A Mechanistic Investigation of Nonlinear Interfacial Instabilities Leading to Slug Formation in Multiphase Flows

by

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Abstract

Many industrial applications involve the transport of multiphase flows through pipes. For instance, the design and operation of oil pipelines and production facilities relies heavily on understanding the hydrodynamics of multiphase flow. Industrial engineers utilizes multiphase flow simulators to aid in the design and flow assurance of such systems; however, the complexity of the physics and the range of scales involved in the problem require that the numerical algorithms invoke phase averaging methods and rely on empirical models. These assumptions and simplifications often result in predictions which are non-physical or are off by orders of magnitude forcing engineers to implement conservative safety factors to accommodate the large uncertainties. The development of physics based models may reduce the empiricism in the simulators allowing for the creation of more robust and cost effective designs.

The work described in this thesis carries out both theoretical and computational investigations of some nonlinear mechanisms governing the interfacial stability and nonlinear evolution of stratified two-phase flows through horizontal channels and pipes. The resulting investigation identifies a strong nonlinear energy transfer mechanism which extracts energy generated by an interfacial instability and transfers it (with possible bi-exponential growth rates) to long wavelength waves which may eventually evolve into large amplitude waves and slugs. Detailed investigations demonstrate the effectiveness of this mechanism in flows ranging from ideal (potential) to turbulent two-phase flows. This thesis consists of three key focus areas.

The first section develops a nonlinear potential flow analysis to identify a mechanism composed of a triad of resonantly interacting interfacial waves which are influenced by the Kelvin-Helmholtz interfacial instability. The mechanism that is identified permits the rapid energy transfer from linearly unstable short waves to stable long waves through nonlinear resonant wave interactions. It was found that, depending on the flow conditions, it is possible for linearly stable waves to achieve bi-exponential growth due to the resonant coupling. Extensions of this mechanism to broadbanded wave interactions were found to be in close agreement with experimental measurements. The analysis was also adjusted to examine the special case of sub-harmonic
resonant interactions which have been observed in many experimental measurements and it was shown that this special case could still effectively create rapid long wave growth with up to bi-exponential growth rates.

The second focus area examines the robustness of the aforementioned potential flow mechanism by identifying if a linear interfacial instability could be effectively coupled with resonant interactions in the presence of viscosity and flow turbulence. Using a linear stability analysis along with direct numerical simulations, comparisons were made against experimental measurements. This analysis was able to accurately identify the bandwidth of unstable interfacial modes as well as predict the existence of the strong sub-harmonic and triad resonances among modes which were reported in the experimental observations. The behavior observed in the numerical simulations demonstrates that the coupled instability-resonance mechanism is capable of existing in more complex two-phase turbulent flows and still permits the rapid exchange of energy from unstable short to linearly stable long wavelength modes. In addition, the numerical simulation results provide high-resolution data sets for which the interfacial stress distributions could quantified and described providing insights into the necessary behavior of future interfacial stress modeling.

The final focus area is dedicated to developing a novel nonlinear slug transition criterion which couples the effects of a linear instability with that of nonlinear resonant interaction theory. An energy bounding condition is proposed for which the number of resonant modes which are linearly unstable is minimized allowing for a critical gas velocity to be identified. Comparisons are made against experiments carried out in horizontal channels and good agreement is observed. A heuristic method is proposed which allows for "equivalent" channel flow conditions to be obtained which are representative of the original pipe flow conditions. Unlike previously developed slug transition conditions, this new nonlinear criterion provides predictions which are significantly more accurate when compared against experimental measurements and maintains its accuracy over a large range of pipe diameters, flow conditions, and fluid combinations.

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Chapter 1

Introduction

1.1 Introduction to Multiphase Flows

A multiphase flow is a flow which includes some combination of gases, liquids, and/or solids. These types of flows are common occurrences in many industrial applications. In many cases, the materials being processed and transported do not flow as a single phase. Instead, depending on the flow conditions, a wide range of flow regimes can be observed with complex dynamic interfacial topologies.

