95}$$

We see that $\text{w}_2\text{w}_5 = \text{w}_3\text{w}_4 = \text{w}_2^2\text{w}_3 = \text{w}_7 = 0$. So $\sigma\text{iTO}_L^7 = 2\mathbb{Z}$.

D. Relation to cobordism groups

Two oriented smooth n-dimensional manifolds M and N are said to be equivalent if $M \cup (-N)$ is a boundary of another manifold, where $-N$ is the N manifold with a reversed orientation. With the multiplication given by the disjoint union, the corresponding equivalence classes has a structure of an Abelian group Ω_n^{SO}, which is called the cobordism group of closed oriented smooth manifolds. For low dimensions, we have the following [94]:

$\Omega_0^{SO} = \mathbb{Z}$, generated by a point.
$\Omega_1^{SO} = 0$, since circles bound disks.
$\Omega_2^{SO} = 0$, since all oriented surfaces bound handlebodies.
$\Omega_3^{SO} = 0$.
$\Omega_4^{SO} = \mathbb{Z}$, generated by $\mathbb{C}P^2$, detected by $\frac{1}{3}\int_M p_1$.
$\Omega_5^{SO} = \mathbb{Z}_2$, generated by the Wu manifold $SU(3)/SO(3)$, detected by the deRham invariant or Stiefel-Whitney number $\int_M \text{w}_2\text{w}_3$.
$\Omega_6^{SO} = 0$.
$\Omega_7^{SO} = 0$.
$\Omega_8^{SO} = 2\mathbb{Z}$ generated by $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$.

The potential gravitational topological invariants give us a map from closed space-time M^d to $U(1)$: $Z_{\text{fixed}}(M^d) = e^{i\int_{M^d} 2\pi W_{\text{top}}^d(\Gamma)} \in U(1)$. For locally null topological invariants, such a map reduces to a map from Ω_d^{SO} to $U(1)$. In fact, $e^{i\int_{M^d} 2\pi W_{\text{top}}^d(\Gamma)}$ is a 1D representation of group Ω_d^{SO}. So the locally null potential gravitational topological invariants are described by 1D representations of the cobordism group Ω_d^{SO}. Since the locally null potential gravitational topological invariants are discrete, so they are actually described by 1D representation of (Ω_d^{SO}). Since, for an Abelian group G_A, the set of its 1D representations also form an Abelian group, which is G_A itself. Therefore, the discrete locally null potential gravitational topological invariants in d-dimensional space-time are described by (Ω_d^{SO}). Since all the locally null potential gravitational topological invariants are realizable, we find

$$\text{Tor}(\sigma\text{iTO}_L^d) = \text{Tor}(\Omega_d^{SO}). \tag{96}$$

The Chern-Simons potential gravitational topological invariants in d-dimensional space-time are described by (Ω_{d+1}^{SO}) since (Ω_{d+1}^{SO}) is a subgroup of $H^{d+1}[BSO,\mathbb{Z}(\frac{1}{n})]$. So the Chern-Simons realizable gravitational topological invariants, described by $\mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z}) = H^{d+1}(BSO,\mathbb{Z})$, form a subgroup of $\text{Free}(\Omega_d^{SO})$:

$$\text{Free}(\sigma\text{iTO}_L^d) \subset \text{Free}(\Omega_d^{SO}). \tag{97}$$

V. PURE AND MIXED SPT STATES

A. A generic result

In this section, we are going to consider L-type SPT states protected by G symmetry (which may contain time-reversal symmetry) in d-dimensional space-time. Those SPT states form an Abelian group LSPT^d_G. We only consider SPT states that are realized by $G \times SO$ NLσMs. The different $G \times SO$ NLσMs are characterized by their topological terms which are classified by $\mathcal{H}^d(G \times SO,\mathbb{R}/\mathbb{Z})$. Those topological terms induced the realizable gauge-gravity topological invariants $W^d_{\text{top}}(A,\Gamma)$ that are also "classified" by $\mathcal{H}^d(G \times SO,\mathbb{R}/\mathbb{Z})$. Therefore, L-type SPT states from NLσMs are "classified" by $\mathcal{H}^d(G \times SO,\mathbb{R}/\mathbb{Z})$, but in a many-to-one fashion; i.e., different elements in $\mathcal{H}^d(G \times SO,\mathbb{R}/\mathbb{Z})$ may correspond to the same gauge-gravity topological invariant $W^d_{\text{top}}(A,\Gamma)$ and the same SPT phase.

To understand this many-to-one correspondence, we note that the gauge-gravity topological invariants $W^d_{\text{top}}(A,\Gamma)$ should be fully detectable in the following sense. The gauge-gravity topological invariants $W^d_{\text{top}}(A,\Gamma)$ can be regarded as map from a pair (M^d,A) to $\mathbb{R}/\mathbb{Z}$:

$$\int_{M^d} W^d_{\text{top}}(A,\Gamma) = \frac{\theta}{2\pi} \bmod 1, \tag{98}$$

where M^d is a close space-time manifold with various topologies and A is a G-symmetry twist on M^d. Two topological invariants are said to be different if they produce different maps $(M^d,A) \to \mathbb{R}/\mathbb{Z}$ that cannot be smoothly connected to each other. However, there indeed exist gauge-gravity topological invariants $ZW^d_{\text{top}}(A,\Gamma)$ whose induced map $(M^d,A) \to \mathbb{R}/\mathbb{Z}$ can be smoothly connected to 0 (see Appendix C). Then, any two topological invariants differing by $ZW^d_{\text{top}}(A,\Gamma)$ should correspond to the same SPT phase and should be identified. We call $ZW^d_{\text{top}}(A,\Gamma) = 0$ a relation between topological invariants. $ZW^d_{\text{top}}(A,\Gamma)$ generate a subgroup of $\mathcal{H}^d(G \times SO,\mathbb{R}/\mathbb{Z})$ which will be called $\Lambda^d(G)$. We see that the distinct SPT phases, plus the iTO phases that are also produced by the NLσMs, are classified by the quotient

$$\text{LSPT}^d_G \oplus \sigma\text{iTO}^d_L = \mathcal{H}^d(G \times SO,\mathbb{R}/\mathbb{Z})/\Lambda^d(G). \tag{99}$$

In the next subsection, we will discuss how to compute the subgroup $\Lambda^d(G)$.

Using the Künneth formula (62), we find that

$$\mathcal{H}^d(G \times SO,\mathbb{R}/\mathbb{Z}) \simeq \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z})$$
$$\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})]. \tag{100}$$

Clearly, the term $\mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z})$ describes iTO phases that do not require any symmetry G. So

$$\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \tag{101}$$

should cover all the SPT states, i.e., every cocycle in $\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ is realizable and describes a SPT state. In other words,

$$\text{LSPT}^d_G = \frac{\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})}{\Lambda^d(G)}. \tag{102}$$

For example, the term $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ describes pure SPT states. Each element in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ corresponds to distinct realizable SPT states (quotient is not needed).

Similarly, the term $H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})]$ describes mixed SPT states. Every cocycle in $H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})]$ describes a mixed SPT state. But different cocycles may correspond to the same SPT state. This can be seen from the dimension reduction discussed in Sec. II D. We put a $G \times SO$ SPT state on $M^k \times M^{d-k}$ which is described by a cocycle ν_d in $\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z})$. The cocycle ν_d can be viewed as a gauge-gravity topological invariant W^d_{top} and vice versa. Here we will consider a mixed SPT state described by $W^{d,k}_{\text{top}} \in H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})]$ in more detail.

Let us put a G-symmetry twist A_G on M^k, but for the time being not any SO-symmetry twist on M^k. The decomposition $\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})]$ implies that, in the large M^{d-k} limit, we get a $(d-k)$-dimensional topological state on M^{d-k}, described by a cocycle ν_{d-k} in $\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})$. Formally, we can express the above dimension reduction as

$$\int_{M^k,A_G} W^{d,k}_{\text{top}} = \nu^{SO}_{d-k} \in \mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z}), \tag{103}$$

where A_G represent the G-symmetry twist on M^d. In particular, if we choose M^d, A_G, and $W^{d,k}_{\text{top}} \in H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})]$ arbitrarily, we can produce any elements in $\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})$.

However, due to the restrictive relation between the SO connection and the topology of M^{d-k}, different cocycles in $\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})$ may correspond to the same topological state. So the distinct topological states are described by a quotient $\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})/\Lambda^{d-k}$. As we have discussed before, the distinct topological states from $\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})$ are nothing but the $(d-k)$-dimensional iTO states that form σiTO^{d-k}_L. Therefore, the distinct iTO states on M^{d-k} imply that the parent SPT states on M^d before the dimension reduction are distinct. However, it is still possible that different parent SPT states on M^d lead to the same iTO state on M^{d-k}. So the SPT states are described by $H^k[BG,\sigma\text{iTO}^{d-k}_L]$ plus something extra. This way, we conclude that the L-type realizable SPT states are described by

$$\sigma\text{LSPT}^d_G = \left[E^d(G) \rtimes \oplus_{k=1}^{d-1} H^k\left(BG,\sigma\text{iTO}^{d-k}_L\right)\right]$$
$$\oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}), \tag{104}$$

which is one of the main results of this paper. We like to point out that if G contains time-reversal transformation, it will have a nontrivial action $\mathbb{R}/\mathbb{Z} \to -\mathbb{R}/\mathbb{Z}$ and $\sigma\text{iTO}^{d-k}_L \to -\sigma\text{iTO}^{d-k}_L$. In the next subsection, we will compute this extra group $E^d(G)$.

However, there is a mistake in the above derivation of Eq. (104). Due to the restrictive relation between the SO connection and the topology of M^k, we cannot set the SO-symmetry twist on M^k to zero. So the dimension reduction is actually given by

$$\int_{M^k,A_G,\Gamma} W^{d,k}_{\text{top}} = \nu^{\text{iTO}}_{d-k} \in \sigma\text{iTO}^{d-k}_L, \tag{105}$$

where Γ represent the SO-symmetry twist on M^d. Due to the restrictive relation between (M^d,A_G) and Γ, it is not clear that if we choose M^d, A_G, and $W_{\text{top}}^{d,k} \in H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})]$ arbitrarily, we can still produce any elements in σiTO_L^{d-k}.

In the following, we will show that we can indeed produce any elements in σiTO_L^{d-k}. (1) We note that the SO tangent bundle of $M^k \times M^{d-k}$ splits into an SO'' tangent bundle on M^{d-k} and a SO' tangent bundle on M^k. So we can rewrite Eq. (105) as

$$\int_{M^k,A_G,\Gamma'} W_{\text{top}}^{d,k} = \nu_{d-k}^{SO} \in \mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z}), \tag{106}$$

where A_G,Γ' is the $G\times SO'$ symmetry twist on M^k and we put the SO'' symmetry twist on M^{d-k}. This motivates us to consider a $G\times SO'\times SO''$ NLσM and its topological terms. (2) The natural group homomorphism $G\times SO'\times SO''\to G\times SO$ via embedding $SO'\times SO''$ into SO leads to a ring homomorphism $H^*[B(G\times SO),\mathbb{Z}]\to H^*[B(G\times SO'\times SO''),\mathbb{Z}]$. (3) Due to the isomorphism $\mathcal{H}^n(G,\mathbb{R}/\mathbb{Z})\simeq H^{n+1}(BG,\mathbb{Z})$, $W_{\text{top}}^{d,k}$ in $H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})]$ can be viewed as an element in $H^k[BG,H^{d-k+1}(BSO,\mathbb{Z})]$. As a result, we can express $W_{\text{top}}^{d,k}$ as a characteristic class in $H^k[BG,H^{d-k+1}(BSO,\mathbb{Z})]$. For example, $W_{\text{top}}^{d,k} = F_k^G F_l^{SO} F_{d-k-l+1}^{SO}$, where F_k^G is a characteristic class in $H^k(BG,\mathbb{Z})$, and F_n^{SO} is a characteristic class in $H^n(BSO,\mathbb{Z})$. (4) Using the above ring homomorphism, we can map $W_{\text{top}}^{d,k}$ into an element in $H^k[BG,H^{d-k+1}[B(SO'\times SO''),\mathbb{Z}]]$:

$$\begin{aligned} W_{\text{top}}^{d,k} &= F_k^G F_l^{SO} F_{d-k-l+1}^{SO} \\ &\to F_k^G\left(F_l^{SO'}+F_l^{SO''}\right)\left(F_{d-k-l+1}^{SO'}+F_{d-k-l+1}^{SO''}\right) \\ &\in H^k[BG,H^{d-k+1}[B(SO'\times SO''),\mathbb{Z}]]. \end{aligned} \tag{107}$$

(5) Since the SO' twist is only on M^k and the SO'' twist is only on M^{d-k}, the above expression allows us to conclude that only the term $F_k^G F_l^{SO''} F_{d-k-l+1}^{SO''}$ contribute to $\int_{M^k,A_G,\Gamma'} W_{\text{top}}^{d,k}$. Thus,

$$\int_{M^k,A_G,\Gamma'} W_{\text{top}}^{d,k} = \int_{M^k,A_G,0} W_{\text{top}}^{d,k}, \tag{108}$$

which reduces Eq. (105) to Eq. (103) that leads to Eq. (104). This completes our proof.

B. A calculation of $\Lambda^d(G)$ and $E^d(G)$

The subgroup $\Lambda^d(G)$ is generated by a set of relations in $\mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z})\oplus\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})] = H^{d+1}(G\times SO,\mathbb{Z})$. To compute such a set of the relations, we can choose a homomorphism $G\to O$, which will lead to a homomorphism $H^*(BO,\mathbb{Z}_2)\to H^*(BG,\mathbb{Z}_2)$ as rings. We know that $H^*(BO,\mathbb{Z}_2)$ is generated by the Stiefel-Whitney classes $\mathrm{w}_1,\mathrm{w}_2,\ldots$. w_i will map into $\mathrm{w}_i^G \in H^i(BG,\mathbb{Z}_2)$. Then, we can treat w_i^G as the Stiefel-Whitney classes and use the Wu formula (74) to compute $Sq^i(\mathrm{w}_j^G)$. The Wu formula and the following defining properties of the Wu classes

$$\begin{aligned} Sq^{d-i}\left(w_i^G\right) &= u_{d-i} w_i^G, \\ Sq^{d-i-j}\left(w_i w_j^G\right) &= u_{d-i-j} w_i w_j^G, \ldots \end{aligned} \tag{109}$$

will generate the relations [denoted as $ZW_{\text{top}}^d(A,\Gamma)$]

$$\begin{aligned} &Sq^{d-i}\left(w_i^G\right)+u_{d-i}w_i^G, \\ &Sq^{d-i-j}\left(w_i^G w_j^G\right)+u_{d-i-j}w_i^G w_j^G, \ldots \end{aligned} \tag{110}$$

in $\oplus_{k=1}^{d-1} H^k[BG,H^{d-k}(O,\mathbb{R}/\mathbb{Z})]\oplus\mathcal{H}^d(O,\mathbb{R}/\mathbb{Z})\oplus\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. Those relations become the relations in $\oplus_{k=1}^{d-1} H^k[BG,H^{d-k}(SO,\mathbb{R}/\mathbb{Z})]\oplus\mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z})\oplus\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) = H^{d+1}[B(G\times SO),\mathbb{Z}]$ through the natural map $\beta: H^d[B(G\times SO),\mathbb{Z}_2]\to H^{d+1}[B(G\times SO),\mathbb{Z}]$, after we set $\mathrm{w}_1=0$.

$\Lambda^d(G)$ also contain another type of relation: if $a\in\mathcal{H}^d(G\times SO,\mathbb{R}/\mathbb{Z})$ can be expressed as a mod 2 reduction of $\bar a\in \text{Free}\mathcal{H}^d(G\times SO,\mathbb{Z})$, then a is in $\Lambda^d(G)$. The reason for such type of relations is discussed below Eq. (120).

The relations will generate $\Lambda^d(G)$ which also allow us to compute $E^d(G)$. Certainly, the subgroup of $\oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})]$, $\oplus_{k=1}^{d-1} H^k(BG,\sigma\text{iTO}_L^{d-k})$, will survive the quotient by $\Lambda^d(G)$. $E^d(G)$ is the subgroup not in $\oplus_{k=1}^{d-1} H^k(BG,\sigma\text{iTO}_L^{d-k})$ that also survive the quotient. Thus, $E^d(G)\rtimes\oplus_{k=1}^{d-1} H^k(BG,\sigma\text{iTO}_L^{d-k})$ describes the distinct SPT phases. Next, we will demonstrate the above approach by computing the pure and mixed SPT states for some simple symmetry groups.

C. $U(1)$ SPT states

From [13]

$$\begin{aligned} H^d[BU(1),\mathbb{Z}] &= 0 \quad \text{if} \quad d=\text{odd}, \\ H^d[BU(1),\mathbb{Z}] &= \mathbb{Z} \quad \text{if} \quad d=\text{even}, \end{aligned} \tag{111}$$

we can obtain

$$\begin{aligned} H^d[BU(1),\mathbb{Z}_2] &= 0 \quad \text{if} \quad d=\text{odd}, \\ H^d[BU(1),\mathbb{Z}_2] &= \mathbb{Z}_2 \quad \text{if} \quad d=\text{even} \end{aligned} \tag{112}$$

using universal coefficient theorem [49,69]. The ring $H^*[BU(1),\mathbb{Z}]$ is generated by the first Chern class c_1.

This allows us to calculate the $U(1)$ mixed SPT described by $\oplus_{k=1}^{d-1} H^k[BU(1),\mathcal{H}^{d-k}(O,\mathbb{R}/\mathbb{Z})]$. We obtain $U(1)$ mixed SPT states in 4+1D described by the group cohomology $H^2[BU(1),\sigma\text{iTO}_L^3]=\mathbb{Z}$. We also obtain mixed SPT states in 6+1D described by $H^4[BU(1),\sigma\text{iTO}_L^3]\oplus H^2[BU(1),\sigma\text{iTO}_L^5]=\mathbb{Z}\oplus Z_2$.

The well-known 2+1D $U(1)$ pure SPT states have the following Chern-Simons topological invariants:

$$W_{\text{top}}^3(A,\Gamma) = \frac{k}{(2\pi)^2} A\,dA, \quad k\in\mathbb{Z} \tag{113}$$

where A is the $U(1)$ gauge connection one-form. Their Hall conductances are given by $\sigma_{xy}=\frac{2k}{2\pi}$.

