Existence of Competitive Equilibria in Combinatorial Auctions

by

Ji Young Lee

B.S. California Institute of Technology (2010)

Submitted to the Department of Electrical Engineering and Computer Science in partial fulfillment of the requirements for the degree of

Master of Science in Electrical Engineering and Computer Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 2015

Signature redacted

Author ....
Department of Electrical Engineering and Computer Science
January 14, 2015

Signature redacted

Certified by ....
Asu Ozdaglar
Professor, Laboratory for Information and Decision Systems
Thesis Supervisor

Signature redacted

Certified by ....
Pablo Parrilo
Professor, Laboratory for Information and Decision Systems
Thesis Supervisor

Signature redacted

Accepted by ....
Professor Leslie A. Kolodziejski
Chair, Committee on Graduate Students
Department of Electrical Engineering and Computer Science

Existence of Competitive Equilibria in Combinatorial Auctions

by

Ji Young Lee

Submitted to the Department of Electrical Engineering and Computer Science
on January 14, 2015, in partial fulfillment of the
requirements for the degree of
Master of Science in Electrical Engineering and Computer Science

Abstract

Competitive equilibrium provides a natural steady state for iterative combinatorial auctions that maximize social welfare, and therefore the first step in auction design is to establish its existence. Recent work by Baldwin and Klemperer (2012) has proved that the "demand type" of valuations being "unimodular" is a necessary and sufficient condition for the existence of a competitive equilibrium, but under the general setting where both buyers and sellers as well as multiple copies of items may exist, and the supply could be any combination of items available. In this work, we investigate the same condition under the more restrictive but standard setting for combinatorial auctions, where only buyers and a single copy of each distinct item are allowed and the supply is fixed to be the set of all available items.

First, we provide an alternative proof of the sufficiency result for unimodular "complements" demand type, which defines a subclass of valuations for which a competitive equilibrium exists according to Baldwin and Klemperer (2012). While their original proof and analysis use tools from tropical geometry, our approach is based on linear programming. Relying on a result from Bikhchandani and Mamer (1999) that a competitive equilibrium exists if and only if a related linear program has an integral optimal solution, we provide a direct proof that the linear program has an integral optimal solution. Our analysis provides a fundamental understanding of the structure of the linear program and leads to various properties which may be helpful in auction design.

Second, we provide an algorithm to determine the demand types of sign-consistent tree graphical valuations, for which competitive equilibria are known to exist due to Candogan et al. (2013). We then analyze the relationship between the set of the demand types of sign-consistent tree graphical valuations and the set of unimodular demand types. Our analysis implies that the unimodularity of demand type is not necessary for the existence of a competitive equilibrium in combinatorial auctions.

Thesis Supervisor: Asu Ozdaglar
Title: Professor, Laboratory for Information and Decision Systems

Thesis Supervisor: Pablo Parrilo
Title: Professor, Laboratory for Information and Decision Systems

Acknowledgments

First and foremost I would like to express my deepest gratitude for my advisors, Prof. Asu Ozdaglar and Prof. Pablo Parrilo, who have supported me throughout my thesis with their patience and knowledge. Without their guidance, I would not have been able to complete my Master's thesis.

Besides my advisors, Dr. Ozan Candogan provided me with the starting point of my thesis, sharing all his expertise in the field of mechanism design. I also thank Dr. Alex Teytelboym for sharing his creative ideas, intuitions, and positive energy. I really appreciate Elie Adam for offering to help all the time; he helped me think more clearly at times of distress, by going through my problem in detail together. Many ideas in Part II were conceived while discussing with him.

I am also grateful to all other current as well as former labmates, Dr. Ermin Wei, Kimon Drakopoulos, Christina Lee, Annie Chen, Ali Makhdoumi, Yonghwan Lim, and Mila Nambiar, for both their advice and emotional support.

At times, I also solicited help from my friend Dongkwan Kim who is a graduate student in mathematics at MIT. I would not have been able to devise many proofs in Part I without his mathematical intuition in tropical geometry.

Kwanjeong Scholarship Foundation from Korea has provided the financial support for my studies, to supplement the funding from my advisors and the MIT EECS department. I appreciate their continued financial support for my graduate studies.

Last but not least, I would like to thank my parents Kyung Hwan Lee and Suk Hee Hong, and my sister Min Young Lee for supporting me spiritually throughout my life, and of course, my fiancé Franck Dernoncourt for cheering me up and standing by my side through the good times and bad.

Contents

List of Figures

Chapter 1

Introduction

In combinatorial auctions, a number of distinct, indivisible items are auctioned simultaneously and bidders are allowed to express preferences on combinations of discrete items, rather than on individual items or continuous quantities. This is preferable to auctioning each item separately when there are dependencies such as complements and substitutes between different items. A spectrum auction is a well-known example of a combinatorial auction, where there are complex complementary relationships among the bands of electronic spectrums being sold. Combinatorial auctions are also commonly used in truckload transportation, bus routes, and industrial procurement.

In many applications of combinatorial auctions, the main objective is to maximize social welfare, which is the sum of the values of the bundles allocated to bidders. Therefore, the existence of a Walrasian equilibrium, a set of allocation that maximizes the welfare and prices that support the allocation, is a prerequisite for designing and executing such auctions. Existing literature established the existence of the Walrasian equilibrium either by imposing highly restrictive conditions on how demand behaves when the price of an item is increased (such as Gross Substitutes (GS), Kelso Jr and Crawford (1982), or Gross Substitutes and Complements (GSC), Sun and Yang (2006)), or by requiring complex pricing structures that involve a different price for each bundle of items and/or for each bidder (such as Bikhchandani and Ostroy (2002), or Sun and Yang (2014)). The conditions in the former category do not support complex complementary relationships between items, which are observed in various auctions. Those in the second category may not be practical, since the number of different prices reported to the bidders at each stage of the auction is exponential in the number of items and/or the different set of prices for each bidder is hard to justify or be accepted. This motivates us to identify the weakest possible conditions on valuations or demands under which a Warasian equilibrium exists, for the simplest possible pricing structure, viz., linear and anonymous pricing.

Notable recent work by Candogan et al. (2013) identified a new class of valuations for which a Walrasian equilibrium exists. This work investigates a special class of valuations called graphical valuations, where the value of a bundle of items is given by the sum of the individual values of items and values for the pairs of items that capture the pairwise complementarity or substitutability between them. This valuation class can naturally be represented in terms of a weighted undirected graph, where the nodes correspond to items and the edges link pairs of items that exhibit complementarity or substitutability. They established that for "sign-consistent tree" graphical valuations, where the underlying graph is a tree and all bidders' valuations are according to the same pairwise complementarity or substitutability relationships among the items, a Walrasian equilibrium exists under linear and anonymous pricing.

Another remarkable recent work by Baldwin and Klemperer (2012) provided a necessary and sufficient condition for the existence of a Walrasian equilibrium with respect to any supply bundle, in a general economy with multiple copies of indivisible items. In order to describe this condition, of valuations having "unimodular demand type," they introduced a new framework of "demand types." Instead of working with the direct utility functions as done in the previous literature, this framework is based on certain properties on the geometric structure of the regions in the price space where an agent demands different bundles. Intuitively, demand type can be thought of as one possible way to represent complex complementary or substitutability relationships among different bundles of items. This framework of "demand type" may be a useful tool for categorizing and understanding demand, effectively incorporating existing definitions (i.e., GS and GSC) from the literature discussed earlier.

In this work, we study this new framework of demand types under the standard setting for combinatorial auctions, in order to understand whether the same necessary and sufficient condition applies to combinatorial auctions. Note that the work by Baldwin and Klemperer (2012) is based on a general economy where both buyers and sellers as well as multiple copies of items may exist, and the supply could be any combination of items available. In contrast, we restrict our attention to the standard economy for combinatorial auctions, where only buyers and a single copy of each distinct item are permitted, and the supply is fixed to be the set of all available items.[1] This would allow our work to be more directly related and applied to combinatorial auctions.

Our work provides two useful insights regarding the existence of a competitive equilibrium in combinatorial auctions. First, we prove the sufficient condition for the existence of a competitive equilibrium for a subset of unimodular demand types, via linear programming. Our analysis leads to a fundamental understanding of the structure of the linear program related to competitive equilibrium, which may provide insights on auction design. Second, we analyze the demand types of sign-consistent tree graphical valuations, for which a competitive equilibrium is known to exist due to Candogan et al. (2013). Our analysis implies that the unimodularity of demand type is not necessary for a competitive equilibrium to exist. In the rest of this section, we explain our main contributions and in more detail.

In the first part of this work, we provide an alternative proof of the sufficiency result from Baldwin and Klemperer (2012) for "unimodular complements" demand types, which define a special subclass of unimodular demand types for which a competitive equilibrium exists according to Baldwin and Klemperer (2012). While their original proof and analysis use tools from tropical geometry, our approach is based on linear programming. Relying on a result from Bikhchandani and Mamer (1997) that a competitive equilibrium exists if and only if a related linear program has an integral optimal solution, we provide a direct proof that the linear program has an integral optimal solution.

[1]For more information about the difference between the model of Baldwin and Klemperer (2012) and the standard model for combinatorial auctions, see Appendix A.

Our analysis provides a fundamental understanding of the structure of the linear program and leads to various properties which may be helpful in auction design. In particular, the so-called "lattice" property from our analysis, which says that the demand set of each bidder at any given prices forms a lattice with respect to the partial order defined via subset inclusion, may be used in the design of auctions with complementary items to reduce the communication complexity of auctions. For example, instead of requiring bidders to report the entire demand set in each iteration, we may ask them to reveal just the smallest and the greatest demanded bundles, since such bundles exist for all bidders at any given prices, due to the lattice property.

In the second part of this work, we provide an algorithm to determine the demand types of sign-consistent tree graphical valuations, for which competitive equilibria are known to exist due to Candogan et al. (2013). Since both demand types and graphical valuations are possible ways to represent complex complementarity and substitutability among items, there exist natural relationships between demand types and graphical valuations in some cases. In particular, we focus on graphical valuations with respect to a "signed tree" graph, where the underlying graph is a tree with signed edges and each edge weight is nonnegative for positive edges and nonpositive for negative edges. Our algorithm generates the demand type of graphical valuations with respect to a signed tree graph. Moreover, we show that any valuation that has the demand type of graphical valuations with respect to a signed tree graph, is graphical with respect to the same graph.

Using this result, we further analyze the relationship between the set of the demand types of signed tree graphical valuations and the set of unimodular demand types. We first identify some signed tree graphical valuations that have unimodular demand types. We then present signed tree graphical valuations that have demand types that are not unimodular. Finally, we show that there exist unimodular demand types that do not correspond to any signed tree graphical valuation. Our analysis implies that the unimodularity of demand type is not necessary for the existence of a competitive equilibrium in combinatorial auctions.[2]

[2]Although it may seem contradictory, this result is due to the fact the supply is fixed in combina-

The outline of this thesis is as follows. In Chapter 2, we introduce the formal model and preliminaries used in this work, such as efficient combinatorial auctions, demand types, unimodularity, and graphical valuations. Chapter 3 is the first part of our work, where we prove the existence of a competitive equilibrium when valuations have a unimodular complements demand type. Chapter 4 is the second part of our work, where we provide the algorithm to determine the demand types of signed tree graphical valuations, and how they relate to unimodular demand types. Finally, Chapter 5 concludes our work and presents possible future research directions.

torial auctions, and hence does not contradict Baldwin and Klemperer (2012). Please see Appendix A for more details.

Chapter 2

Model and Preliminaries

In this section, we describe the formal model and introduce relevant preliminaries used throughout this work. In Section 2.1, we describe our model of efficient combinatorial auctions, and discuss the notion of Walrasian equilibria. In Section 2.2, we present the concept of tropical hypersurfaces and demand types. In Section 2.3, we define unimodular demand types, which are the main focus of this work. In Section 2.4, we present the definition of signed tree graphical valuations, which will be examined in Part II.

2.1 Efficient Auctions and Walrasian Equilibria

In combinatorial auctions, there are single copies of N distinct, indivisible items to be allocated among M bidders. We denote the set of items by $\mathcal{N} = \{1, \ldots, N\}$ and the set of bidders by $\mathcal{M} = \{1, \ldots, M\}$. We denote a bundle of items as a vector $\mathrm{s} \in \{0,1\}^N$, whose i-th component is set to 1 if i-th item is inthe bundle, and 0 otherwise. Each bidder $m \in \mathcal{M}$ has a *value function*, or a *valuation*, $v^m : \{0,1\}^N \to \mathbb{R}^+$, where $v^m(\mathrm{s})$ is the value of any bundle $\mathrm{s} \in \{0,1\}^N$ to the bidder m. We assume that the value functions are *normalized,* i.e., $v^m(\mathbf{0}) = 0$, and *monotonically increasing,* i.e., $v^m(\mathrm{s}_1) \leq v^m(\mathrm{s}_2)$ if $\mathrm{s}_1 \leq \mathrm{s}_2$.

Given bundles of items $\mathrm{s}^m \in \{0,1\}^N$ for all $m \in \mathcal{M}$, $\{\mathrm{s}^m\}_{m \in \mathcal{M}}$ is called a *feasible allocation* if (i) each bidder $m \in \mathcal{M}$ receives a bundle of items s^m, and (ii) each item is assigned to at most one bidder. An *efficient allocation* is a feasible allocation $\{\mathbf{s}^m\}_{m\in\mathcal{M}}$ that maximizes the *welfare*, defined as $\sum_m v^m(\mathbf{s}^m)$.

A simple and common pricing rule used in auctions is *anonymous item pricing rule*, where a price p_i for each item $i \in \mathcal{N}$ is suggested to all bidders and the price of each bundle $\mathbf{s}$ is determined as the sum of prices of items in $\mathbf{s}$. Given a price vector $\mathbf{p} = (p_1, \ldots, p_N)$, the *surplus* of bidder m associated with bundle $\mathbf{s}$ is defined as $v^m(\mathbf{s})-\mathbf{p}^T\mathbf{s}$. We say that a bundle $\mathbf{s}^*$ is *demanded* by bidder m if the maximum surplus $\pi_m = \max_{\mathbf{s}\in\{0,1\}^N}\{v^m(\mathbf{s}) - \mathbf{p}^T\mathbf{s}\}$ is achieved for this bundle. The set of demanded bundles, or the *demand set* of bidder m is denoted by $D^m(\mathbf{p}) = \arg\max_{\mathbf{s}\in\{0,1\}^N}\{v^m(\mathbf{s}) - \mathbf{p}^T\mathbf{s}\}$.

A natural termination point for an iterative combinatorial auction is a set of prices and an allocation where all bidders obtain one of their demanded bundles. This outcome coincides with a Walrasian equilibrium, or a competitive equilibrium, which is a classical equilibrium concept in microeconomic theory.

Definition 2.1.1 (Walrasian equlibrium). *A* Walrasian equilibrium, *or a* competitive equilibrium, *is a tuple* $(\mathbf{p}, \mathbf{S})$, *where* $\mathbf{p} = (p_1, \ldots, p_N)$ *is a nonnegative price vector and* $\mathbf{S} = (\mathbf{s}^1, \ldots, \mathbf{s}^M)$ *is a feasible allocation such that (i) the bundle* $\mathbf{s}^m$ *is demanded by bidder* m, *i.e.,* $\mathbf{s}^m \in D^m(\mathbf{p})$, *and (ii)* $p_i = 0$ *for all unallocated items* i.

According to the first welfare theorem, an allocation $\mathbf{S}$ associated with a Walrasian equilibrium is efficient. Moreover, the second welfare theorem implies that given an efficient allocation $\mathbf{S}$, there exists a price vector $\mathbf{p}$ that support a Warlasian equilibrium whose allocation is $\mathbf{S}$. Therefore, an auction terminating at a Walrasian equilibrium maximizes social welfare. However, the Walrasian equilibrium may fail to exist for some valuation profiles. Thus, establishing the existence of a competitive equilibrium is a prerequisite for designing an efficient auction.

2.2 Tropical Hypersurfaces and Demand Types

Consider a bidder m with the valuation v^m over the set of all items. Recall that the demand set of the bidder is defined as $D^m(\mathbf{p}) = \arg\max_{\mathbf{s}\in\{0,1\}^N}\{v^m(\mathbf{s})-\mathbf{p}^T\mathbf{s}\}$. The set of prices $\mathbf{p}$ at which the bidder demands more than one bundle, i.e. $\{\mathbf{p} \mid |D^m(\mathbf{p})| > 1\}$, is known as a *tropical hypersurface* (TH) in a new sub-discipline of algebraic geometry called tropical geometry. A tropical hypersurface can be thought of as a geometric object that divides the price space into disjoint components, at which only one bundle is demanded. Formally, a tropical hypersurface has a few identifiable components that can be defined as follows.

Definition 2.2.1 (Basic Components of Tropical Hypersurface).