For instance, complex gas-liquid flow are observed in boilers, condensers, refrigerators, heat exchangers, and air-conditioners. The phase changes which occur from boiling in nuclear reactors often leads to complex engineering challenges due to the formation of bubbles. Similarly the chemical and oil industries utilize land and sub-sea pipelines for the transport of oil from wells to processing facilities. These flows contain different components such as hydrocarbons, natural gases, water, wax and hydrate mixtures produced while drilling. In the case of offshore drilling, these fluids are extracted from sub-sea wells, brought up to the offshore platform through risers (which are often more than a mile long), and then are sent to land based processing facilities through sub-sea pipelines that can be \( O(100) \) miles long. While the mixture is in transit, it is possible for the flow to exist in a number of unique hydrodynamic states. Each of these flow regimes produce significant challenges in the transportation, separation, and processing of the flow into its individual components. Being
able to predict which hydrodynamic state the flow is in, along with the conditions under which a flow regime transition will occur is of vital importance to the accurate modeling of multiphase flows. In this work, attention is focused on two-phase (either liquid-liquid or gas-liquid) flow through horizontal channels and pipes. Figure 1-1 demonstrates the possible flow regimes which can exist within the pipe.

The simplest of these regimes is the stratified smooth flow. For low gas and liquid volumetric flow rates, the restoring influence of gravity causes the two phase to remain cleanly separated resulting in the lower density fluid flowing over the second. As the gas velocity is increased, in the interfacial force increase causing rippling to occur on the interface yielding what is referred to as the stratified wavy state. If the volumetric flow rates of the two fluids are further increased, the waves on the interface continue to grow until eventually one of the waves bridges the pipe diameter trapping long bubbles of gas within the liquid. If the two-phases remain clearly separated, this is referred to as an elongated bubble regime. A more violent flow is known as slug flow. In these flows, the blockage propagates through the pipeline and entrains a large amount of gas within the liquid body. These flows result in strong turbulent mixing and the slug body moves at a velocity which is different from both the gas/liquid
mean velocities. In some cases, cross stream perturbations can cause one of the phases to become completely surrounded by the other resulting in core annular flow. Additionally, it is also possible for one phase to become dispersed in the other. In one case this can be due to droplets entrainment within a gas or large volume of gas becoming dispersed as bubbles within the liquid. Due to the importance of these flows in industrial applications, a significant amount of research has been carried out over the past 60 years on characterizing the nature of these flow regimes and determining the conditions under which the flow transitions from one regime to another. To this day, this subject remains an active area of research. Due to the wide range of flow regimes, and the drastically different physics associated with each regime, this work focuses attention on the transition from stratified smooth to stratified wavy and slugs.

Experiments have shown that there are a number of different physical processes which can result in the transition from a stratified to slug flow. Fan et al. [24] carried out experiments with air-water flows through a horizontal 9.5 cm pipe. In their work, they found that for large superficial gas velocities, \( U_{SG} > 4 \text{ m/s} \), wave spectra measurements demonstrated that slugs form through the coalescence of large amplitude irregular waves. For lower superficial gas velocities, \( U_{SG} < 4 \text{ m/s} \), it was found that initially a linear instability would excite the growth of short-wavelength interfacial waves. These waves underwent a nonlinear resonant energy cascade resulting in the growth of large amplitude long waves which may eventually form a slug (assuming that the liquid depth was sufficiently deep).

As the slug body develops, experiments have shown a few important properties. First, the formation of slugs is a highly unsteady and intermittent process. Even when the volumetric flow rates of the two-phases are held constant the formation of slugs can occur at irregular times and locations within the pipe. Additionally, the velocities and pressures at any cross-sectional location can have significant variation over time. Another important property of slugs has to do with the size of the slug bodies. Experiments, such as those carried out by Ujang et al. [76], have shown that the slug body can grow to be \( O(10 - 50) \) diameters. The large size of the slug bodies has a number of important consequence to the design of industrial facilities. As the slugs
travel through the pipeline, the dynamic pressures on the pipe wall can excite resonant vibrations within the pipe system resulting in fatigue which must be monitored and periodically repaired. Furthermore, the presence of the slug can cause significant changes in the pressure drop. This requires designers to accommodate pressure drops which can be an order of magnitude larger than those seen in the stratified flow. Furthermore, the presence of slugs in a pipeline can create significant design challenges to the design of separation facilities. As fluids exit the pipeline, these high velocity slugs can exit the pipe and impact the walls of the processing equipment resulting in damage and fatigue problems. As a result, this requires large pre-separation stage "slug catchers" to safely stop the slugs from damaging the equipment and reduce some of the strong unsteadiness of the incoming flow.

