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ASYMPTOTIC UNBOUNDED ROOT LOCI - FORMULAE AND COMPUTATION

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We present a new geometric way of calculating the asymptotic behavior of unbounded root loci of a strictly proper, linear, time invariant control system as loop gain$\to\infty$. The asymptotic behavior of unbounded root loci has been studied extensively by Kouvaritakis, Shaked, Edmunds, Owens and others [1,2,3,4]. Our approach, we feel, leads to more explicit formulae and our methods have applications in other asymptotic calculations as well for example, we have applied them to the hierarchical multiple time-scales aggregation of Markov chains with weak coupling [5]. The details of our approach are presented in [6] - here we only state the main results.

1. Restrictions of a Linear Map

Given a linear map A from $\mathbb{C}^n$ to $\mathbb{C}^n$ and two subspaces S_1, S_2 of complementary dimension the linear map $A(\text{mod } S_2)\big|_{S_1}$ from S_1 to $\mathbb{C}|S_2$ is defined by

$$\begin{tikzcd} S_1 \arrow[r, hook, "i"] \arrow[drr, dashed] & \mathbb{C}^n \arrow[r, "A"] & \mathbb{C}^n \arrow[d, "P"] \\ & & \mathbb{C}^n|S_2 \end{tikzcd}$$

$$A(\text{mod } S_2)\big|_{S_1} = P.A.i$$

(Here i stands for the inclusion map and P for the canonical projection).

λ is said to be an eigenvalue of $A(\text{mod } S_2)\big|_{S_1}$ if $\exists$ non-zero $x \in S_1$ such that $(A-\lambda I)x \in S_2$. This terminology is motivated by the fact that when $S_1 \oplus S_2 = \mathbb{C}^n$ there is a natural isomorphism $\tilde{I}$ between S_1 and $\mathbb{C}^n/S_2$ as follows:

$$\begin{tikzcd} S_1 \arrow[r, hook, "i"] \arrow[drr, "\tilde{I} = P \cdot I \cdot i"'] & \mathbb{C}^n \arrow[r, "I"] & \mathbb{C}^n \arrow[d, "P"] \\ & & \mathbb{C}^n/S_2 \end{tikzcd}$$

$A(\text{mod } S_2)\big|_{S_1}$ is said to have <u>simple null structure</u> if $\nexists x \in S_1$ such that

$$A(\text{mod } S_2)x \neq 0 \text{ and } A(\text{mod } S_2)\tilde{I}^{-1}A(\text{mod } S_2)x = 0 .$$

(i.e., there are no generalized eigenvectors associated with the eigenvalue $\lambda=0$). It is clear that if $A(\text{mod } S_2)\big|_{S_1}$ has simple null structure, the number (counting multiplicities) of its non-zero eigenvalues is equal to its rank, or the dimension of its range. $A(\text{mod } S_2)\big|_{S_1}$ is said to have <u>simple structure associates with an eigenvalue λ</u> if $A-\lambda I(\text{mod } S_2)\big|_{S_1}$ has simple null structure.

2. System Description, Assumptions and Main Formulae

The system under study is the system of Figure 1, where G(s) is the mxm transfer function matrix of a linear, time-invariant, strictly proper control system <u>assumed</u> to have Taylor expansion about $s=\infty$

$$G(s) = \frac{G_1}{s} + \frac{G_2}{s^2} + \frac{G_2}{s^3} + \ldots$$

The one-parameter curves (parameterized by k) traced on an appropriately defined Riemann surface, by the closed loop eigenvalues are referred to as <u>multivariable root loci</u>. An unbounded multivariable root locus $s_n(k)$ is said to be an <u>nth order unbounded root locus</u> (n=1,2,3,...) if asymptotically

$$s_n(k) = \mu_n(k)^{1/n} + O(k^0)$$

where $\mu_n \neq 0$ and $O(k^0)$ is a term of order 1.

We identify an nth order unbounded root locus with μ_n, the coefficient of its asymptotic value.