1. *The* cell interior *of a TH at a price* $\mathbf{p}$ *consists of points* $\mathbf{p}'$ *such that* $D^m(\mathbf{p}) = D^m(\mathbf{p}')$. *A subset of a TH is a cell interior if it is the cell interior at some point in the TH.*

2. *A subset of a TH is a* cell *if it is the closure of a cell interior of the TH. A cell of dimension* k *is called* k-cell, *and an* $(n-1)$*-cell is called a* facet.

3. *The* boundary *of a cell of a TH consists of those points in the cell that are not in its cell interior.*

4. *A* unique demand region *is a connected component of the complement of a TH, where a unique bundle is demanded.*

A facet, or an $(n-1)$-cell, of a tropical hypersurface is the set of prices at which exactly two bundles are demanded except at the boundary, and is a border between two unique demand regions. We can associate a primitive integer normal vector to each facet, which represents a change in demand as we cross the facet in the price space.

Definition 2.2.2 (Primitive Integer Normal Vector of Facet). *Let* F *be a facet and let* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *be the bundles demanded in the unique demand regions on either side. The vector* $\mathbf{s}_2 - \mathbf{s}_1$ *is called a* primitive integer normal of the facet F.

Note that the vector $\mathbf{s}_2 - \mathbf{s}_1$ is called a "normal" of the facet F in the definition. This is because $\mathbf{p} \cdot (\mathbf{s}_2 - \mathbf{s}_1)$ is constant at all $\mathbf{p} \in F$, since the bidder is indifferent between $\mathbf{s}_1$ and $\mathbf{s}_2$, that is, $v^m(\mathbf{s}_1) - \mathbf{p} \cdot \mathbf{s}_1 = v^m(\mathbf{s}_2) - \mathbf{p} \cdot \mathbf{s}_2$.

Having introduced the elements of tropical hypersurface, we can now define "demand types." The notion of demand types was first introduced in Baldwin and Klemperer (2012). Let $\mathcal{D} = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$ be a set of primitive integer vectors in $\mathbb{Z}^n$, such that if $\mathbf{v} \in \mathcal{D}$ then $-\mathbf{v} \in \mathcal{D}$. Note that we may often abuse the notation and represent a demand type $\mathcal{D}$ by any $n \times \frac{r}{2}$ matrix whose columns comprise one representative $\mathbf{v}$ of each pair $\mathbf{v}, -\mathbf{v} \in \mathcal{D}$, and refer to it as a *demand type matrix*, or the *matrix representation* of a demand type.

Definition 2.2.3 (Demand type). *A bidder or a valuation has a* demand of type $\mathcal{D}$ *if all primitive integer normals to the facets of the corresponding tropical hypersurface lie in the set* $\mathcal{D}$.

As an example, Figure 2-1 shows the tropical hypersurface for the value function $v : \{0,1\}^2 \to \mathbb{R}^+$ defined as $v((1,1)) = 3,\ v((1,0)) = v((0,1)) = 1,\ v((0,0)) = 0$. Note that the line segments $L_1,\ L_2,\ \ldots,\ L_5$ do not contain the points A or B.

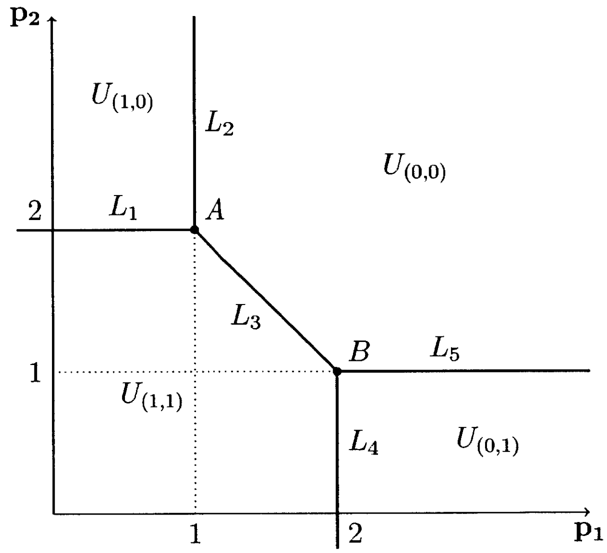


Figure 2-1: The tropical hypersurface of the value function $v : \{0,1\}^2 \to \mathbb{R}^+$ defined as $v((1,1)) = 3,\ v((1,0)) = v((0,1)) = 1,\ v((0,0)) = 0$.

The basic components of tropical hypersurfaces can be identified according to Definition 2.2.1:

1. Cell interiors: L_1, L_2, $\ldots$, L_5, A and B.

2. Cells that are facets: $L_1 \cup A$, $L_2 \cup A$, $L_3 \cup A \cup B$, $L_4 \cup B$, and $L_5 \cup B$.

3. Cells that are not facets: A and B.

4. Boundaries of the cell: A and B are the boundaries of the cells that include them.

5. Unique demand regions: $U_{(0,0)}, U_{(1,0)}, U_{(0,1)}$, and $U_{(0,0)}$, where $U_{\mathbf{s}}$ denotes the unique demand region where $\mathbf{s}$ is the only bundle that is demanded.

The primitive integer normal vectors of facets $L_1 \cup A$ and $L_5 \cup B$ are $\pm(0,1)$, of facets $L_2 \cup A$ and $L_4 \cup A$ are $\pm(1,0)$, and of facets $L_3 \cup A \cup B$ are $\pm(1,1)$, according to Definition 2.2.2. It follows that the demand type of a bidder with value function v is $\left\{ \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. We will often abuse the notation by including only one representative of each pair $\pm(1,0), \pm(0,1)$, and $\pm(1,1)$ and using the matrix form instead: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

2.3 Unimodular Demand Types

All bidders having a unimodular demand type is a sufficient condition for a competitive equilibrium to exist for any supply bundle in a general economy, according to Baldwin and Klemperer (2012). In order to define a unimodular demand type, we first need a definition of totally unimodular matrices, which can be found in standard textbooks on integer optimization such as Schrijver (1998).

Definition 2.3.1 (Totally Unimodular Matrices). *A matrix A is* totally unimodular *if each subdeterminant of A is $0, +1$, or -1.*

In particular, each entry in a totally unimodular matrix is $0, +1$, or -1.

Then unimodular demand types can be defined as follows.[1]

[1] Note that this definition is adopted from Baldwin and Klemperer (2012), but slightly modified for our setup. Please see Appendix A for more details.

Definition 2.3.2 (Unimodular Demand Types). *We say that a demand type* $\mathcal{D}$ *is* unimodular *if the matrix whose columns are the vectors of* $\mathcal{D}$ *is totally unimodular.*

2.4 Signed Tree Graphical Valuations

Signed tree graphical valuations form a special class of valuations that exhibit fixed pairwise complementarity and substitutability between items, predefined by the underlying graph. Remarkable work by Candogan et al. (2013) established that competitive equilibrium exists when bidders have graphical valuations with respect to a signed tree.[2] The definition of graphical valuations below is due to Candogan et al. (2013).

Definition 2.4.1 (Graphical Valuation). *Let* $G = (\mathcal{N}, \mathcal{E})$ *be a graph such that the set of nodes correspond to the set of items* $\mathcal{N}$ *and the edges may exhibit complementarity or substitutability. We say that the value function* $v : 2^{\mathcal{N}} \to \mathbb{R}^+$ *is* graphical *with respect to its* value graph G, *if it satisfies* $v(S) = \sum_{i \in S} w_i + \sum_{(i,j) \in \mathcal{E} | i,j \in S} w_{ij}$, *where* $\{w_i\}_{i \in \mathcal{N}}$ *are the nonnegative node weights, and* $\{w_{ij}\}_{(i,j) \in \mathcal{E}}$ *are the edge weights.*

In other words, a valuation is graphical if there exist node weights and edge weights associated with the underlying value graph G, such that the value of any bundle S is the sum of weights of the nodes and the edges contained in a subgraph of G induced by the set of nodes S. Any edge with a positive weight signifies that the items that correspond to the nodes connected by this edge exhibit pairwise complementarity, and any edge with a negative weight signifies pairwise substitutability. Figure 2-2 shows an example of a graphical valuation.

In addition, let us define a signed graphical valuation. Recall that in graph theory, a signed graph is a graph in which each edge has a positive or negative sign.

Definition 2.4.2 (Signed Graphical Valuation). *A graphical valuation is* signed *if the underlying value graph* $G = (\mathcal{N}, \mathcal{E})$ *is signed, and* $w_{ij} \geq 0$ *for each positive edge* $(i, j) \in \mathcal{E}$, *and* $w_{ij} \leq 0$ *for each negative edge* $(i, j) \in \mathcal{E}$.

[2] In Candogan et al. (2013), they use the term "sign-consistent" instead of "signed."

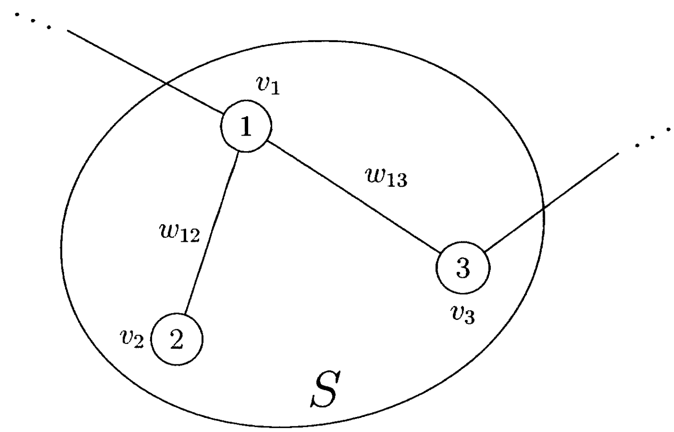


Figure 2-2: An example of a graphical valuation. The value of the bundle S above is $v(S) = v_1 + v_2 + v_3 + w_{12} + w_{13}$.

In other words, in a signed graphical valuation, the weight w_{ij} of each edge $(i, j) \in \mathcal{E}$ has the corresponding sign of the edge in the underlying signed graph. Therefore, all graphical valuations with respect to a signed graph have the same pairwise complementarity/substitutability structure among items.

For the rest of our discussion, when we say that a valuation is graphical with respect to a signed graph, then we implicitly assume that the valuation is a signed graphical valuation, in order to avoid unnecessary repetitions of the word "signed."

Finally, a signed tree graphical valuation is a signed graphical valuation, whose underlying graph is a tree as well as signed.

Chapter 3

Part I: Unimodular Complements Demand Type is Sufficient for the Existence of Competitive Equilibrium

In the first part of this work, we establish the existence of a competitive equilibrium for a special subclass of the sufficient conditions introduced by Baldwin and Klemperer (2012), i.e., when bidders have unimodular complements demand type. While the original proof by Baldwin and Klemperer (2012) was based on tropical geometry, the alternative proof we provide in this work is based on linear programming.

The outline of this part is as follows. In Section 3.1, we formally state the theorem that we prove in this part. In Section 3.2, we introduce some preliminaries on polyhedron that are required for the proof. In Section 3.3, we introduce our proof approach based on linear programming. Using the result from Bikhchandani and Mamer (1997), we demonstrate that proving the the existence of a competitive equilibrium is equivalent to proving the existence of an integral optimal solution to a certain linear program. In Section 3.4 and Section 3.5, we discuss lattice lemma and zonotope construction respectively, which are the main idea and tools for the proof. In Section 3.6, we reformulate the optimal solution set of the linear program as another equivalent system of equation and inequalities, and make use of the lattice lemma and zonotope construction to show that an integral optimal solution exists.

3.1 Problem Statement

The formal problem statement of the theorem we prove in this work is as follows.

Theorem 3.1.1. *Suppose that all bidders have valuations of a demand type $\mathcal{D}$ contained in $\pm\{0,1\}^N$. If the demand type $\mathcal{D}$ is unimodular, then a competitive equilibrium exists in a combinatorial auction with single copies of each indivisible item.*

Note that this theorem is a rectricted version of the original one from Baldwin and Klemperer (2012) for two reasons. First, for direct relevance and applicability to combinatorial auctions, we restrict our economy to have single copies of each item and only buyers, whereas Baldwin and Klemperer (2012) allows for multiple copies of items and both buyers and sellers to be present. Moreover, while their work concerns the existence of a competitive equilibrium with respect to all possible supply bundles, we focused primarily on the existence of a competitive equilibrium with respect to a fixed supply bundle consisting of all available items, which is standard in combinatorial auction literatures. Due to this restriction, the necessary and sufficient condition from Baldwin and Klemperer (2012) is sufficient but not necessary for the existence of competitive equilibrium in our setup, as shown in Part II (Section 4). Please refer to Appendix A for more details.

Second, we assume *complementarity* along with unimodularity of the demand type. That is, for each vector in the demand type, all nonzero elements of the vector are of the same sign. Such restriction would allow our proof using linear programming to be simple, yet provide insights on why the unimodularity of the demand type leads to the existence of competitive equilibrium. Moreover, it turns out that the class of unimodular "complements" demand type has a nice structural property on the demand set, which we call "lattice" property (see Section 3.4). Throughout this work, we say that a demand type is *complements* if it is contained in $\pm\{0,1\}^N$ as in the statement of the theorem above.

3.2 Preliminaries on Polyhedra

In this section, we introduce some basic concepts and results on the geometric structure of polyhedra, which are treated in standard texts such as Schrijver (1998).

Let $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a nonempty polyhedron in n dimensions. Supporting hyperplanes, faces and edges of a polyhedron can be defined as follows.

Definition 3.2.1 (Supporting hyperplane). *If c is a nonzero vector, and $\delta = \max\{c^T x \mid Ax \leq b\}$, then the affine hyperplane $\{x \mid c^T x = \delta\}$ is called a* supporting hyperplane *of P.*

Definition 3.2.2 (Face and Edge). *A subset F of P is called a* face *of P if $F = P$ or if F is the intersection of P with a supporting hyperplane of P. A face of dimension 1 is called an* edge.

The following proposition states the condition under which a subset of a face of a polyhedron is again a face of the same polyhedron.

Proposition 3.2.3. *If F is a face of P and $F' \subseteq F$, then F' is a face of P if and only if F' is a face of F.*

Next, we introduce the definition of convex objects called cone and zonotope. A nonempty set C of points in Euclidean space is called a *(convex) cone* if $\lambda x + \mu y \in C$ whenever $x, y \in C$ and $\lambda, \mu \geq 0$. The (convex) cone of an arbitrary set $X \subseteq \mathbb{R}^n$ is the set

$$\text{cone}(X) := \{\lambda x + \mu y \mid x, y \in C, \lambda, \mu \geq 0\}$$

The cone *generated* by the vectors $x_1, \ldots, x_d$ is the set

$$\text{cone}\{x_1, \ldots, x_d\} := \{\lambda_1 x_1 + \cdots + \lambda_d x_d \mid \lambda_1, \ldots, \lambda_d \geq 0\}.$$

The following is a fundamental result in convex geometry from Carathéodory (1911).

Theorem 3.2.4 (Carathéodory's Theorem). *If $X \subseteq \mathbb{R}^n$ and $x \in \text{cone}(X)$, then $x \in \text{cone}\{x_1, \ldots, x_d\}$ for some linearly independent vectors $x_1, \ldots, x_d$ in X.*

A *zonotope* is a set of points in Euclidean space constructed from vectors $\mathbf{v}_i$ by taking the sum of $a_i\mathbf{v}_i$, where each a_i is a scalar between 0 and 1. Alternatively, it can be viewed as a Minkowski sum of line segments connecting the origin to the endpoint of each vector.

Finally, we introduce a well-known property of totally unimodular matrices, related to the integrality of polytopes.

Theorem 3.2.5. *An integral matrix A is totally unimodular if and only if for all integral vectors a, b, c, d the vertices of the polytope $\{x \mid c \leq x \leq d, a \leq Ax \leq b\}$ are integral.*

3.3 Linear Programming Approach

In contrast with the original work by Baldwin and Klemperer (2012), we use a linear programming approach to prove the main theorem. In particular, we use the theorem from Bikhchandani and Mamer (1997) that a combinatorial auction has a competitive equilibrium if and only if a certain linear program (LP) has an integral optimal solution. We use the following modification of the LP, where the theorem is not affected as a result of modification.[1]

Primal

$$\begin{aligned}
\max \quad & \sum_m \sum_{\mathbf{s}} v^m(\mathbf{s}) x^m(\mathbf{s}) & \\
\text{s.t.} \quad & \sum_{\mathbf{s}} x^m(\mathbf{s}) = 1, & \forall m \\
& \sum_m \sum_{\mathbf{s}} \mathbf{s} \cdot x^m(\mathbf{s}) + \mathbf{t} = \mathbf{e} & \\
& x^m(\mathbf{s}) \geq 0, & \forall m, \mathbf{s} \\
& \mathbf{t} \geq \mathbf{0} &
\end{aligned}$$

Dual

$$\begin{aligned}
\min \quad & \sum_m \pi_m + \mathbf{e}^T \mathbf{p} & \\
\text{s.t.} \quad & \pi_m + \mathbf{s}^T \mathbf{p} \geq v^m(\mathbf{s}), & \forall m, \mathbf{s} \\
& \mathbf{p} \geq \mathbf{0} &
\end{aligned}$$

[1] In effect, we have standardized the linear program from Bikhchandani and Mamer (1997) and put it partially in a vector form to obtain the LP above.