1.2 Theoretical Prediction of Slug Formation

Over the past 40+ years, a significant amount of work has gone into developing theoretical methods capable of predicting interfacial instabilities and the resulting transition from stratified to slug flow. In this section, a brief review of the current literature and the existing theoretical slug transition models is reviewed such that the important assumptions and relevant closure models which have been previously implemented can be examined in more detail.

Many transition models start by considering the stability of stratified flows using the Kelvin-Helmholtz instability theory. This theory considers the linear stability of two inviscid superposed fluids subjected to traveling wave perturbations. This instability gives rise to a pressure which is 180 degrees out of phase with the wave crests resulting in exponential wave growth. The Kelvin-Helmholtz [56] dispersion relationship is found to be of the form

\[
c = \frac{\rho_g C_g U_g + \rho_l C_l U_l}{\rho_g C_g + \rho_l C_l} \pm \sqrt{\frac{\sigma k^2 + (\rho_l - \rho_g) g}{k (\rho_g C_g + \rho_l C_l)} - \frac{\rho_g \rho_l C_g C_l (U_g - U_l)^2}{(\rho_g C_g + \rho_l C_l)^2}}
\]

(1.1)

where \( g \) is gravity, \( \sigma \) is the value of surface tension, \( k \) is the wavenumber, \( C_{g/l} = \)
\[
\coth \left( kh_{g/l} \right), \ U_{g/l} \text{ are the mean currents, and } \rho_{g/l} \text{ is the density with the subscripts } \\
g/l \text{ denoting the gas and liquid respectively. When the flow conditions produce a} \]

complex value for the wave speed \( c \), an instability is said to exist. Neutral stability \( \text{for the wave occurs when the term under the radical in eqn. (1.1) is equal to zero.} \)

Following the works of Wallis & Dobson [78], invoking a long wavelength assumption, \( kh_l, kh_g \ll 1 \) and assuming that \( \frac{\rho_g}{\rho_l} \ll 1 \), produces simplified stability condition \( \text{of the form which dictates that waves will grow if} \)

\[
\rho_g (U_g - U_l)^2 > (\rho_l - \rho_g) gh_g \tag{1.2}
\]

producing a wave with wave speed \( c_r = U_l + U_g \frac{\rho g h_l}{\rho_l h_g} \). Using this linear theory, they suggested that slugs will occur when an interfacial wave becomes unstable and bridges the cross-section of the channel. Comparisons of this theory against experimental measurements found that eqn. (1.2) over predicts the critical gas velocity by approximately a factor of two.

Taitel & Dukler [73] modified this model by considering the case of a finite wave over a flat liquid sheet. In their work, they considered a finite solitary wave on an otherwise horizontal surface. By neglecting the wave motion, they developed the criterion for instability of the form

\[
U_g > C_1 \sqrt{\frac{g (\rho_l - \rho_g) h_g}{\rho_g}} \tag{1.3}
\]

where

\[
C_1 = \left( \frac{2}{h_g h_g' \left( \frac{h_g}{h_g'} + 1 \right)} \right)^{1/2} \tag{1.4}
\]

with \( h_{g/l} \) are the gas and liquid equilibrium depth and \( h_{g/l}' \) are the gas and liquid depths above the maximum wave height. For infinitesimal disturbances, \( C_1 \to 1 \), and eqn. (1.2) is recovered. They further speculate that they can approximate \( C_1 = 1 - \frac{h_l}{h_g} \). Using an average \( \frac{h_l}{h_g} = 0.5 \) recovers the final Wallis & Dobson condition (eqn. (1.2) with the 0.5 weighted correction). In their analysis, the equilibrium liquid depth \( h_l \)
is determined through the equilibrium momentum balance between the two phases

\[-A_l \left( \frac{dP}{dz} \right) - \tau_l S_l + \tau_i S_i = 0 \quad (1.5a)\]

\[-A_g \left( \frac{dP}{dz} \right) - \tau_g S_g - \tau_i S_i = 0 \quad (1.5b)\]

where $S_i$ is the interfacial surface area, $A_{g/l}$ is the cross sectional area occupied by the gas and liquids, and $\tau_{i/g/l}$ is the shear stress interface, gas, and liquid respectively. The pressure gradient was eliminated between the two phases and the shear stresses were evaluated through conventional friction factors. Therefore, the use of friction factors allows the solution to have a weak dependence on viscosity.

