Sudakov safety in perturbative QCD

Citation: Larkoski, Andrew J., Simone Marzani, and Jesse Thaler. "Sudakov Safety in Perturbative QCD." Phys. Rev. D 91, 111501 (June 2015). © 2015 American Physical Society

As Published: http://dx.doi.org/10.1103/PhysRevD.91.111501

Publisher: American Physical Society

Persistent URL: http://hdl.handle.net/1721.1/97389

Version: Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

PHYSICAL REVIEW D 91, 111501(R) (2015)

Sudakov safety in perturbative QCD

Andrew J. Larkoski,* Simone Marzani,† and Jesse Thaler‡
Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 12 February 2015; published 11 June 2015)

Traditional calculations in perturbative quantum chromodynamics (pQCD) are based on an order-by-order expansion in the strong coupling α_s. Observables that are calculable in this way are known as "safe." Recently, a class of unsafe observables was discovered that do not have a valid α_s expansion but are nevertheless calculable in pQCD using all-orders resummation. These observables are called "Sudakov safe" since singularities at each α_s order are regulated by an all-orders Sudakov form factor. In this paper, we give a concrete definition of Sudakov safety based on conditional probability distributions, and we study a one-parameter family of momentum sharing observables that interpolate between the safe and unsafe regimes. The boundary between these regimes is particularly interesting, as the resulting distribution can be understood as the ultraviolet fixed point of a generalized fragmentation function, yielding a leading behavior that is independent of α_s.

DOI: 10.1103/PhysRevD.91.111501 PACS numbers: 13.87.-a, 11.15.Bt, 12.38.-t

Infrared and collinear (IRC) safety has long been a guiding principle for determining which observables are calculable using perturbative quantum chromodynamics (pQCD) [1,2]. IRC-safe observables are insensitive to arbitrarily soft gluon emissions and arbitrarily collinear parton splittings. This property ensures that perturbative singularities cancel between real and virtual emissions, leading to finite cross sections order-by-order in the strong coupling α_s. At the Large Hadron Collider (LHC), IRC-safe jet algorithms like anti-k_T [3] play a key role in almost every analysis, and many jet-related cross sections have been calculated to next-to-leading and even next-to-next-to-leading order [4–7]. Of course, there are observables relevant for collider physics that are not IRC safe, though one can often use nonperturbative objects—like parton distribution functions, fragmentation functions (FFs), and their generalizations [8–11]—to absorb singularities and restore calculational control.

In this paper, we show how to extend the calculational power of pQCD into the IRC unsafe regime using purely perturbative techniques. We study a class of unsafe observables that are not defined at any fixed order in α_s, yet nevertheless have finite cross sections when all-orders effects are included. These observables are known in the literature as "Sudakov safe" [12], since a perturbative Sudakov form factor [13] naturally (and exponentially) regulates real and virtual infrared (IR) divergences. To date, however, the study of Sudakov-safe observables has been limited to specific examples. Here, we achieve a deeper understanding of these observables by providing a concrete definition of Sudakov safety based on conditional probabilities. The techniques in this paper apply to any perturbative quantum field theory, but we focus on pQCD to highlight an example of direct relevance to jet physics at the LHC.

Because Sudakov-safe observables are not defined at any fixed perturbative order, they in general have nonanalytic dependence on α_s. Examples in the literature include observables with an apparent expansion in $\sqrt{\alpha_s}$ [12] and observables which are independent of α_s at sufficiently high energies [14,15]. As a case study, we consider a one-parameter family of momentum sharing observables z_g based on "soft drop declustering" [14], which already appears in many jet substructure studies, e.g. [16–18]. This family not only interpolates between the above two Sudakov-safe behaviors but also includes an IRC-safe regime. We explain how the boundary between the safe and unsafe regimes can be understood using the more familiar language of (generalized) FFs; the renormalization group (RG) evolution of the FF has an ultraviolet (UV) fixed point, suggesting an extended definition of IRC safety.