The 4+1D $U(1)$ mixed SPT states described by $\mathcal{H}^2[U(1),\sigma\text{iTO}_L^3]$ has been discussed in [59]. Its gauge-gravity topological invariant is given by (see a discussion in Appendix I)

$$W_{\text{top}}^5(A,\Gamma) = \omega_3\frac{dA}{2\pi} = \frac{A}{2\pi}p_1. \tag{114}$$

In four spatial dimensions, the $U(1)$ monopole is a 1D loop. In this SPT state, such a 1D loop will carry the gapless edge state of $2+1$D $(E_8)^3$ bosonic quantum Hall state.

The $6+1$D $U(1)$ mixed SPT states described by $\mathcal{H}^4[U(1),\sigma\text{iTO}_L^3]=\mathbb{Z}$ have the following topological invariants:

$$W^7_{\text{top}}(A,\Gamma)=\frac{k}{(2\pi)^2}\omega_3\, dA\, dA,\quad k\in\mathbb{Z}. \tag{115}$$

The $6+1$D $U(1)$ mixed SPT state described by $\mathcal{H}^2[U(1),\sigma\text{iTO}_L^5]=\mathbb{Z}_2$ has

$$W^7_{\text{top}}(A,\Gamma)=\frac{1}{2}\text{w}_2\text{w}_3\frac{dA}{2\pi}. \tag{116}$$

To see if there are extra mixed $U(1)$ SPT phases, let us first note that the ring $H^*[BU(1),\mathbb{Z}_2]$ is generated by f_2, which is the mod 2 reduction (denoted as ρ_2) of the first Chern class c_1: $f_2=\rho_2 c_1$. If we choose the natural embedding $U(1)\to O$, we find that

$$\text{w}_2^{U(1)}=f_2,\quad \text{w}_i^{U(1)}=0,\quad i=1,\quad \text{or}\quad i>2. \tag{117}$$

In $3+1$D, the potential extra mixed SPT phases are described by $H^2[BU(1),\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]=\mathbb{Z}_2$. We note that $\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})$ is generated by the w_2 [see Eq. (67)]. Therefore, the extra $U(1)$ SPT phases described by $H^2[BU(1),\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]=\mathbb{Z}_2$ are generated by $f_2\text{w}_2$, In $3+1$D, we have a relation

$$\begin{aligned} Sq^2\big(\text{w}_2^{U(1)}\big)&=u_2\text{w}_2^{U(1)},\\ Sq^2\big(\text{w}_2^{U(1)}\big)&=u_2^{U(1)}\text{w}_2^{U(1)}, \end{aligned} \tag{118}$$

which gives us

$$\text{w}_2^{U(1)}\text{w}_2^{U(1)}=\big(\text{w}_1^2+\text{w}_2\big)\text{w}_2^{U(1)}. \tag{119}$$

In $3+1$D, we also have a relation $f_2^2=\text{w}_2^{U(1)}\text{w}_2^{U(1)}=0$ mod 2. For oriented space-time $\text{w}_1=0$, so $f_2\text{w}_2$ vanishes. There is no extra $3+1$D $U(1)$ SPT phase.

Here is another way to rephrase the above reasoning. In $3+1$D, there is a potential topological invariant

$$W^4_{\text{top}}(A,\Gamma)=\frac{1}{2}\text{w}_2\frac{dA}{2\pi}=\frac{1}{2}\text{w}_2\rho_2 c_1, \tag{120}$$

where the Chern class $c_1=dA/2\pi$ and ρ_2 is the mod 2 reduction. Using the relation $Sq^2(\rho_2c_1)=u_2\rho_2c_1$ and $Sq^2(\rho_2c_1)=(\rho_2c_1)^2$, we find that $(\text{w}_1^2+\text{w}_2)\rho_2c_1=(\rho_2c_1)^2$. Therefore, on oriented manifold, we have [62]

$$W^4_{\text{top}}(A,\Gamma)=\frac{1}{2}\text{w}_2\frac{dA}{2\pi}=\frac{1}{2}(f_2)^2=\frac{1}{2}c_1^2=\frac{1}{2(2\pi)^2}dA\, dA. \tag{121}$$

Such a topological invariant is not quantized. It can continuously deform into zero via $\frac{\theta}{(2\pi)^2}dA\,dA$ as θ goes from π to 0. This is why there is no $U(1)$ SPT phase in $3+1$D.

We note that on space-time M^4 with spin structure, $\text{w}_2=0$. The above result implies that all the $U(1)$ bundles on such M^4 satisfy

$$\int_{M^4}c_1^2=\text{even} \tag{122}$$

or, in other words, the $\mathbb{Z}_2$ reduction of c_1^2 cannot be probed by any M^4 with spin structure, no matter what $U(1)$-symmetry twists we add.

In $4+1$D, the potential extra SPT phases are described by $H^2[BU(1),\mathcal{H}^3(SO,\mathbb{R}/\mathbb{Z})]\oplus H^4[BU(1),\mathcal{H}^1(SO,\mathbb{R}/\mathbb{Z})]=\mathbb{Z}$. But, $H^2[BU(1),\mathcal{H}^3(SO,\mathbb{R}/\mathbb{Z})]$ is $H^2[BU(1),\sigma\text{iTO}_L^3]$ which has been included before. Thus, the potential extra mixed SPT phases are described by $H^4[BU(1),\mathcal{H}^1(SO,\mathbb{R}/\mathbb{Z})]=0$, i.e., there is no extra $U(1)$ SPT phase in $4+1$D.

It has been pointed out that the following gauge-gravity topological invariant may exist:

$$W^5_{\text{top}}(A,\Gamma)=\frac{1}{2}\text{w}_3\frac{dA}{2\pi}. \tag{123}$$

It may suggest that there is an extra $U(1)$ SPT phase in $4+1$D. Here we would like to show that such a term always vanishes. We start with the relation (78):

$$Sq^1\big(\text{w}_2\text{w}_2^{U(1)}\big)=u_1\text{w}_2\text{w}_2^{U(1)}. \tag{124}$$

The left-hand side gives us

$$\begin{aligned} &Sq^1(\text{w}_2)\text{w}_2^{U(1)}+\text{w}_2Sq^1\big(\text{w}_2^{U(1)}\big)\\ &\quad=(\text{w}_1\text{w}_2+\text{w}_3)\text{w}_2^{U(1)}+\text{w}_2Sq^1\big(\text{w}_2^{U(1)}\big). \end{aligned} \tag{125}$$

Since $\text{w}_2^{U(1)}$ is a Stiefel-Whitney class of an O vector bundle over M^5 (which is not the tangent bundle that gives rise to Stiefel-Whitney classes w_i), we can use the Wu formula (74) to calculate $Sq^1(\text{w}_2^{U(1)})=\text{w}_1^{U(1)}\text{w}_2^{U(1)}+\text{w}_3^{U(1)}=0$. Thus, we have

$$\begin{aligned} Sq^1\big(\text{w}_2\text{w}_2^{U(1)}\big)&=(\text{w}_1\text{w}_2+\text{w}_3)\text{w}_2^{U(1)}=u_1\text{w}_2\text{w}_2^{U(1)}\\ &=\text{w}_1\text{w}_2\text{w}_2^{U(1)}, \end{aligned} \tag{126}$$

which gives us $\text{w}_3f_2=\text{w}_3c_1=0$ mod 2 for any $U(1)$ bundle on M^5 which can even be unorientable. This leads to the vanishing of Eq. (123).

In $5+1$D, we may have extra mixed $U(1)$ SPT phases described by $\mathcal{H}^4[U(1),\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]\oplus\mathcal{H}^2[U(1),\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})]=2\mathbb{Z}_2$, generated by $\text{w}_2f_2^2$, w_4f_2. We have the following relations:

$$\begin{aligned} &Sq^1\big(\text{w}_3\text{w}_2^{U(1)}\big)=u_1\text{w}_3\text{w}_2^{U(1)}\\ &\to 0=0,\\ &Sq^2\big(\text{w}_2^{U(1)}\text{w}_2^{U(1)}\big)=u_2\text{w}_2^{U(1)}\text{w}_2^{U(1)}\\ &\to(\text{w}_1^2+\text{w}_2)\text{w}_2^{U(1)}\text{w}_2^{U(1)}=0,\\ &Sq^2\big(\text{w}_2\text{w}_2^{U(1)}\big)=u_2\text{w}_2\text{w}_2^{U(1)}\\ &\to\text{w}_2^3+\text{w}_1^2\text{w}_2\text{w}_2^{U(1)}=0, \end{aligned} \tag{127}$$

and $\text{w}_2^2=\text{w}_4$. Since $\text{w}_1=\text{w}_2^3=0$ for six-dimensional orientable manifold (see Sec. IV C), we only have one relation $\text{w}_2f_2^2=0$. However, $\text{w}_4f_2=\text{w}_2^2f_2=p_1f_2$ mod 2 (see Appendix F). So, the $\mathbb{Z}_2$ class w_4f_2 is part of an integer class p_1c_1, and the topological invariant from an integer class does not have a quantized coefficient [see the discussion below Eq. (120)]. So the term w_4f_2 can smoothly connect to zero, and there is no extra mixed $U(1)$ SPT phases in $5+1$D.

In $6+1$D, we may have extra $U(1)$ SPT phases described by $H^4[BU(1),\mathcal{H}^3(SO,\mathbb{R}/\mathbb{Z})] \oplus H^2[BU(1),\mathcal{H}^5(SO,\mathbb{R}/\mathbb{Z})] = \mathbb{Z}\oplus\mathbb{Z}_2$, but they are discussed before since $H^4[BU(1),\mathcal{H}^3(SO,\mathbb{R}/\mathbb{Z})] \oplus H^2[BU(1),\mathcal{H}^5(SO,\mathbb{R}/\mathbb{Z})] = H^4[BU(1),\sigma\text{iTO}_L^3]\oplus H^2[BU(1),\sigma\text{iTO}_L^5]$. So, there is no extra $U(1)$ SPT phase in $6+1$D.

D. Z_n SPT states

From [13,49,69]

$$H^d(BZ_n,\mathbb{Z}) = 0 \quad \text{if} \quad d=\text{odd},$$
$$H^d(BZ_n,\mathbb{Z}) = \mathbb{Z}_n \quad \text{if} \quad d=\text{even}, \tag{128}$$

we obtain

$$H^d(BZ_n,\mathbb{Z}_2) = \mathbb{Z}_2 \quad \text{if} \quad d=0,$$
$$H^d(BZ_n,\mathbb{Z}_2) = \mathbb{Z}_{\langle n,2\rangle} \quad \text{if} \quad d>0, \tag{129}$$

where $\langle m,n\rangle$ is the greatest common divisor of m,n. This allows us to obtain Z_n mixed SPT states described by $\oplus_{k=1}^{d-1} H^k[BZ_n,\mathcal{H}^{d-k}(O,\mathbb{R}/\mathbb{Z})]$. There are no such Z_n mixed SPT states in $3+1$D. The $4+1$D Z_n mixed SPT states are described by the group cohomology $H^2(BZ_n,\sigma\text{iTO}_L^3) = \mathbb{Z}_n$. We also obtain mixed SPT states in $5+1$D described by $H^1(BZ_n,\sigma\text{iTO}_L^5)=\mathbb{Z}_{\langle n,2\rangle}$, in $6+1$D described by $H^4(BZ_n,\sigma\text{iTO}_L^3)\oplus H^2(BZ_n,\sigma\text{iTO}_L^5) = \mathbb{Z}_n\oplus\mathbb{Z}_{\langle n,2\rangle}$, and in $7+1$D described by $H^3(BZ_n,\sigma\text{iTO}_L^5)=\mathbb{Z}_{\langle n,2\rangle}$.

$H^2(BZ_n,\sigma\text{iTO}_L^3) = \mathbb{Z}_n$ is generated by $W_{\text{top}}^5 = \beta(A_{Z_n}/2\pi)\omega_3$ where $A_{Z_n}/2\pi$ is the generator of $\mathcal{H}^1(Z_n,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ (or $\oint A_{Z_n}/2\pi = \frac{1}{n}$ mod 1). According to Appendix I, $H^2(BZ_n,\sigma\text{iTO}_L^3)=\mathbb{Z}_n$ is generated by

$$W_{\text{top}}^5 = \frac{A_{Z_n}}{2\pi}p_1. \tag{130}$$

The structure of the above results also leads to a physical probe of the corresponding SPT states by dimension reduction [49,59]. We put the system on a 4D space of a form $S^2\times D^2$ and put n identical monodromy defects on S^2. In the small S^2 limit, the effective $2+1$D state on D^2 will be an $(E_8)^3$ bosonic quantum Hall state, with gapless excitations on the boundary of D^2. We may also replace S^2 by $\tilde{D}^2$ and break the Z_n symmetry on the boundary of $\tilde{D}^2$. We then create n identical Z_n domain walls on the boundary of $\tilde{D}^2$. This will have the same effect as n identical monodromy defects on S^2. We get an $(E_8)^3$ bosonic quantum Hall state on D^2 in the small $\tilde{D}^2$ limit. In fact, all the mixed SPT states described by $H^2(BG,\sigma\text{iTO}_L^{d-2})$ and all the $G_1\times G_2$ pure SPT states described by $H^2[BG_1,\mathcal{H}^{d-2}(G_2,\mathbb{R}/\mathbb{Z})]$ can be probed in this way.

In the following, we will consider if there are extra mixed Z_n SPT phases. We find that there is no extra mixed Z_n SPT phase if $n=$ odd. So in the following, we will assume $n=$ even. We first note that the ring $H^*[BZ_n,\mathbb{Z}_2]$ is generated by a_1. If we choose the natural embedding $Z_n\to O$ via $Z_n/Z_{n/2}\to O/SO$, we find that

$$\mathrm{w}_1^{Z_n} = a_1, \quad \mathrm{w}_i^{Z_n} = 0, \quad i>1. \tag{131}$$

In $2+1$D, the potential extra mixed Z_n SPT phases are described by $H^1[BZ_n,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})] = \mathbb{Z}_{\langle n,2\rangle}$. We note that $\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})$ is generated by the w_2 [see Eq. (67)]. Therefore, the potential extra Z_n SPT phases described by $H^1[BZ_n,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]$ are generated by $a_1\mathrm{w}_2$. In $2+1$D, we have the following relations:

$$Sq^1[(\mathrm{w}_1^{Z_n})^2] = u_1(\mathrm{w}_1^{Z_n})^2 \to \mathrm{w}_1(\mathrm{w}_1^{Z_n})^2 = 0;$$
$$u_2 = \mathrm{w}_1^2+\mathrm{w}_2 = 0. \tag{132}$$

We see that $\mathrm{w}_2=0$ for orientable $2+1$D space-time and $a_1\mathrm{w}_2$ vanishes. Thus, there is no extra Z_n SPT phase in $2+1$D.

In $3+1$D, the potential extra mixed Z_n SPT phases are described by $H^2[BZ_n,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})] = \mathbb{Z}_{\langle n,2\rangle}$, which are generated by $a_1^2\mathrm{w}_2$. In $3+1$D, we have the following relations (setting $\mathrm{w}_1=0$):

$$\mathrm{w}_2^2+\mathrm{w}_4 = 0,$$
$$a_1^2\mathrm{w}_2 + a_1\mathrm{w}_3 = a_1^4 = a_1^4 + a_1^2\mathrm{w}_2 = 0. \tag{133}$$

We see that $a_1^2\mathrm{w}_2=0$ and there is no extra Z_n SPT phase in $3+1$D.

In the above, we also see that $a_1^4=0$. What is the physical meaning of this relation? In fact, in $1+1$D, we have $a_1^2=0$. Let us discuss this simpler $1+1$D situation. The relation $a_1^2=0$ comes from

$$Sq^1(a_1) = u_1a_1 \to a_1^2 = \mathrm{w}_1a_1. \tag{134}$$

We see that on nonorientable M^2, a_1^2 do not have to be zero. This means that $\int_{M^2} a_1^2$ can be nonzero if M^2 is nonorientable. But, $\int_{M^2} a_1^2$ must be zero mod 2 if M^2 is orientable. For Z_n SPT state without time-reversal symmetry, we cannot use the nonorientable M^2 to probe it. So a_1^2 cannot produce any measurable topological invariant, and should be quotient out. This is why $H^2(BZ_n,\mathbb{R}/\mathbb{Z})$ is trivial since its potential generator a_1^2 is not measurable on any orientable space-time for any symmetry twist.

In $4+1$D, the potential extra mixed Z_n SPT phases are described by $H^3[BZ_n,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]\oplus H^1[BZ_n,\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})] = 2\mathbb{Z}_{\langle n,2\rangle}$, which are generated by $a_1^3\mathrm{w}_2$, $a_1\mathrm{w}_4$. In $4+1$D, we have the following relations (setting $\mathrm{w}_1=0$):

$$\mathrm{w}_2^2+\mathrm{w}_4 = \mathrm{w}_5 = 0,$$
$$a_1^2\mathrm{w}_3 = a_1^5 + a_1^3\mathrm{w}_2 = 0. \tag{135}$$

We see that $a_1^3\mathrm{w}_2 = a_1^5$ which is already included by $\mathcal{H}^5(Z_n,\mathbb{R}/\mathbb{Z})$. But, nothing restricts $a_1\mathrm{w}_4$ (except $\mathrm{w}_4=\mathrm{w}_2^2$). So there is an Z_n SPT phase in $4+1$D generated by $a_1\mathrm{w}_2^2$ for $n=$ even. Its topological invariant is given by

$$W_{\text{top}}^5(A,\Gamma) = \frac{n}{2}\frac{A_{Z_n}}{2\pi}p_1, \tag{136}$$

where A_{Z_n} is the flat connection that describes the Z_n twist [59]

$$\oint A_{Z_n} = 0 \text{ mod } 2\pi/n. \tag{137}$$

But, the above topological invariant has been included by Z_n SPT phases described by $H^2(BZ_n,\sigma\text{iTO}_L^3)$ [see Eq. (130)]. So there is no extra Z_n SPT phase in $4+1$D.

In $5+1$D, the potential extra mixed Z_n SPT phases are described by $H^4[BZ_n,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]\oplus$

$H^2[BZ_n,\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})] = 2\mathbb{Z}_{\langle n,2\rangle}$, which are generated by $a_1^4 w_2$, $a_1^2 w_4$. In $5+1$D, we have the following relations (setting $w_1 = 0$):

$$\begin{aligned} &w_2^2 + w_4 = w_5 = w_3^2 + w_2 w_4 = w_3^2 = w_6 = 0,\\ &a_1^2 w_4 = a_1^4 w_2 = a_1^3 w_3 = a_1^6 = 0. \end{aligned} \tag{138}$$

We see that $a_1^2 w_4 = a_1^4 w_2 = 0$. So there is no extra Z_n SPT phase in $5+1$D.

