We present a new geometric way of calculating the asymptotic behavior of unbounded root loci of a strictly proper, linear, time invariant control system as loop gain \( \lim s \to \infty \). The asymptotic behavior of unbounded root loci has been studied extensively by Kouvaritakis, Shaked, Edmunds, Owens and others [1,2,3,4]. Our approach, we feel, leads to more explicit formulae and our methods have applications in other asymptotic calculations as well for example, we have applied them to the hierarchical aggregation of Markov chains with weak coupling [5]. The details of our approach are presented in [6] - here we only state the main results.

1. Restrictions of a Linear Map

Given a linear map \( A \) from \( \mathbb{C}^n \) to \( \mathbb{C}^n \) and two subspaces \( S_1, S_2 \) of complementary dimension the linear map \( A(mod S_2) \) from \( S_1 \) to \( \mathbb{C}|S_2 \) is defined by

\[
A(mod S_2) = P \cdot A \cdot i \quad \text{and} \quad \mathbb{C}|S_2
\]

(Here \( i \) stands for the inclusion map and \( P \) for the canonical projection).

\( \lambda \) is said to be an eigenvalue of \( A(mod S_2) \) if \( \exists \) non-zero \( x \in S_1 \) such that \( (A-\lambda I)x \in S_2 \). This terminology is motivated by the fact that when \( S_1 \cap S_2 = \mathbb{C}^n \) there is a natural isomorphism \( I \) between \( S_1 \) and \( \mathbb{C}^n/S_2 \) as follows:

\[
\begin{align*}
S_1 &\xrightarrow{i} \mathbb{C}^n \xrightarrow{A} \mathbb{C}^n \xrightarrow{P} \mathbb{C}|S_2 \\
I &= P \cdot A \cdot i
\end{align*}
\]

\( A(mod S_2) \) is said to have simple null structure if \( \exists \) \( x \in S_1 \) such that

\[
A(mod S_2)x \neq 0 \quad \text{and} \quad A(mod S_2)^{-1}A(mod S_2)x = 0
\]

(\( i.e., there are no generalized eigenvectors associated with the eigenvalue \( \lambda=0 \)). It is clear that if \( A(mod S_2) \) has simple null structure, the number

\[
\text{nullity of } A(mod S_2) \quad \text{is equal to its rank, or the dimension of its range.}
\]

\( A(mod S_2) \) is said to have simple structure associated with an eigenvalue \( \lambda \) if \( A-\lambda I(mod S_2) \) has simple null structure.

2. System Description, Assumptions and Main Formulae

The system under study is the system of Figure 1, where \( G(s) \) is the \( m \times m \) transfer function matrix of a linear, time-invariant, strictly proper control system assumed to have Taylor expansion about \( s=\infty \):

\[
G(s) = \frac{G_1}{s} + \frac{G_2}{s^2} + \frac{G_3}{s^3} + \ldots
\]

The one-parameter curves (parameterized by \( k \)) traced on an appropriately defined Riemann surface, by the closed loop eigenvalues are referred to as multivariable root loci. An unbounded multivariable root locus \( \lambda (k) \) is said to be an \( n \)-th order unbounded root locus if asymptotically

\[
\lambda_n (k) = \mu_n (k) \frac{1}{n} + O(k^0)
\]

where \( \mu_n \neq 0 \) and \( O(k^0) \) is a term of order 1.

We identify an \( n \)-th order unbounded root locus with \( \mu_n \), the coefficient of its asymptotic value.

**Theorem** (Generalized eigenvalue problem for \( n \)-th order unbounded root locus).

\[
\mu_n = (-\lambda)^{1/n} \quad \text{and} \quad \mathbb{C} \text{ is the coefficient of the asymptotic value of an } n \text{-th order unbounded root locus iff } \lambda \text{ is a non-zero solution of}
\]

\[
\begin{bmatrix}
G_n - \lambda I & G_{n-1} & \cdots & G_1 & e_1 \\
G_{n-1} & G_{n-2} & \cdots & 0 & e_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
G_1 & 0 & \cdots & 0 & e_n
\end{bmatrix} = 0
\]

with \( e_1 \neq 0, e_2, \ldots, e_n \in \mathbb{C}^m \).

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Assumption 1 (non-degeneracy)

\( G(s) \) has normal (or generic) rank \( m \).

Assumption 2 (Simple Null Structure)

Assume

\[
\hat{G}_2 = G_2 \mod R(G_1) \bigg|_{N(G_1)}
\]

\[
\hat{G}_3 = G_3 - G_2 \hat{G}_2 \mod R(G_1) \mod R(\hat{G}_2) \bigg|_{N(\hat{G}_2)}
\]

to have simple null structure.

Comment: Under Assumption 2, \( I_1, I_2, I_3 \) in the figure below are isomorphisms.

![Diagram]

Theorem

Under Assumptions 1 and 2, the only unbounded root loci of the system of Figure 1 are the 1st, 2nd, ..., \( n_0 \)th order unbounded root loci.

Remarks:

(i) \( n_0 \) in the above theorem is the degree of the Smith McMillan zero of \( G(1/s) \) at \( s=0 \) (see [6] and [7] for details).

(ii) Formulae for non-integral order root loci under failure of Assumption 2 are discussed in [6].

3. Pivots for the Asymptotic Root Loci

Under a further assumption (Assumption 3) the asymptotic series for the integral order unbounded root loci is given by

\[
S_{1,n} = (-k\lambda)^{1/n}_1 + c_1 + o(1) \quad n=1,\ldots,n_0 \quad (3.1)
\]

The leading term in the expansion of the \( O(1) \) term in (3.1) is referred to as the pivot of the asymptotic root locus.

Assumption 3 (Simple Structure)

Assume that \( \hat{G}_1, \hat{G}_2, \hat{G}_3, \ldots, \hat{G}_n \) have simple structure associated with all their eigenvalues.

Theorem (Expression for the pivots)

Under assumptions 1, 2, and 3, the nth order asymptotic unbounded root loci for the system of Figure 1 have the form (3.1) with \( c_1 \) given by

\[
S_{1,n} = (-k\lambda)^{1/n}_1 + c_1 + o(1) \quad n=1,\ldots,n_0 \quad (3.1)
\]
3.1 Formulae for the Pivots of the Unbounded Root Loci

First Order

c_1 is \( \frac{1}{\lambda_1} \) times an eigenvalue of

\[
\begin{bmatrix}
G_{n+1} - c_1 G_n - \lambda_1 I & \cdots & G_1 \\
G_n - \lambda_1 I & \cdots & 0 \\
\vdots & \ddots & \vdots \\
G_1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_{n+1}
\end{bmatrix}
= 0
\]

Second Order

c_1 is \( \frac{1}{2\lambda_1} \) times an eigenvalue of

\[
\begin{bmatrix}
G_3 - (G_2 - \lambda_1 I) G_1 (G_2 - \lambda_1 I) \\
G_2 - \lambda_1 I \\
G_1
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_{n+1}
\end{bmatrix}
= 0
\]

where \( \hat{G} = G_2 - \lambda_1 I \mod R(G_1) \) and so on.

In [6] we have discussed how the computation of the formulae given above can be mechanized in a numerically robust fashion using orthogonal projections and the singular value decomposition.

References:


