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Infrared and collinear (IRC) safety has long been a guiding principle for determining which observables are calculable using perturbative quantum chromodynamics (pQCD) [1,2]. IRC-safe observables are insensitive to arbitrarily soft gluon emissions and arbitrarily collinear parton splittings. This property ensures that perturbative singularities cancel between real and virtual emissions, leading to finite cross sections order-by-order in the strong coupling \( \alpha_s \). At the Large Hadron Collider (LHC), IRC-safe jet algorithms like anti-\( k_T \) [3] play a key role in almost every analysis, and many jet-related cross sections have been calculated to next-to-leading and even next-to-next-to-leading order [4–7]. Of course, there are observables relevant for collider physics that are not IRC safe, though one can often use nonperturbative objects—like parton distribution functions, fragmentation functions (FFs), and their generalizations [8–11]—to absorb singularities and restore calculational control.

In this paper, we show how to extend the calculational power of pQCD into the IRC unsafe regime using purely perturbative techniques. We study a class of unsafe observables that are not defined at any fixed order in \( \alpha_s \), yet nevertheless have finite cross sections when all-orders effects are included. These observables are known in the literature as “Sudakov safe” [12], since a perturbative Sudakov form factor [13] naturally (and exponentially) regulates real and virtual infrared (IR) divergences. To date, however, the study of Sudakov-safe observables has been limited to specific examples. Here, we achieve a deeper understanding of these observables by providing a concrete definition of Sudakov safety based on conditional probabilities. The techniques in this paper apply to any perturbative quantum field theory, but we focus on pQCD to highlight an example of direct relevance to jet physics at the LHC.

Because Sudakov-safe observables are not defined at any fixed perturbative order, they in general have nonanalytic dependence on \( \alpha_s \). Examples in the literature include observables with an apparent expansion in \( \sqrt{\alpha_s} \) [12] and observables which are independent of \( \alpha_s \) at sufficiently high energies [14,15]. As a case study, we consider a one-parameter family of momentum sharing observables that interpolate between the safe and unsafe regimes. The boundary between these regimes is particularly interesting, as the resulting distribution can be understood as the ultraviolet fixed point of a generalized fragmentation function, yielding a leading behavior that is independent of \( \alpha_s \).

To begin our general discussion of Sudakov safety, consider an IRC unsafe observable \( u \) and a companion IRC-safe observable \( s \). The observable \( s \) is chosen such that its measured value regulates all singularities of \( u \). That is, even though the probability of measuring \( u \),

\[
p(u) = \frac{1}{\sigma} \frac{d\sigma}{du},
\]

is ill defined at any fixed perturbative order, the probability of measuring \( u \) given \( s \), \( p(u|s) \), is finite at all perturbative orders, except possibly at isolated values of \( s \); e.g., \( s = 0 \). Given this companion observable \( s \), we want to know whether \( p(u) \) can be calculated from pQCD.

Because \( s \) is IRC safe, \( p(s) \) is well defined at all perturbative orders (although resummation may be required

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*eight.png*
to regulate isolated singularities, see below). This allows us
to define the joint probability distribution
\[ p(s, u) = p(s)p(u|s), \]
which is also finite at all perturbative orders, except
possibly at isolated values of s. To calculate \( p(u) \), we
can simply marginalize over s:
\[ p(u) = \int ds p(s)p(u|s). \]
If \( p(s) \) regulates all (isolated) singularities of \( p(u|s) \), thus
ensuring that the above integral is finite, then we deem u to
be Sudakov safe. In the case that one IRC-safe observable
is insufficient to regulate all singularities in \( u \), we can
measure a vector of IRC-safe observables \( s = \{s_1, \ldots, s_n\} \).
This gives a more general definition of Sudakov safety:
\[ p(u) = \int d^n sp(s)p(u|s). \]

All previous examples of Sudakov safety fall in the
category of (3) above where only a single IRC-safe
companion \( s \) was required. In [14], the energy loss
distribution from soft drop grooming was defined precisely
as in (3), where \( u \) was the factional energy loss \( \Delta E \) and \( s \)
was the groomed jet radius \( r_g \) (see below). In [12], ratio
observables \( r = a/b \) were originally defined in terms of a
double-differential cross section [19,20] as
\[ p(r) = \int dadb p(a,b)\delta(r-a/b), \]
where \( a \) and \( b \) are IRC safe but \( r \) is not, because there are
singularities at \( b = 0 \) at every finite perturbative order,
leading to a divide-by-zero issue for \( r \). Integrating over \( a \)
and using the definition of conditional probability (2), we
can write (5) as
\[ p(r) = \int db p(b)p(r|b), \]
and \( r \) is Sudakov safe because \( p(b) \) has an all-orders
Sudakov form factor that renders \( p(r) \) finite.