In $6+1$D, the potential extra mixed Z_n SPT phases are described by $H^5[BZ_n,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})] \oplus H^3[BZ_n,\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})] \oplus H^1[BZ_n,\mathcal{H}^6(SO,\mathbb{R}/\mathbb{Z})] = 4\mathbb{Z}_{\langle n,2\rangle}$, which are generated by $a_1^5 w_2$, $a_1^3 w_4, a_1 w_6, a_1 w_3^2$. In $6+1$D, we have the following relations (setting $w_1 = 0$):

$$\begin{aligned} &w_2^2 + w_4 = w_5 = w_3^2 + w_2 w_4 = w_6 = 0,\\ &a_1^2 w_2 w_3 + a_1 w_3^2 = a_1^5 w_2 = a_1^4 w_3 = 0. \end{aligned} \tag{139}$$

We see that $a_1^5 w_2 = a_1 w_6 = 0$, but nothing restricts $a_1^2 w_2 w_3 = a_1 w_3^2$ and $a_1^3 w_4$. However, $a_1^2 w_2 w_3$ is already included by $H^2(BZ_n,\sigma\mathrm{iTO}_L^5)$. So there is an Z_n SPT phase in $6+1$D generated by $a_1^3 w_2^2$ for $n =$ even. Its topological invariant is given by

$$W_{\text{top}}^7(A,\Gamma) = \frac{1}{2\pi^3} A_{Z_n}^3 p_1. \tag{140}$$

The above topological invariant has been included by Z_n SPT phases described by $H^4(BZ_n,\sigma\mathrm{iTO}_L^3) \simeq \mathcal{H}^3(Z_n,\mathbb{R}/\mathbb{Z})$ (see Appendix I). So there is no extra Z_n SPT phase in $6+1$D.

E. $U(1) \rtimes Z_2 = O(2)$ SPT states

In Ref. [13], it was shown that

$$H^d[BO_2,\mathbb{Z}] \subset \begin{cases} \mathbb{Z} \oplus \frac{d}{4}\mathbb{Z}_2, & d = 0 \bmod 4\\ \frac{d-1}{4}\mathbb{Z}_2, & d = 1 \bmod 4\\ \frac{d+2}{4}\mathbb{Z}_2, & d = 2 \bmod 4\\ \frac{d+1}{4}\mathbb{Z}_2, & d = 3 \bmod 4. \end{cases} \tag{141}$$

In Refs. [95,96], it was shown that

$$H[BO_2,\mathbb{Z}] = \mathbb{Z}[x_2,x_3,x_4]/\big(2x_2,2x_3,x_3^2 - x_2x_4\big), \tag{142}$$

where $x_2 = \beta w_1^{O_2}$, $x_3 = \beta w_2^{O_2}$, and $x_4 = p_1^{O_2}$ is the Pontryagin class. Here β is the natural map $H^d(BG,\mathbb{Z}_2) \to H^{d+1}(G,\mathbb{Z})$. In other words, we have a relation $(\beta w_2^{O_2})^2 = \beta w_1^{O_2} p_1^{O_2}$. We find that

$$\begin{aligned}
&H^0(BO_2,\mathbb{Z}) = \mathbb{Z},\\
&H^1(BO_2,\mathbb{Z}) = 0,\\
&H^2(BO_2,\mathbb{Z}) = \mathbb{Z}_2, \text{ basis } \big[\beta w_1^{O_2}\big],\\
&H^3(BO_2,\mathbb{Z}) = \mathbb{Z}_2, \text{ basis } \big[\beta w_2^{O_2}\big],\\
&H^4(BO_2,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2, \text{ basis } \big[(\beta w_1^{O_2})^2, p_1^{O_2}\big],\\
&H^5(BO_2,\mathbb{Z}) = \mathbb{Z}_2, \text{ basis } \big[\beta w_1^{O_2} \beta w_2^{O_2}\big],\\
&H^6(BO_2,\mathbb{Z}) = 2\mathbb{Z}_2, \text{ basis } \big[\beta w_1^{O_2} p_1^{O_2}, (\beta w_1^{O_2})^3\big],\\
&H^7(BO_2,\mathbb{Z}) = 2\mathbb{Z}_2, \text{ basis } \big[\beta w_2^{O_2} p_1^{O_2}, (\beta w_1^{O_2})^2 \beta w_2^{O_2}\big],\\
&H^8(BO_2,\mathbb{Z}) = \mathbb{Z} \oplus 2\mathbb{Z}_2, \text{ basis } \big[(p_1^{O_2})^2, (\beta w_1^{O_2})^2 p_1^{O_2}, (\beta w_1^{O_2})^4\big],
\end{aligned} \tag{143}$$

which agrees with Eq. (141) with $\subset$ replaced by $=$. So, we actually have

$$H^d[BO_2,\mathbb{Z}] = \begin{cases} \mathbb{Z} \oplus \frac{d}{4}\mathbb{Z}_2, & d = 0 \bmod 4\\ \frac{d-1}{4}\mathbb{Z}_2, & d = 1 \bmod 4\\ \frac{d+2}{4}\mathbb{Z}_2, & d = 2 \bmod 4\\ \frac{d+1}{4}\mathbb{Z}_2, & d = 3 \bmod 4 \end{cases} \tag{144}$$

which allow us to get [49,69]

$$H^d[BO_2,\mathbb{Z}_2] = \begin{cases} \frac{d+2}{2}\mathbb{Z}_2, & d = 0 \bmod 2\\ \frac{d+1}{2}\mathbb{Z}_2, & d = 1 \bmod 2. \end{cases} \tag{145}$$

We also have

$$\begin{aligned}
&\mathcal{H}^0(O_2,\mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z},\\
&\mathcal{H}^1(O_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2, \text{ basis } \left[\frac{1}{2} w_1^{O_2}\right],\\
&\mathcal{H}^2(O_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2, \text{ basis } \left[\frac{1}{2} w_2^{O_2}\right],\\
&\mathcal{H}^3(O_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2, \text{ basis } \left[\frac{1}{2}(w_1^{O_2})^3, \frac{1}{2\pi} A\, dA\right],\\
&\mathcal{H}^4(O_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2, \text{ basis } \left[\frac{1}{2}(w_1^{O_2})^2 w_2^{O_2}\right],\\
&\mathcal{H}^5(O_2,\mathbb{R}/\mathbb{Z}) = 2\mathbb{Z}_2, \text{ basis } \left[\frac{1}{2} w_1^{O_2}(w_2^{O_2})^2, \frac{1}{2}(w_1^{O_2})^5\right],\\
&\mathcal{H}^6(O_2,\mathbb{R}/\mathbb{Z}) = 2\mathbb{Z}_2, \text{ basis } \left[\frac{1}{2}(w_2^{O_2})^3, \frac{1}{2}(w_1^{O_2})^4 w_2^{O_2}\right],\\
&\mathcal{H}^7(O_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z} \oplus 2\mathbb{Z}_2,\\
&\qquad \text{basis } \left[\frac{A(dA)^3}{(2\pi)^3}, \frac{(w_1^{O_2})^3 (w_2^{O_2})^2}{2}, \frac{(w_1^{O_2})^7}{2}\right].
\end{aligned} \tag{146}$$

The above basis gives rise to the basis in Eq. (143) after the natural map $\tilde\beta: \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \to H^{d+1}(BG,\mathbb{Z})$, which becomes the Steenrod square Sq^1 when acting on $w_i^{O_2}$'s. One can use the properties in Eq. (E1) to do the calculation (see Appendix E).

This allows us to obtain O_2 mixed SPT states which are given by the following:

in $3+1$D: $H^1(BO_2,\sigma\mathrm{iTO}_L^3) = 0$,

in $4+1$D: $H^2(BO_2,\sigma\mathrm{iTO}_L^3) = \mathbb{Z}_2$,

in $5+1$D: $H^3(BO_2,\sigma\mathrm{iTO}_L^3) \oplus H^1(BO_2,\sigma\mathrm{iTO}_L^5) = 2\mathbb{Z}_2$,

in $6+1$D: $H^4(BO_2,\sigma\mathrm{iTO}_L^3) \oplus H^2(BO_2,\sigma\mathrm{iTO}_L^5) = \mathbb{Z} \oplus 3\mathbb{Z}_2$,

in $7+1$D: $H^5(BO_2,\sigma\mathrm{iTO}_L^3) \oplus H^3(BO_2,\sigma\mathrm{iTO}_L^5) \oplus H^1(BO_2,\sigma\mathrm{iTO}_L^7) = 3\mathbb{Z}_2$.

In the following, we will consider if there are extra mixed O_2 SPT phases. We first note that the ring $H^*[BO_2,\mathbb{Z}_2]$ is generated by a_1, f_2. If we choose the natural embedding $O_2 \to O$ via $O(2) \to O$, we find that

$$w_1^{O_2} = a_1, \quad w_2^{O_2} = f_2, \quad w_i^{O_2} = 0,\ i > 2. \tag{147}$$

In $2+1$D, the potential extra mixed O_2 SPT phases are described by $H^1[BO_2,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})] = \mathbb{Z}_2$. Therefore, the potential extra O_2 SPT phases are generated by $a_1 w_2$. In

$2+1$D, we have the following relations:

$$Sq^1\left[\left(w_1^{O_2}\right)^2\right] = u_1\left(w_2^{O_2}\right)^2 \to w_1\left(w_1^{O_2}\right)^2 = 0;$$
$$u_2 = w_1^2 + w_2 = 0. \quad (148)$$

We see that $w_1 = w_2 = 0$ for orientable $2+1$D space-time and $a_1 w_2$ vanishes. Thus, there is no extra O_2 SPT phase in $2+1$D.

In $3+1$D, the potential extra mixed O_2 SPT phases are described by $H^2[BO_2,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})] = 2\mathbb{Z}_2$ generated by $f_2 w_2$, $a_1^2 w_2$. In $3+1$D, we have the following relations (setting $w_1 = 0$):

$$\begin{aligned} &w_2^2 + w_4 = 0,\\ &a_1^4 = w_3 a_1 = w_2 a_1^2 = f_2^2 + w_2 f_2 = 0. \end{aligned} \quad (149)$$

We see that $w_2 a_1^2 = 0$ and $w_2 f_2 = f_2^2 = c_1^2$ mod 2. So $w_2 f_2$, as part of c_1^2, can be smoothly deformed to zero. Thus, there is no extra O_2 SPT phase in $3+1$D.

In $4+1$D, the potential extra mixed O_2 SPT phases are described by $H^3[BO_2,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})] \oplus H^1[BO_2,\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})] = 3\mathbb{Z}_2$ generated by $w_2 a_1 f_2$, $w_2 a_1^3$, $w_4 a_1$. In $4+1$D, we have the following relations (setting $w_1 = 0$):

$$\begin{aligned} &w_2^2 + w_4 = w_5 = 0,\\ &a_1^3 f_2 = w_3 a_1^2 = w_3 f_2 + w_2 a_1 f_2\\ &\quad = a_1 f_2^2 + a_1 f_2 w_2 = 0. \end{aligned} \quad (150)$$

We see that $w_2 a_1^3 = 0$, $w_2 a_1 f_2 = w_3 f_2 = w_3 f_2$, and $a_1 f_2^2 = a_1 f_2 w_2$. But, $a_1 f_2^2 = w_1^{O_2}(w_2^{O_2})^2$ is already included in $H^5(BO_2,\mathbb{R}/\mathbb{Z})$ [see Eq. (146)]. So, $a_1 w_4 = a_1 w_2^2 = a_1 p_1$ is not restricted to zero. There is an O_2 SPT phase in $4+1$D with topological invariant

$$W_{\text{top}}^5(A,\Gamma) = \frac{A_{Z_2}}{2\pi} p_1, \quad (151)$$

where A_{Z_2} is the flat connection that describes the Z_2 twist [59]

$$\oint A_{Z_2} = 0 \bmod \pi. \quad (152)$$

The above topological invariant has been included by O_2 SPT phases described by $H^2(BO_2,\sigma\text{iTO}_L^3) \simeq \mathcal{H}^1(O_2,\mathbb{R}/\mathbb{Z})$ (see Appendix I). So, there is no extra O_2 SPT phase in $4+1$D.

In $5+1$D, the potential extra mixed O_2 SPT phases are described by $H^4[BO_2,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})] \oplus H^2[BO_2,\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})] = 5\mathbb{Z}_2$ generated by $w_2 a_1^2 f_2, w_2 f_2^2, w_2 a_1^4$, $w_4 a_1^2$, $w_4 f_2$. In $5+1$D, we have the following relations (setting $w_1 = 0$):

$$\begin{aligned} &w_2^2 + w_4 = w_3^2 + w_2 w_4 = w_3^2 = w_5 = w_6 = 0,\\ &w_4 a_1^2 = w_2 a_1^4 = w_2 f_2^2 = w_2 a_1^2 f_2 + a_1^4 f_2 = 0. \end{aligned} \quad (153)$$

We see that $w_2 a_1^4 = w_4 a_1^2 = w_2 f_2^2 = 0$ and $w_2 a_1^2 f_2 = a_1^4 f_2$. But, $w_2 a_1^2 f_2 = a_1^4 f_2 = (w_1^{O_2})^4 w_2^{O_2}$ is already included in $H^6(BO_2,\mathbb{R}/\mathbb{Z})$ [see Eq. (146)]. Also $w_4 f_2 = w_2^2 f_2 = p_1 c_1$ mod 2 is connected to zero. There are no extra O_2 SPT phases in $5+1$D.

In $6+1$D, the potential extra mixed O_2 SPT phases are described by $H^5[BO_2,\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})] \oplus H^3[BO_2,\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})] \oplus H^1[BO_2,\mathcal{H}^6(SO,\mathbb{R}/\mathbb{Z})] = 7\mathbb{Z}_2$ generated by $w_2 a_1^5$, $w_2 a_1^3 f_2$, $w_2 a_1 f_2^2$, $w_4 a_1^3$, $w_4 a_1 f_2$, $w_6 a_1$, $w_3^2 a_1$. In $6+1$D, we have the following relations (setting $w_1 = 0$):

$$w_2^2 + w_4 = w_3^2 + w_2 w_4 = w_5 = w_6 = 0, \quad (154)$$
$$\begin{aligned} &w_2 w_3 a_1^2 + w_3^2 a_1 = w_4 a_1 f_2 = w_2 a_1^5 = w_2 a_1^3 f_2 + w_2 a_1 f_2^2\\ &= a_1^3 f_2^2 + w_2 a_1 f_2^2 = a_1^3 f_2^2 + w_3 a_1^2 f_2 = a_1^5 f_2 = a_1 f_2^3 = 0. \end{aligned}$$

We see that $w_2 a_1^5 = w_4 a_1 f_2 = 0$ and $w_2 w_3 a_1^2 = w_3^2 a_1$, $w_2 a_1 f_2^2 = w_2 a_1^3 f_2 = w_3 a_1^2 f_2 = a_1^3 f_2^2$. But, $w_3^2 a_1 = w_2 w_3 a_1^2$ is already included in $H^2(BO_2,\sigma\text{iTO}_L^5)$, and $w_2 a_1 f_2^2 = w_2 a_1^3 f_2 = w_3 a_1^2 f_2 = a_1^3 f_2^2$ are already included in $H^7(BO_2,\mathbb{R}/\mathbb{Z})$ [see Eq. (146)]. However, $a_1^3 w_4 = a_1^3 w_2^2 = a_1^3 p_1$ mod 2 is not restricted. There is an O_2 SPT phase in $6+1$D described by a topological invariant

$$W_{\text{top}}^7(A,\Gamma) = \frac{1}{2\pi^3} A_{Z_2}^3 p_1. \quad (155)$$

The above topological invariant has been included by O_2 SPT phases described by $H^4(BO_2,\sigma\text{iTO}_L^3) \simeq \mathcal{H}^3(O_2,\mathbb{R}/\mathbb{Z})$ (see Appendix I). So, there is no extra O_2 SPT phase in $6+1$D.

F. Z_2^T SPT states

Note that [13,49,69]

$$\begin{aligned} &H^d\left(BZ_2^T,\mathbb{Z}\right) = 0 \quad \text{if} \quad d = \text{even},\\ &H^d\left(BZ_2^T,\mathbb{Z}\right) = \mathbb{Z}_2 \quad \text{if} \quad d = \text{odd}, \end{aligned} \quad (156)$$

$$\begin{aligned} &H^d\left(BZ_2^T,\mathbb{Z}_2\right) = \mathbb{Z}_2 \quad \text{if} \quad d = 0,\\ &H^d(BZ_2^T,\mathbb{Z}_2) = \mathbb{Z}_2 \quad \text{if} \quad d > 0, \end{aligned} \quad (157)$$

where the time-reversal has a nontrivial action $\mathbb{Z} \to -\mathbb{Z}$. This allows us to obtain, in $3+1$D, Z_2^T pure SPT states described by the group cohomology $\mathcal{H}^4(Z_2^T,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$, and Z_2^T mixed SPT states described by the group cohomology $H^1(BZ_2^T,\sigma\text{iTO}_L^3) = \mathbb{Z}_2$. We also obtain mixed SPT states in $5+1$D described by $H^3(BZ_2^T,\sigma\text{iTO}_L^3) \oplus H^1(BZ_2^T,\sigma\text{iTO}_L^5) = 2\mathbb{Z}_2$, in $6+1$D described by $H^2(BZ_2^T,\sigma\text{iTO}_L^5) = \mathbb{Z}_2$, and in $7+1$D described by $H^5(BZ_2^T,\sigma\text{iTO}_L^3) \oplus H^2(BZ_2^T,\sigma\text{iTO}_L^5) \oplus H^1(BZ_2^T,\sigma\text{iTO}_L^7) = 4\mathbb{Z}_2$.

The $3+1$D Z_2^T mixed SPT state described by $H^1(BZ_2^T,\sigma\text{iTO}_L^3)$ may be produced in the following way [29,59,77]: We start with a system with Z_2^T symmetry whose ground state breaks the Z_2^T symmetry. Then, we allow the fluctuations of the domain walls of the Z_2^T order parameter to restore a Z_2^T symmetry. We may bound a $2+1$D $(E_8)^3$ bosonic quantum Hall state to such domain wall. In this case, the restored Z_2^T symmetric state is the mixed SPT state described by $H^1(BZ_2^T,\sigma\text{iTO}_L^3)$. In fact, all the mixed SPT states described by $H^1(BG,\sigma\text{iTO}_L^{d-2})$ and all the $G_1 \times G_2$ pure SPT states described by $H^1[BG_1,\mathcal{H}^{d-2}(G_2,\mathbb{R}/\mathbb{Z})]$ can be constructed in this way [29].