An additional slug transition model was developed by Lin & Hanratty [43]. Their work differs from the Kelvin-Helmholtz theory by by including viscous and interial terms. They also account for the shear stress at the interface along with the interfacial pressure which is out of phase with the wave height. Their model accounts for either turbulent gas-turbulent liquid or turbulent gas-laminar liquid flows. They also modified the theory to account for either channel or pipe geometries.

Over the past 50+ years of research, many other slug transition criteria have been proposed. Each of which makes a range of assumptions, utilizes different empirical models, often with conflicting assumptions such as long wave assumption while including the effects of surface tension. The work by Mata et al. [52] carried out a comparison of a number of existing slug transition criterion and showed that a wide range of stability predictions are made. Their work also showed that none of the methods demonstrated consistently good agreement against experimental measurements.

### 1.3 Nonlinear Resonant Interaction Theory

One commonality of the methods described in §1.2 was that the transition criteria which were developed were used to determine whether long-wavelength disturbances
were linearly unstable. The experimental observations described in §1.1 has shown that slugs form through either the evolution of short waves into large-amplitude long waves or wave coalescence. Fan, Lusseyran & Hanratty [24] studied the formation of slugs in horizontal pipes and used power spectra from the wave heights along the pipe to demonstrate the presence of a mechanism which creates a cascade of energy from short to long-wavelength components. Similarly, Jurman, Deutsch & McCready [40], carried out experiments for two-fluid stratified flows through a horizontal channel and used wave probes to recover the spectral evolution of the interface as a function of fetch. Strong energy transfer from short to long waves was observed and in some cases there was a strong subharmonic energy transfer.

The observations of energy transfer across the wave spectrum is similar to the trends observed in ocean surface wave environments. Phillips[65] was the first to consider the effects of weak, nonlinear resonant wave-wave interactions in an ocean wave field. His work determined that these nonlinear interactions were responsible for transferring significant amounts of energy across the wavenumber spectrum and provided significant insight into the mechanics of the evolution of surface-gravity waves. Phillips’ work, and a large number of follow up papers, such as Longuet-Higgins[81] works, apply a regular perturbation scheme to determine modal growth rates and quantify the rate of energy transfer among interacting waves. However, the time range over which this scheme is applicable is steepness limited. Subsequent work by Benney[7] and McGoldrick[53] applied the method of multiple scales to extend the theory of resonant wave-wave interactions by developing coupled nonlinear interaction equations for a discrete set of resonant wave modes. McGoldrick derived closed-form solutions in terms of Jacobi elliptic functions, which described the interfacial elevation of these discrete modes and demonstrated energy conservation (for nondissipative conditions). This multi-scale expansion was demonstrated to be accurate for times up to an order of magnitude longer than the traditional regular perturbation scheme. Since Phillips’ 1960 paper, the theory of resonant wave-wave interactions among surface gravity waves has been a subject of active research and has reached maturity. By incorporating the effects of nonlinearity, which have been neglected from
the majority of the previous slug transition studies, it may be shown that resonant wave-wave interaction play a dominant role in the interfacial evolution and must be accounted for in predicting the development of large-amplitude long interfacial waves in stratified flows.

1.4 Previous Modeling Efforts

Due to the frequency for which multiphase flows occur within industrial systems, a significant amount of effort has been dedicated to developing effective and robust numerical simulators. However, since gas-liquid flows offer a wide range of flow regimes, the accurate prediction of the phase behavior and pressure gradient across the pipe remains an active area of development. In 1969, Wallis developed a unit cell method which consists of a long air-bubble and an arrested liquid slug. The method describes the flow by the average properties of both the bubbles and slugs. The problem was further reduced by approximating the intermittent flow physics by a periodic assumptions allowing for continuity and momentum equations to be in the frame of reference of the slug resulting in a steady flow model. The model is further supplemented by closure models for variables such as the bubble velocity, void fraction, slug length, slug frequency, interfacial and wall shear stresses, etc.