It should be stressed that the definition of Sudakov safety
in (4) is not vacuous and it does not save all IRC unsafe
observables. As a counterexample, consider particle multi-

ciplicity; because perturbation theory allows an arbitrary
number of soft or collinear emissions, one would need to
measure an infinite number of IRC-safe observables to
regulate all singularities to all orders. Also, it should be
stressed that just because an observable is Sudakov safe,
that does not imply that nonperturbative aspects of QCD
are automatically suppressed. While a detailed discussion is
beyond the scope of this paper, both [12] and [14] include
an estimate of nonperturbative effects, which are analogous
to power corrections and underlying event corrections
familiar from the IRC-safe case. In some cases, these

corrections are known to scale away as a (fractional)
inverse power of the collision energy.

Crucially, one needs some kind of all-orders information
to obtain finite distributions for \( p(u) \). If a fixed-order
expansion of \( p(s) \) and \( p(u|s) \) were sufficient, then \( p(u) \)
would have a series expansion in \( \alpha_s \), contradicting the
assumption that \( u \) is IRC unsafe. In this paper, we use
logarithmic resummation to capture all-orders information
about \( p(s) \), which regulates isolated singularities at \( s = 0 \)
to ensure the integral in (3) is finite. In all cases we have
encountered, a finite \( p(u|s) \) with a resummed \( p(s) \) is
sufficient to calculate \( p(u) \), though this may not be the case
generally.

Unlike IRC-safe distributions which have a unique \( \alpha_s \)
expansion, the formal perturbative accuracy of a Sudakov-
safe distribution is potentially ambiguous. First, there are
different choices for \( s \) that can regulate the singularities
in \( u \). This is analogous to the choice of evolution variables
in a parton shower, as each choice gives a finite (albeit
different) answer at a given perturbative accuracy. Second,
the probability distributions \( p(s) \) and \( p(u|s) \) can be
calculated to different formal accuracies. Below we use
leading logarithmic resummation for \( p(s) \), but only work
to lowest order in \( \alpha_s \) for \( p(u|s) \). Thus, when discussing
the accuracy of \( p(u) \), one must specify the choice of \( s \)
and the accuracy of \( p(s) \) and \( p(u|s) \) separately. We stress,
however, that the accuracy of both objects is systematically
improvable.

We now study an instructive example that demonstrates
the complementarity of Sudakov safety and IRC safety.
This example is based on soft drop declustering [14], which
we briefly review. Consider a jet clustered with the
Cambridge-Aachen (C/A) algorithm [21,22] with jet radius
\( R_0 \). One can decluster through the jet’s branching history,
grooming away the softer branch until one finds a branch
that satisfies the condition
\[ \frac{\min(p_{T1}, p_{T2})}{p_{T1} + p_{T2}} > z_{cut} \left( \frac{R_{12}}{R_0} \right)^\beta, \]
where 1 and 2 denote the branches at that step in the
clustering, \( p_{Ti} \) are the corresponding transverse momenta,
and \( R_{12} \) is their rapidity-azimuth separation. The kinematics
of this branch defines the groomed jet radius \( r_g \) and
the groomed momentum sharing \( z_g \),
\[ r_g = \frac{R_{12}}{R_0}, \quad z_g = \frac{\min(p_{T1}, p_{T2})}{p_{T1} + p_{T2}}; \]
\( r_g \) is IRC safe and its distribution was studied in [14].