Note that the summations for $\mathbf{s}$ is over all possible bundles including the empty bundle, all $\mathbf{t} = (t_1, \ldots, t_N)$ and $\mathbf{p} = (p_1, \ldots, p_N)$ are vectors, and $\mathbf{e}$ and $\mathbf{0}$ are the vectors of all ones and all zeros, respectively.

In order to prove the existence of a competitive equilibrium, it is sufficient to show that the primal LP above has an integral optimal solution, due to Bikhchandani and Mamer (1997). Moreover, in an integral optimal solution to this LP, the values of $x^m(\mathbf{s})$ can be interpreted as an efficient allocation: $x^m(\mathbf{s})$ is 1 if a bundle $\mathbf{s}$ is allocated to bidder m, and 0 otherwise. In addition, $\mathbf{p}$ are the prices that support the allocation in the competitive equilibrium, and π_m is a surplus of each bidder m.

We now take one step further by deriving a set of linear equations and inequalities that is exactly the set of optimal solutions to the primal LP. To do this, we fix an optimal basic feasible solution to the dual, and use the fact that the set of primal optimal solutions is exactly the set of primal feasible solutions that satisfy complementary slackness with the dual optimal solution, as follows.

First, note that the dual problem has at least one feasible solution since the feasible set does not contain a line. Let $(\boldsymbol{\pi}, \mathbf{p})$ be an optimal basic feasible solution to the dual, where $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_M)$. It is clear that $\pi_m = \max_{\mathbf{s} \in \{0,1\}^N} v^m(\mathbf{s}) - \mathbf{s}^T \mathbf{p}$, since $(\boldsymbol{\pi}, \mathbf{p})$ is optimal. Let $D^m(\mathbf{p}) = \arg\max_{\mathbf{s} \in \{0,1\}^N} (v^m(\mathbf{s}) - \mathbf{s}^T \mathbf{p})$, so $m, \mathbf{s} \in D^m(\mathbf{p})$ are the indices of the active constraints among $\pi_m + \mathbf{s}^T \mathbf{p} \geq v^m(\mathbf{s})$ at $(\boldsymbol{\pi}, \mathbf{p})$. Also let $Z(\mathbf{p}) = \{i \mid p_i = 0\}$, so $Z(\mathbf{p})$ is the set of indices of active constraints among $p_i \geq 0$.

Let $\mathbf{x}$ denote the vector $(x^m(\mathbf{s}))_{m,\mathbf{s}}$. Suppose $(\mathbf{x}, \mathbf{t})$ is a feasible solution to the primal. Then complementary slackness states that $(\mathbf{x}, \mathbf{t})$ and $(\boldsymbol{\pi}, \mathbf{p})$ are the optimal solutions to their respective problems if and only if

$$
\text{(CS)} \qquad
\begin{aligned}
&\pi_m\Big(\sum_{\mathbf{s}} x^m(\mathbf{s}) - 1\Big) = 0, && \forall m \\
&p_i\Big(\sum_{m} \sum_{\mathbf{s}} s_i \cdot x^m(\mathbf{s}) + t_i - 1\Big) = 0, && \forall i \\
&x^m(\mathbf{s})(\pi_m - \mathbf{s}^T \mathbf{p} - v^m(\mathbf{s})) = 0, && \forall m, \mathbf{s} \\
&t_i p_i = 0, && \forall i.
\end{aligned}
$$

The first two lines of equations is always satisfied since $(\mathbf{x}, \mathbf{t})$ is feasible. The last two lines of equations are equivalent to $x^m(\mathbf{s}) = 0$ for $m, \mathbf{s} \notin D^m(\mathbf{p})$, and $t_i = 0$ for $i \notin Z(\mathbf{p})$.

Let us rewrite the primal feasibility constraints with complementary slackness conditions in the feasibility problem, (FP).

$$\text{(FP)} \qquad \sum_m \sum_{\mathbf{s} \in D^m(\mathbf{p})} \mathbf{s} \cdot x^m(\mathbf{s}) + I\mathbf{t} = \mathbf{e} \tag{3.1}$$

$$\sum_{\mathbf{s} \in D^m(\mathbf{p})} x^m(\mathbf{s}) = 1, \qquad \forall m \tag{3.2}$$

$$x^m(\mathbf{s}) \geq 0, \qquad \forall m, \mathbf{s} \in D^m(\mathbf{p}) \tag{3.3}$$

$$t_i \geq 0, \qquad \forall i \in Z(\mathbf{p}) \tag{3.4}$$

$$x^m(\mathbf{s}) = 0, \qquad \forall m, \mathbf{s} \notin D^m(\mathbf{p}) \tag{3.5}$$

$$t_i = 0, \qquad \forall i \notin Z(\mathbf{p}) \tag{3.6}$$

Following the discussion above, to introduce the next lemma.

Lemma 3.3.1. *The solution set of (FP) is the set of optimal solutions to the primal LP.*

Proof. If $(\mathbf{x}, \mathbf{t})$ is a solution to (FP), then it is primal feasible and satisfies complementary slackness with the dual (optimal) feasible solution $(\boldsymbol{\pi}, \mathbf{p})$. Therefore, it is an optimal solution to the primal. Conversely, if $(\mathbf{x}, \mathbf{t})$ is a primal optimal solution, then it is primal feasible and satisfies complementary slackness with $(\mathbf{x}, \mathbf{t})$. Thus, it satisfies all equations and inequalities in (FP). □

The immediate consequence of this lemma is the following corollaries, which will be useful for our proof.

Corollary 3.3.2. *The solution set of (FP) is nonempty,*

Proof. It follows from Lemma 3.3.1, since there exists at least one optimal solution to the primal. This is because the feasible set of the primal is nonempty (as

$x^m(\mathbf{0}) = 1$,$x^m(\mathbf{s}) = 0$ for all $\mathbf{s} \neq \mathbf{0}, m \in \mathcal{M}$ and $\mathbf{t} = \mathbf{e}$ is feasible) and the objective value is bounded above. □

The next corollary is the most important result of this section.

Corollary 3.3.3. *If there exists an integral solution to (FP), then a competitive equilibrium exists.*

Proof. Follows from Lemma 3.3.1, since an integral solution to (FP) is an integral optimal solution to the LP. □

Therefore, in order to show that a competitive equilibrium exists, it is sufficient to show that there exists an integral solution to (FP). We show this in the next few sections. In Section 3.4, we show "Lattice" lemma, which states that at any given prices, each bidder's demand set forms a lattice if the demand type is complements. In Section 3.5, we use Lattice lemma to introduce how to construct a zonotope that is equivalent to the convex hull of a demand set for each bidder, within the unit hypercube whose vertex set is $\{0, 1\}^N$. These two sections provide the necessary tools to prove that (FP) has an integral solution. In Section 3.6, we reformulate (FP) as an "equivalent" system of equations and inequalities called (FP*), by replacing the expressions for the convex hulls of demand sets with those for the corresponding zonotopes as constructed in Section 3.5. The unimodularity of demand type will then imply that (FP*) is an integral polyhedron, i.e., all its vertices are integral. This will imply that (FP) has an integral solution by the "equivalence" of (FP) and (FP*), which follows from the equivalence of the convex hulls of the demand sets and the corresponding zonotopes.

3.4 Demand Set of a Bidder is a Lattice

In this section, we present an interesting structural property on the demand set of a bidder when his demand type is complements in the "Lattice" lemma. The lemma states that at any given prices, the demand set of a bidder with complements demand type forms a lattice with respect to the partial order defined by subset inclusion (i.e.,

$\mathbf{s} < \mathbf{t}$, if $\mathbf{s} \subset \mathbf{t}$). In this partial order, the meet (or infimum) $\vee$ and the join (or supremum) $\wedge$ are equivalent to the intersection and the union, respectively: that is, $\mathbf{s} \vee \mathbf{t} = \mathbf{s} \cap \mathbf{t}$ and $\mathbf{s} \wedge \mathbf{t} = \mathbf{s} \cup \mathbf{t}$.

Lemma 3.4.1 ("Lattice" Lemma). *If a bidder has a demand type contained in $\pm\{0,1\}^N$, then for any given prices his demand set is a lattice with respect to the partial order defined by subset inclusion.*

Before we present the proof for this lemma, we first introduce a proposition needed for the proof. The following proposition states that any vector that corresponds to an edge in the convex hull of the demand set of a bidder, is contained in his demand type.

Proposition 3.4.2. *If there is an edge between bundles $\mathbf{s}_1$ and $\mathbf{s}_2$ in the convex hull of bundles demanded by a single bidder at some prices, then $\mathbf{s}_1 - \mathbf{s}_2$ is in his demand type.*

Proof. If $\mathbf{s}_1$ and $\mathbf{s}_2$ are the only demanded bundles at the given prices, then the price vector is on the facet between the unique demand regions for $\mathbf{s}_1$ and $\mathbf{s}_2$ in the tropical hypersurface. Since $\mathbf{s}_1 - \mathbf{s}_2$ is a normal to the facet, it is in the demand type by definition.

Suppose there are other demanded bundles at the given prices $\mathbf{p}$. Since there is an edge between $\mathbf{s}_1$ and $\mathbf{s}_2$ in the convex hull $\mathcal{C}$ of demanded bundles, there exists a supporting hyperplane $\mathbf{c}^T\mathbf{s} = \delta$ of $\mathcal{C}$, whose intersection with $\mathcal{C}$ is the edge. It follows that $\mathbf{c}^T\mathbf{s}_1 = \mathbf{c}^T\mathbf{s}_2 = \delta$, and $\mathbf{c}^T\mathbf{s}_0 < \delta$ for all other demanded bundles $\mathbf{s}_0 \neq \mathbf{s}_1, \mathbf{s}_2$.

Let us now perturb the price vector to $\mathbf{p} - \epsilon\mathbf{c}$ for an arbitrarily small $\epsilon > 0$. If we let π_0 be the surplus of the bidder at the given prices $\mathbf{p}$, then the surplus of the bidder associated with a demanded bundle $\mathbf{s}$ at the perturbed prices is $v^m(\mathbf{s}) - (\mathbf{p} - \epsilon\mathbf{c})^T\mathbf{s} = \pi_0 + \epsilon\mathbf{c}^T\mathbf{s}$. So the maximum surplus is associated only with the bundles $\mathbf{s}_1$ and $\mathbf{s}_2$ among the demanded bundles, as $\mathbf{c}^T\mathbf{s}$ is maximized at $\mathbf{s}_1$ and $\mathbf{s}_2$. Moreover, for an arbitrarily small ϵ, this value is greater than the surplus associated with any bundle that is not demanded. Therefore, the surplus is maximized only for the bundles $\mathbf{s}_1$ and $\mathbf{s}_2$ at the perturbed prices. This implies that at the perturbed prices $\mathbf{p} - \epsilon\mathbf{c}$, the only demanded bundles are $\mathbf{s}_1$ and $\mathbf{s}_2$, so the same argument follows as in the earlier case. □

A direction consequence of Proposition 3.4.2 above is the following.

Corollary 3.4.3. *If a bidder has a demand type contained in $\pm\{0,1\}^N$, then every vector that corresponds to an edge in the convex hull of his demand set at any given prices is of the form $\pm\{0,1\}^N$.*

Having introduced the proposition and its corollary needed for the proof, we now present the proof of the Lattice lemma.

Proof of Lattice lemma (Lemma 3.4.1). If the bidder demands only one bundle, there is nothing to prove. Assume that the bidder demands two or more bundles.

Let $\mathbf{s}$ and $\mathbf{t}$ be a pair of bundles that a bidder demands at a given price, denoted as a $0/1$ vector. We need to prove that $\mathbf{s}\wedge\mathbf{t}$ and $\mathbf{s}\vee\mathbf{t}$ is also demanded by the bidder. If either $\mathbf{s} < \mathbf{t}$ or $\mathbf{t} < \mathbf{s}$, then since $\{\mathbf{s}\wedge\mathbf{t}, \mathbf{s}\vee\mathbf{t}\} = \{\mathbf{s}, \mathbf{t}\}$, both $\mathbf{s}\wedge\mathbf{t}$ and $\mathbf{s}\vee\mathbf{t}$ are demanded. So let us assume that $\mathbf{s}$ and $\mathbf{t}$ do not include each other, and prove that $\mathbf{s}\wedge\mathbf{t}$ and $\mathbf{s}\vee\mathbf{t}$ are also demanded by induction.

Consider a pair of demanded bundles $\mathbf{s}$ and $\mathbf{t}$ that differ by just two items. Without loss of generality, assume that $\mathbf{s} = (1,0,\mathbf{v})$ and $\mathbf{t} = (0,1,\mathbf{v})$ where $\mathbf{v} \in \{0,1\}^{N-2}$. Then $\mathbf{s}$ and $\mathbf{t}$ are on a 2-dimensional face F of a unit hypercube, and $\mathbf{s}\vee\mathbf{t} = (0,0,\mathbf{v})$ and $\mathbf{s}\wedge\mathbf{t} = (1,1,\mathbf{v})$ are the other bundles on F. Suppose that either $\mathbf{s}\vee\mathbf{t} = (0,0,\mathbf{v})$ or $\mathbf{s}\wedge\mathbf{t} = (1,1,\mathbf{v})$ is not demanded. Then there is an edge between $\mathbf{s}$ and $\mathbf{t}$ on the face F, so there is an edge between $\mathbf{s}$ and $\mathbf{t}$ in the convex hull of demanded bundles, by Proposition 3.2.3. However, the edge between $\mathbf{s}$ and $\mathbf{t}$ have the form $\mathbf{s}-\mathbf{t} = (1,-1,0)$, which is a contradiction to Corollary 3.4.3.

Assume that for every pair of demanded bundles $\mathbf{s}$ and $\mathbf{t}$ that differ by less than j items, both $\mathbf{s}\vee\mathbf{t}$ and $\mathbf{s}\wedge\mathbf{t}$ are also demanded. Now, consider demanded bundles $\mathbf{s}$ and $\mathbf{t}$ that differ by j items, and let us prove that both $\mathbf{s}\vee\mathbf{t}$ and $\mathbf{s}\wedge\mathbf{t}$ are demanded. Without loss of generality, let $\mathbf{s} = (\underbrace{1,\ldots,1}_{i},\underbrace{0,\ldots,0}_{j-i},v)$ and $\mathbf{t} = (\underbrace{0,\ldots,0}_{i},\underbrace{1,\ldots,1}_{j-i},v)$, where $1 < i < j$ (since otherwise $\mathbf{s} < \mathbf{t}$ or $\mathbf{t} < \mathbf{s}$) and $v \in \{0,1\}^{N-j}$.

If some bundle $\mathbf{x} = (\mathbf{u}, \mathbf{v})$ different from $\mathbf{s}, \mathbf{t}, \mathbf{s} \vee \mathbf{t}$, or $\mathbf{s} \wedge \mathbf{t}$ but shares the last $(N-j)$ items with $\mathbf{s}$ and $\mathbf{t}$ is demanded, then it is easy to show that both $\mathbf{s} \vee \mathbf{t}$ and $\mathbf{s} \wedge \mathbf{t}$ must also be demanded. Let $\mathbf{x} = (\overbrace{\underbrace{1, \dots, 1}_{p}, \underbrace{0, \dots, 0}_{q}}^{i}, \overbrace{\underbrace{1, \dots, 1}_{r}, \underbrace{0, \dots, 0}_{s}}^{j-i}, v)$ without loss of generality. Then $\mathbf{s} \vee \mathbf{x} = (\overbrace{\underbrace{1, \dots, 1}_{p}, \underbrace{0, \dots, 0}_{q}}^{i}, \overbrace{0, \dots, 0}^{j-i}, v)$ and $\mathbf{t} \vee \mathbf{x} = (\overbrace{0, \dots, 0}^{i}, \overbrace{\underbrace{1, \dots, 1}_{r}, \underbrace{0, \dots, 0}_{s}}^{j-i}, v)$ are both demanded, by the induction hypothesis. Since either $q \geq 1$ or $r \geq 1$ (otherwise $\mathbf{x} = \mathbf{s}$), either $\mathbf{s} \vee \mathbf{x}$ and $\mathbf{t}$, or $\mathbf{t} \vee \mathbf{x}$ and $\mathbf{s}$, differ by at most $(j-1)$ items. Therefore, either $(\mathbf{s} \vee \mathbf{x}) \vee \mathbf{t}$ or $(\mathbf{t} \vee \mathbf{x}) \vee \mathbf{s}$ must be demanded, by the induction hypothesis. But in fact, $(\mathbf{s} \vee \mathbf{x}) \vee \mathbf{t} = (\mathbf{t} \vee \mathbf{x}) \vee \mathbf{s} = \mathbf{s} \vee \mathbf{t}$. Therefore, we proved that $\mathbf{s} \vee \mathbf{t}$ is demanded, and we can similarly prove that $\mathbf{s} \wedge \mathbf{t}$ is also demanded.