To begin our general discussion of Sudakov safety, consider an IRC unsafe observable u and a companion IRC-safe observable s. The observable s is chosen such that its measured value regulates all singularities of u. That is, even though the probability of measuring u,

$$p(u) = \frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}u}, \tag{1}$$

is ill defined at any fixed perturbative order, the probability of measuring u given s, $p(u|s)$, is finite at all perturbative orders, except possibly at isolated values of s; e.g., $s = 0$. Given this companion observable s, we want to know whether $p(u)$ can be calculated from pQCD.

Because s is IRC safe, $p(s)$ is well defined at all perturbative orders (although resummation may be required

*larkoski@mit.edu
†smarzani@mit.edu
‡jthaler@mit.edu

to regulate isolated singularities, see below). This allows us to define the joint probability distribution

$$p(s,u) = p(s)p(u|s), \tag{2}$$

which is also finite at all perturbative orders, except possibly at isolated values of s. To calculate $p(u)$, we can simply marginalize over s:

$$p(u) = \int \mathrm{d}s\, p(s)p(u|s). \tag{3}$$

If $p(s)$ regulates all (isolated) singularities of $p(u|s)$, thus ensuring that the above integral is finite, then we deem u to be Sudakov safe. In the case that one IRC-safe observable is insufficient to regulate all singularities in u, we can measure a vector of IRC-safe observables $\mathbf{s} = \{s_1, \ldots, s_n\}$. This gives a more general definition of Sudakov safety:

$$p(u) = \int \mathrm{d}^n\mathbf{s}\, p(\mathbf{s})p(u|\mathbf{s}). \tag{4}$$

All previous examples of Sudakov safety fall in the category of (3) above where only a single IRC-safe companion s was required. In [14], the energy loss distribution from soft drop grooming was defined precisely as in (3), where u was the factional energy loss Δ_E and s was the groomed jet radius r_g (see below). In [12], ratio observables $r = a/b$ were originally defined in terms of a double-differential cross section [19,20] as

$$p(r) = \int \mathrm{d}a\mathrm{d}b\, p(a,b)\delta\left(r - \frac{a}{b}\right), \tag{5}$$

where a and b are IRC safe but r is not, because there are singularities at $b = 0$ at every finite perturbative order, leading to a divide-by-zero issue for r. Integrating over a and using the definition of conditional probability (2), we can write (5) as

$$p(r) = \int \mathrm{d}b\, p(b)p(r|b), \tag{6}$$

and r is Sudakov safe because $p(b)$ has an all-orders Sudakov form factor that renders $p(r)$ finite.

It should be stressed that the definition of Sudakov safety in (4) is not vacuous and it does not save all IRC unsafe observables. As a counterexample, consider particle multiplicity; because perturbation theory allows an arbitrary number of soft or collinear emissions, one would need to measure an infinite number of IRC-safe observables to regulate all singularities to all orders. Also, it should be stressed that just because an observable is Sudakov safe, that does not imply that nonperturbative aspects of QCD are automatically suppressed. While a detailed discussion is beyond the scope of this paper, both [12] and [14] include an estimate of nonperturbative effects, which are analogous to power corrections and underlying event corrections familiar from the IRC-safe case. In some cases, these corrections are known to scale away as a (fractional) inverse power of the collision energy.

Crucially, one needs some kind of all-orders information to obtain finite distributions for $p(u)$. If a fixed-order expansion of $p(s)$ and $p(u|s)$ were sufficient, then $p(u)$ would have a series expansion in α_s, contradicting the assumption that u is IRC unsafe. In this paper, we use logarithmic resummation to capture all-orders information about $p(s)$, which regulates isolated singularities at $s = 0$ to ensure the integral in (3) is finite. In all cases we have encountered, a finite $p(u|s)$ with a resummed $p(s)$ is sufficient to calculate $p(u)$, though this may not be the case generally.