Our observable of interest is \( z_g \), and the angular exponent
\( \beta \) determines whether or not \( z_g \) is IRC safe. For \( \beta < 0 \), \( z_g \) is
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IRC safe, because \( z_g > z_{\text{cut}} \) for any branch that passes (7); if this condition is never satisfied, the jet is simply removed from the analysis. For \( \beta > 0 \), \( z_g \) is IRC unsafe, since measuring \( z_g \) does not regulate collinear singularities. The boundary case \( \beta = 0 \) corresponds to the (modified) mass drop tagger [16–18] which also has collinear divergences, but we will show that it actually satisfies an extended version of IRC safety.

In our calculations, we work to lowest nontrivial order to illustrate the physics, though we provide supplemental materials [23] for the interested reader that include higher-order (and nonperturbative) effects. We take the parameter \( z_{\text{cut}} \) to be small, but large enough that \( \log z_{\text{cut}} \) terms need not be resummed, with a benchmark of \( z_{\text{cut}} \approx 0.1 \).

We now use the strategy in (3) to calculate the momentum sharing \( z_g \) for all values of \( \beta \), using the groomed radius \( r_g \) to regulate collinear singularities:

\[
p(z_g) = \frac{1}{\sigma} \frac{d\sigma}{dz_g} = \int dr_g p(r_g) p(z_g|r_g). \tag{9}
\]

We use all-orders resummation to determine \( p(r_g) \) and regulate the isolated \( r_g = 0 \) singularity. This has been carried out to next-to-leading-logarithmic accuracy in [14]. Here, it is sufficient to consider the fixed-coupling limit:

\[
p(r_g) = \frac{d}{dr_g} \exp \left[ -\frac{2\alpha_s C_i}{\pi} \int_{r_g}^{1} \frac{d\theta}{\theta} \int_{0}^{1} dz P_i(z) \Theta_{\text{cut}} \right], \tag{10}
\]

where \( C_i \) is the color factor of the jet, \( P_i(z) \) is the appropriate splitting function (summed over final states), and the phase space cut is

\[
\Theta_{\text{cut}} = \Theta(1/2 - z) \Theta(z - z_{\text{cut}} \theta^\beta) + \Theta(z - 1/2) \Theta((1 - z) - z_{\text{cut}} \theta^\beta). \tag{11}
\]

The exponential part of (10) is the \( r_g \)-Sudakov form factor, where \( \Theta_{\text{cut}} \) defines the no-emission criteria. To calculate \( p(z_g|r_g) \), note that \( z_g \) is defined by a single emission in the jet. For small \( R_0 \), the lowest-order matrix element is well approximated by a \( 1 \rightarrow 2 \) splitting function:

\[
p(z_g|r_g) = \frac{\bar{P}_i(z_g)}{\int_{z_{\text{cut}} r_g}^{1/2} dz \bar{P}_i(z)} \Theta(z_g - z_{\text{cut}} r_g^\beta), \tag{12}
\]

where \( 0 < z_g < 1/2 \) and we have introduced the notation

\[
\bar{P}_i(z) = P_i(z) + P_i(1 - z). \tag{13}
\]

In the double-logarithmic limit, we simply have \( \bar{P}_i(z) = 1/z \), allowing an explicit evaluation of (9):

\[
p(z_g) = \sqrt{\frac{\alpha_s C_i}{\beta}} \exp \left[ \frac{\alpha_s C_i}{\pi \beta} \log^2 \left( \frac{1}{2 z_{\text{cut}}} \right) \right] \bar{P}_i(z_g)
\times \left( \text{erf} \left[ \sqrt{\frac{\alpha_s C_i}{\pi \beta} \log \left( \frac{1}{a_1} \right)} \right] - \text{erf} \left[ \sqrt{\frac{\alpha_s C_i}{\pi \beta} \log \left( \frac{1}{a_2} \right)} \right] \right), \tag{14}
\]

where

\[
\beta > 0: \quad a_1 = 0, \quad a_2 = \min \{2z_{\text{cut}}, 2z_g\}, \tag{15}
\]

\[
\beta < 0: \quad a_1 = 2z_g, \quad a_2 = 2z_{\text{cut}}. \tag{16}
\]

Because (14) is finite, we see that \( z_g \) is at least Sudakov safe for all \( \beta \). Distributions of \( z_g \) calculated with (9) at fixed \( \alpha_s \) are shown in Fig. 1.