Let us now consider the case where every bundle other than $\mathbf{s}, \mathbf{t}, \mathbf{s} \wedge \mathbf{t}$, and $\mathbf{s} \vee \mathbf{t}$ that share the last $(N-j)$ items with $\mathbf{s}$ and $\mathbf{t}$ is not demanded. Assume, for a contradiction, that either $\mathbf{s} \vee \mathbf{t}$ or $\mathbf{s} \wedge \mathbf{t}$ is not demanded. We will show that there must be an edge between $\mathbf{s}$ and $\mathbf{t}$ in the convex hull of demanded bundles, by constructing a supporting hyperplane of the convex hull that contains only $\mathbf{s}$ and $\mathbf{t}$ among all demanded bundles. This would contradict Corollary 3.4.3, since the edge between $\mathbf{s}$ and $\mathbf{t}$ is of the form $\pm(\underbrace{1, \dots, 1}_{i}, \underbrace{-1, \dots, -1}_{j-i}, \underbrace{0, \dots, 0}_{N-j})$.

Suppose that $\mathbf{s} \vee \mathbf{t}$ is not demanded, and let $v = (\underbrace{1, \dots, 1}_{k-j}, \underbrace{0, \dots, 0}_{N-k})$ without loss of generality. Then

$$\mathbf{s} = (\underbrace{1, \dots, 1}_{i}, \underbrace{0, \dots, 0}_{j-i}, \underbrace{1, \dots, 1}_{k-j}, \underbrace{0, \dots, 0}_{N-k}), \text{ and } \mathbf{t} = (\underbrace{0, \dots, 0}_{i}, \underbrace{1, \dots, 1}_{j-i}, \underbrace{1, \dots, 1}_{k-j}, \underbrace{0, \dots, 0}_{N-k}).$$

We can construct a supporting hyperplane of the convex hull of demanded bundles that contains only $\mathbf{s}$ and $\mathbf{t}$ among the demanded bundles as follows:

$$\frac{(x_1 + \cdots + x_i)}{i} + \frac{(x_{i+1} + \cdots + x_j)}{j-i} + a((1-x_{j+1}) + \cdots + (1-x_k) + x_{k+1} + \cdots + x_N) = 1,$$

where $a \gg 0$. It is easy to check that $\mathbf{s}$ and $\mathbf{t}$ are on the hyperplane, while $\mathbf{s} \wedge \mathbf{t}$ and all bundles that do not share the last $(N-j)$ elements with $\mathbf{s}$ and $\mathbf{t}$ are above the hyperplane. All bundles other than $\mathbf{s}, \mathbf{t}, \mathbf{s} \wedge \mathbf{t}$, and $\mathbf{s} \vee \mathbf{t}$ that share the last $(N-j)$ items with $\mathbf{s}$ and $\mathbf{t}$ are irrelevant, since they are not demanded.

In the case where $\mathbf{s} \wedge \mathbf{t}$ instead of $\mathbf{s} \vee \mathbf{t}$ is not demanded, the same hyperplane with $a \ll 0$ instead of $a \gg 0$ is the supporting hyperplane that contains $\mathbf{s}$ and $\mathbf{t}$. □

A direct consequence of the Lattice lemma (Lemma 3.4.1) is the following:

Corollary 3.4.4. *If a bidder has a demand type contained in $\pm\{0,1\}^N$, then there exists the smallest bundle in his demand set.*

This property is the starting point for constructing the zonotope, as discussed in the next section.

3.5 Zonotope Equivalent to the Convex Hull of a Demand Set

In this section, we describe how to construct a zonotope that coincides with the convex hull of each bidder's demand set, within the unit hypercube whose vertex set is $\{0,1\}^N$. Note that the demand type is assumed to be unimodular complements throughout this section.

Let $\mathbf{s}_0^m$ be the smallest bundle in $D^m(\mathbf{p})$ for each m. Let $\mathbf{s}_j^m, j = 1, \ldots, J_m$ be the bundles in $D^m(\mathbf{p})$ that are connected to $\mathbf{s}_0^m$ by an edge in the convex hull of $D^m(\mathbf{p})$. Let $\mathbf{d}_j^m = \mathbf{s}_j^m - \mathbf{s}_0^m$, for all $j = 1, \ldots, J_m$. Then we immediately note the following:

Proposition 3.5.1. *Each component of $\mathbf{d}_j^m$ for $j = 1, \ldots, J_m$ is nonnegative.*

Proof. For each $j = 1, \ldots, J_m$, $\mathbf{s}_j^m$ is greater than $\mathbf{s}_0^m$ by definition. It follows that $\mathbf{s}_j^m$ contains $\mathbf{s}_0^m$, so $\mathbf{d}_j^m = \mathbf{s}_j^m - \mathbf{s}_0^m$ must be a nonnegative vector. □

The following lemma states that the vertex set of the zonotope constructed from $\mathbf{d}_1^m, \ldots, \mathbf{d}_{J_m}^m$ and translated by $\mathbf{s}_0^m$, is the same as the demand set of a bidder, within the unit hypercube whose vertex set is $\{0,1\}^N$.

Lemma 3.5.2. *Consider* $\mathbf{s} \in \{0,1\}^N$. *Then* $\mathbf{s} \in D^m(\mathbf{p})$ *if and only if* $\mathbf{s} = \mathbf{s}_0^m + \sum_{j\in\mathcal{J}} \mathbf{d}_j^m$ *for some index set* $\mathcal{J} \subseteq \{1,\ldots,J_m\}$.

Proof. Let us first prove the sufficient condition. Observe that the bundle $\mathbf{s}^m$ is a supremum of $\mathbf{s}_0^m + \mathbf{d}_j^m$ for all $j \in \mathcal{J}_m$ in the partial order defined by subset inclusion, as $\mathbf{s}_0^m$ and $\mathbf{d}_j^m$'s are nonnegative. Since $\mathbf{s}_0^m + \mathbf{d}_j^m$ is in the demand set $D^m(\mathbf{p})$ for each $j \in \mathcal{J}_m$, their supremum is also in $D^m(\mathbf{p})$ by Lattice lemma (Lemma 3.4.1).

Let us now prove the necessary condition. Let us translate all demanded bundles by $-\mathbf{s}_0^m$ so that $\mathbf{s}_0^m$ is at the origin. After the translation, the convex hull of demanded bundles is contained in the cone generated by $\mathbf{d}_1^m, \ldots, \mathbf{d}_{J_m}^m$. By Caratheodory's theorem, for each $\mathbf{s} \in D^m(\mathbf{p})$, the vector $\mathbf{s} - \mathbf{s}_0^m$ is in the cone generated by a set of linearly independent vectors $\{\mathbf{d}_j^m\}_{j\in L}$, where $L \subseteq \{1,\ldots,J_m\}$. In other words, there exists nonnegative numbers $a_j, j \in L$ such that $\mathbf{s} - \mathbf{s}_0^m = \sum_{j\in L} a_j \mathbf{d}_j^m$.

In fact, it is easy to see that a_j's are integers. This is because $\{\mathbf{d}_j^m\}_{j\in L}$ form an integral basis for the subspace they span, since they are in the demand type, which was defined to be unimodular.

Moreover, we claim that a_j's cannot be greater than 1. Suppose, for a contradiction, that $a_j > 1$ for some $j \in L$. Since $\mathbf{s}_j^m$ and $\mathbf{s}_0^m$ are two different bundles, there exists at least one nonzero component, say k-th component, of $\mathbf{d}_j^m = \mathbf{s}_j^m - \mathbf{s}_0^m$. In fact, this component is either $+1$ or -1, since all components of both $\mathbf{s}_j^m$ and $\mathbf{s}_0^m$ are 0 or 1. So we have that the k-th component of $|a_j \mathbf{d}_j^m|$ is greater than 1. Also, observe that k-th components of all d_j's have the same sign, and hence so do those of all $a_j \mathbf{d}_j^m$'s. Therefore, the k-th component of $|\mathbf{s} - \mathbf{s}_0^m| = \sum_{j\in L} |a_j \mathbf{d}_j^m|$ is greater than 1. This leads to a contradiction, since $\mathbf{s}$ and $\mathbf{s}_0^m$ are vertices of a unit hypercube, and therefore each component of $|\mathbf{s} - \mathbf{s}_0^m|$ cannot be greater than 1.

Therefore, we have that $\mathbf{s} - \mathbf{s}_0^m = \sum_{j\in L} a_j \mathbf{d}_j^m$ with $a_j = 0$ or 1. The claim follows by letting $\mathcal{J} = \{j \mid a_j = 1\}$. □

This implies that the convex hull of the demand set of a bidder is the same as the corresponding zonotope intersected with the unit hypercube, since the zonotope is also convex. Such equivalence between the convex hull of the demand set and the zonotope will be used in reformulating (FP) as an "equivalent" system of equations and inequalities, as discussed in the next section.

3.6 Equivalent Reformulation to an Integral Polyhedron

In this section, we complete the proof of the main theorem of Part I (Theorem 3.1.1), by showing that (FP) has an integral solution. This is done by reformulating it to an "equivalent" system (FP*) of equations and inequalities, using the equivalence of the convex hull of demand sets and the corresponding zonotopes as in Section 3.5. After the reformulation, the vectors in the demand type show up in the coefficient matrix of (FP*). Since demand type is unimodular, this will imply the integrality of (FP*). It follows that (FP) has an integral solution, due to the "equivalence" between (FP) and (FP*). Note that that the demand type is assumed to be unimodular complements throughout this section.

First, we first reformulate (FP) as an "equivalent" system (FP*) of equations and inequalities, by replacing the expressions for the convex hull of $D^m(\mathbf{p})$ for each bidder m in (FP), with the expressions for the zonotope that is equivalent to the convex hull within the unit hypercube with vertex set $\{0,1\}^N$, as in Section 3.5.

$$\text{(FP*)} \qquad \sum_m \sum_j \mathbf{d}_j^m \cdot x_j^m + I\mathbf{t} = \mathbf{e} - \sum_m \mathbf{s}_0^m \tag{3.1}$$

$$0 \le x_j^m \le 1, \qquad \forall m, j \tag{3.2}$$

$$t_i \ge 0, \qquad \forall i \in Z(\mathbf{p}) \tag{3.3}$$

$$t_i = 0, \qquad \forall i \notin Z(\mathbf{p}) \tag{3.4}$$

By construction, (FP) and (FP*) is "equivalent" in a sense that, if there exists a solution to (FP), then so does (FP*); and moreover, if there exists an integral solution to (FP*), then so does (FP). The following lemma proves this.

Lemma 3.6.1.

1. *If there exists a solution to (FP), then so does (FP*)*

2. *If there exists an integral solution to (FP*), then so does (FP).*

Proof. 1. By Lemma 3.5.2, for each $\mathbf{s} \in D^m(\mathbf{p})$, there exists $a_j^m \in \{0,1\}$ for $j = 1, \ldots, J_m$ such that $\mathbf{s} = \mathbf{s}_0^m + \sum_{j=1}^{J_m} a_j^m \mathbf{d}_j^m$. Let us fix one such representation for each $\mathbf{s} \in D^m(\mathbf{p})$, and define $a_j^m(\mathbf{s})$ to be the value of a_j^m in this representation.

Suppose that $(\mathbf{x}, \mathbf{t})$ is a solution to (FP), where $\mathbf{x} = (x^m(\mathbf{s}))_{m,\mathbf{s}}$ and $\mathbf{t} = (t_1, \ldots, t_N)$. Let $x_j^m = \sum_{\mathbf{s} \in D^m(\mathbf{p})} a_j^m(\mathbf{s}) x^m(\mathbf{s})$.

We claim that $\big(\langle x_j^m \rangle_{m,j},\ \mathbf{t}\big)$ is a solution to (FP*). Lines (3.2), (3.3) and (3.4) are clearly satisfied due to all constraints except for line (3.1) of (FP). Line (3.1) can be verified as follows:

$$
\begin{aligned}
&\sum_m \left(\mathbf{s}_0^m + \sum_j \mathbf{d}_j^m \cdot x_j^m \right) + I\mathbf{t} \\
&= \sum_m \left(\mathbf{s}_0^m \left(\sum_{\mathbf{s} \in D^m(\mathbf{p})} x^m(\mathbf{s}) \right) + \sum_j \mathbf{d}_j^m \left(\sum_{\mathbf{s} \in D^m(\mathbf{p})} a_j^m(\mathbf{s}) x^m(\mathbf{s}) \right) \right) + I\mathbf{t} \\
&= \sum_m \sum_{\mathbf{s} \in D^m(\mathbf{p})} \left(\mathbf{s}_0^m + a_j^m(\mathbf{s}) \mathbf{d}_j^m \right) x^m(\mathbf{s}) + I\mathbf{t} \\
&= \sum_m \sum_{\mathbf{s} \in D^m(\mathbf{p})} \mathbf{s} \cdot x^m(\mathbf{s}) + I\mathbf{t} = \mathbf{e},
\end{aligned}
$$

where the first equation holds since $\sum_{\mathbf{s} \in D^m(\mathbf{p})} x^m(\mathbf{s}) = 1$ by line (1) of (FP), and the last equation follows from line (3.1) of (FP).

2. Suppose that $\big(\langle x_j^m \rangle_{m,j},\ \mathbf{t}\big)$ is an integral solution to (FP*).
We claim that $\mathbf{s}^m := \mathbf{s}_0^m + \sum_{j=1}^{J_m} x_j^m \mathbf{d}_j^m$ is in the demand set $D^m(\mathbf{p})$. To see this, first observe that x_j^m is either 0 or 1, due to line 2. If we let $\mathcal{J}_m = \{j \mid x_j^m = 1\}$, then $\mathbf{s}^m = \mathbf{s}_0^m + \sum_{j \in \mathcal{J}_m} \mathbf{d}_j^m$, which is nonnegative and integral by definition and Proposition 3.5.1. In fact, $\mathbf{s}^m$ is must be in $\{0,1\}^N$, since we have that $\sum_m \mathbf{s}^m + I\mathbf{t} = \mathbf{e}$ from line (3.1) of (FP*) and that $\mathbf{s}^m, \mathbf{t}$ and I are nonnegative. Therefore, this bundle is in the demand set $D^m(\mathbf{p})$, due to Lemma 3.5.2.

For each m, set $x^m(\mathbf{s}) = 1$ only for $\mathbf{s} = \mathbf{s}^m$, and set $x^m(\mathbf{s}) = 0$ for all other $\mathbf{s}$. It is easy to verify all conditions in (FP) are satisfied. Therefore, $(\langle x^m(\mathbf{s})\rangle_{m,\mathbf{s}}, \mathbf{t})$ is an integral solution to (FP). □

Next, we show that the new system (FP*) is an integral polyhedron in the next lemma. This is because the vectors that correspond to the edges of the convex hulls of bidders' demand sets are in the unimodular demand type, and these vectors show up as columns in the coefficient matrix of (FP*). Based on this, we show that the coefficient matrix of the equations (3.1) in (FP*) is totally unimodular, which implies that (FP*) is an integral polyhedron.

Lemma 3.6.2. *The polyhedron described by (FP*) is integral.*

Proof. Let us show that the coefficient matrix A of the equations (3.1) in (FP*) is totally unimodular. Then by Theorem 3.2.5, it follows that (FP*) is an integral polyhedron.

The coefficient matrix A is of the following form.

$$A = \begin{array}{c} \begin{array}{ccccc|c|ccc|ccc} x_1^1 & x_2^1 & \cdots & x_{J_1}^1 & & \cdots & x_1^M & \cdots & x_{J_M}^M & t_1 & \cdots & t_N \end{array} \\ \left(\begin{array}{cccc|c|ccc|c} | & | & & | & & | & & | & \\ \mathbf{d}_1^1 & \mathbf{d}_2^1 & \cdots & \mathbf{d}_{J_1}^1 & \cdots & \mathbf{d}_1^M & \cdots & \mathbf{d}_{J_M}^M & I \\ | & | & & | & & | & & | & \end{array} \right) \end{array}$$

Since $\mathbf{d}_j^m$ is an edge in the convex hull of $D^m(\mathbf{p})$ for all m, j, by Proposition 3.4.2, it is in the demand type. Since demand type is unimodular, it follows that the coefficient matrix A is totally unimodular (as appending an identity matrix preserves total unimodularity). □

Using the lemmas we introduced in this section, we now prove that the main theorem of this part (Theorem 3.1.1), by showing that (FP) has at least one integral solution.