Unlike IRC-safe distributions which have a unique α_s expansion, the formal perturbative accuracy of a Sudakov-safe distribution is potentially ambiguous. First, there are different choices for s that can regulate the singularities in u. This is analogous to the choice of evolution variables in a parton shower, as each choice gives a finite (albeit different) answer at a given perturbative accuracy. Second, the probability distributions $p(s)$ and $p(u|s)$ can be calculated to different formal accuracies. Below we use leading logarithmic resummation for $p(s)$, but only work to lowest order in α_s for $p(u|s)$. Thus, when discussing the accuracy of $p(u)$, one must specify the choice of s and the accuracy of $p(s)$ and $p(u|s)$ separately. We stress, however, that the accuracy of both objects is systematically improvable.

We now study an instructive example that demonstrates the complementarity of Sudakov safety and IRC safety. This example is based on soft drop declustering [14], which we briefly review. Consider a jet clustered with the Cambridge-Aachen (C/A) algorithm [21,22] with jet radius R_0. One can decluster through the jet's branching history, grooming away the softer branch until one finds a branch that satisfies the condition

$$\frac{\min(p_{T1}, p_{T2})}{p_{T1} + p_{T2}} > z_{\mathrm{cut}}\left(\frac{R_{12}}{R_0}\right)^{\beta}, \tag{7}$$

where 1 and 2 denote the branches at that step in the clustering, p_{Ti} are the corresponding transverse momenta, and R_{12} is their rapidity-azimuth separation. The kinematics of this branch defines the groomed jet radius r_g and the groomed momentum sharing z_g,

$$r_g = \frac{R_{12}}{R_0}, \qquad z_g = \frac{\min(p_{T1}, p_{T2})}{p_{T1} + p_{T2}}; \tag{8}$$

r_g is IRC safe and its distribution was studied in [14].

Our observable of interest is z_g, and the angular exponent β determines whether or not z_g is IRC safe. For $\beta < 0$, z_g is

IRC safe, because $z_g > z_{\rm cut}$ for any branch that passes (7); if this condition is never satisfied, the jet is simply removed from the analysis. For $\beta > 0$, z_g is IRC unsafe, since measuring z_g does not regulate collinear singularities. The boundary case $\beta = 0$ corresponds to the (modified) mass drop tagger [16–18] which also has collinear divergences, but we will show that it actually satisfies an extended version of IRC safety.

In our calculations, we work to lowest nontrivial order to illustrate the physics, though we provide supplemental materials [23] for the interested reader that include higher-order (and nonperturbative) effects. We take the parameter $z_{\rm cut}$ to be small, but large enough that $\log z_{\rm cut}$ terms need not be resummed, with a benchmark of $z_{\rm cut} \simeq 0.1$.

We now use the strategy in (3) to calculate the momentum sharing z_g for all values of β, using the groomed radius r_g to regulate collinear singularities:

$$p(z_g) = \frac{1}{\sigma}\frac{d\sigma}{dz_g} = \int dr_g\, p(r_g)\, p(z_g|r_g). \tag{9}$$

We use all-orders resummation to determine $p(r_g)$ and regulate the isolated $r_g = 0$ singularity. This has been carried out to next-to-leading-logarithmic accuracy in [14]. Here, it is sufficient to consider the fixed-coupling limit:

$$p(r_g) = \frac{d}{dr_g}\exp\left[-\frac{2\alpha_s C_i}{\pi}\int_{r_g}^{1}\frac{d\theta}{\theta}\int_0^1 dz P_i(z)\Theta_{\rm cut}\right], \tag{10}$$

where C_i is the color factor of the jet, $P_i(z)$ is the appropriate splitting function (summed over final states), and the phase space cut is

$$\begin{aligned}\Theta_{\rm cut} = {}& \Theta(1/2 - z)\Theta(z - z_{\rm cut}\theta^\beta) \\ & + \Theta(z - 1/2)\Theta((1-z) - z_{\rm cut}\theta^\beta).\end{aligned} \tag{11}$$