By expanding \( p(z_g) \) in small \( \alpha_s \), we can better understand the difference between IRC-safe and Sudakov-safe behavior. For \( \beta < 0 \), \( z_g \) is IRC safe, so \( z_g \) should have a well-defined expansion in \( \alpha_s \). To the accuracy calculated, (9) is fully valid to \( O(\alpha_s^0) \) in the collinear limit, and the expansion of (9) yields the expected IRC-safe result:

\[
\beta < 0: \quad p(z_g) = \frac{2\alpha_s C_i}{\pi |\beta|} \bar{P}_i(z_g) \log \frac{z_g}{z_{\text{cut}}} - \Theta(z_g - z_{\text{cut}}) + O(\alpha_s^0). \tag{17}
\]

For \( \beta > 0 \), \( z_g \) is only Sudakov safe and its distribution should not have a valid Taylor series in \( \alpha_s \). Indeed, for \( \beta > 0 \), the distribution has the expansion

\[
\beta > 0: \quad p(z_g) = \sqrt{\frac{\alpha_s C_i}{\beta}} \bar{P}_i(z_g) + O(\alpha_s^0), \tag{18}
\]

and the presence of \( \sqrt{\alpha_s} \) implies nonanalytic dependence on \( \alpha_s \). To \( O(\sqrt{\alpha_s}) \), the only phase space constraint is \( 0 < z_g < 1/2 \), and the kink visible in Fig. 1 at \( z_g = z_{\text{cut}} \) first appears at

\[
\begin{array}{c}
\text{FIG. 1 (color online). Distributions of } z_g \text{ for various } \beta \text{ values, obtained from (9) at fixed } \alpha_s = 0.1 \text{ and } z_{\text{cut}} = 0.1.
\end{array}
\]
As $\beta$ is adjusted, $p(z_g)$ interpolates between IRC-safe and two Sudakov-safe behaviors, related to the divergences in $z_g$. Here, $n \geq 1$ ranges over positive integers.

<table>
<thead>
<tr>
<th>Safety</th>
<th>Divergences</th>
<th>Expansion</th>
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<tbody>
<tr>
<td>$\beta &lt; 0$</td>
<td>IRC</td>
<td>$\alpha_s^0$</td>
</tr>
<tr>
<td>$\beta = 0$</td>
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appears at $\mathcal{O}(\alpha_s)$. Finally, for the boundary case $\beta = 0$, $p(z_g)$ is independent of $r_\beta$ (in the fixed-coupling approximation), and (14) is independent of $\alpha_s$:

$$\beta = 0: \quad p(z_g) = \frac{P_1(z_g)}{\int_{z_{cut}}^{1/2} dz \bar{P}_1(z)} \Theta(z_g - z_{cut}). \quad (19)$$

We will later show that the $\beta = 0$ case does have a valid perturbative expansion in $\alpha_s$, despite being $\alpha_s$-independent at lowest order. The behavior of $z_g$ for different $\beta$ values is summarized in Table I.

The $\beta = 0$ distribution of $z_g$ is fascinating (and simpler than previous $\alpha_s$-independent examples [14,15]). Because $z_g$ only has collinear divergences, we can understand $p(z_g)$ in a different and illuminating way using FFs. As is well known, FFs absorb collinear divergences in final-state parton evolution, and we can introduce a generalized FF, $F(z_g)$, to play the same role for $z_g$. In the standard case, FFs are nonperturbative objects with perturbative RG evolution. In the $z_g$ case, $F(z_g)$ is still a nonperturbative object, but it has a perturbative UV fixed point, becoming independent of IR boundary conditions at sufficiently high energies.

At Born level, the jet has a single parton, so $z_g$ is undefined. We can, however, define $F(z_g)$ to be the one-prong $z_g$ distribution, such that $F(z_g)$ acts like a nontrivial measurement function that is independent of the kinematics. Working to $\mathcal{O}(\alpha_s)$ in the collinear limit,

$$p(z_g) = F(z_g) + \frac{\alpha_s C_i}{\pi} \int_0^1 d\theta \left( \frac{\bar{P}_1(z_g)}{\bar{P}_1(z)} \Theta(z_g - z_{cut}) - F(z_g) \int_{z_{cut}}^{1/2} dz \bar{P}_1(z) \right) + \mathcal{O}(\alpha_s^2). \quad (20)$$

There are two terms at $\mathcal{O}(\alpha_s)$. The first term accounts for the resolved case where the jet is composed of two prongs from a $1 \rightarrow 2$ splitting. The second term corresponds to additional one-prong configurations [with the same $F(z_g)$ measurement function as the Born case], arising either because the other prong has been removed by soft drop grooming or from one-prong virtual corrections.