Such a Z_2^T mixed SPT state can be probed by surface symmetry breaking [16]. The Z_2^T symmetry breaking domain wall on the boundary will carry the gapless edge state of $2+1$D $(E_8)^3$ bosonic quantum Hall state. In fact, all the mixed SPT states described by $H^1(BG,\sigma\text{iTO}_L^{d-2})$ and all the $G_1\times G_2$ pure SPT states described by $H^1[BG_1,\mathcal{H}^{d-2}(G_2,\mathbb{R}/\mathbb{Z})]$ can be probed in this way.

In the following, we will consider if there are extra mixed Z_2^T SPT phases. Let a_1 be the generator of the ring $H^*(BZ_2^T,\mathbb{Z}_2)$. Let us choose the natural embedding $Z_2^T\to O$ via $Z_2^T\to O/SO$, and we find that

$$\mathrm{w}_1^{Z_2^T}=a_1,\quad \mathrm{w}_i^{Z_2^T}=0,\quad i>1. \tag{158}$$

However, since the Z_2^T twist can only be implemented by reversing the space-time orientation, we need to identify

$$\mathrm{w}_1\to \mathrm{w}_1^{Z_2^T}. \tag{159}$$

In $7+1$D and below, we did not find any extra mixed Z_2^T SPT phases.

G. $U(1)\times Z_2^T$ SPT states

In [13], we obtained

$$H^d[B(U(1)\times Z_2^T),\mathbb{Z}]=\begin{cases}0, & d=0 \bmod 2\\ \frac{d+1}{2}\mathbb{Z}_2, & d=1 \bmod 2\end{cases} \tag{160}$$

which allows us to get [49,69]

$$H^d[B(U(1)\times Z_2^T),\mathbb{Z}_2]=\begin{cases}\frac{d+2}{2}\mathbb{Z}_2, & d=0 \bmod 2\\ \frac{d+1}{2}\mathbb{Z}_2, & d=1 \bmod 2.\end{cases} \tag{161}$$

This allows us to obtain $U(1)\times Z_2^T$ mixed SPT states which are given by the following:

in $3+1$D: $H^1[B[U(1)\times Z_2^T],\sigma\text{iTO}_L^3]=\mathbb{Z}_2$,

in $4+1$D: $H^2[B[U(1)\times Z_2^T],\sigma\text{iTO}_L^3]=0$,

in $5+1$D: $H^3[B[U(1)\times Z_2^T],\sigma\text{iTO}_L^3]\oplus H^1[B[U(1)\times Z_2^T],\sigma\text{iTO}_L^5]=3\mathbb{Z}_2$,

in $6+1$D: $H^4[B[U(1)\times Z_2^T],\sigma\text{iTO}_L^3]\oplus H^2[B[U(1)\times Z_2^T],\sigma\text{iTO}_L^5]=2\mathbb{Z}_2$,

in $7+1$D: $H^5[B[U(1)\times Z_2^T],\sigma\text{iTO}_L^3]\oplus H^2[B[U(1)\times Z_2^T],\sigma\text{iTO}_L^5]\oplus H^1[B[U(1)\times Z_2^T],\sigma\text{iTO}_L^7]=6\mathbb{Z}_2$.

In the following, we will consider if there are extra mixed $U(1)\times Z_2^T$ SPT phases. We first note that the ring $H^*[B[U(1)\times Z_2^T],\mathbb{Z}_2]$ is generated by a_1,f_2 {the same as $H^*[B[U(1)\times Z_2],\mathbb{Z}_2]$ and $H^*[B[U(1)\rtimes Z_2],\mathbb{Z}_2]$}. Note that $H^d[B[U(1)\times Z_2^T],\mathbb{Z}]$ for $d=$ odd is generated by $a_1c_1^{\frac{d-1}{2}}$, $a_1^3c_1^{\frac{d-3}{2}}$, etc., or $H^d[B[U(1)\times Z_2^T],\mathbb{R}/\mathbb{Z}]$ for $d=$ even is generated by $\frac{1}{2}c_1^{\frac{d}{2}}$, $\frac{1}{2}a_1^2c_1^{\frac{d-2}{2}}$, etc.

If we choose the natural embedding $U(1)\times Z_2^T\to O$ via $U(1)\times Z_2\to SO(3)$ which map $U(1)$ to the rotation in the $x-y$ plane and Z_2 to the $z\to -z$ reflection, we find that

$$\mathrm{w}_1^{O_2}=a_1,\quad \mathrm{w}_2^{O_2}=f_2,\quad \mathrm{w}_i^{O_2}=0,\ i>2. \tag{162}$$

Also, since the time-reversal twist is implemented by the orientation reversal, we need to set $\mathrm{w}_1=a_1$.

In $2+1$D, the potential extra mixed $U(1)\times Z_2^T$ SPT phases are described by $H^1[B[U(1)\times Z_2^T],\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]=\mathbb{Z}_2$ generated by $a_1\mathrm{w}_2$. In $2+1$D, we have the following relations (setting $\mathrm{w}_1=a_1$):

$$\mathrm{w}_1^2+\mathrm{w}_2=\mathrm{w}_1\mathrm{w}_2=0. \tag{163}$$

We see that $\mathrm{w}_2\mathrm{w}_1=0$, and there is no extra $U(1)\times Z_2^T$ SPT phase in $2+1$D.

In $3+1$D, the potential extra mixed $U(1)\times Z_2^T$ SPT phases are described by $H^2[B[U(1)\times Z_2^T],\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]=2\mathbb{Z}_2$ generated by $a_1^2\mathrm{w}_2$, $f_2\mathrm{w}_2$. In $3+1$D, we have the following relations (setting $\mathrm{w}_1=a_1$):

$$\begin{aligned}&\mathrm{w}_1\mathrm{w}_2=\mathrm{w}_1\mathrm{w}_3=\mathrm{w}_1^4+\mathrm{w}_2^2+\mathrm{w}_1\mathrm{w}_3+\mathrm{w}_4=0,\\ &\mathrm{w}_1^2f_2=\mathrm{w}_1^2f_2+f_2^2+f_2\mathrm{w}_2=0.\end{aligned} \tag{164}$$

We see that $\mathrm{w}_1^2\mathrm{w}_2=0$ and $f_2\mathrm{w}_2=f_2^2$. But, $f_2\mathrm{w}_2=f_2^2$ can be deformed to zero smoothly. Thus, there is no extra $U(1)\times Z_2^T$ SPT phase in $3+1$D.

In $4+1$D, the potential extra mixed $U(1)\times Z_2^T$ SPT phases are described by $H^3[B[U(1)\times Z_2^T],\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]\oplus H^1[B[U(1)\times Z_2^T],\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})]=3\mathbb{Z}_2$ generated by $\mathrm{w}_2\mathrm{w}_1f_2,\mathrm{w}_2\mathrm{w}_1^3$, $\mathrm{w}_4\mathrm{w}_1$. In $4+1$D, we have the following complete set of relations (setting $\mathrm{w}_1=a_1$):

$$\begin{aligned}&\mathrm{w}_1^5=\mathrm{w}_1f_2^2=\mathrm{w}_1^3\mathrm{w}_2=\mathrm{w}_1f_2\mathrm{w}_2=\mathrm{w}_1\mathrm{w}_2^2=\mathrm{w}_1^2\mathrm{w}_3\\ &=f_2\mathrm{w}_3=\mathrm{w}_1\mathrm{w}_4=\mathrm{w}_5=0.\end{aligned} \tag{165}$$

There is no extra $U(1)\times Z_2^T$ SPT phase in $4+1$D.

In $5+1$D, the potential extra mixed $U(1)\times Z_2^T$ SPT phases are described by $H^4[B[U(1)\times Z_2^T],\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]\oplus H^2[B[U(1)\times Z_2^T],\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})]=5\mathbb{Z}_2$ generated by $\mathrm{w}_2a_1^2f_2,\mathrm{w}_2f_2^2$, $\mathrm{w}_2a_1^4$, $\mathrm{w}_4a_1^2$, w_4f_2. In $5+1$D, we have the following complete set of relations (setting $\mathrm{w}_1=a_1$):

$$\begin{aligned}&\mathrm{w}_1^4f_2=\mathrm{w}_1^6+\mathrm{w}_1^4\mathrm{w}_2=\mathrm{w}_1^2f_2^2+\mathrm{w}_1^2f_2\mathrm{w}_2=f_2^2\mathrm{w}_2\\ &=\mathrm{w}_1f_2\mathrm{w}_3=\mathrm{w}_1\mathrm{w}_3\mathrm{w}_2+\mathrm{w}_2^3=\mathrm{w}_2^3+\mathrm{w}_3^2=\mathrm{w}_1^6+\mathrm{w}_2\mathrm{w}_4\\ &=\mathrm{w}_1^2\mathrm{w}_2^2+\mathrm{w}_1^2\mathrm{w}_4=f_2\mathrm{w}_2^2+f_2\mathrm{w}_4=\mathrm{w}_1^2\mathrm{w}_2^2+\mathrm{w}_1\mathrm{w}_5\\ &=\mathrm{w}_1^2\mathrm{w}_2^2+\mathrm{w}_6=0.\end{aligned} \tag{166}$$

We see that $f_2^2\mathrm{w}_2=0$. Also, $\mathrm{w}_1^4\mathrm{w}_2=\mathrm{w}_1^6$ and $\mathrm{w}_1^2f_2\mathrm{w}_2=\mathrm{w}_1^2f_2^2$ are already included in $\mathcal{H}^6[U(1)\times Z_2^T,\mathbb{R}/\mathbb{Z}]$. $f_2\mathrm{w}_4=f_2\mathrm{w}_2^2=f_2p_1 \bmod 2$ is a part of integer class c_1p_1 and can be smoothly deformed to zero. However, $\mathrm{w}_1^2\mathrm{w}_4=\mathrm{w}_1^2\mathrm{w}_2^2=\mathrm{w}_1^2p_1 \bmod 2$ is not restricted. There is an $U(1)\times Z_2^T$ SPT phase in $5+1$D described by

$$W_{\text{top}}^6(A,\Gamma)=\tfrac{1}{2}\mathrm{w}_1^2p_1. \tag{167}$$

The above topological invariant has been included by $U(1)\times Z_2^T$ SPT phases described by $H^3[B[U(1)\times Z_2^T],\sigma\text{iTO}_L^3]\simeq\mathcal{H}^2[U(1)\times Z_2^T,\mathbb{R}/\mathbb{Z}]$ (see Appendix I). So, there is no extra $U(1)\times Z_2^T$ SPT phase in $5+1$D.

In $6+1$D, the potential extra mixed $U(1)\times Z_2^T$ SPT phases are described by $H^5[B[U(1)\times Z_2^T],\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})]\oplus H^3[B[U(1)\times Z_2^T],\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z})]\oplus H^1[B[U(1)\times Z_2^T],\mathcal{H}^6(SO,\mathbb{R}/\mathbb{Z})]=7\mathbb{Z}_2$ generated by $\mathrm{w}_2a_1^5$, $\mathrm{w}_2a_1^3f_2$, $\mathrm{w}_2a_1f_2^2$, $\mathrm{w}_4a_1^3$, $\mathrm{w}_4a_1f_2$, w_6a_1, $\mathrm{w}_3^2a_1$. In $6+1$D, we have the following complete set of relations (setting $\mathrm{w}_1=a_1$):

$$\begin{aligned}&\mathrm{w}_1^7=\mathrm{w}_1^3f_2^2=\mathrm{w}_1^5\mathrm{w}_2=\mathrm{w}_1^5f_2+\mathrm{w}_1^3f_2\mathrm{w}_2=\mathrm{w}_1f_2^2\mathrm{w}_2\\ &=\mathrm{w}_1^3\mathrm{w}_2^2=\mathrm{w}_1\mathrm{w}_2^3=\mathrm{w}_1^4\mathrm{w}_3=\mathrm{w}_1^5f_2+\mathrm{w}_1^2f_2\mathrm{w}_3=f_2^2\mathrm{w}_3\end{aligned}$$

$$= \mathrm{w}_2^2\mathrm{w}_3 = \mathrm{w}_1\mathrm{w}_3^2 = \mathrm{w}_1^3\mathrm{w}_4 = \mathrm{w}_1 f_2\mathrm{w}_2^2 + \mathrm{w}_1 f_2\mathrm{w}_4$$
$$= \mathrm{w}_1^2\mathrm{w}_2\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_2\mathrm{w}_4 = \mathrm{w}_3\mathrm{w}_4 = \mathrm{w}_1 f_2\mathrm{w}_2^2 + f_2\mathrm{w}_5$$
$$= \mathrm{w}_1^2\mathrm{w}_5 = \mathrm{w}_1^2\mathrm{w}_2\mathrm{w}_3 + \mathrm{w}_2\mathrm{w}_5 = \mathrm{w}_1\mathrm{w}_6 = \mathrm{w}_7 = 0. \quad (168)$$

We see that $\mathrm{w}_2\mathrm{w}_1^5 = \mathrm{w}_1 f_2^2\mathrm{w}_2 = \mathrm{w}_4\mathrm{w}_1^3 = \mathrm{w}_1\mathrm{w}_6 = \mathrm{w}_1\mathrm{w}_3^2 = 0$ and $\mathrm{w}_1^3 f_2\mathrm{w}_2 = \mathrm{w}_1^5 f_2$, $\mathrm{w}_1 f_2\mathrm{w}_4 = \mathrm{w}_1 f_2\mathrm{w}_2^2 = f_2\mathrm{w}_5$. But, $\mathrm{w}_1^3 f_2\mathrm{w}_2 = \mathrm{w}_1^5 f_2$ is already included in $\mathcal{H}^7[U(1)\times Z_2^T,\mathbb{R}/\mathbb{Z}]$. Only $\mathrm{w}_1 f_2\mathrm{w}_2^2$ is not restricted. Thus, there is an extra $U(1)\times Z_2^T$ SPT phase in 6+1D described by a topological invariant

$$W_{\mathrm{top}}^7(A,\Gamma) = \frac{1}{2}\mathrm{w}_1 p_1 \frac{dA}{2\pi}. \quad (169)$$

H. $U(1) \rtimes Z_2^T$ SPT states

In [13], we also obtained

$$H^d[B[U(1)\rtimes Z_2^T],\mathbb{Z}] \subset \begin{cases} \frac{d}{4}\mathbb{Z}_2, & d = 0 \bmod 4\\ \frac{d+3}{4}\mathbb{Z}_2, & d = 1 \bmod 4\\ \mathbb{Z}\oplus\frac{d-2}{4}\mathbb{Z}_2, & d = 2 \bmod 4\\ \frac{d+1}{4}\mathbb{Z}_2, & d = 3 \bmod 4. \end{cases} \quad (170)$$

If we assume

$$H^d[B[U(1)\rtimes Z_2^T],\mathbb{Z}] = \begin{cases} \frac{d}{4}\mathbb{Z}_2, & d = 0 \bmod 4\\ \frac{d+3}{4}\mathbb{Z}_2, & d = 1 \bmod 4\\ \mathbb{Z}\oplus\frac{d-2}{4}\mathbb{Z}_2, & d = 2 \bmod 4\\ \frac{d+1}{4}\mathbb{Z}_2, & d = 3 \bmod 4 \end{cases} \quad (171)$$

then, we will obtain [49,69]

$$H^d[B[U(1)\rtimes Z_2^T],\mathbb{Z}_2] = \begin{cases} \frac{d+2}{2}\mathbb{Z}_2, & d = 0 \bmod 2\\ \frac{d+1}{2}\mathbb{Z}_2, & d = 1 \bmod 2 \end{cases} \quad (172)$$

which should agree with Eq. (145). Indeed, it agrees, implying that Eq. (171) is correct.

Equations (171) and (172) allow us to obtain $U(1)\rtimes Z_2^T$ mixed SPT states which are given by the following:

in 3+1D: $H^1[B[U(1)\rtimes Z_2^T],\sigma\mathrm{iTO}_L^3] = \mathbb{Z}_2$,

in 4+1D: $H^2[B[U(1)\rtimes Z_2^T],\sigma\mathrm{iTO}_L^3] = \mathbb{Z}$,

in 5+1D: $H^3[B[U(1)\rtimes Z_2^T],\sigma\mathrm{iTO}_L^3]\oplus H^1[B[U(1)\rtimes Z_2^T],\sigma\mathrm{iTO}_L^5] = 2\mathbb{Z}_2$,

in 6+1D: $H^4[B[U(1)\rtimes Z_2^T],\sigma\mathrm{iTO}_L^3]\oplus H^2[B[U(1)\rtimes Z_2^T],\sigma\mathrm{iTO}_L^5] = 3\mathbb{Z}_2$,

in 7+1D: $H^5[B[U(1)\rtimes Z_2^T],\sigma\mathrm{iTO}_L^3]\oplus H^2[B[U(1)\rtimes Z_2^T],\sigma\mathrm{iTO}_L^5]\oplus H^1[B[U(1)\rtimes Z_2^T],\sigma\mathrm{iTO}_L^7] = 6\mathbb{Z}_2$.

In the following, we will consider if there are extra mixed $U(1)\rtimes Z_2^T$ SPT phases. We first note that the ring $H^*[B[U(1)\rtimes Z_2^T],\mathbb{Z}_2]$ is generated by a_1, f_2 {the same as $H^*[B[U(1)\times Z_2],\mathbb{Z}_2]$ and $H^*[B[U(1)\rtimes Z_2],\mathbb{Z}_2]$}, where f_2 is c_1 mod 2. As discussed in Appendix H, $\mathcal{H}^d[U(1)\rtimes Z_2^T,\mathbb{R}/\mathbb{Z}]$ is generated by the subgroup of the factor group of $\mathcal{H}^k[Z_2^T,\mathcal{H}^{d-k}(U(1),\mathbb{R}/\mathbb{Z})]$. We find that in our case here, $\mathcal{H}^d[U(1)\rtimes Z_2^T,\mathbb{R}/\mathbb{Z}]$ is generated by the full $\mathcal{H}^k[Z_2^T,\mathcal{H}^{d-k}[U(1),\mathbb{R}/\mathbb{Z}]]$. $\mathcal{H}^n[U(1),\mathbb{R}/\mathbb{Z}] = \mathbb{Z}$ is generated by the Chern-Simons term $ac_1^{\frac{n-1}{2}}$. Z_2^T acts on $\mathcal{H}^n[U(1),\mathbb{R}/\mathbb{Z}]$ by $\mathcal{H}^n[U(1),\mathbb{R}/\mathbb{Z}]\to\mathcal{H}^n[U(1),\mathbb{R}/\mathbb{Z}]$ if $\frac{n-1}{2}$ = even, and by $\mathcal{H}^n[U(1),\mathbb{R}/\mathbb{Z}]\to-\mathcal{H}^n[U(1),\mathbb{R}/\mathbb{Z}]$ if $\frac{n-1}{2}$ = odd. So $\mathcal{H}^d[U(1)\rtimes Z_2^T,\mathbb{R}/\mathbb{Z}]$ is generated by $a_1^m ac_1^n$, with $m+2n+1 = d$ and $(m,n) = $ (even, even) or $(m,n) = $ (odd, odd). As discussed in Appendix I, $a_1^m ac_1^n$ can be viewed as $a_1^{m-1}c_1^{n+1}$. This allows us to obtain the generators of $\mathcal{H}^d[U(1)\rtimes Z_2^T,\mathbb{R}/\mathbb{Z}]$.