The exponential part of (10) is the r_g Sudakov form factor, where $\Theta_{\rm cut}$ defines the no-emission criteria. To calculate $p(z_g|r_g)$, note that z_g is defined by a single emission in the jet. For small R_0, the lowest-order matrix element is well approximated by a $1 \to 2$ splitting function:

$$p(z_g|r_g) = \frac{\bar{P}_i(z_g)}{\int_{z_{\rm cut} r_g^\beta}^{1/2} dz \bar{P}_i(z)}\Theta(z_g - z_{\rm cut} r_g^\beta), \tag{12}$$

where $0 < z_g < 1/2$ and we have introduced the notation

$$\bar{P}_i(z) = P_i(z) + P_i(1-z). \tag{13}$$

In the double-logarithmic limit, we simply have $\bar{P}_i(z) = 1/z$, allowing an explicit evaluation of (9):

$$\begin{aligned}p(z_g) = {}& \sqrt{\frac{\alpha_s C_i}{\beta}}\exp\left[\frac{\alpha_s C_i}{\pi\beta}\log^2\frac{1}{2z_{\rm cut}}\right]\bar{P}_i(z_g) \\ & \times\left(\mathrm{erf}\left[\sqrt{\frac{\alpha_s C_i}{\pi\beta}}\log\frac{1}{a_1}\right] - \mathrm{erf}\left[\sqrt{\frac{\alpha_s C_i}{\pi\beta}}\log\frac{1}{a_2}\right]\right),\end{aligned} \tag{14}$$

where

$$\beta \geq 0: \quad a_1 = 0, \qquad a_2 = \min[2z_{\rm cut}, 2z_g], \tag{15}$$

$$\beta < 0: \quad a_1 = 2z_g, \qquad a_2 = 2z_{\rm cut}. \tag{16}$$

Because (14) is finite, we see that z_g is at least Sudakov safe for all β. Distributions of z_g calculated with (9) at fixed α_s are shown in Fig. 1.

By expanding $p(z_g)$ in small α_s, we can better understand the difference between IRC-safe and Sudakov-safe behavior. For $\beta < 0$, z_g is IRC safe, so z_g should have a well-defined expansion in α_s. To the accuracy calculated, (9) is fully valid to $\mathcal{O}(\alpha_s)$ in the collinear limit, and the expansion of (9) yields the expected IRC-safe result:

$$\beta<0: \; p(z_g) = \frac{2\alpha_s C_i}{\pi|\beta|}\bar{P}_i(z_g)\log\frac{z_g}{z_{\rm cut}}\Theta(z_g - z_{\rm cut}) + \mathcal{O}(\alpha_s^2). \tag{17}$$

For $\beta > 0$, z_g is only Sudakov safe and its distribution should not have a valid Taylor series in α_s. Indeed, for $\beta > 0$, the distribution has the expansion

$$\beta > 0: \quad p(z_g) = \sqrt{\frac{\alpha_s C_i}{\beta}}\bar{P}_i(z_g) + \mathcal{O}(\alpha_s), \tag{18}$$

and the presence of $\sqrt{\alpha_s}$ implies nonanalytic dependence on α_s. To $\mathcal{O}(\sqrt{\alpha_s})$, the only phase space constraint is $0 < z_g < 1/2$, and the kink visible in Fig. 1 at $z_g = z_{\rm cut}$ first

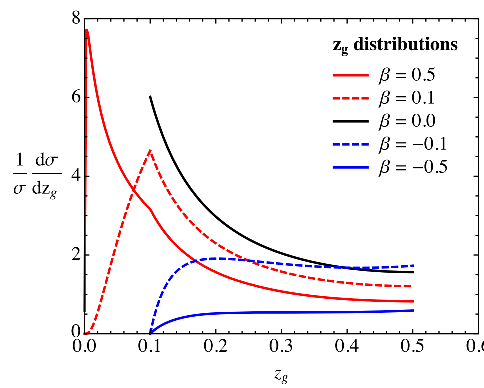

FIG. 1 (color online). Distributions of z_g for various β values, obtained from (9) at fixed $\alpha_s = 0.1$ and $z_{\rm cut} = 0.1$.