Proof of Theorem 3.1.1. In order to show that a competitive equilibrium exists, it is sufficient to show that (FP) has an integral solution due to Corollary 3.3.3. To do this, we show that (FP*) has an integral solution. Then by Lemma 3.6.1, it follows that (FP) also has an integral solution.

We claim that (FP*) is nonempty. To see this, recall that (FP) is nonempty from Lemma 3.3.2. Since (FP) has a solution, it follows that (FP*) also has a solution by Lemma 3.6.1. Also, recall that (FP*) is integral, from Lemma 3.6.2.

Since (FP*) is a nonempty and integral polyhedron, it has at least one vertex that is integral. This vertex is an integral solution to (FP*). □

To summarize, we have shown the existence of competitive equilibrium when every bidder has the same unimodular complements demand type, in combinatorial auctions. Our proof is based on linear programming, in contrast with Baldwin and Klemperer (2012), which uses tools from tropical geometry. The LP-based approach provides insights in two different perspectives.

First, we introduce a structural property of the demand set, i.e., the demand set forms a lattice with respect to the partial order defined by subset inclusion. This maybe used to in the design of combinatorial auctions, e.g., to simplify the demand report by requiring bidders to report only the smallest and the greatest demanded bundles.

Second, we gain an understanding of the fundamental structure of the LP related to the competitive equilibrium. More specifically, we show that the LP has an integral solution, due to the "hidden" unimodularity in the coefficient matrix of the system of equations and inequalities (FP) corresponding to the set of optimal solutions. Such "hidden" unimodularity is revealed by transforming (FP) so that the vectors corresponding to the edges of the convex hull of the demand set appear in the coefficients, since the edges are in the unimodular demand type.

Chapter 4

Part II: Demand Types of Signed Tree Graphical Valuations

In the second part of this work, we analyze the demand types of signed tree graphical valuations in Section 4.1, and the relationship between the set of demand types of signed tree graphical valuations and the set of unimodular demand types in Section 4.2. Our analysis implies that for combinatorial auctions, the unimodularity of demand type is not necessary for a competitive equilibrium to exist.

4.1 Determining the Demand Type of Graphical Valuations with Respect to a Signed Tree

In this section, we provide a constructive algorithm for determining the demand type of graphical valuations with respect to a signed tree. Recall that the signed tree graphical valuations are the class of valuations for which competitive equilibria are known to exist, due to Candogan et al. (2013). Therefore, analyzing the demand types of signed tree graphical valuations will provide insights on whether it is necessary for the demand type to be unimodular for a competitive equilibrium to exist, as discussed in the next section. Before we present our result, we introduce some results from graph theory that are required for our analysis.

Let us first discuss a well-known result on signed graphs. A signed graph is an undirected graph where each edge has a positive or a negative sign. A positive cycle of a signed graph is the one in which the number of negative edges is even, and a negative cycle is the one that is not positive. A signed graph is balanced if all its cycles are positive. A theorem by Harary (1953) states that a signed graph G is balanced if and only if the set of nodes of G can be partitioned into two disjoint sets such that each positive edge joins nodes in the same set and each negative edge joins nodes in different sets. Moreover, his constructive proof includes how to form such a partition.

Note that a signed tree graph is balanced, so the theorem above can be applied. Below is an algorithm to partition the set of nodes $\mathcal{N}$ into the two disjoint sets $\mathcal{N}_1$ and $\mathcal{N}_2$ with the desirable property described above for a signed tree graph, adopted from Harary (1953).

Algorithm 1 Partition of the nodes of a signed tree graph into two disjoint sets

Input: a signed tree graph $G = (\mathcal{N}, \mathcal{E})$.

Output: two disjoint sets $\mathcal{N}_1$ and $\mathcal{N}_2$, which is a partition of $\mathcal{N}$, such that any positive edge joins nodes in the same set and any negative edge joins nodes in different sets.

Pick an arbitrary node i in the graph, and assign it to the set $\mathcal{N}_1$.
for each node j **do**
  if the number of negative edges in the path from i to j is even **then**
    Assign j it to the set $\mathcal{N}_1$.
  else
    Assign j to the set $\mathcal{N}_2$.
  end if
end for

Figure 4-1 shows a signed tree graph and the partition of nodes generated by Algorithm 1, after assigning node 1 to the set of black nodes.

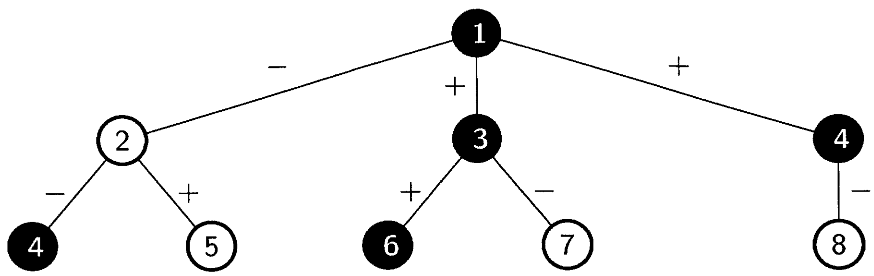


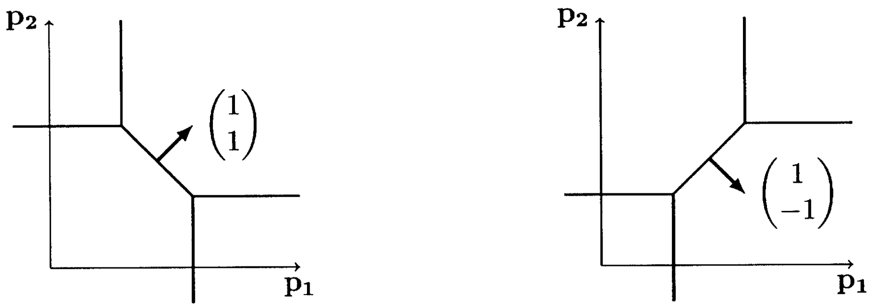
Figure 4-1: A signed tree graph where the set of black node and the set of white nodes form a partition of the nodes, such that any positive edge joins nodes in the same set and each negative edge joins nodes in different sets.

Note that as long as the graph is connected, there is a unique partition that has the desired property that positive edges connect nodes in the same set while the negative edges connect nodes in different sets, ignoring the ordering of $\mathcal{N}_1$ and $\mathcal{N}_2$. This ensures that the demand type of graphical valuations with respect to a given signed tree graph is determined uniquely.

Note that in the partition generated by Algorithm 1 above, the items that correspond to the nodes in the same set can only exhibit (pairwise) complementary relationships among them, since any edge between the nodes that correspond to these items is positive. Similarly, the items that correspond to the nodes in different sets can only exhibit (pairwise) substitutability relationships between them. Therefore, this partition, among all possible partitions, maximizes the sum of the values of the sets in the partition for any graphical valuation with respect to the signed tree graph used as input.

Such partition of nodes is used in the algorithm for constructing the demand type of signed tree graphical valuations (Algorithm 2), to assign appropriate signs to each element in the demand type. At each step of the algorithm, we add to the demand type a pair of vectors $\pm\mathbf{d}_S$ corresponding to each subset S of nodes, when the subgraph induced by S is connected. In $\mathbf{d}_S$, only the elements that correspond to the nodes S are nonzero. Among these elements, the ones that correspond to the nodes in one set of the partition will be $+1$, and the ones in the other set will be -1. In other words, for each vector that is added to the demand type, any two nonzero elements corresponding to the items in the same set of the partition are of the same sign, while any two nonzero elements corresponding to the items in different sets of the partition are of different signs.

To see why this intuitively makes sense, let us consider how pairwise complementarity and substitutability are represented in the demand type. If two items are complements to each other, the nonzero elements corresponding to these items in a vector in the demand type have the same sign. On the other hand, if two items are substitutes to each other, the nonzero elements corresponding to these items in a vector in the demand type have the opposite sign. Figure 4-2 illustrates this for two items that are complements and substitutes.



(a) Tropical hypersurface for complements, i.e., $v((1,0)) + v((0.1)) > v((1,1))$

(b) Tropical hypersurface for substitutes, i.e., $v((1,0)) + v((0.1)) > v((1,1))$

Figure 4-2: Tropical hypersurfaces for two items that are complements and substitutes. For complements, $(1,1)$ is the normal to a facet; for substitutes, $(1,-1)$ is the normal to a facet. Note that the two nonzero elements have the same sign for complements, and different signs for substitutes.

Since any pair of complementary items are in the same set of the partition generated by Algorithm 1, any vector added to the demand type by Algorithm 2 has the same sign for elements corresponding to these items. Similarly, since any pair of substitutable items are in different sets of the partition, any vector added to the demand type by Algorithm 2 has different signs for elements corresponding to these items. This is consistent with the representation of complementarity and substitutability in the demand type.

Let us now formally present an algorithm to construct the demand type of any graphical valuation with respect to a given signed tree graph.

Algorithm 2 Demand type of signed tree graphical valuation

Input: a signed tree graph $G = (\mathcal{N}, \mathcal{E})$.

Output: the demand type of any graphical valuation with respect to the input graph.

Partition the set of nodes $\mathcal{N}$ into $\mathcal{N}_1$ and $\mathcal{N}_2$ using Algorithm 1.

for each nonempty subset S of nodes **do**

Let G_S be the subgraph of G induced by S

if G_S is connected **then**

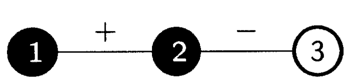
Add the vectors $\pm\mathbf{d}_S = \pm(d_1, \ldots, d_N)$ to the demand type, where each component of $\mathbf{d}_S$ is defined as:

$$d_i = \begin{cases} +1, & \text{if } i \in S \cap \mathcal{N}_1, \\ -1, & \text{if } i \in S \cap \mathcal{N}_2, \\ 0, & \text{otherwise.} \end{cases}$$

end if

end for

Figure 4-3 shows a tree graph and its demand type generated by Algorithm 2.



(a) A partitioned signed tree graph.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix}$$

(b) The demand type of (a)

Figure 4-3: A signed tree graph with a partition of nodes into the sets of black and white nodes, and the corresponding demand type that Algorithm 2 generates.

Note that in the demand type of graphical valuations with respect to a signed tree graph generated by this algorithm, each row corresponds to an item, and each column corresponds to a subset of the nodes whose corresponding induced subgraph is connected. Therefore, the number of rows is equal to the number of items, and the number of columns is equal to the number of subsets of the nodes whose corresponding induced subgraph is connected.

Let us now introduce two lemmas regarding the conditions on the valuation function for which there does not exist a facet between the unique demand regions of S_1 and S_2 in the corresponding tropical hypersurface, where the two bundles S_1 and S_2 differs by at least two items. Since the existence of the facet between the unique demand regions of S_1 and S_2 implies that the normal vector $S_1 - S_2$ to the facet is in the demand type, these lemmas specify how the conditions on the valuation function relate to the demand type.[1] Therefore, these lemmas are the main ideas used in the proof of our main result that Algorithm 2 generates the demand type of graphical valuations with respect to a signed tree graph.

Lemma 4.1.1 states that for any two bundles S_1 and S_2 that differ by at least two items, if there exist another pair of bundles T_1 and T_2 whose sum is equal to the sum of S_1 and S_2, such that $v(T_1) + v(T_2) \geq v(S_1) + v(S_2)$, then there is no facet between the unique demand regions of S_1 and S_2.[1]

Lemma 4.1.1. *Suppose that bundles S_1 and S_2 differ by at least 2 items, i.e., $|S_1 - S_2| \geq 2$. Then there is no facet between the unique demand regions of S_1 and S_2, if there exists some bundles T_1 and T_2 such that (i) $\{T_1, T_2\} \neq \{S_1, S_2\}$, (ii) $T_1 + T_2 = S_1 + S_2$, and (iii) $v(T_1) + v(T_2) \geq v(S_1) + v(S_2)$.*

Proof. Suppose, for a contradiction, that there exists a facet between S_1 and S_2. Then there exists a price vector $\mathbf{p}$ such that $v(S_1) - \mathbf{p} \cdot S_1 = v(S_2) - \mathbf{p} \cdot S_2 > v(S) - \mathbf{p} \cdot S$ for all bundles $S \neq S_1, S_2$. It follows that for each pair of bundles T_1 and T_2 that satisfies the conditions (i) and (ii) above, we have that $v(S_1) - \mathbf{p} \cdot S_1 > v(T_1) - \mathbf{p} \cdot T_1$ and $v(S_2) - \mathbf{p} \cdot S_2 = v(T_2) - \mathbf{p} \cdot T_2$. Summing up these two inequalities and using condition (ii), we have that $v(S_1) + v(S_2) > v(T_1) + v(T_2)$ for all pairs of bundles T_1 and T_2 satisfying the conditions (i) and (ii). Therefore, there does not exist bundles T_1 and T_2 satisfying all three conditions (i), (ii), and (iii) above, which is a contradiction. □

[1]Note that throughout this section, we abuse the notation when describing bundles, by using both the set notation and the vector notation interchangeably, which will be clear from the context.

An equivalent statement to the lemma above is that if there is a facet between the unique demand regions of S_1 and S_2, then $v(S_1) + v(S_2) > v(T_1) + v(T_2)$ holds for any other pair of bundles T_1 and T_2 whose sum is equal to the sum of S_1 and S_2. From this, the conditions for a valuation function to have a certain demand type can be obtained by examining each vector in the demand type.

Suppose that a vector $\mathbf{d}_S$ is in the demand type where S denotes the set of indices corresponding to nonzero elements in $\mathbf{d}_S$, and let S_1 and S_2 are the set of indices corresponding to entries $+1$ and -1 respectively. Then there exists some $X \subseteq \mathcal{N} \setminus (S_1 \cup S_2)$ such that S_1, S_2 maximizes $v(S_1 \cup X) + v(S_2 \cup X)$, among the partition of S into two sets, for any valuation v that has this demand type. That is, there exists $X \subseteq \mathcal{N} \setminus (S_1 \cup S_2)$ such that

$$\{S_1, S_2\} = \arg\max_{\substack{T_1, T_2 \\ T_1 \cup T_2 = S \\ T_1 \cap T_2 = \emptyset}} v(T_1 \cup X) + v(T_2 \cup X),$$

for any valuation that has the demand type containing $\mathbf{d}_S$.

This provides an intuition for our use of the partition of nodes in constructing the demand type of graphical valuations with respect to a signed tree graph. Recall that in the partition $\mathcal{N}_1, \mathcal{N}_2$ of nodes generated by Algorithm 1, positive edges join nodes in the same set and negative edges link nodes in different sets. This implies that for any set $S \subseteq \mathcal{N}$ of bundles, $S \cap \mathcal{N}_1$ and $S \cap \mathcal{N}_2$ is the partition of S that maximizes the sum of the values of the sets in the partition. Therefore, among any partition of S with two sets, if there exists a facet between the unique demand regions of the sets that comprise a partition, then this partition must be $S \cap \mathcal{N}_1$ and $S \cap \mathcal{N}_2$. The normal vector to this facet (if it exists) is $\pm(S \cap \mathcal{N}_1 - S \cap \mathcal{N}_2)$. As a result, if there exists any vector $\mathbf{d}_S$ in the demand type, then $\mathbf{d}_S$ must have $+1$ entries for the elements corresponding to one set of the partition $S \cap \mathcal{N}_1, S \cap \mathcal{N}_2$, and -1 entries for the elements corresponding to the other set.

Lemma 4.1.2 is the special case of Lemma 4.1.1, for when S_1 and S_2 differ by exactly 2 items. In this case, the sufficient condition from Lemma 4.1.1 for the existence of the facet between the unique demand regions of S_1 and S_2 is also necessary.

Lemma 4.1.2. *Suppose that bundles S_1 and S_2 differ by exactly 2 items, i.e., $|S_1 - S_2| = 2$. Then there does not exist a facet between the unique demand regions of S_1 and S_2, if and only if $v(S_1) + v(S_2) \leq v(T_1) + v(T_2)$ for the bundles T_1, and T_2 such that (i) $\{T_1, T_2\} \neq \{S_1, S_2\}$, and (ii) $T_1 + T_2 = S_1 + S_2$.*

Proof. The sufficient condition follows from Lemma 4.1.1. Let us prove the necessary condition. Suppose the contrary, that $v(S_1) + v(S_2) > v(T_1) + v(T_2)$ for the bundles T_1, and T_2 that satisfy conditions (i) and (ii). Let $S_1 + S_2 = (x_1, x_2, \ldots, x_N)$. Due to the condition that S_1 and S_2 must differ by exactly 2 items, only the two elements of $S_1 + S_2$ that correspond to the items that are included in exactly one of S_1 and S_2 must be 1, while all other elements are either 0 or 2. Without loss of generality, let the first two elements be the ones that are 1, i.e., $x_1 = x_2 = 1$, and $x_i = 0$ or 2 for all $i = 3, \ldots, N$.