If we choose the natural embedding $U(1)\rtimes Z_2^T\to O$ via $U(1)\rtimes Z_2 = O(2)\to O$, we find that

$$\mathrm{w}_1^{O_2} = a_1, \quad \mathrm{w}_2^{O_2} = f_2, \quad \mathrm{w}_i^{O_2} = 0,\ i > 2. \quad (173)$$

Since the time-reversal twist is implemented by the orientation reversal, we need to set $\mathrm{w}_1 = a_1$. We also note that $H^k[B[U(1)\rtimes Z_2^T],\mathbb{Z}_2] = H^k[B[U(1)\times Z_2^T],\mathbb{Z}_2]$ since when the coefficient is $\mathbb{Z}_2$, there is no distinction between $U(1)\rtimes Z_2^T$ and $U(1)\times Z_2^T$. The above results imply that the extra mixed $U(1)\rtimes Z_2^T$ SPT phases are the same as the extra mixed $U(1)\times Z_2^T$ SPT phases, so we can use the results from the last section.

VI. SPT STATES PROTECTED BY MIRROR REFLECTION SYMMETRY

In this section, we are going to consider SPT state protected by mirror reflection symmetry Z_2^M, which can be probed by the fixed-point partition function on space-time M^d with symmetry twist. However, here, the symmetry twists make the space-time unoriented [62–64,81]. So, the Z_2^M SPT states are described by the gravitational topological invariants $W_{\mathrm{top}}^d(\Gamma)$, which take nontrivial values for unoriented space-times.

Here we would like to remark that the symmetry twists of the time reversal Z_2^T can be implemented by unoriented space-time since the action amplitude for cells with opposite orientation differ by a complex conjugation [see Eq. (40)]. This suggests that the L-type Z_2^T SPT states and the L-type Z_2^M SPT states are the same.

To study the potential Z_2^M SPT states, we note that the ring of the cobordism group Ω_d^O of closed unoriented smooth manifolds is [97]

$$\Omega^O = \sum_d \Omega_d^O = \mathbb{Z}_2[\{x_d\}], \quad d > 1,\ d \neq 2^i - 1 \quad (174)$$

where $\mathbb{M}[\{x_d\}]$ is the polynomial ring generated by x_d's with $\mathbb{M}$ as coefficient. Also, $x_{2i} = \mathbb{R}P^{2i}$. In lower dimensions, we have the following:

$\Omega_1^O = 0$, since circles bound disks.

$\Omega_2^O = \mathbb{Z}_2$, generated by x_2.

$\Omega_3^O = 0$.

$\Omega_4^O = 2\mathbb{Z}_2$, generated by x_4 and x_2^2.

$\Omega_5^O = \mathbb{Z}_2$, generated by $x_5 = H_{2,4}$.

$\Omega_6^O = 3\mathbb{Z}_2$, generated by x_6, x_2x_4, and x_2^3.

$\Omega_7^O = \mathbb{Z}_2$, generated by $x_2x_5 = H_{2,4}\times\mathbb{R}P^2$.

$\Omega_8^O = 5\mathbb{Z}_2$, generated by x_8, x_2x_6, $x_2^4x_4$, x_4^2, x_2^4.

$H_{m,n}$ is a manifold of dimension $m+n-1$ defined as the subset of $\mathbb{R}P^m\times\mathbb{R}P^n$ of points satisfying the homogeneous equation $x_0y_0+\ldots+x_my_m = 0$.

The potential gravitational topological invariants for Z_2^M SPT phases have been obtained in Refs. [62,63]. However, their realizations have not been discussed systematically. In this paper, we show that all the Z_2^M potential gravitational topological invariants are realizable by the $O(\infty)$ NLσMs. This is because the unoriented cobordism group has no free parts. Thus, there is no Chern-Simon potential gravitational topological invariants. As discussed in Appendix C, the locally null potential gravitational topological invariants are all realizable.

In the following, we will calculate the corresponding locally null gravitational topological invariants for those Z_2^M SPT phases. Many results have been obtained in Refs. [62,63].

In $1+1$D, the gravitational topological invariants $W_{\text{top}}^d(\Gamma)$ are generated by $\pi \mathrm{w}_1^2 = \pi \mathrm{w}_2$ since the condition $u_2 = 0$ requires $\mathrm{w}_1^2 = \mathrm{w}_2$. In $3+1$D, the gravitational topological invariants $W_{\text{top}}^d(\Gamma)$ are generated by $\pi \mathrm{w}_1^3$ and $\pi \mathrm{w}_3$ since the condition $u_3 = 0$ requires $\mathrm{w}_1\mathrm{w}_2 = 0$.

In $4+1$D, there are seven Stiefel-Whitney classes w_1^5, $\mathrm{w}_1^3\mathrm{w}_2$, $\mathrm{w}_1^2\mathrm{w}_3$, $\mathrm{w}_1\mathrm{w}_2^2$, $\mathrm{w}_1\mathrm{w}_4$, $\mathrm{w}_2\mathrm{w}_3$, w_5. The Wu classes $u_3 = u_4 = u_5 = 0$ give us

$$\begin{aligned}\mathrm{w}_1\mathrm{w}_2 &= \mathrm{w}_1^4 + \mathrm{w}_2^2 + \mathrm{w}_1\mathrm{w}_3 + \mathrm{w}_4\\ &= \mathrm{w}_1^3\mathrm{w}_2 + \mathrm{w}_1\mathrm{w}_2^2 + \mathrm{w}_1^2\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_4 = 0. \end{aligned} \tag{175}$$

Other relations can be obtained by applying the Steenrod squares

$$\begin{aligned} Sq^1(u_3) &= \mathrm{w}_1\mathrm{w}_3 = 0,\\ Sq^1(u_4) &= \mathrm{w}_1\mathrm{w}_4 + \mathrm{w}_5 = 0,\\ Sq^2(u_3) &= \mathrm{w}_1^2\mathrm{w}_2 + \mathrm{w}_1\mathrm{w}_2^2 + \mathrm{w}_1^2\mathrm{w}_3 = 0. \end{aligned} \tag{176}$$

Additional relations can be obtained from Eq. (78):

$$\begin{aligned} Sq^1(\mathrm{w}_1^4) + u_1\mathrm{w}_1^4 &= \mathrm{w}_1^5 = 0,\\ Sq^1(\mathrm{w}_4) + u_1\mathrm{w}_4 &= \mathrm{w}_5 = 0,\\ Sq^2(\mathrm{w}_3) + u_2\mathrm{w}_3 &= \mathrm{w}_1\mathrm{w}_4 + \mathrm{w}_5 + \mathrm{w}_1^2\mathrm{w}_3 = 0. \end{aligned} \tag{177}$$

We find that $\mathrm{w}_1^5 = \mathrm{w}_1\mathrm{w}_2 = \mathrm{w}_1\mathrm{w}_3 = \mathrm{w}_1\mathrm{w}_4 = \mathrm{w}_5 = 0$. So we have a realizable gauge-gravity topological invariant in $4+1$D:

$$W_{\text{top}}^5(\Gamma) = \tfrac{1}{2}\mathrm{w}_2\mathrm{w}_3. \tag{178}$$

But, such a topological invariant can exist even if we break the time-reversal symmetry [see (89)]. So, it actually describes a topologically ordered phase. There is no L-type time-reversal SPT in $4+1$D. In general, the L-type realizable Z_2^M SPT phases in d-dimensional space-time are not described by Ω_d^O, but by a quotient of Ω_d^O:

$$\text{PSPT}^d_{Z_2^M} = \Omega_d^O/\bar{\Omega}_d^{SO}, \tag{179}$$

where $\bar{\Omega}_d^{SO}$ is the orientation invariant subgroup of Ω_d^{SO} (i.e., the manifold M^d and its orientation reversal $-M^d$ belong to the same oriented cobordism class).

VII. SUMMARY

In this paper, we use $G \times SO(\infty)$ nonlinear NLσMs to construct pure SPT and mixed SPT states, as well as iTO states. We find that those topological states are classified by a quotient of $\mathcal{H}^d(G\times SO,\mathbb{R}/\mathbb{Z})$. For example, the quotient of $\mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z})$ gives rise to iTO phases: $\mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z})/\Lambda^d = \sigma\text{iTO}_L^d$. Writing $\mathcal{H}^d(G\times SO,\mathbb{R}/\mathbb{Z})$ as $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z}) \oplus \oplus_{k=1}^{d-1} H^k[BG,\mathcal{H}^{d-k}(SO,\mathbb{R}/\mathbb{Z})]$ and using the quotient to reduce $\mathcal{H}^d(SO,\mathbb{R}/\mathbb{Z})$ to σiTO_L^d, we find that L-type realizable G SPT phases are classified by $E^d(G) \rtimes \oplus_{k=1}^{d-1} H^k(BG,\sigma\text{iTO}_L^{d-k}) \oplus \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. This classification includes both the pure states [classified by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$] and the mixed SPT states [classified by $E^d(G) \rtimes \oplus_{k=1}^{d-1} H^k(BG,\sigma\text{iTO}_L^{d-k})$]. (Some of the mixed SPT states were also referred to as the beyond-group-cohomology SPT states. In this paper, we see that those beyond-group-cohomology SPT states are actually within another type of group cohomology classification.)

More general SPT states exist, which cannot be obtained from $G\times SO(\infty)$ nonlinear NLσMs. Those SPT states are described by $E^d(G) \rtimes \oplus_{k=1}^{d-1} H^k(BG,\text{iTO}_L^{d-k})$. We note that, as Abelian groups, σiTO_L^d is isomorphic to iTO_L^d, although $\sigma\text{iTO}_L^d \subset \text{iTO}_L^d$ (see Table I). As a result, $E^d(G) \rtimes \oplus_{k=1}^{d-1} H^k(BG,\text{iTO}_L^{d-k})$ is isomorphic to $E^d(G) \rtimes \oplus_{k=1}^{d-1} H^k(BG,\sigma\text{iTO}_L^{d-k})$, as Abelian groups.

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APPENDIX A: GROUP COHOMOLOGY THEORY

1. Homogeneous group cocycle

In this section, we will briefly introduce group cohomology. The group cohomology class $\mathcal{H}^d(G,\mathbb{M})$ is an Abelian group constructed from a group G and an Abelian group $\mathbb{M}$. We will use "+" to represent the multiplication of the Abelian groups. Each element of G also induces a mapping $\mathbb{M}\to\mathbb{M}$, which is denoted as

$$g\cdot m = m', \quad g\in G,\ m,m'\in\mathbb{M}. \tag{A1}$$

The map $g\cdot$ is a group homomorphism:

$$g\cdot(m_1+m_2) = g\cdot m_1 + g\cdot m_2. \tag{A2}$$

The Abelian group $\mathbb{M}$ with such a G-group homomorphism is called a G module.

A homogeneous d-cochain is a function $\nu_d : G^{d+1}\to\mathbb{M}$, that satisfies

$$\nu_d(g_0,\dots,g_d) = g\cdot\nu_d(gg_0,\dots,gg_d), \quad g,g_i\in G. \tag{A3}$$

We denote the set of d-cochains as $\mathcal{C}^d(G,\mathbb{M})$. Clearly $\mathcal{C}^d(G,\mathbb{M})$ is an Abelian group homogeneous group cocycle.

Let us define a mapping d (group homomorphism) from $\mathcal{C}^d(G,\mathbb{M})$ to $\mathcal{C}^{d+1}(G,\mathbb{M})$:

$$(d\nu_d)(g_0,\dots,g_{d+1}) = \sum_{i=0}^{d+1}(-)^i \nu_d(g_0,\dots,\hat{g}_i,\dots,g_{d+1}), \tag{A4}$$

where $g_0,\dots,\hat{g}_i,\dots,g_{d+1}$ is the sequence $g_0,\dots,g_i,\dots,g_{d+1}$ with g_i removed. One can check that $d^2=0$. The homogeneous d-cocycles are then the homogeneous d-cochains that also satisfy the cocycle condition

$$d\nu_d = 0. \tag{A5}$$

We denote the set of d-cocycles as $\mathcal{Z}^d(G,\mathbb{M})$. Clearly, $\mathcal{Z}^d(G,\mathbb{M})$ is an Abelian subgroup of $\mathcal{C}^d(G,\mathbb{M})$.

Let us denote $\mathcal{B}^d(G,\mathbb{M})$ as the image of the map $d: \mathcal{C}^{d-1}(G,\mathbb{M}) \to \mathcal{C}^d(G,\mathbb{M})$ and $\mathcal{B}^0(G,\mathbb{M})=\{0\}$. The elements in $\mathcal{B}^d(G,\mathbb{M})$ are called d-coboundary. Since $d^2=0$, $\mathcal{B}^d(G,\mathbb{M})$ is a subgroup of $\mathcal{Z}^d(G,\mathbb{M})$:

$$\mathcal{B}^d(G,\mathbb{M}) = \{d\nu_{d-1}|\nu_{d-1}\in\mathcal{C}^{d-1}(G,\mathbb{M})\} \subset \mathcal{Z}^d(G,\mathbb{M}). \tag{A6}$$

The group cohomology class $\mathcal{H}^d(G,\mathbb{M})$ is then defined as

$$\mathcal{H}^d(G,\mathbb{M}) = \mathcal{Z}^d(G,\mathbb{M})/\mathcal{B}^d(G,\mathbb{M}). \tag{A7}$$

We note that the d operator and the cochains $\mathcal{C}^d(G,\mathbb{M})$ (for all values of d) form a so-called cochain complex

$$\dots \xrightarrow{d} \mathcal{C}^d(G,\mathbb{M}) \xrightarrow{d} \mathcal{C}^{d+1}(G,\mathbb{M}) \xrightarrow{d} \dots, \tag{A8}$$

which is denoted as $C(G,\mathbb{M})$. So, we may also write the group cohomology $\mathcal{H}^d(G,\mathbb{M})$ as the standard cohomology of the cochain complex $H^d[C(G,\mathbb{M})]$.

2. Inhomogeneous group cocycle

The above definition of group cohomology class can be rewritten in terms of inhomogeneous group cochains/cocycles. An inhomogeneous group d-cochain is a function $\omega_d: G^d \to M$. All $\omega_d(g_1,\dots,g_d)$ form $\mathcal{C}^d(G,\mathbb{M})$. The inhomogeneous group cochains and the homogeneous group cochains are related as

$$\nu_d(g_0,g_1,\dots,g_d) = \omega_d(g_{01},\dots,g_{d-1,d}), \tag{A9}$$

with

$$g_0=1,\ g_1=g_0g_{01},\ g_2=g_1g_{12},\ \dots\ g_d=g_{d-1}g_{d-1,d}. \tag{A10}$$

Now the d map has a form on ω_d:

$$\begin{aligned}(d\omega_d)&(g_{01},\dots,g_{d,d+1})\\ &= g_{01}\cdot\omega_d(g_{12},\dots,g_{d,d+1})\\ &\quad+\sum_{i=1}^{d}(-)^i\omega_d(g_{01},\dots,g_{i-1,i}g_{i,i+1},\dots,g_{d,d+1})\\ &\quad+(-)^{d+1}\omega_d(g_{01},\dots,\tilde{g}_{d-1,d}).\end{aligned} \tag{A11}$$

This allows us to define the inhomogeneous group d-cocycles which satisfy $d\omega_d=0$ and the inhomogeneous group d-coboundaries which have a form $\omega_d = d\mu_{d-1}$. In the following, we are going to use inhomogeneous group cocycles to study group cohomology. Geometrically, we may view g_i as living on the vertex i, while g_{ij} as living on the edge connecting the two vertices i to j.

APPENDIX B: L-TYPE POTENTIAL GAUGE TOPOLOGICAL INVARIANTS

In Sec. II, we introduced the gauge topological invariant $W^d_{\text{top}}(A)$. In fact, the gauge invariance (47) put a strong constraint on the quantized class of the gauge topological invariant $W^d_{\text{top}}(A)$. In this section, we will solve those self-consistent conditions and obtain the *potential gauge topological invariants* directly without going through the NLσM (i.e., we do not concern about if a gauge topological invariant can be generated/realized by a well-defined local bosonic model or not).

First, it appears that all gauge topological invariants are trivial since we can always rescale them $W^d_{\text{top}}(A)\in\mathbb{R}$ to $\tilde{W}^d_{\text{top}}(A)=\lambda W^d_{\text{top}}(A)$ and send $\lambda\to 0$. The new rescaled topological invariant $\tilde{W}^d_{\text{top}}(A)$ will vanish. This way, we showed that there is no nontrivial gauge topological invariant that does not smoothly connect to zero.

There are two related ways to see the mistake in the above argument. First, we note that gauge topological invariants $W^d_{\text{top}}(A)$ can be gauge invariant only up to a 2π phase. If we scale $W^d_{\text{top}}(A)$ by an arbitrary real number, it will not be gauge invariant.