TABLE I. As β is adjusted, $p(z_g)$ interpolates between IRC-safe and two Sudakov-safe behaviors, related to the divergences in z_g. Here, $n \geq 1$ ranges over positive integers.

	Safety	Divergences	Expansion
$\beta < 0$	IRC	None	α_s^n
$\beta = 0$	IRC via FF	Collinear only	α_s^{n-1}
$\beta > 0$	Sudakov	Collinear and soft-collinear	$\alpha_s^{n/2}$

appears at $\mathcal{O}(\alpha_s)$. Finally, for the boundary case $\beta = 0$, $p(z_g|r_g)$ is independent of r_g (in the fixed-coupling approximation), and (14) is independent of α_s:

$$\beta = 0: \quad p(z_g) = \frac{\bar{P}_i(z_g)}{\int_{z_{\rm cut}}^{1/2} dz \bar{P}_i(z)} \Theta(z_g - z_{\rm cut}). \tag{19}$$

We will later show that the $\beta = 0$ case does have a valid perturbative expansion in α_s, despite being α_s-independent at lowest order. The behavior of z_g for different β values is summarized in Table I.

The $\beta = 0$ distribution of z_g is fascinating (and simpler than previous α_s-independent examples [14,15]). Because z_g only has collinear divergences, we can understand $p(z_g)$ in a different and illuminating way using FFs. As is well known, FFs absorb collinear divergences in final-state parton evolution, and we can introduce a generalized FF, $F(z_g)$, to play the same role for z_g. In the standard case, FFs are nonperturbative objects with perturbative RG evolution. In the z_g case, $F(z_g)$ is still a nonperturbative object, but it has a perturbative UV fixed point, becoming independent of IR boundary conditions at sufficiently high energies.

At Born level, the jet has a single parton, so z_g is undefined. We can, however, define $F(z_g)$ to be the one-prong z_g distribution, such that $F(z_g)$ acts like a nontrivial measurement function that is independent of the kinematics. Working to $\mathcal{O}(\alpha_s)$ in the collinear limit,

$$\begin{aligned} p(z_g) = F(z_g) &+ \frac{\alpha_s C_i}{\pi} \int_0^1 \frac{d\theta}{\theta} \\ &\times \left(\bar{P}_i(z_g)\Theta(z_g - z_{\rm cut}) - F(z_g) \int_{z_{\rm cut}}^{1/2} dz \bar{P}_i(z) \right) \\ &+ \mathcal{O}(\alpha_s^2). \end{aligned} \tag{20}$$

There are two terms at $\mathcal{O}(\alpha_s)$. The first term accounts for the resolved case where the jet is composed of two prongs from a $1 \to 2$ splitting. The second term corresponds to additional one-prong configurations [with the same $F(z_g)$ measurement function as the Born case], arising either because the other prong has been removed by soft drop grooming or from one-prong virtual corrections.

For a general $F(z_g)$, (20) is manifestly collinearly divergent because of the θ integral, and $F(z_g)$ must be renormalized. But there is a unique choice of $F(z_g)$ for which collinear divergences are absent (at this order), without requiring renormalization:

$$F_{\rm UV}(z_g) = \frac{\bar{P}_i(z_g)}{\int_{z_{\rm cut}}^{1/2} dz \bar{P}_i(z)} \Theta(z_g - z_{\rm cut}). \tag{21}$$

Plugging this into (20), the $\mathcal{O}(\alpha_s)$ term vanishes, and we recover precisely the distribution in (19).

In this way, z_g at $\beta = 0$ exhibits an extended version of IRC safety, where a nontrivial (and finite) measurement function is introduced in a region of phase space where the measurement would be otherwise undefined. Similar measurement functions appeared (without discussion) in the early days of jet physics [24,25], where symmetries determined their form. Here, we used the cancellation of collinear divergences order-by-order in α_s to find an appropriate $F(z_g)$. We can also extend (20) beyond the collinear limit by considering full real and virtual matrix elements, leading to finite $\mathcal{O}(\alpha_s)$ corrections to $p(z_g)$.