<u>Theorem</u> (Generalized eigenvalue problem for n th order unbounded root locus).

$\mu_n = (-\lambda)^{1/n} \in \mathbb{C}$ is the coefficient of the asymptotic value of an nth order unbounded root locus iff λ is a non-zero solution of

$$\begin{bmatrix} G_n - \lambda I & G_{n-1} & \cdot & G_1 \\ G_{n-1} & G_{n-2} & \cdot & 0 \\ \vdots & \vdots & \cdot & \cdot \\ & G_1 & & \\ G_1 & 0 & \cdot & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = 0 \qquad (2.1)$$

with $e_1 \neq 0$, $e_2, \ldots, e_n \in \mathbb{C}^m$.

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Comments: (i) There are n separate nth order root loci corresponding to the distinct nth roots of 1 associated with each solution of (2.1).

(ii) The matrix of (2.1) is a triangular, block Toeplitz matrix and so must admit of simplication:

2.1 Formulaue for the asymptotic values of the unbounded root loci first-order.

These are clearly the negatives of the non-zero eigenvalues of G_1.

Second Order

These are given by

$$s_{i,2} = (-\lambda_{i,2}k)^{1/2} + O(k^0)$$

where $\lambda_{i,2}$ is a non-zero eigenvalue of $G_2(\mathrm{mod}\ R(G_1))\Big|_{N(G_1)} =: \hat{G}_2$, i.e.

$$\begin{tikzcd} N(G_1) \arrow[r, hook, "i_1"] \arrow[drr, dashed, "\hat{G}_2"'] & \mathbb{R}^m \arrow[r, "G_2"] & \mathbb{R}^m \arrow[d] \\ & & \mathbb{R}^m | R(G_1) \end{tikzcd}$$

Third Order

These are given by

$$s_{i,3} = (-\lambda_{i,3}k)^{1/3} + O(k^0)$$

where $\lambda_{i,3}$ is a non-zero eigenvalue of

$$G_3 - G_2G_1^+G_2(\mathrm{mod}\ R(G_1))(\mathrm{mod}\ R(\hat{G}_2))\Big|_{N(\hat{G}_2)} = :\hat{G}_3$$

(Here G_1^+ stands for (any) right pseudo-inverse of G_1), i.e.

$$\begin{tikzcd} N(\hat{G}_2) \arrow[r, hook, "i_2"] \arrow[ddrrr, dashed, "\hat{G}_3"'] & N(G_1) \arrow[r, hook, "i_1"] & \mathbb{R}^m \arrow[r, "G_3-G_2G_1^+G_2"] & \mathbb{R}^m \arrow[d] \\ & & & \mathbb{R}^m | R(G_1) \arrow[d] \\ & & & \mathbb{R}^m | R(G_1) | R(\hat{G}_2) \end{tikzcd}$$

Higher order root loci formulae may also be written. The basic idea is to solve the triangular system using right pseudo-inverses. The Fredholm alternative is kept track off through the computation by restricting in the domain successively to $N(G_1) \supset N(\hat{G}_2)$ ... and modding out in the range $R(G_1)$, $R(\hat{G}_2)$,... .

2.2 Simple Null Structure and Integer Order for All Unbounded Root Loci

In general, the branches at $s=\infty$ of the algebraic function defined by $\det(I+kG(s)) = 0$ have asymptotic expansion with leading term of the form $\lambda(k)^{m/n}$ showing possible non-integral order unbounded root loci. Using techniques of van Dooren, et al [7] and some simple assumptions it may be shown that there are only integral order root loci.

Assumption 1 (non degeneracy)

$G(s)$ has normal (or generic) rank m.

Assumption 2 (Simple Null Structure)

Assume

G_1

$\hat{G}_2 : = G_2 \bmod R(G_1)\Big|_{N(G_1)}$

$\hat{G}_3 : = G_3 - G_2G_1^+G_2 \bmod R(G_1) \bmod R(\hat{G}_2)\Big|_{N(\hat{G}_2)}, \ldots$

to have simple null structure.