One of the leading industry slug simulators is a dynamic two-phase flow model called OLGA[6]. In this method, separate gas and liquid continuity equations are solved with interphasial mass transfer terms which account for droplet dispersion, phase changes, etc. Additionally, the momentum equations are formulated for the gas (and possible dispersed droplets) and another for the liquid film layer. To reduce the complexity of the governing equations, the momentum equations are expressed as 1-D phase averaged equations. As a result, the numerical scheme solves the reduced order equations using large time steps and spatial discretizations. Closure relationships for interfacial friction factors and wetted perimeters are flow regime dependent and classified as either a distributed phase (slug and bubble flows) or separated flow (stratified, annular). The friction factors are closed based on specifying if the flow

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is laminar or turbulent (though in practice, the maximum value is chosen) in which case

\[ \lambda_{turbulent} = 0.0055 \left[ 1 + \left( \frac{2 \times 10^4 \epsilon}{d_h} + \frac{10^6}{Re} \right)^{1/2} \right] \]  \hspace{1cm} (1.6)

and

\[ \lambda_{laminar} = \frac{64}{Re} \]  \hspace{1cm} (1.7)

where \( \epsilon \) is the absolute pipe roughness and \( d_h \) is the hydraulic diameter.

It is apparent that the complexity of the multiphase flow problem requires industrial flow simulators to invoke strong simplifying assumptions which may have drastic impact on the resulting flow predictions. Experimental measurements and empirical correlations are often used as guidance for code performance tuning; however, simply changing the fluid densities, viscosities, pipe diameters, and so on can result in significantly different flow behavior making the existing empirical correlations ineffective. As such, flow assurance engineers invoke large safety factors to account for the design uncertainty.

To alleviate the reliance on non-physics based prediction methods, detailed investigations into the underlying fundamental physics must be undertaken. The root cause of the flow intermittency and nonlinear flow physics must be examined and quantified such that more reliable physics based models can be developed and used to supplement commercial slug simulators. Even without solving the entire multiphase flow problem, simple common modeling terms can be examined and replaced with more robust models. Physics based estimations of the slug length and slug frequency could prove to be valuable for designers of liquid-gas separation facilities as well as provide guidance to the proper initial conditions for OLGA simulations. Examination of the detailed interfacial stresses in multiphase flows could provide more accurate and robust interfacial friction factors resulting in significantly improved momentum conservation equations.
1.5 Viscous Stability Analysis

§1.2 discussed that traditional slug modeling was initially formulated around the classical Kelvin-Helmholtz instability. As it became evident that that mechanism was unable to generate instability among long wave modes, the theory was supplemented with additional physics and closure models. However, as was mentioned in §1.4, the resulting theoretical predictions rely on empirical correlations which are not universally valid. As such, a physics based investigation into the viscous stability may provide insight into underlying mechanics and provide guidance in the development of physics based closure models.

The work of Yih [84] provided insights into the interfacial instability in a viscous two-phase channel flow. It was shown that within each fluid, the problem is governed by the classical Orr-Sommerfeld equation along with the appropriate wall and interfacial boundary conditions. Asymptotic analysis uncovered that viscosity stratification alone could provide a source of instability and further demonstrated the complexity of predicting the interfacial evolution. Over the years, a number of different groups carried out similar investigations developing asymptotic and numerical methods for the accurate evaluation of the coupled Orr-Sommerfeld solution[17][82]. As computing capabilities became more advanced, direct numerical simulations of laminar two-phase flows were carried out allowing for the linear theory to be confirmed and the resulting deviation from linear theory to nonlinearity to be examined [17][77].

Further investigations proceeded to examined the influence of near-interface turbulence on the behavior of coupled gas-liquid flows [48][44]. It was found that the coupling between the two fluids forces the turbulence to satisfy interfacial velocity and stress boundary conditions. The presence of the viscosity and density discontinuities result in different turbulent behaviors in the gas/liquid interfacial layers. The wall layers appeared to maintain resemblance to the classical single-phase wall turbulence; however, in the vicinity of the interface, the Reynolds stresses, and other statistical quantities, were significantly altered compared to the wall region of the channel flow. Vortex structures became coupled across the interface resulting in complex vortex
Additional studies were carried out involving direct numerical simulations of turbulent flow over flexible wavy walls [70][2]. Within this field, it was found that the wavy boundary could have significant effects on the mean flow and turbulent statistics. The mean velocity profile shows a strong departure from the flat-wall case (especially in the vicinity of the interface). As the wave steepness increases, these differences become more pronounced. High shear stress regions form downstream of the wave troughs and remain coupled to vortex structures.