As alluded to above, $F_{\rm UV}(z_g)$ also has the interpretation of being a UV fixed point from RG evolution. The collinear divergence of (20) can be absorbed into a renormalized FF, $F^{(\rm ren)}(z_g;\mu)$, at the price of introducing explicit dependence on the $\overline{\rm MS}$ renormalization scale μ. Requiring (20) to be independent of μ through $\mathcal{O}(\alpha_s)$ results in the following RG equation for $F^{(\rm ren)}(z_g;\mu)$:

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} F^{(\rm ren)}(z_g;\mu) = \frac{\alpha_s C_i}{\pi} \times \Bigg(&\bar{P}_i(z_g)\Theta(z_g - z_{\rm cut}) \\ &- F^{(\rm ren)}(z_g;\mu) \int_{z_{\rm cut}}^{1/2} dz \bar{P}_i(z) \Bigg). \end{aligned} \tag{22}$$

As μ goes to $+\infty$, the IR boundary condition is suppressed and $F^{(\rm ren)}(z_g;\mu)$ asymptotes to $F_{\rm UV}(z_g)$.

This UV asymptotic behavior can be tested using parton shower Monte Carlo generators. In Fig. 2 we show the z_g distribution for $\beta = 0$ for HERWIG++ 2.6.3 [26] at the 13 TeV LHC, using FASTJET 3.1 [27] and the RECURSIVETOOLS contrib [28]. As shown in the supplement [23], other generators give similar results. As the jet p_T increases, $p(z_g)$ asymptotes to the form in (21) (which happens to be nearly identical for quark and gluon jets). This is due both to the RG flow in (22), which suppresses nonperturbative corrections, and the decrease of α_s with energy, which suppresses $\mathcal{O}(\alpha_s)$ corrections to $p(z_g)$.

In this paper, we gave a concrete definition of Sudakov safety, which extends the reach of pQCD beyond the traditional domain of IRC-safe observables. Even at lowest perturbative order, the z_g example highlights the different analytic structures possible in the Sudakov-safe regime,

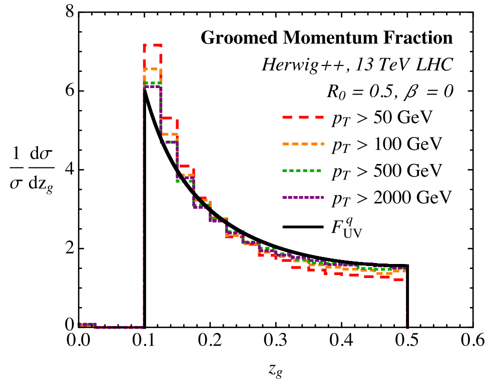

FIG. 2 (color online). Distributions of z_g for $\beta = 0$ and $z_{\rm cut} = 0.1$ at the 13 TeV LHC, as simulated by HERWIG++ 2.6.3. The p_T of the jets ranges from 50 GeV to 2 TeV, and the asymptotic distribution for quark jets, $F^q_{\rm UV}$ in (21), is solid black.

and the FF approach to the IRC-safe/unsafe boundary yields new insights into the structure of perturbative singularities. In addition to being an interesting conceptual result in perturbative field theory, (4) offers a concrete prescription for how to leverage the growing catalog of high-accuracy pQCD calculations (both fixed order and resummed) to make predictions in the IRC unsafe regime. This can be done without have to rely (solely) on nonperturbative modeling, enhancing the prospects for precision jet physics in the LHC era.

ACKNOWLEDGMENTS

We thank Gregory Soyez for collaborating in the early stages of this work and Bob Jaffe, Ian Moult, Duff Neill, Michael Peskin, Gavin Salam, George Sterman, and Iain Stewart for enlightening discussions. This work was supported by the U.S. Department of Energy (DOE) under Cooperative Research Agreement No. DE-FG02-05ER-41360. J. T. is also supported by the DOE Early Career Research Program No. DE-FG02-11ER-41741 and by a Sloan Research Fellowship from the Alfred P. Sloan Foundation. S. M. is supported by the U.S. National Science Foundation, under Grant No. PHY-0969510, the LHC Theory Initiative.