Consider a price vector $\mathbf{p} = (p_1, p_2, \ldots, p_N)$, where we fix arbitrarily high prices for items i such that $x_i = 0$, and arbitrarily low prices for items j such that $x_j = 2$. Then at this price vector $\mathbf{p}$, only the bundles S_1, S_2, T_1 and T_2 may be demanded, where T_1 and T_2 are the bundles that satisfy conditions (i) and (ii). From the inequality $v(S_1)+v(S_2) > v(T_1)+v(T_2)$ and condition (ii), we have that $(v(S_1)-\mathbf{p}\cdot S_1)+(v(S_2)-\mathbf{p}\cdot S_2) > (v(T_1) - \mathbf{p}\cdot T_1) + (v(T_2) - \mathbf{p}\cdot T_2)$ for any price vector $\mathbf{p}$, and hence for the price vector $\mathbf{p}$ we selected. In addition to the previously chosen elements $p_3, \ldots, p_N$ of the price vector $\mathbf{p}$, let us choose p_1 and p_2 so that $v(S_1) - \mathbf{p}\cdot S_1 = v(S_2) - \mathbf{p}\cdot S_2$ and $v(T_1) - \mathbf{p}\cdot T_1 = v(T_2) - \mathbf{p}\cdot T_2$, which exists since the ratios of the coefficients of p_1 to p_2 in both equalities are different after moving all terms to one side. At this price vector $\mathbf{p}$, we have that $v(S_1) - \mathbf{p}\cdot S_1 = v(S_2) - \mathbf{p}\cdot S_2 > v(T_1) - \mathbf{p}\cdot T_1 = v(T_2) - \mathbf{p}\cdot T_2$, and therefore $v(S_1) - \mathbf{p}\cdot S_1 = v(S_2) - \mathbf{p}\cdot S_2 > v(S) - \mathbf{p}\cdot S$ for all bundles $S \neq S_1, S_2$. Therefore, there exists a facet between the unique demand regions of S_1 and S_2. □

Having introduced all required lemmas, we now present the main theorems of this section. Theorem 4.1.3 states that the output of the algorithm is the demand type of any graphical valuation with respect to the signed tree input graph, and conversely, any valuation that has this demand type must be graphical with respect to the same graph.

Theorem 4.1.3. *Let G be a signed tree graph. Suppose that $\mathcal{D}$ is the demand type generated by Algorithm 2 for the input graph G. Then:*

1. *$\mathcal{D}$ is the demand type of any graphical valuation with respect to G.*

2. *Any valuation with demand type $\mathcal{D}$ is graphical with respect to G.*

The proof of the first part of the theorem is proceeded as follows. In order to show that $\mathcal{D}$ is the demand type of any graphical valuation v with respect to G, we need to show that for any vector $\mathbf{d} \in \{0, \pm 1\}^N$ that is not in the demand type, for any pair of bundles S_1 and S_2 such that $S_1 - S_2 = \mathbf{d}$, there does not exist a facet between the unique demand regions of S_1 and S_2 in the tropical hypersurface of v. This is because $S_1 - S_2$ is the normal to the facet between the unique demand regions of S_1 and S_2, which must exist in the demand type if such facet exists. Let $S, \mathcal{I}_1$ and $\mathcal{I}_2$ be the set of indices of $\mathbf{d}$ the correspond to nonzero, +1, and -1 entries, respectively. Since $S_1 - S_2 = \mathbf{d}$, there exists $X \subseteq \mathcal{N} \setminus S$ such that $S_1 = \mathcal{I}_1 \cup X$ and $S_2 = \mathcal{I}_2 \cup X$.

In order to prove that there does not exist a facet between the unique demand regions of S_1 and S_2, it is sufficient to find the pair of bundles T_1 and T_2 that are different from S_1 and S_2 and satisfy $T_1 + T_2 = S_1 + S_2$ and $v(T_1) + v(T_2) \geq v(S_1) + v(S_2)$, due to Lemma 4.1.1. Note that according to Algorithm 2, any $\mathbf{d} \in \{0, \pm 1\}^N$ that is not in the demand type must fall into one of the two cases: (i) the subgraph G_S of G induced by S is not connected, or (ii) the subgraph G_S is connected, but $\mathbf{d} \neq \pm \mathbf{d}_S$, where $\mathbf{d}_S$ is defined as in Algorithm 2. For case (i), we can find T_1 and T_2 with the desired property, by using the fact that G_S is not connected. For case (ii), if we let $\mathcal{N}_1, \mathcal{N}_2$ be the partition of the nodes $\mathcal{N}$ generated by Algorithm 1, then $T_1 = (S \cap \mathcal{N}_1) \cup X, T_2 = (S \cap \mathcal{N}_2) \cup X$ are the desired pair of bundles. This is because $\mathcal{N}_1, \mathcal{N}_2$ is the unique partition of the nodes such that each positive edge joins nodes in the same set and each negative edge joins nodes in different sets. It follows that $S \cap \mathcal{N}_1, S \cap \mathcal{N}_2$ is the partition of S that maximizes the sum of the values of each set, which implies that $v(T_1) + v(T_2) \geq v(S_1) + v(S_2)$.

The proof of the second part of the theorem proceeds by induction. We assume that the statement holds for any valuation whose domain is $k \geq 2$ available items, and show that it holds for any valuation whose domain has $k+1 \geq 3$ items. Suppose that G is a signed tree graph with $k+1$ nodes corresponding to the items, and $\mathcal{D}$ is the demand type generated by Algorithm 2 for the input graph G. Let v be any valuation that has the demand type $\mathcal{D}$, and we will show that v is graphical with respect to G using induction. Let us pick two nodes n_1, n_2 in G that are not connected by an edge. Let G_{-n_1} be the graphs that are obtained from G by eliminating the node n_1 and its adjacent edges, and let G_{-n_2} be similarly defined for n_2. We use the inductive assumption on G_{-n_1} and G_{-n_2} to argue that valuation v is graphical with respect to G, for the bundles that are contained in either $\mathcal{N} \setminus \{n_1\}$ or $\mathcal{N} \setminus \{n_1\}$. For the bundles that are not contained in these sets, i.e., the bundles that contain both items n_1 and n_2, observe that a vector with only n_1-th n_2-th entry being nonzero does not exist in the demand set, since n_1 and n_2 were defined to be not connected by an edge. Therefore, due to Lemma 4.1.2, we have that $v(X \cup \{n_1, n_2\}) + v(X) = v(X \cup \{n_1\}) + v(X \cup \{n_2\})$ holds for any $X \subseteq \mathcal{N} \setminus \{n_1, n_2\}$. Since we already have that $v(X), v(X \cup \{n_1\})$, and $v(X \cup \{n_2\})$ is graphical with respect to G, from this equation we can deduce that the valuation v for any bundle of the form $X \cup \{n_1, n_2\}$, i.e. any bundle that contains both n_1 and n_2 is also graphical with respect to G, completing the proof.

Proof of Theorem 4.1.3. 1. Let us first prove that the demand type $\mathcal{D}$ generated by Algorithm 2 for the input graph G is the correct demand type of any graphical valuation with respect to G. We need to show that all primitive integer normals to the facets of the tropical hypersurface of v are contained in the the demand type $\mathcal{D}$. It is sufficient to show that for every possible primitive integer normal $\mathbf{d} \in \{0, \pm 1\}^N$ that does not exist in $\mathcal{D}$, for every possible pair $\{S_1, S_2\}$ of bundles such that $S_1 - S_2 = \mathbf{d}$, there is no facet between the unique demand regions of S_1 and S_2.

Consider a possible primitive integer normal $\mathbf{d} = (d_1, \ldots, d_N) \in \{0, \pm 1\}^N$ that is not contained in $\mathcal{D}$. Let $\mathcal{I}_+ := \{i \mid d_i = +1\}$, $\mathcal{I}_- := \{i \mid d_i = -1\}$, and $\mathcal{I}_0 := \{i \mid d_i = 0\}$. Consider any pair of bundles S_1 and S_2 that satisfy $S_1 - S_2 = \mathbf{d}$. Then there exists a bundle $X \subseteq \mathcal{I}_0$ such that $S_1 = \mathcal{I}_+ \cup X, S_2 = \mathcal{I}_- \cup X$.

Let S be the set of indices i such that d_i is nonzero, i.e., $S := \mathcal{I}_+ \cup \mathcal{I}_-$, and let G_S be the subgraph of G induced by S. Since $\mathbf{d}$ is not added in any step of Algorithm 2, it must be the case that either (i) G_S is not connected, or (ii) G_S is connected, but $\mathbf{d} \neq \pm\mathbf{d}_S$. First, suppose that case (i) holds. Since G_S is not connected, the set of nodes S can be partitioned into two nonempty, disjoint set of nodes A and B so that there is no edge that joins nodes in different sets. Let $T_1 = (\mathcal{I}_+ \cap A) \cup (\mathcal{I}_- \cap B) \cup X$ and $T_2 = (\mathcal{I}_+ \cap B) \cup (\mathcal{I}_- \cap A) \cup X$. Then $T_1 + T_2 = S_1 + S_2$, and $\{T_1, T_2\} \neq \{S_1, S_2\}$. Moreover, it can be easily shown that $v(T_1) + v(T_2) = v(S_1) + v(S_2)$, since the edges contained in either T_1 or T_2 are exactly those in S_1 and S_2. Therefore, by Lemma 4.1.1, there is no facet between the unique demand regions of S_1 and S_2.

Next, suppose that case (ii) holds. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be the partition of $\mathcal{N}$ that the algorithm used for determining the demand type. Consider the bundles $T_1 := (S \cap \mathcal{N}_1) \cup X$ and $T_2 := (S \cap \mathcal{N}_2) \cup X$. Then $T_1 + T_2 = S_1 + S_2$, and $\{T_1, T_2\} \neq \{S_1, S_2\}$ since $\mathbf{d} \neq \pm\mathbf{d}_S$. Moreover, $v(T_1) + v(T_2) \geq v(S_1) + v(S_2)$, since $S \cap \mathcal{N}_1$ and $S \cap \mathcal{N}_2$ is the partition of S that maximizes the sum of edge weights contained in either of the sets, and the weight of any edge contained in X or between X and S is counted exactly once for both $v(T_1) + v(T_2)$ and $v(S_1) + v(S_2)$. Therefore, by Lemma 4.1.1, there is no facet between the unique demand regions of S_1 and S_2.

2. Next, let us prove for any demand type $\mathcal{D}$ that is a possible output of Algorithm 2, any valuation with demand type $\mathcal{D}$ must be a graphical valuation with respect to the corresponding input graph. We prove this by induction on N, the number of items in the signed tree graph G that is used as an input to the algorithm for the output $\mathcal{D}$.

Base Case: For $N = 2$, the possible output demand types of the algorithm for signed tree graphical valuations are $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ for the connected two-node graph with positive edge, and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ for the connected two-node graph with negative edges. Suppose that v is a valuation with demand type $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Since $(1, -1) \notin \mathcal{D}_2$, there must not exist a facet between the unique demand regions of $(1, 0)$ and $(0, 1)$. Thus, by Lemma 4.1.2, we must have that $v((1, 1)) + v((0, 0)) \geq v((1, 0)) + v((0, 1))$. Let $v_1 := v((1, 0)), v_2 := v((0, 1))$, and $w_{12} := v((1, 1)) - v_1 - v_2$. Then $v_1, v_2, w_{12} \geq 0$ by the monotonicity and the preceding inequality. Therefore, this valuation is a graphical valuation with respect to a connected two-node graph with positive edge. we can similarly verify that a valuation with demand type $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ is a graphical valuation with respect to a connected two-node graph with negative edge.

Induction Step: Suppose that $N = k, k \geq 2$, if a valuation is of a demand type that is a possible output of the algorithm for some signed tree graphical valuation with N items, then it must be a graphical valuation with respect to the corresponding input graph. Let us prove the same for $N = k + 1$. Let $\mathcal{D}$ be a demand type that is an output of the algorithm for a signed tree input graph $G = (\mathcal{N}, \mathcal{E})$ with $k + 1$ items, and let v^* be a graphical valuation with respect to G.

Consider a valuation v that is of the demand type $\mathcal{D}$. Since $k + 1 \geq 3$, We can find two nodes n_1, n_2 of G that are not connected by an edge.

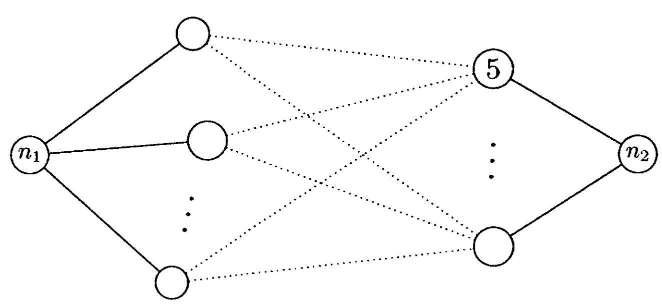


Figure 4-4: Two nodes n_1 and n_2 that are not connected by an edge

Let v_{-n_1} be the valuation over all items except for item n_1, which can be obtained by setting $v_{-n_1}(S) = v(S)$ for each $S \subseteq \mathcal{N} \setminus \{n_1\}$. If we fix an arbitrarily high price for item n_1 for all price vectors $\mathbf{p}$, then only the bundles that do not contain item n_1 will be demanded at these prices. All facets that exist at these prices must be in the demand type, so the valuation v_{-n_1} must have a demand type $\mathcal{D}_{-n_1}$, which can be obtained by eliminating the n_1-th row of the original demand type $\mathcal{D}$.

Let G_{-n_1} be a graph that is obtained from G by eliminating the node n_1 and its adjacent edges, and let $v^*_{-n_1}$ be the valuation over all items except for item n_1, where $v^*_{-n_1}(S) = v^*(S)$ for each bundle $S \subseteq \mathcal{N} \setminus \{n_1\}$. It is clear that G_{-n_1} is still a signed tree graph, and that $v^*_{-n_1}$ is still a graphical valuation with respect to G_{-n_1}. Moreover, $\mathcal{D}_{-n_1}$ is the output of the algorithm for the graphical valuation $v^*_{-n_1}$ with respect to the signed tree graph G_{-n_1}.

Therefore, by the inductive assumption, the valuation v_{-n_1} is a graphical valuation with respect to the signed tree graph G_{-n_1}. Let $\{v_i\}_{i\in\mathcal{N}\setminus\{n_1\}}$ and $\{w_{ij}\}_{(i,j)\in\mathcal{E}\,|\,i,j\neq n_1}$ be the node and edge weights, respectively, associated with the graphical valuation v_{-n_1}. Using an analogous argument by setting an arbitrarily high price for item n_2 instead of n_1, we can also establish that v_{-n_2} is a graphical valuation with respect to G_{-n_2} with node weights $\{v'_i\}_{i\in\mathcal{N}\setminus\{n_2\}}$ and edge weights $\{w'_{ij}\}_{(i,j)\in\mathcal{E}\,|\,i,j\neq n_2}$.

It can be easily seen that all node and edge weights shared by the graphical valuations v_{-n_1} and v_{-n_2} are consistent: $v_i = v'_i$ for all $i \in \mathcal{N} \setminus \{n_1, n_2\}$ and $w_{ij} = w'_{ij}$ for all $(i,j) \in \mathcal{E}$ such that $\{i,j\} \cap \{n_1, n_2\} = \emptyset$. This is because $v(S) = v_{-n_1}(S) = \sum_{i\in S} v_i + \sum_{(i,j)\in\mathcal{E}\,|\,i,j\in S} w_{ij} = v_{-n_2}(S) = \sum_{i\in S} v'_i + \sum_{(i,j)\in\mathcal{E}\,|\,i,j\in S} w'_{ij}$ for each $S \subseteq \mathcal{N} \setminus \{n_1, n_2\}$. Note that we have already defined the weights on all nodes and edges in the signed tree graph G, since n_1, n_2 were defined as two leaf nodes that are not connected by an edge. From now on, let us denote all the node and edge weights defined so far, as $\{v_i\}_{i\in\mathcal{N}}$ and $\{w_{ij}\}_{(i,j)\in\mathcal{E}}$, without an apostrophe($'$).

Thus far, we have established that the valuation for all bundles S such that $\subseteq \mathcal{N} \setminus \{n_1\}$ or $S \subseteq \mathcal{N} \setminus \{n_2\}$ is graphical with respect to $G = (\mathcal{N}, \mathcal{E})$ with node weights $\{v_i\}_{i\in\mathcal{N}}$ and edge weights $\{w_{ij}\}_{(i,j)\in\mathcal{E}}$. It remains to show that the valuation for all bundles S that contain both item n_1 and item n_2 is also graphical with respect to the same value graph.