These investigations into the influence of turbulence on coupled two-phase flows suggest that the common modeling practices of borrowing from single phase theory and directly applying to a two-phase flow may introduce a significant amount of error. Advanced models which account for interfacial wave mechanics are necessary in order to capture the correct stresses, flow profiles, and resulting nonlinear evolution. While a large body of work exists within the literature, there remains a large number of questions which need to be addressed before this theory can be applied to the subject of slug transition.

1.6 Scope of Thesis

The purpose of this thesis is to examine the role of nonlinearity in the evolution of interfacial waves in the transition from stratified to wavy and slug flow. This is accomplished through three key contributions.

Chapter 2 is dedicated to investigating the initial growth and nonlinear evolution of the interface under the influence of a linear instability. §1.1 cited that one of the common mechanisms which causes slug formation is a strong spectral energy cascade such as the one depicted in the experiments by Fan et al. [24]. Since linear theory is incapable of accounting for such a process, this work investigates the effects of coupling an interfacial instability with nonlinear wave-wave interactions. In doing so, a new energy cascade mechanism is found which permits the rapid transfer of energy from linearly unstable short waves to stable long waves. An analytical and numerical
investigation supporting this theory is derived in §2. Since several experiments report a strong sub-harmonic bifurcation in the energy cascade resulting in slug formation, an additional analytical study is carried out in §3 in which sub-harmonic resonant interactions under the influence of the Kelvin-Helmholtz instability.

The second thesis contribution examines the role of viscosity and turbulence on the nonlinear stability of stratified flows. In §2 & 3, the potential flow assumptions were invoked in order to make the analytical nonlinear analysis tractable. While the theoretical predictions were shown to be consistent with experiments, the implications of the inviscid assumption remained to be demonstrated. Experimental measurements carried out by Jurman et al., for high viscosity liquids, exhibited wave spectra with strong nonlinear behavior. The work described in §6 carries out both theoretical and numerical studies for the purpose of examining the nonlinear interfacial development under the influence of gas turbulence and high liquid viscosity. In doing so, it is possible to justify if the coupled instability-resonance model derived from potential flow is still valid for more complex (non-ideal) flows. Additionally, the high resolution simulations which are carried out in support of this task allow for the examination of common modeling techniques used in industrial slug simulators and simple slug transition models. The interfacial stresses can be recovered from the numerical simulations allowing for the dominant flow features to be identified and quantified. Such a result would allow for the validation of current closure model techniques or identify the necessary mechanisms which need to be accounted for in future interfacial stress models.

The final area of investigation in §7 is dedicated to taking the concepts identified by the coupled instability-resonant interaction mechanism and incorporating it into a novel nonlinear transition criterion which is capable of predicting the formation of slugs in horizontal channels and pipes.
Chapter 2

Triad Resonant Interaction Theory

We consider the problem of nonlinear resonant interactions of interfacial waves with the presence of a linear interfacial instability in an inviscid two-fluid stratified flow through a horizontal channel. The resonant triad consists of a (linearly) unstable wave and two stable waves, one of which has a wavelength that can be much longer than that of the unstable component. Of special interest is the development of the long wave by energy transfer from the base flow due to the coupled effect of nonlinear resonance and interfacial instability. By use of the method of multiple scales, we derive the interaction equations which govern the time evolution of the amplitudes of the interacting waves including the effect of interfacial instability. The solution of the evolution equations shows that depending on the flow conditions, the (stable) long wave can achieve a bi-exponential growth rate through the resonant interaction with the unstable wave. Moreover, the unstable wave can grow unboundedly even when the nonlinear self-interaction effect is included, as do the stable waves in the associated resonant triad. For the verification of the theoretical analysis and the practical application involving a broadband spectrum of waves, we develop an effective direct simulation method, based on a high-order pseudo-spectral approach, which accounts for nonlinear interactions of interfacial waves up to an arbitrary high order. The direct numerical simulations compare well with the theoretical analysis for all the characteristic flows considered, and agree qualitatively with the experimental observation of slug development near the entrance of two-phase flow into a pipe.
2.1 Introduction

In this work, we investigate a nonlinear mechanism for the generation and evolution of long waves on the interface in a two-layer density-stratified flow through a horizontal channel under the influence of a linear interfacial instability and nonlinear resonant wave interactions. This work is motivated by the observations of a unique class of large wave disturbances which can occur in horizontal channels and pipes. Under certain flow conditions, it is possible for short waves to form at the interface and grow into large amplitude long waves which bridge the channel and touch the top trapping long bubbles of one fluid within the other. This phenomena, known as slug flow, has been well documented experimentally, but theoretically understanding the underlying mechanisms and properly defining the critical flow conditions for slug formation remains an active area of research.