[1] G. F. Sterman and S. Weinberg, Phys. Rev. Lett. **39**, 1436 (1977).
[2] R. K. Ellis, W. J. Stirling, and B. Webber, *QCD and Collider Physics* (Cambridge University Press, Cambridge, 1996), Vol. 8, pp. 1–435.
[3] M. Cacciari, G. P. Salam, and G. Soyez, J. High Energy Phys. 04 (2008) 063.
[4] A. Gehrmann-De Ridder, T. Gehrmann, E. Glover, and J. Pires, Phys. Rev. Lett. **110**, 162003 (2013).
[5] J. Currie, A. Gehrmann-De Ridder, E. Glover, and J. Pires, J. High Energy Phys. 01 (2014) 110.
[6] R. Boughezal, F. Caola, K. Melnikov, F. Petriello, and M. Schulze, J. High Energy Phys. 06 (2013) 072.
[7] X. Chen, T. Gehrmann, E. Glover, and M. Jaquier, Phys. Lett. B **740**, 147 (2015).
[8] D. Krohn, M. D. Schwartz, T. Lin, and W. J. Waalewijn, Phys. Rev. Lett. **110**, 212001 (2013).
[9] W. J. Waalewijn, Phys. Rev. D **86**, 094030 (2012).
[10] H.-M. Chang, M. Procura, J. Thaler, and W. J. Waalewijn, Phys. Rev. Lett. **111**, 102002 (2013).
[11] H.-M. Chang, M. Procura, J. Thaler, and W. J. Waalewijn, Phys. Rev. D **88**, 034030 (2013).
[12] A. J. Larkoski and J. Thaler, J. High Energy Phys. 09 (2013) 137.
[13] V. Sudakov, Sov. Phys. JETP **3**, 65 (1956).
[14] A. J. Larkoski, S. Marzani, G. Soyez, and J. Thaler, J. High Energy Phys. 05 (2014) 146.
[15] A. J. Larkoski and J. Thaler, Phys. Rev. D **90**, 034010 (2014).
[16] J. M. Butterworth, A. R. Davison, M. Rubin, and G. P. Salam, Phys. Rev. Lett. **100**, 242001 (2008).
[17] M. Dasgupta, A. Fregoso, S. Marzani, and G. P. Salam, J. High Energy Phys. 09 (2013) 029.
[18] M. Dasgupta, A. Fregoso, S. Marzani, and A. Powling, Eur. Phys. J. C **73**, 2623 (2013).
[19] A. J. Larkoski, I. Moult, and D. Neill, J. High Energy Phys. 09 (2014) 046.
[20] M. Procura, W. J. Waalewijn, and L. Zeune, J. High Energy Phys. 02 (2015) 117.
[21] M. Wobisch and T. Wengler, arXiv:hep-ph/9907280.
[22] Y. L. Dokshitzer, G. Leder, S. Moretti, and B. Webber, J. High Energy Phys. 08 (1997) 001.
[23] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.91.111501 for additional calculations and analyses to highlight the behavior of Sudakov safe observables.
[24] S.-Y. Pi, R. Jaffe, and F. Low, Phys. Rev. Lett. **41**, 142 (1978).
[25] G. Kramer, G. Schierholz, and J. Willrodt, Phys. Lett. **79B**, 249 (1978).
[26] M. Bahr, S. Gieseke, M. Gigg, D. Grellscheid, K. Hamilton *et al.*, Eur. Phys. J. C **58**, 639 (2008).
[27] M. Cacciari, G. P. Salam, and G. Soyez, Eur. Phys. J. C **72**, 1896 (2012).
[28] http://fastjet.hepforge.org/contrib/.