Note that any vector $\mathbf{d}_{\{n_1,n_2\}} \in \{0, \pm 1\}^N$, where only the n_1-th and n_2-th elements are nonzero, is not contained in the output demand type $\mathcal{D}$ of the algorithm for the graphical valuation v^* with respect to G as input. This is because the subgraph of G induced by $\{n_1, n_2\}$ is not connected, by our definition that the nodes n_1 and n_2

are not connected by an edge in G. Let $X \subseteq \mathcal{N} \setminus \{n_1, n_2\}, S = \{n_1, n_2\} \cup X, T_1 = \{n_1\} \cup X, T_2 = \{n_2\} \cup X$. Then T_1 and T_2 are the only pair of bundles such that $T_1 + T_2 = S + X$, and $\{T_1, T_2\} \neq \{S, X\}$. There does not exist a facet between the unique demand regions of S and X, nor between those of T_1 and T_2, since the normal vectors $S - X$ and $T_1 - T_2$ of these facets, with only the n_1-th and n_2-th element as being nonzero, is not contained in $\mathcal{D}$. Therefore, by Lemma 4.1.2 we must have that

$$v(S) + v(X) = v(T_1) + v(T_2). \tag{4.1}$$

Note that we already know $v(X), v(T_1)$ and $v(T_2)$ in terms of the graphical valuation with respect to G with node weights $\{v_i\}_{i \in \mathcal{N}}$ and edge weights $\{w_{ij}\}_{(i,j) \in \mathcal{E}}$. Moreover, we have that

$$v(T_1) = v(X) + v_{n_1} + \sum_{i \in X \mid (i, n_1) \in \mathcal{E}} w_{in_1}, \quad v(T_2) = v(X) + v_{n_2} + \sum_{i \in X \mid (i, n_2) \in \mathcal{E}} w_{in_2}. \tag{4.2}$$

Plugging (4.2) into (4.1) and rearranging, we obtain

$$v(S) = v(X \cup \{n_1, n_2\}) = v(X) + v_{n_1} + v_{n_2} + \sum_{i \in X \mid (i, n_1) \in \mathcal{E}} w_{in_1} + \sum_{i \in X \mid (i, n_2) \in \mathcal{E}} w_{in_2}$$

Since the valuation for each bundle $X \subseteq \mathcal{N} \setminus \{n_1, n_2\}$ is graphical with respect to G, we conclude that the valuation for all bundles that contain both item n_1 and item n_2 is also graphical with respect to the same graph G. □

Finally, the next theorem states that the output of Algorithm 2 is the "minimal" demand type of a *strictly signed* graphical valuation with respect to G, in a sense that for each vector in the demand type, there exists a facet whose normal is this vector in the tropical hypersurface of the valuation. Here, *strictly signed* graphical valuation with respect to G means that each edge weight is nonzero. This theorem ensures that the demand type of signed tree graphical valuations constructed by Algorithm 2 does not contain any unnecessary columns, when we analyze whether the demand type of graphical valuations with respect to a particular signed tree graph is unimodular in the next section.

Theorem 4.1.4. *Suppose v is a* strictly signed *graphical valuation with respect to a signed tree graph G, i.e., the associated edge weights are strictly positive for positive edges and strictly negative for negative edges. Let $\mathcal{D}$ be the demand type that is generated by Algorithm 2 for the input G. Then for each vector $\mathbf{d}$ in the demand type, there exist a facet with normal $\mathbf{d}$ in the tropical hypersurface of v.*

Proof of Theorem 4.1.4. Let us prove that for each vector $\mathbf{d}$ in the demand type $\mathcal{D}$ produced by Algorithm 2 for a strictly signed tree input graph G, there exist a facet with normal $\mathbf{d}$ in the tropical hypersurface of a strictly signed graphical valuation v with respect to G. Consider a vector $\mathbf{d} \in \mathcal{D}$. Let $\mathcal{I}_+ := \{i \mid d_i = +1\}$ and $\mathcal{I}_- := \{i \mid d_i = -1\}$. It is sufficient to prove that there exists a facet between bundles $\mathcal{I}_+$ and $\mathcal{I}_-$. To prove this, we will construct a price vector $\mathbf{p} = (p_1, \ldots, p_N)$ at which only these two bundles are demanded.

Let S be the set of indices i such that d_i is nonzero, i.e., $S := \mathcal{I}_+ \cup \mathcal{I}_-$. Consider the subgraph G_S of the original graph G induced by S. Since $\mathbf{d} \in \mathcal{D}$, the graph G_S must be a connected tree. Moreover, if we let $\mathcal{N}_1$ and $\mathcal{N}_2$ be the partition of $\mathcal{N}$ that the algorithm used for determining the demand type, then $\mathcal{I}_+ \subseteq \mathcal{N}_1$ and $\mathcal{I}_- \subseteq \mathcal{N}_2$, without loss of generality.

Let L the leaf nodes of G_S. For each node $i \in L$, let $p_i = v(\{i\}) = v_i$. In the graph G_S, let $N^+(i)$ be the set of neighbors j of node i, such that the edge weight w_{ij} is positive. For each node $i \in G_S$, let $p_i = v_i + \frac{1}{2}\sum_{j \in N^+(i)} w_{ij}$. For each node $i \notin G_S$, set p_i to be arbitrarily high. Let $G_S \cap \mathcal{N}_1 = \cup_{i=1}^{r_1} A_i$ and $G_S \cap \mathcal{N}_2 = \cup_{i=1}^{r_2} B_i$ be the decomposition of $G_S \cap \mathcal{N}_1$ and $G_S \cap \mathcal{N}_2$ into the connected sets of nodes in the subgraph of G_S induced by $G_S \cap \mathcal{N}_1$ and $G_S \cap \mathcal{N}_2$, respectively. Then at the prices $\mathbf{p}$, only the unions of some A_i's, the unions of some B_j's, and the empty set are demanded. Let us perturb $\mathbf{p}$ for an arbitrarily small $\boldsymbol{\epsilon} > 0$ to obtain $\mathbf{p}(\boldsymbol{\epsilon}) = \mathbf{p} - \boldsymbol{\epsilon}$, where $\sum_{i \in G_S \cap \mathcal{N}_1} \epsilon_i = \sum_{j \in G_S \cap \mathcal{N}_2} \epsilon_j$. Then at the prices $\mathbf{p}(\boldsymbol{\epsilon})$, only the two bundles $G_S \cap \mathcal{N}_1$ and $G_S \cap \mathcal{N}_2$ are demanded, as long as all edge weights are nonzero. □

4.2 Demand Types of Signed Tree Graphical Valuations vs. Unimodular Demand Types

In this section, we study the relationship between the set of the demand types of signed tree graphical valuations and the set of unimodular demand types. We will present three examples to illustrate this relationship. First, we show an example of some signed tree graphical valuations whose corresponding demand types are unimodular. However, another example shows that not all signed tree graphical valuations have unimodular demand types. This would imply that the unimodularity of demand type is not a necessary condition for the existence of competitive equilibrium under the standard setting for combinatorial auctions. Finally, we present a unimodular demand type that no signed tree graphical valuation can have.

Let us now present the first example of signed path graphical valuations, whose demand types are unimodular. A signed path graphical valuation is a graphical valuation with respect to a path graph that is signed, where the path graph with N nodes can be defined as the graph $G = (\mathcal{N}, \mathcal{E})$ with nodes $\mathcal{N} = \{i\}_{i=1}^{N}$ and edges $\mathcal{E} = \{(i, i+1)\}_{i=1}^{N-1}$. In order to show that the corresponding demand type is unimodular, we argue that the demand type is equivalent to a network matrix, which is known to be totally unimodular due to a remarkable work by Tutte (1965). The network matrix can be defined as follows.

Definition 4.2.1 (Network Matrix). *Let $T = (V, A)$ be a directed tree graph that is connected. Let A_0 be another set of arcs on the same node set V. Let us define a matrix M, where the rows correspond to the arcs A and the columns correspond to the arcs A_0. To compute each entry M_{ij} of M at row $i \in A$ and column $j = (s, t) \in A_0$, look at the $s - t$ path P in T; then the entry can be defined as*

$$M_{ij} = \begin{cases} +1 & \textit{if arc } i \textit{ appears forward in } P, \\ -1 & \textit{if arc } i \textit{ appears backward in } P, \\ 0 & \textit{if arc } i \textit{ does not appear in } P. \end{cases}$$

Such matrices are called network matrices. *We say that T and A_0 represent M.*

We now present the example of signed path graphical valuations, which have unimodular demand types.

Example 4.2.2 (Signed Tree Graphical Valuations with Unimodular Demand Type). *A signed path graphical valuation has a unimodular demand type, since the demand type of a signed path graphical valuation is a network matrix. Given a graphical valuation with respect to a signed path graph G of demand type $\mathcal{D}$, we can construct a directed tree graph T, which represents the network matrix that is equivalent the matrix representation of $\mathcal{D}$, as follows. Refer to Figure 4-5 for an example.*

1. *Create $N+1$ nodes for the directed tree graph T.*
2. *Create an arc $(1,2)$ from node 1 to 2. This arc corresponds to the first node of the signed path graph G.*
3. *For $i = 2, 3, \ldots, N$, if the edge $(i-1, i)$ in G is positive, then add an arc between node i and $i+1$ in the same direction as the arc between node $i-1$ and i in T. Otherwise, add the arc in the opposite direction. This arc corresponds to the i-th node in G.*

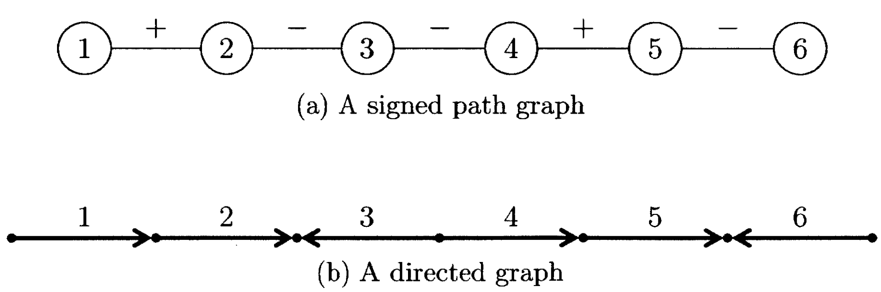


Figure 4-5: Any graphical valuation with respect to the signed path graph in (a) has the demand type whose matrix representation is equivalent to the network matrix represented by the directed graph in (b) and all possible arcs on its nodes.

It can be easily verified that the network matrix represented by T and the set of all possible arcs on its nodes is equivalent to the demand type matrix of the graphical *valuation with respect to G. Since any network matrix is totally unimodular, this implies that the demand type of a signed path graphical valuation is unimodular.*

The next example demonstrates that graphical valuations with respect to a star graph with 3 positive edges has a demand type that is not unimodular. It follows that any graphical valuation with respect to a signed tree graph, where some node has more than two adjacent edges of the same sign has a non-unimodular demand type.

Example 4.2.3 (Signed Tree Graphical Valuations with Non-Unimodular Demand Type). *Consider graphical valuations with respect to a positive-edge star graph with 4 nodes, where node 1 is the center node.*

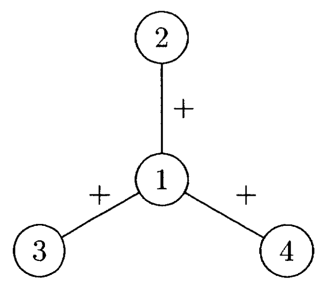


Figure 4-6: Example of a signed tree graphical valuation that has a non-unimodular demand type.

The demand type corresponding to this signed star graph can be determined by Algorithm 2 as:

$$\left(\begin{array}{ccccccc|ccc|c} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cline{8-10} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ \cline{8-10} \end{array}\right)$$

This demand type is not unimodular, since the determinant of the submatrix indicated by the square box above is -2. It is easy to see that this submatrix exist in the demand type of graphical valuations with respect to any signed tree graph, where some node has more than two adjacent edges of the same sign. Therefore, such signed tree graphical valuations have non-unimodular demand types.

This example implies that the unimodularity of demand type is not a necessary condition for the existence of competitive equilibrium under the standard setting of combinatorial auctions. This is because there exists a competitive equilibrium if all bidders have graphical valuations with respect to the same signed tree graph, according to Candogan et al. (2013). Since the demand type corresponding to the example above is not unimodular, this is a case where there exists a competitive equilibrium, even though all bidders have a non-unimodular demand type.

Our final example shows that there exists a unimodular demand type that cannot correspond to any signed tree graphical valuation.

Example 4.2.4 (Unimodular Demand Type that Does Not Correspond to Any Signed Tree Graphical Valuation). *Any valuation with the following unimodular demand type $\mathcal{D}$ cannot be a signed tree graphical valuation:*

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix} \tag{4.3}$$

This is a valid demand type, since the valuation $v(\{1\}) = 2, v(\{2\}) = 2, v(\{3\}) = 2, v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 3$ is of this demand type.

Suppose that a valuation v with demand type $\mathcal{D}$ is a graphical valuation with respect to a signed tree graph G, for a contradiction. By analyzing the demand types after eliminating each row of $\mathcal{D}$ and using Theorem 4.1.3, we can easily see that there must be a negative edge between the nodes $\{1$ and $2\}$, $\{2$ and $3\}$, and $\{1$ and $3\}$, respectively, in the signed tree graph G. However, these edges comprise a cycle, which contradicts the assumption that G is a tree.

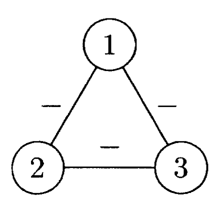


Figure 4-7: If we assume that there exists a signed tree graphical valuation that has the unimodular demand type **D** in (4.3), then the underlying graph must have a cycle, which is a contradiction.

The Venn diagram shown in Figure 4-8 illustrates the relationship between the set $\mathcal{T}$ of the demand types of signed tree graphical valuations and the set $\mathcal{U}$ of unimodular demand types, as implied by the three examples above. Example 4.2.2 belongs to the intersection $\mathcal{T} \cap \mathcal{U}$, and Example 4.2.3 and Example 4.2.4 belong to $\mathcal{T} \setminus \mathcal{U}$ and $\mathcal{U} \setminus \mathcal{T}$, respectively. Note that $\mathcal{C}$ is the set of demand types $\mathcal{D}$ such that a competitive equilibrium exists if each bidder has demand type $\mathcal{D}$.

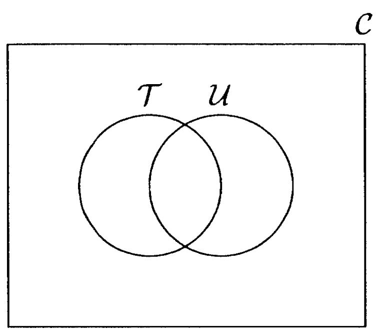


Figure 4-8: Relationship between the set $\mathcal{T}$ of demand types of signed tree graphical valuations and the set $\mathcal{U}$ of unimodular demand types, where $\mathcal{C}$ is the set of demand types $\mathcal{D}$ such that a competitive equilibrium exists if each bidder has demand type $\mathcal{D}$.

Chapter 5

Conclusion

This work analyzes the necessary and sufficient condition of having unimodular demand type for the existence of a competitive equilibrium introduced by Baldwin and Klemperer (2012). While they investigate this in a general setting where both buyers and sellers as well as multiple copies of items may exist, and the supply could be any combination of items available, we analyze the same condition under the more restrictive but standard setting for combinatorial auctions, where only buyers and a single copy of each distinct item are allowed and the supply is fixed to be the set of all available items.

On the one hand, we provide an alternative proof via linear programming that all bidders having a unimodular demand type is sufficient for a competitive equilibrium to exist, when the demand type is restricted to be complements. Relying on a result from Bikhchandani and Mamer (1997) that a competitive equilibrium exists if and only if a related linear program (LP) has an integral optimal solution, we provide a direct proof that the LP has an integral optimal solution. Our analysis illustrates why the unimodularity is related to the existence of a competitive equilibrium, by showing how the unimodularity of demand type contributes to the integrality of the optimal solution set of the LP, resulting in the existence of a integral optimal solution to the LP. Moreover, for the case of unimodular complements demand type, our analysis unveils a property that the demand set of each bidder at any given prices forms a lattice, which may be helpful in auction design.

On the other hand, we provide a constructive algorithm that determines the demand type of graphical valuations with respect to any signed tree graph, for which a competitive equilibrium is known to exist due to Candogan et al. (2013). We show that any graphical valuation with respect to a signed tree graph has the demand type generated by this algorithm, and conversely, any valuation of this demand type must be graphical with respect to the same graph. Using this result, we analyze the relationship between the set of the demand types of signed tree graphical valuations and the set of unimodular demand types. Our study shows that these two sets are different: there exists demand types of signed tree graphical valuations that are unimodular, and some that are not unimodular; there also exists a demand type that is unimodular, but does not correspond to any signed tree graphical valuation. The existence of signed tree graphical valuations whose demand type is not unimodular implies that the unimodularity of demand type is not necessary for a competitive equilibrium to exist in combinatorial auctions.