For a general $F(z_g)$, (20) is manifestly collinearly divergent because of the $\theta$ integral, and $F(z_g)$ must be renormalized. But there is a unique choice of $F(z_g)$ for which collinear divergences are absent (at this order), without requiring renormalization:

$$F_{UV}(z_g) = \frac{\bar{P}_1(z_g)}{\int_{z_{cut}}^{1/2} dz \bar{P}_1(z)} \Theta(z_g - z_{cut}). \quad (21)$$

Plugging this into (20), the $\mathcal{O}(\alpha_s)$ term vanishes, and we recover precisely the distribution in (19).

In this way, $z_g$ at $\beta = 0$ exhibits an extended version of IRC safety, where a nontrivial (and finite) measurement function is introduced in a region of phase space where the measurement would be otherwise undefined. Similar measurement functions appeared (without discussion) in the early days of jet physics [24,25], where symmetries determined their form. Here, we used the cancellation of collinear divergences order-by-order in $\alpha_s$ to find an appropriate $F(z_g)$. We can also extend (20) beyond the collinear limit by considering full real and virtual matrix elements, leading to finite $\mathcal{O}(\alpha_s)$ corrections to $p(z_g)$.

As alluded to above, $F_{UV}(z_g)$ also has the interpretation of being a UV fixed point from RG evolution. The collinear divergence of (20) can be absorbed into a renormalized FF, $F^{(ren)}(z_g; \mu)$, at the price of introducing explicit dependence on the $\overline{\text{MS}}$ renormalization scale $\mu$. Requiring (20) to be independent of $\mu$ through $\mathcal{O}(\alpha_s)$ results in the following RG equation for $F^{(ren)}(z_g; \mu)$:

$$\mu \frac{\partial}{\partial \mu} F^{(ren)}(z_g; \mu) = \frac{\alpha_s C_i}{\pi} \times \left( \frac{\bar{P}_1(z_g)}{\bar{P}_1(z)} \Theta(z_g - z_{cut}) \right. \left. - F^{(ren)}(z_g; \mu) \int_{z_{cut}}^{1/2} dz \bar{P}_1(z) \right). \quad (22)$$

As $\mu$ goes to $+\infty$, the IR boundary condition is suppressed and $F^{(ren)}(z_g; \mu)$ asymptotes to $F_{UV}(z_g)$.

This UV asymptotic behavior can be tested using parton shower Monte Carlo generators. In Fig. 2 we show the $z_g$ distribution for $\beta = 0$ for HERWIG++ 2.6.3 [26] at the 13 TeV LHC, using FASTJET 3.1 [27] and the RECURSIVE TOOLS contrib [28]. As shown in the supplement [23], other generators give similar results. As the jet $p_T$ increases, $p(z_g)$ asymptotes to the form in (21) (which happens to be nearly identical for quark and gluon jets). This is due both to the RG flow in (22), which suppresses nonperturbative corrections, and the decrease of $\alpha_s$ with energy, which suppresses $\mathcal{O}(\alpha_s)$ corrections to $p(z_g)$.

In this paper, we gave a concrete definition of Sudakov safety, which extends the reach of pQCD beyond the traditional domain of IRC-safe observables. Even at lowest perturbative order, the $z_g$ example highlights the different analytic structures possible in the Sudakov-safe regime.
and the FF approach to the IRC-safe/unsafe boundary yields new insights into the structure of perturbative singularities. In addition to being an interesting conceptual result in perturbative field theory, (4) offers a concrete prescription for how to leverage the growing catalog of high-accuracy pQCD calculations (both fixed order and resummed) to make predictions in the IRC unsafe regime. This can be done without have to rely (solely) on non-perturbative modeling, enhancing the prospects for precision jet physics in the LHC era.

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