So, different nontrivial gauge topological invariants that do not smoothly connect to zero are classified by their quantized changes under gauge transformations:

$$\int_{M^d} W^d_{\text{top}}(A^g) - \int_{M^d} W^d_{\text{top}}(A) = 0 \bmod 1. \tag{B1}$$

We note that the change of the gauge topological invariant $\int_{M^d}[W^d_{\text{top}}(A+A) - W^d_(A)]$ can be expressed as [93]

$$\int_{M^d}\left[W^d_{\text{top}}(A+\delta A) - W^d_{\text{top}}(A)\right] = \int_{\tilde{N}^{d+1}} P_{\text{top}}(F^N)$$
$$\times P_{\text{top}}(F^N) = dW^d_{\text{top}}(A), \tag{B2}$$

where M^d is closed $\partial M^d=\emptyset$, $\tilde{N}^{d+1}=M^d\times I$, and the gauge connection A^N on $\tilde{N}^{d+1}$ satisfies that on one boundary of $\tilde{N}^{d+1}$ $A^N=A$ and on the other boundary $A^N=A+\delta A$. We call A^N an extension of $A, A+\delta A$ on the boundary $M^d\cup(-M^d)=\partial\tilde{N}^{d+1}$ to $\tilde{N}^{d+1}$. Therefore, Eq. (B1) can be rewritten as

$$\int_{N^{d+1}} P_{\text{top}}(F^N) = 0 \bmod 1,\quad N^{d+1}=M^d\times S^1, \tag{B3}$$

where the G bundle on $N^{d+1}=M^d\times S^1$ has a twist generated by g around S^1. We note that $P_{\text{top}}(F^N)$ is a closed form (or a cocycle) $dP_{\text{top}}(F^N)=0$. Its change under a gauge transformation on N^{d+1} is given by

$$\int_{N^{d+1}\times S^1} dP_{\text{top}} = 0. \tag{B4}$$

Thus, $\int_{N^{d+1}} P_{\text{top}}(F^N)$ is gauge invariant. So, we can express $P_{\text{top}}(F^N)$ as a function of the field strength. Also, a smooth change of the local bosonic Lagrangian will change $W^d_{\text{top}}(A)$

by a gauge invariant term $\Delta W(F^N)$ and change $P_{\text{top}}(F^N)$ by an exact form $P_{\text{top}}(F^N) \to P'_{\text{top}}(F^N) = P_{\text{top}}(F^N) + d\Delta W(F^N)$.

Also, when g is trivial on M^d or when the G bundle on $N^{d+1} = M^d \times S^1$ can be reduced to a G bundle on M^d, we have

$$\int_{N^{d+1}} P_{\text{top}}(F^N) = 0, \quad N^{d+1} = M^d \times S^1. \tag{B5}$$

In other words, when the G bundle on $N^{d+1} = M^d \times S^1$ can be extended to a G bundle on $\tilde{D}^{d+2} = M^d \times D^2$, where D^2 is a disk, we have

$$\int_{\partial(M^d\times D^2)} P_{\text{top}}(F^N) = 0. \tag{B6}$$

The above also implies that

$$\int_{\partial D^{d+2}} P_{\text{top}}(F^N) = 0, \tag{B7}$$

where D^{d+2} is a $(d+2)$-dimensional disk.

To see the second mistake, we note that $W^d_{\text{top}}(A)$ is only required to be well defined when A is deformable to $A = 0$. In general, only the difference $\int_{M^d}[W^d_{\text{top}}(\tilde{A}) - W^d_{\text{top}}(A)]$ is well defined, and only up to a 2π phase. If we scale $W^d_{\text{top}}(A)$ by an arbitrary real number, it will not be well defined. In this case, we need to use Eq. (B2) to define the difference. More generally, if we want to define the difference of the topological invariant on spaces with different geometry, we need to generalize Eq. (B2) to

$$\int_{\tilde{M}^d} W^d_{\text{top}}(\tilde{A}) - \int_{M^d} W^d_{\text{top}}(A) = \int_{N^{d+1}} P_{\text{top}}(F^N)$$
$$\times P_{\text{top}}(F^N) = dW^d_{\text{top}}(A), \tag{B8}$$

where $\partial N^{d+1} = \tilde{M}^d \cup (-M^d)$ and the gauge connection A^N on N^{d+1} satisfies that on one boundary $-M^d$, $A^N = A$, and on the other boundary $\tilde{M}^d$, $A^N = \tilde{A}$. In order for the above difference to be well defined, we require that

$$\int_{N^{d+1}} P_{\text{top}}(F^N) = 0 \text{ mod } 1, \text{ for any closed } N^{d+1}, \tag{B9}$$

which is a stronger quantization condition on $P_{\text{top}}(F^N)$.

Now, we would like to retell the above story in terms of classifying space, and following [93], try to understand the different quantized topological invariants $P_{\text{top}}(F)$ from the classifying space point of view.[2] We first note that all the gauge configurations on N^{d+1} can be understood through classifying space BG and universal bundles EG (with a connection): all G bundles on N^{d+1} with all the possible connections can be obtained by choosing a suitable map of N^{d+1} into BG, $\gamma : N^{d+1} \to BG$ [93]. BG is a very large space, often infinite dimensional. If we pick a connection in the universal bundle EG, even the different connections in the same G bundle on M^d can be obtained by different maps γ. Therefore, we can express $P_{\text{top}}(F)$ as

$$\int_{N^{d+1}} P_{\text{top}}(F) = Q^{d+1}_{\text{top}}(\gamma). \tag{B10}$$

We will further assume that we can express $P_{\text{top}}(F)$ as

$$\int_{N^{d+1}} P_{\text{top}}(F) = Q^{d+1}_{\text{top}}\big(N^{d+1}_\gamma\big), \tag{B11}$$

where N^{d+1}_γ is the image of N^{d+1} in the classifying space BG under the map γ. We will come back to this point later. Here we use the superscript $d+1$ to stress that $Q^{d+1}_{\text{top}}(\dots)$ is function of $(d+1)$-dimensional manifolds.

We see that once we specify a connection on BG, every map $\gamma : M^d \to BG$ will define a connection F on N^{d+1}. Thus, we can view the function of N^{d+1}_γ, $Q^{d+1}_{\text{top}}(N^{d+1}_\gamma)$ as a function of the connection $P_{\text{top}}(F)$. Therefore, we can study the properties (such the quantization condition) of gauge topological invariant $P_{\text{top}}(F)$ via the function $Q^{d+1}_{\text{top}}(N^{d+1}_\gamma)$ in the classifying space BG.

The function $Q^{d+1}_{\text{top}}(N^{d+1})$ has the following defining properties [see Eq. (B6)]:

$$Q^{d+1}_{\text{top}}(N^{d+1}) \in \mathbb{R},$$
$$Q^{d+1}_{\text{top}}\big(N^{d+1}_1\big) + Q^{d+1}_{\text{top}}\big(N^{d+1}_2\big) = Q^{d+1}_{\text{top}}\big(N^{d+1}_1 \cup N^{d+1}_2\big),$$
$$Q^{d+1}_{\text{top}}(N^{d+1}) = 0 \text{ if } N^{d+1} = \partial(M^d \times D^2). \tag{B12}$$

Here, $N^{d+1}_1 \cup N^{d+1}_2$ is an algebraic union of N^{d+1}_1 and N^{d+1}_2. For example, if N^{d+1}_2 is N^{d+1}_1 with an opposite orientation, then $N^{d+1}_1 \cup N^{d+1}_2 = \emptyset$. (More precisely, N^{d+1}_1 and N^{d+1}_2 should be viewed as chains, and $N^{d+1}_1 \cup N^{d+1}_2$ as the addition of chains in homological theory.) The above also implies that

$$Q^{d+1}_{\text{top}}(N^{d+1}) = 0 \text{ if } N^{d+1} = \partial(D^{d+2}). \tag{B13}$$

Then using the additive property of Q^{d+1}_{top}, we can show that

$$Q^{d+1}_{\text{top}}(N^{d+1}) \in \mathbb{R},$$
$$Q^{d+1}_{\text{top}}\big(N^{d+1}_1\big) + Q^{d+1}_{\text{top}}\big(N^{d+1}_2\big) = Q^{d+1}_{\text{top}}\big(N^{d+1}_1 \cup N^{d+1}_2\big),$$
$$Q^{d+1}_{\text{top}}(N^{d+1}) = 0 \quad \text{if} \quad N^{d+1} = \partial O^{d+2}. \tag{B14}$$

Also, from Eq. (B9), we obtain

$$Q^{d+1}_{\text{top}}(N^{d+1}) = 0 \text{ mod } 1 \quad \text{if} \quad \partial N^{d+1} = \emptyset. \tag{B15}$$

From the condition (B14), we see that the function $Q^{d+1}_{\text{top}}(N^{d+1})$ can be described by a cocycle $\omega_{d+1} \in Z^{d+1}(BG,\mathbb{R})$, where $Z^{d+1}(BG,\mathbb{R})$ is the space of all cocycles on the classifying space BG with coefficient $\mathbb{R}$:

$$Q^{d+1}_{\text{top}}(N^{d+1}) = \langle \omega_{d+1}, N^{d+1} \rangle. \tag{B16}$$

Certainly not every cocycle in $C^{d+1}(BG,\mathbb{R})$ satisfies the quantization condition (B15). Let us use $Z^{d+1}_{\mathbb{Z}}(BG,\mathbb{R})$ to denote the set of cocycles that satisfy the quantization condition (B15), and use $B^{d+1}(BG,\mathbb{R})$ to denote the set of coboundaries. Since the coboundaries are all connected and represent local smooth changes of the bosonic Lagrangian, $Z^{d+1}_{\mathbb{Z}}(BG,\mathbb{R})/B^{d+1}(BG,\mathbb{R})$ describes the quantized topological invariants, which are not smoothly connected to each other

[2] For a simple introduction on classifying space, see the Wiki article "Classifying space." For a continuous group G, the classifying space BG in this paper is defined with real manifold topology on G.

by the local smooth changes of the bosonic Lagrangian. It turns out that

$$\text{Free}[H^{d+1}(BG,\mathbb{Z})] \equiv Z_{\mathbb{Z}}^{d+1}(BG,\mathbb{R})/B^{d+1}(BG,\mathbb{R}). \quad \text{(B17)}$$

Thus, $\text{Free}[H^{d+1}(BG,\mathbb{Z})]$ describes a set of the quantized potential topological invariants.

But, $\text{Free}[H^{d+1}(BG,\mathbb{Z})]$ does not describe all the potential topological invariants. $\text{Free}[H^{d+1}(BG,\mathbb{Z})]$ only describe a type of topological invariants that change their value under a smooth change of the gauge configuration $\int_{M^d}[W_{\text{top}}^d(A+\delta A) - W_{\text{top}}^d(A)] \neq 0$. We will call such type of topological invariants as Chern-Simons topological invariants. However, there are another type of topological invariants that do not change under a smooth change of the gauge configuration $\int_{M^d}[W_{\text{top}}^d(A+\delta A) - W_{\text{top}}^d(A)] = 0$. We will call such type of topological invariants as locally null topological invariants. The locally null topological invariants correspond to $P(F^N) = 0$, so it is missed by our discussion above. In the classifying space approach, the locally null topological invariants $e^{\int_{M^d} W_{\text{top}}^d(A)}$ are described by cocycles in $H^d(BG,\mathbb{R}/\mathbb{Z})$ [67–69]. However, $H^d(BG,\mathbb{R}/\mathbb{Z})$ may contain continuous part, such as $\mathbb{R}/\mathbb{Z}$. So the quantized potential locally null topological invariants are described by $\text{Dis}[H^d(BG,\mathbb{R}/\mathbb{Z})]$, the discrete part of $H^d(BG,\mathbb{R}/\mathbb{Z})$.

This way, we show that the potential gauge topological invariants that cannot connect to zero and cannot connect to each other are described by $\text{Free}[H^{d+1}(BG,\mathbb{Z})] \oplus \text{Dis}[H^d(BG,\mathbb{R}/\mathbb{Z})]$. Since $\text{Dis}[H^d(BG,\mathbb{R}/\mathbb{Z})] = \text{Tor}[H^{d+1}(BG,\mathbb{Z})]$, we may say that the potential gauge topological invariants are described by $H^{d+1}(BG,\mathbb{Z})$.

Since the different gauge transformation properties

$$2\pi \int_{M^d} W_{\text{top}}^d(A^g) - W_{\text{top}}^d(A) = \Big|_{A=0} \int_{M^d} L_{\text{top}}^d(g^{-1}\partial g) \quad \text{(B18)}$$

are classified by group cohomology $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ [where $L_{\text{top}}^d(g)$ is a cocycle in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$] and since $H^{d+1}(BG,\mathbb{Z}) = \mathcal{H}_B^d(G,\mathbb{R})$ (see, for example, Ref. [69]), we find that the L-type potential gauge topological invariants coincide with the realizable L-type gauge topological invariants produced by the NLσM. This means that the L-type potential gauge topological invariants described by $H^{d+1}(BG,\mathbb{Z})$ can all be produced by L-type local bosonic models (i.e., the NLσMs with fields in G), if we "gauge" the symmetry G. In other words, all the L-type potential gauge topological invariants described by $H^{d+1}(BG,\mathbb{Z})$ are realizable by L-type local bosonic systems.

For example, when $G = U(1)$, $H^4[BU(1),\mathbb{Z}] = \mathbb{Z}$, whose generator is c_1^2 with $c_1 = \frac{1}{2\pi}F$ and $c_1^2 = \frac{1}{4\pi^2}FF = P_{\text{top}}(F)$. The corresponding gauge topological invariant is $W_{\text{top}}^3(A) = \frac{1}{(2\pi)^2}AF$, where F is the curvature two-form of the $U(1)$ connection one-form A. Such a gauge topological invariant describes a $U(1)$ SPT state in $\mathcal{H}^3[U(1),\mathbb{R}/\mathbb{Z}]$ in 2 + 1D.

APPENDIX C: L-TYPE POTENTIAL GAUGE-GRAVITY TOPOLOGICAL INVARIANTS

In Sec. III B, we introduced L-type realizable gauge-gravity topological invariants $W_{\text{top}}^d(A,\Gamma)$. In this section, we will discuss the L-type potential gauge-gravity topological invariants $W_{\text{top}}^d(A,\Gamma)$, by repeating the discussion in Appendix B. We can use a $(d+1)$-form P_{top} to define difference of the potential gauge-gravity topological invariant $W_{\text{top}}^d(A,\Gamma)$ [93]:

$$\int_{\tilde{M}^d} W_{\text{top}}^d(\tilde{A},\Gamma) - \int_{M^d} W_{\text{top}}^d(A,\Gamma) = \int_{N^{d+1}} P_{\text{top}}(F^N,R^N), \text{ with } \partial N^{d+1} = \tilde{M}^d \cup (-M^d). \quad \text{(C1)}$$

In the classifying space approach, $P_{\text{top}}(F^N,R^N)$ is expressed as

$$\int_{N_\gamma^{d+1}} P_{\text{top}}(F^N,R^N) = Q_{\text{top}}^{d+1}(N_\gamma^{d+1}), \quad \text{(C2)}$$

where N_γ^{d+1} is the image of the map $\gamma : N^{d+1} \to B(G \times SO)$. We find that $Q_{\text{top}}^{d+1}(N_\gamma^{d+1})$ satisfies

$$\begin{aligned} &Q_{\text{top}}^{d+1}(N_\gamma^{d+1}) \in \mathbb{R}, \\ &Q_{\text{top}}^{d+1}(N_\gamma^{d+1}) + Q_{\text{top}}^{d+1}(\tilde{N}_\gamma^{d+1}) = Q_{\text{top}}^{d+1}(N_\gamma^{d+1} \cup \tilde{N}_\gamma^{d+1}), \\ &Q_{\text{top}}^{d+1}(N_\gamma^{d+1}) = 0 \quad \text{if} \quad N_\gamma^{d+1} = \partial O. \\ &Q_{\text{top}}^{d+1}(N_\gamma^{d+1}) = 0 \text{ mod } 1 \quad \text{if} \quad \partial N_\gamma^{d+1} = \emptyset. \end{aligned} \quad \text{(C3)}$$

However, the quantization condition $Q_{\text{top}}^{d+1}(N_\gamma^{d+1}) = 0 \text{ mod } 1$ is required only for a subset of cycles N_γ^{d+1} in $B(SO \times G)$. This is because for a closed N^{d+1} with a given topology, a generic map $\gamma : N^{d+1} \to B(G \times SO)$ can give rise to an arbitrary $G \times SO$ principle bundle over N^{d+1} (whose fiber is the $G \times SO$ group). The corresponding SO vector bundle (whose fiber is the vector space that forms the fundamental representation of SO) may not be the tangent bundle over N^{d+1}. Such a map is not allowed. The quantization condition $Q_{\text{top}}^{d+1}(N_\gamma^{d+1}) = 0 \text{ mod } 1$ is required only for the maps γ that give rise to the tangent bundle over N^{d+1}.