Comment: Under Assumption 2, I_1, I_2, I_3 in the figure below are isomorphisms.

$$\begin{tikzcd} N(\hat{G}_3) \arrow[r, hook] \arrow[dddrrrr, "I_3"'] & N(\hat{G}_2) \arrow[r, hook] \arrow[ddrrr, "I_2"'] & N(G_1) \arrow[r, hook] \arrow[drr, "I_1"'] & \mathbb{R}^m \arrow[r, "I"] & \mathbb{R}^m \arrow[d] \\ & & & & \mathbb{R}|R(G_1) \arrow[d] \\ & & & & (\mathbb{R}^m|R(G_1))|R(\hat{G}_2) \arrow[d] \\ & & & & ((\mathbb{R}^m|R(G_1))|R(\hat{G}_2))|R(\hat{G}_3) \end{tikzcd}$$

Theorem

Under Assumptions 1 and 2 the only unbounded root loci of the system of Figure 1 are the 1st, 2nd ..., n_0th order unbounded root loci.

Remarks: (i) n_0 in the above theorem is the degree of the Smith McMillan zero of $G(1/s)$ at $s=0$ (see [6] and [7] for details).

(ii) formulae for non-integral order root loci under failure of Assumption 2 are discussed in [6].

3. Pivots for the Asymptotic Root Loci

Under a further assumption (Assumption 3) the asymptotic series for the integral order unbounded root loci is given by

$$s_{i,n} = (-k\lambda_i)^{1/n} + c_i + o(1) \qquad n=1,\ldots,n_0 \quad (3.1)$$

The leading term in the expansion of the $O(1)$ term in (3.1) is referred to as the pivot of the asymptotic root locus.

Assumption 3 (Simple Structure)

Assume that G_1, $\hat{G}_2$, $\hat{G}_3$,...,$\hat{G}_{n_0}$ have simple structure associated with all their eigenvalues.

Theorem (Expression for the pivots)

Under assumptions 1,2,3 the nth order asymptotic unbounded root loci for the system of Figure 1 have the form (3.1) with c_i given by

$$\begin{bmatrix} G_{n+1} - c_i G_n & G_n - \lambda_i I & \cdot & \cdot & G_1 \\ G_n - \lambda_i I & G_{n-1} & & & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ G_1 & 0 & & & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_{n+1} \end{bmatrix} = 0$$

with $e_1 \neq 0,\ e_2, \ldots, e_{n+1} \in \mathbb{C}^m$.

3.1 Formulae for the Pivots of the Unbounded Root Loci

First Order

c_i is $\frac{1}{\lambda_i}$ times an eigenvalue of

$G_2 \bmod \mathcal{R}(G_1 - \lambda_i I)\Big|_{\mathcal{N}(G_1 - \lambda_i I)}$.

Second Order

c_i is $\frac{1}{2\lambda_i}$ times an eigenvalue of

$(G_3 - (\hat{G}_2 - \lambda_i I) G_1^{\dagger} (G_2 - \lambda_i I))$

$\bmod \mathcal{R}(\hat{G}_2 - \lambda_i \hat{I}) \bmod \mathcal{R}(G_1)\Big|_{\mathcal{N}(\hat{G}_2 - \lambda_i \hat{I})}$

where $\hat{G}_2 - \lambda_i \hat{I} := G_2 - \lambda_i I \bmod \mathcal{R}(G_1)\Big|_{\mathcal{N}(G_1)}$ and so on.

In [6] we have discussed how the computation of the formulae given above can be mechanized in a numerically robust fashion using orthogonal rojections and the singular value decomposition.

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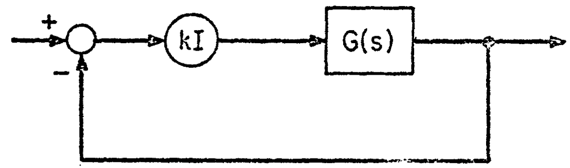

Fig. 1: System Configuration