Our work suggests a few possible future research directions. One possible direction is to study what conditions on demand types are necessary for the existence of competitive equilibrium in combinatorial auctions. Such study may allow one to extend the set of demand types or signed graphs for which a competitive equilibrium exists, if all bidders have the same demand type or graphical valuations with respect to the same signed graph. Another interesting direction is to design an auction for unimodular complements demand type by utilizing the lattice property. For example, one may consider an iterative auction in which each bidder reports the smallest and the greatest demanded bundles instead of the entire demand set, which would reduce the communication complexity of the auction.

Appendix A

Differences Between the Model of Baldwin and Klemperer (2012) and the Standard Model for Combinatorial Auctions

In this section, we describe the differences between the model of Baldwin and Klemperer (2012) and the standard model for combinatorial auctions. Understanding these differences will help us resolve the results of our work that might appear contradictory to that of Baldwin and Klemperer (2012).

Consider an economy with N items, which come in indivisible units. In the model of Baldwin and Klemperer (2012), each agent has a valuation $v : A \to \mathbb{R}$ over a finite set $A \subsetneq \mathbb{Z}^N$ of possible bundles, called the *domain* of the valuation. Note that either negative bundles are permitted to allow for sellers as well as buyers.

In contrast, in the standard model for combinatorial auctions that we use, we assume that there is only a single copy of each item, so the set of available items can be defined as $\mathcal{N} = \{1, 2, \ldots, N\}$. Naturally, each bidder has a valuation $v : 2^{\mathcal{N}} \to \mathbb{R}^+$, where the domain is restricted to all possible subsets of available items $\mathcal{N}$. Allowing only a single copy of each item ensures that each bidder's valuation is *concave*. This is discussed in more detail in Section A.1. Moreover, restricting to have only a single copy of each item allows us to adopt a slightly different definition of unimodular demand type than in Baldwin and Klemperer (2012), which is equivalent, but easier to describe. This is discussed in more detail in Section A.2.

In combinatorial auctions, the valuation for each bundle is assumed to be nonnegative. We pose additional assumptions that the valuation is *normalized,* i.e., $v(\emptyset) = 0$, and *monotonic*, i.e., $v(S) \leq v(T)$, if $S \subseteq T$. These two assumptions would allow us to restrict our attention to nonnegative price vectors, since all critical points of the tropical hypesurface will be contained in the nonnegative orthant.

Besides the difference between the models, the statements of the theorem are also different regarding for which supply bundle the competitive equilibrium exists. In Baldwin and Klemperer (2012), they say that a concave demand type $\mathcal{D}$ *always* has a competitive equilibrium, if for every set of agents with concave valuation of demand type $\mathcal{D}$, and for any supply bundle in the domain of aggregate valuation, a competitive equilibrium exists. Here, the domain of aggregate valuation is defined as the Minkowski sum of the domain of all agents. With this definition of *always* having a competitive equilibrium, their actual theorem is the following:

Theorem A.0.5 (Baldwin and Klemperer (2012)). *A concave demand type* $\mathcal{D}$ always *has a competitive equilibrium if and only if it is unimodular.*

However, the standard model for combinatorial auctions only concerns a competitive equilibrium for a *fixed* supply bundle, namely the set of all available items $\mathcal{N}$. We may restate the theorem from Baldwin and Klemperer (2012) according to this model as follows.[1]

Theorem A.0.6 (Restatement of Baldwin and Klemperer (2012) for Combinatorial Auctions). *Suppose that all M bidders have a valuation of demand type $\mathcal{D}$. A demand type $\mathcal{D}$ is unimodular if and only if a competitive equilibrium exists for each supply bundle with at most M copies of each item.*

[1] Note that we drop the "concavity" condition here since each valuation is guaranteed to be concave under our setting.

Clearly, if a competitive equilibrium exists for each supply bundle with at most M copies of each item, then a competitive equilibrium exists in the standard sense of combinatorial auctions. Therefore, Theorem A.0.6 implies our main theorem (Theorem 3.1.1) of Part I (Section 3) that the competitive equilibrium exists if the demand type is unimodular, but the converse may not be true.

Recall that the main objective of Part I was to provide an alternative proof of this result via linear programming. In Part II, we have shown that the converse is not true, but this does not necessarily contradict the original necessary and sufficient result of Baldwin and Klemperer (2012). This is because, Baldwin and Klemperer (2012) prove the necessary and sufficient condition for the existence of competitive equilibrium with respect to all *possible supply bundles,* whereas we show that the same condition is only sufficient for the existence of competitive equilibrium with respect to a *fixed* supply bundle.

A.1 Concavity of Valuations

The theorem from Baldwin and Klemperer (2012) requires the valuations to be concave. In this section, we show any valuation over all possible subsets of $\mathbf{N} = \{1, 2, \ldots, N\}$ as in the standard setting for combinatorial auctions is concave. We first introduce the definition of concavity in this context.

Definition A.1.1. *A function $v : A \to \mathbb{R}$ is concave if $A \subsetneq \mathbb{Z}^N$ is convex (as a subset of $\mathbb{Z}^N$) and if v can be extended to a weakly concave function on $\mathbb{R}^N$.*

It has been proven by Milgrom and Strulovici (2009) that concave functions are precisely those for which there are no bundles in A that are never demanded. That is:

Lemma A.1.2. *Let $A \subsetneq \mathbb{Z}^N$. A function $v : A \to \mathbb{R}$ is concave if only if for all $\mathbf{x} \in A$, their exists $\mathbf{p} \in \mathbb{R}^N$ such that $\mathbf{x} \in D(\mathbf{p}) := \arg\max_{\mathbf{s} \in A}\{v(\mathbf{s}) - \mathbf{p}^T\mathbf{s}\}$.*

In the following proposition, we show that any valuation over the set $\{0, 1\}^N$ is concave by using Lemma A.1.2

Proposition A.1.3. *Any function $v : \{0,1\}^N \to \mathbb{R}$ is concave.*

Proof. Is sufficient to show that for all $\mathbf{x} \in \{0,1\}^N$, their exists $\mathbf{p} \in \mathbb{R}^N$ such that $\mathbf{x} \in D(\mathbf{p}) := \arg\max_{\mathbf{s}\in\{0,1\}^N}\{v(\mathbf{s}) - \mathbf{p}^T\mathbf{s}\}$, due to Lemma A.1.2. Consider a bundle $\mathbf{x} \in \{0,1\}^N$. Set the price of every item i contained in the bundle $\mathbf{x}$, i.e. $x_i = 1$, to be arbitrarily low and the price of all other items to be arbitrarily high. Then at this price $\mathbf{p}$, only the bundle $\mathbf{x}$ will be demanded; that is, $\mathbf{x} \in D(\mathbf{p})$. □

Therefore, we do not have to be concerned about ensuring the concavity of valuations under the standard combinatorial auctions setting.

A.2 Definition of Unimodular Demand Types

Our definition of unimodular demand types is slightly different from the original definition in Baldwin and Klemperer (2012). However, the two definitions are consistent in combinatorial auctions with only a single copy of each items. Before we discuss the differences between the two definitions, let us introduce the definition of a unimodular matrix, which is required for the discussion.

Definition A.2.1 (Unimodular Matrices).

1. *Let U be a nonsingular square matrix. Then U is called* unimodular *if U is integral and has determinant ± 1.*

2. *For any $m \times n$ matrix A of full row (or column) rank, let a basis of A be a nonsingular submatrix of order m (or n). A matrix A of full row (or column) rank is* unimodular *if A is integral, and each basis of A has determinant ± 1.*

3. *A matrix A of rank r is called* unimodular *if A is integral and if for each submatrix B consisting of r linearly independent columns of A, the greatest common divisor of the subdeterminants of B of order r is 1.*

The definitions above are consistent with each other and are provided in the order of increasing generality. Note that total unimodularity implies unimodularity, but the converse may be false.

The following is a well-known property of unimodular matrices.

Theorem A.2.2. *A matrix A is totally unimodular if and only if the matrix $[I \;\; A]$ is unimodular.*

Baldwin and Klemperer (2012) defines a demand type $\mathcal{D}$ to be *unimodular* if any linearly independent set of vectors in $\mathcal{D}$ is an integer basis for the subspace they span[2]. This condition is slightly weaker than the matrix representation D of demand type $\mathcal{D}$ being totally unimodular, and stronger than D being unimodular:

$$D \text{ is totally unimodular} \Rightarrow \mathcal{D} \text{ is a unimodular demand type} \Rightarrow D \text{ is unimodular.}$$

However, under the standard setting for combinatorial auction with a single copy of each item, all three conditions above turn out to be equivalent, as the next proposition shows. Proposition A.2.3 says that the demand type matrix D must contain the identity matrix I. This implies that if D is unimodular, then D is totally unimodular due to Theorem A.2.2. Therefore, it follows that all three conditions above are equivalent.

Proposition A.2.3. *Given an economy endowed with N distinct indivisible items, consider the set V of valuations over all possible bundles. Then the set $V_{\mathcal{D}} \subseteq V$ of valuations that correspond to a demand type $\mathcal{D}$ is nonempty if and only if $\mathcal{D}$ contains the set of coordinate vectors $\mathcal{I} = \{\mathbf{e}_i \mid i = 1, \ldots, N\}$.*

Proof. First, let us prove the sufficient statement. Suppose that $\mathcal{D}$ contains $\mathcal{I}$. Then any linear valuation v that satisfies $v(S \cup T) = v(S) + v(T)$ for all S, T such that $S \cap T = \emptyset$ is of demand type $\mathcal{D}$. Therefore, $V_{\mathcal{D}}$ is not empty.

Now, for the necessary statement, it is sufficient to prove that the tropical hypersurface of any valuation v in $V_{\mathcal{D}}$ has facets with the normal $\mathbf{e}_i$ for each i. Let us set the price $\mathbf{p}_i$ of an item i to be -1 and all other prices to be arbitrarily high. Then

[2]A set of s linearly independent vectors in $\mathbb{Z}^n$ is an integer basis for the subspace they span, if and only if among the determinants of all the $s \times s$ matrices consisting of s rows of the $n \times s$ matrix whose columns are these s vectors, the greatest common factor is 1.

these prices correspond to a unique demand region for the bundle of single item i. If we keep increasing the price of item i, then at some point we will cross a facet and reach a unique demand region for the empty bundle. The normal of this facet is $\pm\mathbf{e}_i$, so $\mathbf{e}_i$ must be in the demand type $\mathcal{D}$. Thus $\mathcal{D}$ contains $\mathcal{I}$. $\square$

Bibliography

Abraham, I., Babaioff, M., Dughmi, S., and Roughgarden, T. (2012). Combinatorial auctions with restricted complements. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, pages 3–16. ACM.

Ausubel, L. M. (2004). An efficient ascending-bid auction for multiple objects. *American Economic Review*, pages 1452–1475.

Ausubel, L. M. (2006). An efficient dynamic auction for heterogeneous commodities. *The American economic review*, pages 602–629.

Ausubel, L. M. and Milgrom, P. (2006). The lovely but lonely vickrey auction. *Combinatorial auctions*, 17:22–26.

Ausubel, L. M. and Milgrom, P. R. (2002). Ascending auctions with package bidding. *Advances in Theoretical Economics*, 1(1).

Baldwin, E. and Klemperer, P. (2012). Tropical geometry to analyse demand. Technical report, Working paper, Oxford University.

Bertsimas, D. and Tsitsiklis, J. N. (1997). *Introduction to linear optimization*. Athena Scientific.

Bikhchandani, S., de Vries, S., Schummer, J., and Vohra, R. V. (2002). Linear programming and vickrey auctions. *Mathematics of the Internet: E-auction and Markets*, 127:75–116.

Bikhchandani, S., de Vries, S., Schummer, J., and Vohra, R. V. (2011). An ascending vickrey auction for selling bases of a matroid. *Operations research*, 59(2):400–413.

Bikhchandani, S. and Mamer, J. W. (1997). Competitive equilibrium in an exchange economy with indivisibilities. *Journal of economic theory*, 74(2):385–413.

Bikhchandani, S. and Ostroy, J. M. (2002). The package assignment model. *Journal of Economic theory*, 107(2):377–406.

Candogan, O., Ozdaglar, A., and Parrilo, P. (2013). Iterative auction design for graphical valuations part i: Tree valuations. Technical report, Working Paper.

Carathéodory, C. (1911). Über den variabilitätsbereich der fourierâĂŹschen konstanten von positiven harmonischen funktionen. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 32(1):193–217.

Cramton, P., Shoham, Y., and Steinberg, R. (2006). *Combinatorial auctions.* MIT press.

de Vries, S., Schummer, J., and Vohra, R. V. (2007). On ascending vickrey auctions for heterogeneous objects. *Journal of Economic Theory*, 132(1):95–118.

De Vries, S. and Vohra, R. V. (2003). Combinatorial auctions: A survey. *INFORMS Journal on computing*, 15(3):284–309.

Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. *American Mathematical Monthly*, pages 9–15.

Grünbaum, B. (1967). Convex polytopes. 1967. *Interscience, New York*, page 138.

Gul, F. and Stacchetti, E. (1999). Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, 87(1):95–124.

Gul, F. and Stacchetti, E. (2000). The english auction with differentiated commodities. *Journal of Economic theory*, 92(1):66–95.

Harary, F. (1953). On the notion of balance of a signed graph. *Michigan Math. J.*, 2(2):143–146.

Itenberg, I., Mikhalkin, G., and Shustin, E. I. (2009). *Tropical algebraic geometry.* Springer.

Kelso Jr, A. S. and Crawford, V. P. (1982). Job matching, coalition formation, and gross substitutes. *Econometrica: Journal of the Econometric Society*, pages 1483–1504.

Krishna, V. (2009). *Auction theory.* Academic press.

Lehmann, B., Lehmann, D., and Nisan, N. (2001). Combinatorial auctions with decreasing marginal utilities. In *Proceedings of the 3rd ACM conference on Electronic Commerce*, pages 18–28. ACM.

Maclagan, D. and Sturmfels, B. (2009). Introduction to tropical geometry. *Book in preparation*, 34.

Mas-Collel, A., Whinston, M. D., and Green, J. R. *Microeconomic theory.* Oxford University Press.

Mikhalkin, G. (2004). Decomposition into pairs-of-pants for complex algebraic hypersurfaces. *Topology*, 43(5):1035–1065.

Milgrom, P. and Strulovici, B. (2009). Substitute goods, auctions, and equilibrium. *Journal of Economic theory*, 144(1):212–247.

Milgrom, P. R. (2004). *Putting auction theory to work.* Cambridge University Press.

Mishra, D. and Parkes, D. C. (2007). Ascending price vickrey auctions for general valuations. *Journal of Economic Theory*, 132(1):335–366.

Mishra, D. and Parkes, D. C. (2009). Multi-item vickrey–dutch auctions. *Games and Economic Behavior*, 66(1):326–347.

Nisan, N., Roughgarden, T., Tardos, E., and Vazirani, V. V. (2007). *Algorithmic game theory.* Cambridge University Press.

Nisan, N. and Segal, I. (2006). The communication requirements of efficient allocations and supporting prices. *Journal of Economic Theory*, 129(1):192–224.

Parkes, D. (2006). Iterative combinatorial auctions. *Combinatorial auctions*, pages 41–77.

Parkes, D. C. (1999). i bundle: an efficient ascending price bundle auction. In *Proceedings of the 1st ACM conference on Electronic commerce*, pages 148–157. ACM.

Schrijver, A. (1998). *Theory of linear and integer programming.* John Wiley & Sons.

Sun, N. and Yang, Z. (2006). Equilibria and indivisibilities: gross substitutes and complements. *Econometrica*, 74(5):1385–1402.

Sun, N. and Yang, Z. (2009). A double-track adjustment process for discrete markets with substitutes and complements. *Econometrica*, 77(3):933–952.

Sun, N. and Yang, Z. (2014). An efficient and incentive compatible dynamic auction for multiple complements. *Journal of Political Economy*, 122(2):422–466.

Tutte, W. T. (1965). Lectures on matroids. *J. Res. Nat. Bur. Standards Sect. B*, 69(1-47):468.

Vohra, R. V. (2011). *Mechanism Design: a linear programming approach.* Cambridge University Press.

Zhou, H., Berry, R. A., Honig, M. L., and Vohra, R. (2009). Complementarities in spectrum markets. In *Communication, Control, and Computing, 2009. Allerton 2009. 47th Annual Allerton Conference on*, pages 499–506. IEEE.