Let $Z_G^{d+1}[B(G \times SO),\mathbb{R}]$ be the space of quantized cocycles, and let $B_{d+1}[B(G \times SO),\mathbb{R}]$ be the space of coboundaries. Then the potential gauge-gravity topological invariants are described by

$$H_G^{d+1}[B(G \times SO),\mathbb{R}] \equiv Z_G^{d+1}[B(G \times SO),\mathbb{R}]/B_{d+1}[B(G \times SO),\mathbb{R}]. \quad \text{(C4)}$$

Since the quantization condition is enforced only one of a subset of $(d+1)$-cycles N_γ^{d+1}, $H_G^{d+1}[B(G \times SO),\mathbb{R}]$ may contain unquantized continuous part $\mathbb{R}$. It may also contain quantized discrete part $\mathbb{Z}$. In other words,

$$H_G^{d+1}[B(G \times SO),\mathbb{R}] = \left(\oplus_{i=1}^{n_{\mathbb{R}}^G} \mathbb{R}\right) \oplus \left(\oplus_{i=1}^{n_{\mathbb{Z}}^G} \mathbb{Z}\right). \quad \text{(C5)}$$

We note that the cocycles in $H^{d+1}(BG,\mathbb{Z})$ also satisfy all the conditions in Eq. (C3), thus we have a group homomorphism

$$\text{Free}[H^{d+1}[B(G \times SO),\mathbb{Z}]] \to H_G^{d+1}[B(G \times SO),\mathbb{R}]. \quad \text{(C6)}$$

The image of the map is formed by realizable gauge-gravity topological invariants. We also note that there is another group homomorphism (an exact sequence)

$$0 \to \text{Dis}\big[H_G^{d+1}(BG,\mathbb{R})\big] \to \text{Free}[H^{d+1}[B(G \times SO),\mathbb{Z}(1/n)]] \quad \text{(C7)}$$

for a certain n, where $\mathbb{Z}(\frac{1}{n})$ is the fractional integer $\{0,\pm\frac{1}{n},\pm\frac{2}{n},\ldots\}$. This is because all unquantized cocycles are dropped, and a quantized cocycle corresponds an element of $H^{d+1}[B(G\times SO),\mathbb{Z}(\frac{1}{n})]$. Also different quantized cocycles correspond different elements of $H^{d+1}[B(G\times SO),\mathbb{Z}(\frac{1}{n})]$. If we write

$$\text{Free}[H^{d+1}[B(G\times SO),\mathbb{Z}]] = \left(\oplus_{i=1}^{n_\mathbb{Z}}\mathbb{Z}\right), \tag{C8}$$

we have

$$n_\mathbb{Z} = n_\mathbb{R}^G + n_\mathbb{Z}^G. \tag{C9}$$

Note that $H_G^{d+1}(BG,\mathbb{R})$ only describe Chern-Simons gauge-gravity topological invariants. The locally null gauge-gravity topological invariants are described by

$$H_G^d[B(G\times SO),\mathbb{R}/\mathbb{Z}] \equiv \text{Dis}\big(H^d[B(G\times SO),\mathbb{R}/\mathbb{Z}]/\Lambda_G^d\big), \tag{C10}$$

where Λ_G^d is a subgroup of $H^d[B(G\times SO),\mathbb{R}/\mathbb{Z}]$ form by cocycles ω^d that satisfy

$$\langle\omega^d,N_\gamma^d\rangle = 0, \tag{C11}$$

where N_γ^d is all the close d-manifolds in $B(G\times SO)$ such that the SO bundle on N_γ^d is smoothly connected to the tangent bundle of N_γ^d. Since Dis $\big(H^d[B(G\times SO),\mathbb{R}/\mathbb{Z}]\big)\simeq$ Tor $\big(H^{d+1}[B(G\times SO),\mathbb{Z}]\big)$, we have

$$H_G^d[B(G\times SO),\mathbb{R}/\mathbb{Z}] \subset \text{Tor}(H^{d+1}[B(G\times SO),\mathbb{Z}]), \tag{C12}$$

that describes the locally null potential gauge-gravity topological invariants. Those locally null gauge-gravity topological invariants are all realizable. We also have

$$\text{Dis}\left(H_G^{d+1}[B(G\times SO),\mathbb{R}]\right) \subset \text{Free}(H^{d+1}[B(G\times SO),\mathbb{Z}(1/n)]), \tag{C13}$$

that describes the Chern-Simons potential gauge-gravity topological invariants. A subset of those Chern-Simons gauge-gravity topological invariants that are also in Free$(H^{d+1}[B(G\times SO),\mathbb{Z}])$ are realizable.

We like to remark that, in general, the image of the map (C6) is not the whole $H_G^{d+1}[B(G\times SO),\mathbb{R}]$. This means that some potential gauge-gravity topological invariants cannot be generated from NLσM construction discussed in Sec. II B. However, it is not clear if there are some other bosonic path integrals that can generate the missing potential topological invariants.

APPENDIX D: THE RING OF $H^*(BSO,\mathbb{Z})$

According to [96], the ring $H^*(BSO,\mathbb{Z})$ is a polynomial ring generated by p_i and $\beta(\text{w}_{2i_1}\text{w}_{2i_2}\ldots)$, $0<i_1<i_2<\ldots$, with integer coefficients. Here $p_i\in H^{4i}(BSO,\mathbb{Z})$ is the Pontryagin class of dimension $4i$ and $\text{w}_i\in H^i(BSO,\mathbb{Z}_2)$ is the Stiefel-Whitney class of dimension i. Since Tor$H^d(BG,\mathbb{R}/\mathbb{Z})=$ Tor$H^{d+1}(BG,\mathbb{Z})$ (see, for example, [69]), the natural map $H^d(BG,\mathbb{Z}_2)\to$ Tor$H^d(BG,\mathbb{R}/\mathbb{Z})$ induces a natural map $H^d(BG,\mathbb{Z}_2)\to H^{d+1}(BG,\mathbb{Z})$: $\beta: H^i(BSO,\mathbb{Z}_2)\to H^{i+1}(BSO,\mathbb{Z})$. Therefore, $\beta(\text{w}_{2i_1}\text{w}_{2i_2}\ldots)$ has a dimension $1+2i_1+2i_2+\ldots$.

More precisely, to obtain the ring $H^*(BSO,\mathbb{Z})$ from a polynomial ring generated by p_i and $\beta(\text{w}_{2i_1}\text{w}_{2i_2}\ldots)$, we need to quotient out certain equivalence relations:

$$H^*(BSO,\mathbb{Z}) = \mathbb{Z}[\{p_i\},\{\beta(\text{w}_{2i_1}\text{w}_{2i_2}\ldots)\}]/\sim, \tag{D1}$$

where the equivalence relations $\sim$ contain

$$\begin{aligned}&2\beta(\text{w}_{2i_1}\text{w}_{2i_2}\ldots)=0,\\ &\beta w(I)\beta w(J)\\ &=\sum_{k\in I}\beta\text{w}_{2k}\beta\text{w}[(I-k)\cup J-(I-k)\cap J]p[(I-k)\cap J],\end{aligned} \tag{D2}$$

where $I=\{i_1,i_2,\ldots\}$, $\text{w}(I)=\text{w}_{2i_1}\text{w}_{2i_2}\ldots$ and $p(I)=p_{i_1}p_{i_2}\ldots$. Here, we list all the second kinds of the equivalence relations for low dimensions:

$$\begin{aligned}\beta\text{w}_2\beta\text{w}_2&=\beta\text{w}_2\beta\text{w}_2,\\ \beta\text{w}_2\beta\text{w}_4&=\beta\text{w}_2\beta\text{w}_4,\\ \beta(\text{w}_2\text{w}_4)\beta\text{w}_2&=\beta\text{w}_2\beta(\text{w}_2\text{w}_4).\end{aligned} \tag{D3}$$

We see that those relations are identities (mod 2), and thus there are no effective equivalence relations of the second kind for dimensions less than 12. So, for low dimensions,

$$\begin{aligned}&H^1(BSO,\mathbb{Z})=0,\\ &H^2(BSO,\mathbb{Z})=0,\\ &H^3(BSO,\mathbb{Z})=\mathbb{Z}_2=\{m\beta\text{w}_2\},\\ &H^4(BSO,\mathbb{Z})=\mathbb{Z}=\{np_1\},\\ &H^5(BSO,\mathbb{Z})=\mathbb{Z}_2=\{m\beta\text{w}_4\},\\ &H^6(BSO,\mathbb{Z})=\mathbb{Z}_2=\{m\beta\text{w}_2\beta\text{w}_2\},\\ &H^7(BSO,\mathbb{Z})=2\mathbb{Z}_2=\{m_1\beta\text{w}_6+m_2\beta\text{w}_2p_1\},\\ &H^8(BSO,\mathbb{Z})=2\mathbb{Z}\oplus\mathbb{Z}_2=\{n_1p_1^2+n_2p_2+m\beta\text{w}_2\beta\text{w}_4\},\end{aligned} \tag{D4}$$

where m's are in Z_2 and n's in $\mathbb{Z}$.

Also, according to Ref. [96], the ring $H^*(BO_n,\mathbb{Z}_2)$ is given by

$$H^*(BO_n,\mathbb{Z}_2)=\mathbb{Z}_2[\text{w}_1,\text{w}_2,\ldots]. \tag{D5}$$

For low dimensions, we find that

$$\begin{aligned}&H^1(BO,\mathbb{Z}_2)=\mathbb{Z}_2=\{m\text{w}_1\},\\ &H^2(BO,\mathbb{Z}_2)=2\mathbb{Z}_2=\{m_1\text{w}_2+m_2\text{w}_1^2\},\\ &H^3(BO,\mathbb{Z}_2)=3\mathbb{Z}_2=\{m_1\text{w}_3+m_2\text{w}_1\text{w}_2+m_3\text{w}_1^3\},\\ &H^4(BO,\mathbb{Z}_2)=5\mathbb{Z}_2,\\ &H^5(BO,\mathbb{Z}_2)=7\mathbb{Z}_2,\\ &H^6(BO,\mathbb{Z}_2)=11\mathbb{Z}_2,\end{aligned} \tag{D6}$$

where m's are in Z_2.

APPENDIX E: CALCULATE THE GENERATORS IN EQS. (67) AND (146) FROM EQS. (66) AND (143)

The basis in Eqs. (67) and (146) gives rise to the basis in Eqs. (66) and (143) after the one-to-one natural map $\tilde\beta$: $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})\to H^{d+1}(BG,\mathbb{Z})$. We also have a natural map $\pi : \mathcal{H}^d(G,\mathbb{Z}_2) \to \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, such as $\pi \mathrm{w}_i = \frac{1}{2}\mathrm{w}_i$. The Bockstein homomorphism $\beta : \mathcal{H}^d(G,\mathbb{Z}_2) = H^d(BG,\mathbb{Z}_2) \to H^{d+1}(BG,\mathbb{Z})$ is given by $\beta = \tilde{\beta}\pi$, which is equal to the Steenrod square Sq^1. One can use the properties (see Sec. IV B)

$$Sq^1 Sq^1 = 0, \quad Sq^1(xy) = Sq^1(x)y + xSq^1(y),$$
$$Sq^1(w_i) = \mathrm{w}_1\mathrm{w}_i + (i+1)\mathrm{w}_{i+1}, \quad Sq^1(w_2^2) = 0 \tag{E1}$$

for $x,y \in H^*(X,\mathbb{Z}_2)$ to compute the action of Sq^1.

Let us first calculate the generators in Eq. (67) from those in Eq. (66). In two-dimensional space-time $\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z}) = H^3(BSO,\mathbb{Z}) = \mathbb{Z}_2$. $H^3(BSO,\mathbb{Z})$ is generated by the promoted three-dimensional topological invariant $K^3(\Gamma) = Sq^1(\mathrm{w}_2) = \mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3$. $\mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z})$ is generated by $W^2_{\text{top}}(\Gamma)$ which is the pull back of $K^3(\Gamma) = \mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3$ by the natural map $\tilde{\beta} : \mathcal{H}^2(SO,\mathbb{R}/\mathbb{Z}) \to H^3(BSO,\mathbb{Z})$:

$$\tilde{\beta}\big[W^d_{\text{top}}(\Gamma)\big] = K^{d+1}(\Gamma). \tag{E2}$$

Using $\tilde{\beta} = Sq^1\pi^{-1}$, we find that $W^2_{\text{top}} = \frac{1}{2}\mathrm{w}_2$ since $\pi^{-1}\frac{1}{2}\mathrm{w}_2 = \mathrm{w}_2 =$ and $Sq^1(\mathrm{w}_2) = \mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3$. In 2+1D space-time, the corresponding $H^4(BSO,\mathbb{Z}) = \mathbb{Z}$ is generated by $K^4(\Gamma) = p_1$. The pull back of the promoted generator p_1 by the natural map $\tilde{\beta} : \mathcal{H}^3(SO,\mathbb{R}/\mathbb{Z}) \to H^4(BSO,\mathbb{Z})$ is the gauge-gravity topological invariant $W^3_{\text{top}} = \omega_3$. In 3+1D space-time, the corresponding $\mathcal{H}^4(SO,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$ is generated by the gauge-gravity topological invariant $W^4_{\text{top}} = \frac{1}{2}\mathrm{w}_4$ since $\tilde{\beta}\frac{1}{2}\mathrm{w}_4 = Sq^1\pi^{-1}\frac{1}{2}\mathrm{w}_4 = Sq^1\mathrm{w}_4 = \beta\mathrm{w}_4$. In 4+1D space-time, the corresponding $\mathcal{H}^5(SO,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$ is generated by the gauge-gravity topological invariant $W^4_{\text{top}} = \frac{1}{2}\mathrm{w}_2(\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3)$. This is because $\tilde{\beta}\frac{1}{2}\mathrm{w}_2(\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3) = Sq^1\pi^{-1}\frac{1}{2}\mathrm{w}_2(\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3) = Sq^1\mathrm{w}_2(\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3) = Sq^1\mathrm{w}_2Sq^1\mathrm{w}_2 = \beta\mathrm{w}_2\beta\mathrm{w}_2$, where we have used Eq. (E1). In 5+1D space-time, again, using $\tilde{\beta} = Sq^1\pi^{-1}$ and Eq. (E1), we can show that the corresponding $\mathcal{H}^6(SO,\mathbb{R}/\mathbb{Z}) = 2\mathbb{Z}_2$ is generated by the gauge-gravity topological invariant $W^4_{\text{top}} = \frac{1}{2}\mathrm{w}_6, \frac{1}{2}\mathrm{w}_2^3$. Similarly, we can show that, in 6+1D space-time, the corresponding $\mathcal{H}^8(SO,\mathbb{R}/\mathbb{Z}) = 2\mathbb{Z} \oplus \mathbb{Z}_2$ is generated by the gauge-gravity topological invariant $W^4_{\text{top}} = \omega_7^{p_1^2}, \omega_7^{p_2}, \frac{1}{2}(\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3)\mathrm{w}_4$.

We would like to remark that $\tilde{\beta}$ maps both $\frac{1}{2}(\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3)\mathrm{w}_4$ and $\frac{1}{2}\mathrm{w}_2(\mathrm{w}_1\mathrm{w}_4 + \mathrm{w}_5)$ in $\mathcal{H}^7(SO,\mathbb{R}/\mathbb{Z})$ to the same $\beta\mathrm{w}_2\beta\mathrm{w}_4$ in $H^8(BSO,\mathbb{Z})$ since $\beta = Sq^1$ maps both $(\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3)\mathrm{w}_4$ and $\mathrm{w}_2(\mathrm{w}_1\mathrm{w}_4 + \mathrm{w}_5)$ in $\mathcal{H}^7(SO,\mathbb{Z}_2)$ to the same $\beta\mathrm{w}_2\beta\mathrm{w}_4$. Since both β and π are many-to-one maps, the above fact does not contradict with the facts that $\tilde{\beta}$ is a one-to-one map and $\beta = \tilde{\beta}\pi$. Although $(\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3)\mathrm{w}_4$ and $\mathrm{w}_2(\mathrm{w}_1\mathrm{w}_4 + \mathrm{w}_5)$ are different cocycles in $\mathcal{H}^7(SO,\mathbb{Z}_2)$, their images under π, $\frac{1}{2}(\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3)\mathrm{w}_4$ and $\frac{1}{2}\mathrm{w}_2(\mathrm{w}_1\mathrm{w}_4 + \mathrm{w}_5)$, belong to the same cocycle in $\mathcal{H}^7(SO,\mathbb{R}/\mathbb{Z})$ (i.e., differ only by a coboundary).

Using a similar approach, we can calculate the generators in Eq. (146) from those in Eq. (143). For example, we can show $\tilde{\beta}\frac{1}{2}(\mathrm{w}_1^{O_2})^3(\mathrm{w}_2^{O_2})^2 = (\beta\mathrm{w}_1^{O_2})^2 p_1^{O_2}$. This is because $\tilde{\beta}\frac{1}{2}(\mathrm{w}_1^{O_2})^3(\mathrm{w}_2^{O_2})^2 = Sq^1[(\mathrm{w}_1^{O_2})^3(\mathrm{w}_2^{O_2})^2] = Sq^1[(\mathrm{w}_1^{O_2})^3](\mathrm{w}_2^{O_2})^2 = Sq^1[(\mathrm{w}_1^{O_2})^3]p_1^{O_2}$, where we have used $(\mathrm{w}_2^{O_2})^2 = p_1^{O_2}$ mod 2. Then, using $Sq^1[(\mathrm{w}_1^{O_2})^3] = (Sq^1\mathrm{w}_1^{O_2})^2 = (\beta\mathrm{w}_1^{O_2})^2$, we find $\tilde{\beta}\frac{1}{2}(\mathrm{w}_1^{O_2})^3(\mathrm{w}_2^{O_2})^2 = (\beta\mathrm{w}_1^{O_2})^2 p_1^{O_2}$, where we have used $Sq^1\mathrm{w}_1^{O_2} = (\mathrm{w}_1^{O_2})^2$.

APPENDIX F: RELATION BETWEEN PONTRYAGIN CLASSES AND STIEFEL-WHITNEY CLASSES

There is a result due to Wu that relates Pontryagin classes and Stiefel-Whitney classes (see [98], Theorem C): Let B be a vector bundle over a manifold X, w_i be its Stiefel-Whitney classes and p_i its Pontryagin classes. Let ρ_4 be the reduction modulo 4 and θ_2 be the embedding of $\mathbb{Z}_2$ into $\mathbb{Z}_4$ (as well as their induced actions on cohomology groups). Then,

$$\mathcal{P}_2(\mathrm{w}_{2i}) = \rho_4(p_i) + \theta_2\Bigg(\mathrm{w}_1 Sq^{2i-1}\mathrm{w}_{2i} + \sum_{j=0}^{i-1}\mathrm{w}_{2j}\mathrm{w}_{4i-2j}\Bigg), \tag{F1}$$

where $\mathcal{P}_2$ is the Pontryagin square [99], which maps $x \in H^{2n}(X,\mathbb{Z}_2)$ to $\mathcal{P}_2(x) \in H^{4n}(X,\mathbb{Z}_4)$. Let ρ_2 be the reduction modulo 2. The Pontryagin square has a property that $\rho_2\mathcal{P}_2(x) = x^2$. Therefore,

$$\rho_2\mathcal{P}_2(\mathrm{w}_{2i}) = \mathrm{w}_{2i}^2 = \rho_2(p_i). \tag{F2}$$

APPENDIX G: KÜNNETH FORMULA

The Künneth formula is a very helpful formula that allows us to calculate the cohomology of chain complex $X \times X'$ in terms of the cohomology of chain complex X and chain complex X'. The Künneth formula is expressed in terms of the tensor-product operation $\otimes_R$ and the torsion-product operation $\boxtimes_R \equiv \mathrm{Tor}_1^R$, which have the following properties:

$$\begin{aligned}
&\mathbb{M} \otimes_{\mathbb{Z}} \mathbb{M}' \simeq \mathbb{M}' \otimes_{\mathbb{Z}} \mathbb{M},\\
&\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{M} \simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{M},\\
&\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{M} \simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{M}/n\mathbb{M},\\
&\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \simeq \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0,\\
&\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{Z}_{\langle m,n\rangle},\\
&\mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} = 0,\\
&\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} = 0,\\
&\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R},\\
&(\mathbb{M}' \oplus \mathbb{M}'') \otimes_R \mathbb{M} = (\mathbb{M}' \otimes_R \mathbb{M}) \oplus (\mathbb{M}'' \otimes_R \mathbb{M}),\\
&\mathbb{M} \otimes_R (\mathbb{M}' \oplus \mathbb{M}'') = (\mathbb{M} \otimes_R \mathbb{M}') \oplus (\mathbb{M} \otimes_R \mathbb{M}'')
\end{aligned} \tag{G1}$$

and

$$\begin{aligned}
&\mathrm{Tor}_1^R(\mathbb{M},\mathbb{M}') \equiv \mathbb{M} \boxtimes_R \mathbb{M}',\\
&\mathbb{M} \boxtimes_R \mathbb{M}' \simeq \mathbb{M}' \boxtimes_R \mathbb{M},\\
&\mathbb{Z} \boxtimes_{\mathbb{Z}} \mathbb{M} = \mathbb{M} \boxtimes_{\mathbb{Z}} \mathbb{Z} = 0,\\
&\mathbb{Z}_n \boxtimes_{\mathbb{Z}} \mathbb{M} = \{m \in \mathbb{M} | nm = 0\},\\
&\mathbb{Z}_n \boxtimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} = \mathbb{Z}_n,\\
&\mathbb{Z}_m \boxtimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{Z}_{\langle m,n\rangle},\\
&\mathbb{M}' \oplus \mathbb{M}'' \boxtimes_R \mathbb{M} = \mathbb{M}' \boxtimes_R \mathbb{M} \oplus \mathbb{M}'' \boxtimes_R \mathbb{M},\\
&\mathbb{M} \boxtimes_R \mathbb{M}' \oplus \mathbb{M}'' = \mathbb{M} \boxtimes_R \mathbb{M}' \oplus \mathbb{M} \boxtimes_R B,
\end{aligned} \tag{G2}$$

where $\langle m,n\rangle$ is the greatest common divisor of m and n. These expressions allow us to compute the tensor product $\otimes_R$ and the torsion product $\boxtimes_R$. Here R is a ring and $\mathbb{M},\mathbb{M}',\mathbb{M}''$ are R modules. An R module is like a vector space over R (i.e., we can "multiply" a vector by an element of R.)

The Künneth formula itself is given by (see Ref. [100], page 247)

$$\begin{aligned} &H^d(X\times X',\mathbb{M}\otimes_R\mathbb{M}') \\ &\simeq \big[\oplus_{k=0}^{d} H^k(X,\mathbb{M})\otimes_R H^{d-k}(X',\mathbb{M}')\big] \\ &\quad \big[\oplus_{k=0}^{d+1} H^k(X,\mathbb{M})\boxtimes_R H^{d-k+1}(X',\mathbb{M}')\big]. \end{aligned} \tag{G3}$$

Here R is a principal ideal domain and $\mathbb{M},\mathbb{M}'$ are R modules such that $\mathbb{M}\boxtimes_R\mathbb{M}'=0$. We also require either

(1) $H_d(X,\mathbb{Z})$ and $H_d(X',\mathbb{Z})$ are finitely generated, or

(2) $\mathbb{M}'$ and $H_d(X',\mathbb{Z})$ are finitely generated. For example, $\mathbb{M}'=\mathbb{Z}\oplus\ldots\oplus\mathbb{Z}\oplus\mathbb{Z}_n\oplus\ldots\oplus\mathbb{Z}_m$.

For more details on principal ideal domain and R module, see the corresponding Wiki articles. Note that $\mathbb{Z}$ and $\mathbb{R}$ are principal ideal domains, while $\mathbb{R}/\mathbb{Z}$ is not. Also, $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are not finitely generate R modules if $R=\mathbb{Z}$. The Künneth formula works for topological cohomology where X and X' are treated as topological spaces. But, it does not work for group cohomology by treating H^d as $\mathcal{H}^d$ and X and X' as groups, $X=G$ and $X'=G'$, as demonstrated by the example $\mathbb{M}=\mathbb{M}'=\mathbb{R}/\mathbb{Z}$ and $X=X'=Z_n$. However, since $\mathcal{H}^d(G,\mathbb{Z})=H^d(BG,\mathbb{Z})$, the above Künneth formula works for group cohomology when $\mathbb{M}=\mathbb{M}'=\mathbb{Z}$. The above Künneth formula also works for group cohomology when G,G' are finite or when G' is finite and $\mathbb{M}'$ is finitely generate (such as $\mathbb{M}'$ is $\mathbb{Z}$ or $\mathbb{Z}_n$).

As the first application of Künneth formula, we like to use it to calculate $H^*(X',\mathbb{M})$ from $H^*(X',\mathbb{Z})$, by choosing $R=\mathbb{M}'=\mathbb{Z}$. In this case, the condition $\mathbb{M}\boxtimes_R\mathbb{M}'=\mathbb{M}\boxtimes_{\mathbb{Z}}\mathbb{Z}=0$ is always satisfied. $\mathbb{M}$ can be $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}_n$ etc. So, we have

$$\begin{aligned} &H^d(X\times X',\mathbb{M}) \\ &\simeq \big[\oplus_{k=0}^{d} H^k(X,\mathbb{M})\otimes_{\mathbb{Z}} H^{d-k}(X',\mathbb{Z})\big] \\ &\quad \oplus\big[\oplus_{k=0}^{d+1} H^k(X,\mathbb{M})\boxtimes_{\mathbb{Z}} H^{d-k+1}(X',\mathbb{Z})\big]. \end{aligned} \tag{G4}$$

The above is valid for topological cohomology. It is also valid for group cohomology:

$$\begin{aligned} &\mathcal{H}^d(G\times G',\mathbb{M}) \\ &\simeq \big[\oplus_{k=0}^{d} \mathcal{H}^k(G,\mathbb{M})\otimes_{\mathbb{Z}} \mathcal{H}^{d-k}(G',\mathbb{Z})\big] \\ &\quad \oplus\big[\oplus_{k=0}^{d+1} \mathcal{H}^k(G,\mathbb{M})\boxtimes_{\mathbb{Z}} \mathcal{H}^{d-k+1}(G',\mathbb{Z})\big] \end{aligned} \tag{G5}$$

provided that G' is a finite group. Using Eq. (G13), we can rewrite the above as

$$\begin{aligned} &\mathcal{H}^d(G\times G',\mathbb{M})\simeq\mathcal{H}^d(G,\mathbb{M}) \\ &\quad \oplus\big[\oplus_{k=0}^{d-2} \mathcal{H}^k(G,\mathbb{M})\otimes_{\mathbb{Z}} \mathcal{H}^{d-k-1}(G',\mathbb{R}/\mathbb{Z})\big] \\ &\quad \oplus\big[\oplus_{k=0}^{d-1} \mathcal{H}^k(G,\mathbb{M})\boxtimes_{\mathbb{Z}} \mathcal{H}^{d-k}(G',\mathbb{R}/\mathbb{Z})\big], \end{aligned} \tag{G6}$$

where we have used

$$\mathcal{H}^1(G',\mathbb{Z})=0. \tag{G7}$$

If we further choose $\mathbb{M}=\mathbb{R}/\mathbb{Z}$, we obtain

$$\begin{aligned} &\mathcal{H}^d(G\times G',\mathbb{R}/\mathbb{Z}) \\ &\simeq\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})\oplus\mathcal{H}^d(G',\mathbb{R}/\mathbb{Z}) \\ &\quad \oplus\big[\oplus_{k=1}^{d-2} \mathcal{H}^k(G,\mathbb{R}/\mathbb{Z})\otimes_{\mathbb{Z}} \mathcal{H}^{d-k-1}(G',\mathbb{R}/\mathbb{Z})\big] \\ &\quad \oplus\big[\oplus_{k=1}^{d-1} \mathcal{H}^k(G,\mathbb{R}/\mathbb{Z})\boxtimes_{\mathbb{Z}} \mathcal{H}^{d-k}(G',\mathbb{R}/\mathbb{Z})\big], \end{aligned} \tag{G8}$$

where G' is a finite group.

We can further choose X to be the space of one point (or the trivial group of one element) in Eq. (G4), and use

$$H^d(X,\mathbb{M}))=\begin{cases}\mathbb{M} & \text{if}\quad d=0,\\ 0 & \text{if}\quad d>0,\end{cases} \tag{G9}$$

to reduce Eq. (G4) to

$$H^d(X,\mathbb{M})\simeq\mathbb{M}\otimes_{\mathbb{Z}}H^d(X,\mathbb{Z})\oplus\mathbb{M}\boxtimes_{\mathbb{Z}}H^{d+1}(X,\mathbb{Z}), \tag{G10}$$

where X' is renamed as X. The above is a form of the universal coefficient theorem which can be used to calculate $H^*(X,\mathbb{M})$ from $H^*(X,\mathbb{Z})$ and the module $\mathbb{M}$. The universal coefficient theorem works for topological cohomology where X is a topological space. The universal coefficient theorem also works for group cohomology when X is a finite group.

Using the universal coefficient theorem, we can rewrite Eq. (G4) as

$$H^d(X\times X',\mathbb{M})\simeq\oplus_{k=0}^{d}H^k[X,H^{d-k}(X',\mathbb{M})]. \tag{G11}$$

The above is valid for topological cohomology. It is also valid for group cohomology:

$$\mathcal{H}^d(G\times G',\mathbb{M})\simeq\oplus_{k=0}^{d}\mathcal{H}^k[G,\mathcal{H}^{d-k}(G',\mathbb{M})], \tag{G12}$$

provided that both G and G' are finite groups.

We may apply the above to the classifying spaces of group G and G'. Using $B(G\times G')=BG\times BG'$, we find

$$H^d[B(G\times G'),\mathbb{M}]\simeq\oplus_{k=0}^{d}H^k[BG,H^{d-k}(BG',\mathbb{M})].$$

Choosing $\mathbb{M}=\mathbb{R}/\mathbb{Z}$ and using

$$\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})=\mathcal{H}^{d+1}(G,\mathbb{Z})=H^{d+1}(BG,\mathbb{Z}), \tag{G13}$$

we have

$$\begin{aligned} \mathcal{H}_B^d(G\times G',\mathbb{R}/\mathbb{Z})&=H^{d+1}[B(G\times G'),\mathbb{Z}] \\ &=\oplus_{k=0}^{d+1}H^k[BG,H^{d+1-k}(BG',\mathbb{Z})] \\ &=\mathcal{H}_B^d(G,\mathbb{R}/\mathbb{Z})\oplus\mathcal{H}_B^d(G',\mathbb{R}/\mathbb{Z}) \\ &\quad\oplus_{k=1}^{d-1}H^k[BG,\mathcal{H}_B^{d-k}(G',\mathbb{R}/\mathbb{Z})], \end{aligned} \tag{G14}$$

where we have used $H^0(BG',\mathbb{Z})=\mathbb{Z}$, $\mathcal{H}_B^1(G',\mathbb{Z})=H^1(BG',\mathbb{Z})$, and $\mathcal{H}_B^1(G',\mathbb{Z})=0$ if G' is compact (or finite). Equation (G14) is valid for any groups G and G'. If G also satisfies (for example when G is finite)

$$H^d(BG,\mathbb{Z})=\mathcal{H}_B^d(G,\mathbb{Z}),\quad H^d(BG,\mathbb{Z}_n)=\mathcal{H}_B^d(G,\mathbb{Z}_n), \tag{G15}$$

we can rewrite the above as

$$\mathcal{H}^d(G\times G',\mathbb{R}/\mathbb{Z})=\oplus_{k=0}^{d}\mathcal{H}^k[G,\mathcal{H}^{d-k}(G',\mathbb{R}/\mathbb{Z})]. \tag{G16}$$

Such a result is consistent with Eq. (H1) for arbitrary G,G'.

Choosing $X=BG$, $\mathbb{M}=\mathbb{Z}_n$, Eq. (G10) becomes

$$\mathcal{H}^d(G,\mathbb{Z}_n)\simeq\mathbb{Z}_n\otimes_{\mathbb{Z}}\mathcal{H}^d(G,\mathbb{Z})\oplus\mathbb{Z}_n\boxtimes_{\mathbb{Z}}\mathcal{H}^{d+1}(G,\mathbb{Z}), \tag{G17}$$

where we have used Eq. (G15). Using Eq. (G17), we find that

$$\begin{aligned} \mathcal{H}^d[G,\mathcal{H}^{d'}(G',\mathbb{R}/\mathbb{Z})]&\simeq\mathcal{H}^{d'}(G',\mathbb{R}/\mathbb{Z})\otimes_{\mathbb{Z}}\mathcal{H}^{d-1}(G,\mathbb{R}/\mathbb{Z}) \\ &\quad\oplus\mathcal{H}^{d'}(G',\mathbb{R}/\mathbb{Z})\boxtimes_{\mathbb{Z}}\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}). \end{aligned} \tag{G18}$$

APPENDIX H: LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCE

The Lyndon-Hochschild-Serre spectral sequence (see Ref. [101], pages 280 and 291, and Ref. [102]) allows us to understand the structure of $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ to a certain degree. (Here G is a group extension of SG by GG: $SG = G/GG$). We find that $\mathcal{H}^d(G,\mathbb{M})$, when viewed as an Abelian group, contains a chain of subgroups

$$\{0\} = H_{d+1} \subset H_d \subset \ldots \subset H_0 = \mathcal{H}^d(G,\mathbb{M}) \quad \text{(H1)}$$

such that H_k/H_{k+1} is a subgroup of a factor group of $\mathcal{H}^k[SG,\mathcal{H}^{d-k}(GG,\mathbb{M})_{SG}]$, i.e., $\mathcal{H}^k[SG,\mathcal{H}^{d-k}(GG,\mathbb{M})_{SG}]$ contains a subgroup Γ^k, such that

$$H_k/H_{k+1} \subset \mathcal{H}^k[SG,\mathcal{H}^{d-k}(GG,\mathbb{R}/\mathbb{Z})_{SG}]/\,\Gamma^k, \quad k = 0,\ldots,d. \quad \text{(H2)}$$

Note that G may have a nontrivial action on $\mathbb{M}$ and SG may have a nontrivial action on $\mathcal{H}^{d-k}(GG,\mathbb{M})$ as determined by the structure $1 \to GG \to GG \leftthreetimes SG \to SG \to 1$. We add the subscript SG to $\mathcal{H}^{d-k}(GG,\mathbb{R}/\mathbb{Z})$ to stress this point. We also have

$$H_0/H_1 \subset \mathcal{H}^0[SG,\mathcal{H}^d(GG,\mathbb{R}/\mathbb{Z})_{SG}],$$
$$H_d/H_{d+1} = H_d = \mathcal{H}^d(SG,\mathbb{R}/\mathbb{Z})/\,\Gamma^d. \quad \text{(H3)}$$

In other words, all the elements in $\mathcal{H}^d(GG \leftthreetimes SG,\mathbb{R}/\mathbb{Z})$ can be one-to-one labeled by $(x_0,x_1,\ldots,x_d)$ with

$$x_k \in H_k/H_{k+1} \subset \mathcal{H}^k[SG,\mathcal{H}^{d-k}(GG,\mathbb{R}/\mathbb{Z})_{SG}]/\,\Gamma^k. \quad \text{(H4)}$$

Note that here $\mathbb{M}$ can be $\mathbb{Z},\mathbb{Z}_n,\mathbb{R},\mathbb{R}/\mathbb{Z}$, etc. Let $x_{k,\alpha}$, $\alpha = 1,2,\ldots$, be the generators of H^k/H^{k+1}. Then we say $x_{k,\alpha}$ for all k,α are the generators of $\mathcal{H}^d(G,\mathbb{M})$. We also call H_k/H_{k+1}, $k = 0,\ldots,d$, the generating subfactor groups of $\mathcal{H}^d(G,\mathbb{M})$.

The above result implies that we can use $(m_0,m_1,\ldots,m_d)$ with

$$m_k \in \mathcal{H}^k[SG,\mathcal{H}^{d-k}(GG,\mathbb{R}/\mathbb{Z})_{SG}] \quad \text{(H5)}$$

to label all the elements in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. However, such a labeling scheme may not be one-to-one, and it may happen that only some of $(m_0,m_1,\ldots,m_d)$ correspond to the elements in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. But, on the other hand, for every element in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, we can find a $(m_0,m_1,\ldots,m_d)$ that corresponds to it.

APPENDIX I: GENERATORS OF $H^k(BG,\sigma\text{iTO}_L^3)$

The Abelian group $H^k(BG,\sigma\text{iTO}_L^3)$ is generated by $W^k_{\text{top}}(A,\Gamma)/2\pi = x\omega_3$ where x are the generators of $H^k(BG,\mathbb{Z})$. Since $\sigma\text{iTO}_L^3 = \mathcal{H}^3(SO,\mathbb{R}/\mathbb{Z}) = H^4(BSO,\mathbb{Z})$ and since $H^4(BSO,\mathbb{Z})$ is generated by the first Pontryagin class p_1, we may also say that $H^k(BG,\sigma\text{iTO}_L^3)$ is generated by xp_1 in $H^k[BG,H^4(BSO,\mathbb{Z})]$. We also know that $H^k(BG,\mathbb{Z}) \simeq \mathcal{H}^{k-1}(G,\mathbb{R}/\mathbb{Z})$, thus we can further say that $H^k(BG,\sigma\text{iTO}_L^3)$ is generated by ap_1 in $H^{k+3}[B(G\times SO),\mathbb{R}/\mathbb{Z}]$ where a are the generators of $\mathcal{H}^{k-1}(G,\mathbb{R}/\mathbb{Z})$ and $\beta(ap_1) = xp_1$ under the natural map $\beta : H^{k+3}[B(G\times SO),\mathbb{R}/\mathbb{Z}] \to H^{k+4}[B(G\times SO),\mathbb{Z}]$.

For example, when $k = 2$ and $G = U(1)$, $H^2[BU(1), \sigma\text{iTO}_L^3]$ is generated by $W^5_{\text{top}}(A,\Gamma) = c_1\omega_3$. W can also say that it is generated by $W^5_{\text{top}}(A,\Gamma) = ap_1$ where $da = c_1$ and a generates $\mathcal{H}^1[U(1),\mathbb{R}/\mathbb{Z}]$. So we can write

$$c_1\omega_3 = \beta(a)\omega_3 = -a\,d\omega_3 = -ap_1. \quad \text{(I1)}$$

We see that the natural map $\beta : \mathcal{H}^k(G,\mathbb{R}/\mathbb{Z}) \to \mathcal{H}^{k+1}(BG,\mathbb{Z})$ behaves like a derivative d. Similarly, we can do

$$\beta(a_1)\omega_3 = -\tfrac{1}{2}a_1\beta(\omega_3) = -\tfrac{1}{2}a_1 p_1. \quad \text{(I2)}$$

Note that when acting on the cocycles with $\mathbb{Z}_2$ coefficient, $\beta = Sq^1$.

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