Early theoretical work on slug prediction was based on the classical Kelvin-Helmholtz instability criteria for infinitesimal waves at the interface of a stratified flow. Experimental trials found that this criteria made poor predictions of the upper fluid velocity at which the slug transition occurs. Numerous works were dedicated to modifying that criteria by including additional physical effects such as the interfacial and wall friction (e.g. Lin & Hanratty[43]; Barnea & Taitel[5]) and normal viscous stresses at the interface (e.g. Funada & Joseph[27]). The work by Taitel & Dukler[73] attempted to improve the predictions by examining the effects of finite-amplitude waves. However, that work simply assumed the existence of a finite amplitude state within the channel and did not examine the mechanism(s) leading to the wave's formation. The results of these previous efforts have been a wide range of stability predictions as demonstrated in the survey by Mata et al.[52].

One commonality with these methods was that the transition criteria which were developed were used to determine if long wavelength disturbances were unstable in the stratified flow. Experimental observation has shown that slugs form through either the evolution of short waves into large amplitude long waves or wave coalescence. Fan, Lusseyran & Hanratty[24] studied the formation of slugs in horizontal pipes and
used power spectra from the wave heights along the pipe to demonstrate the presence of a mechanism which was creating a cascade of energy from short to long wavelength components. Jurman, Deutsch & McCready[40], carried out experiments for two-fluid stratified flows through a horizontal channel and used bi-coherence spectrum to examine the spectral evolution of the interface. Strong energy transfer from short to long waves was observed and in some cases there was a strong subharmonic energy transfer. These characteristic behaviors are impossible to see from linear theory because it does not permit wave interactions. This suggests that nonlinear interactions, which have been neglected from the majority of the previous studies, may play a dominant role in the interfacial evolution and must be accounted for in predicting the development of large-amplitude long interfacial waves in stratified flows.

Nayfeli & Saric[61] used the method of multiple scales to develop a third-order amplitude equation which governed the nonlinear evolution of a finite-amplitude wave on the interface of a two fluid density stratified flow of infinite depth. Their analysis considered a single linearly unstable mode and found that depending on the flow conditions it was possible for the nonlinear solution to grow unboundedly. Similar work was also carried out by Maslowe & Kelly [51] and Drazin[23]. Pedlosky[64] also studied the nonlinear evolution of an interface in the presence of a linear instability within the context of baroclinic waves. These methods provided a basis for understanding the nonlinear effects upon the growth of linearly unstable waves.

While the results of Nayfeli & Saric[61] provided the methods necessary to examine the nonlinear evolution of linearly unstable waves, the results lacked the means to generate large amplitude long waves from unstable short waves. The observations of energy transfer across the wave spectrum is similar to the effects observed in ocean surface wave environments. Phillips[65] was the first to consider the effects of weak, nonlinear resonant wave-wave interactions in an ocean wave field. His work determined that these nonlinear interactions were responsible for transferring significant amounts of energy across the wavenumber spectrum and provided significant insight into the mechanics of the evolution of surface-gravity waves.

Phillips' work, and a large number of follow up papers, such as Longuet-Higgins[81]
works, apply a regular perturbation scheme to determine modal growth rates and quantify the rate of energy transfer among interacting waves. However, the time range over which this scheme is applicable is steepness limited. Subsequent work by Benney[7] and McGoldrick[53] applied the method of multiple scales to extend the theory of resonant wave-wave interactions by developing coupled nonlinear interaction equations for a discrete set of resonant wave modes. McGoldrick derived closed-form solutions in terms of Jacobi elliptic functions, which described the interfacial elevation of these discrete modes and demonstrated energy conservation (for non-dissipative conditions). This multi-scale expansion was demonstrated to be accurate for times up to an order of magnitude longer than the traditional regular perturbation scheme. Since Phillips’ 1960 paper, the theory of resonant wave-wave interactions among surface gravity waves has been a subject of active research and has reached maturity.

Janssen[35, 36] considered the effects of resonant interactions between a primary wave and its second harmonic (referred to as a second harmonic resonance or an overtone resonance). His work found that this class of resonant interactions is responsible for the observed period doubling behavior seen in spectral measurements. More recently, Bontozoglou & Hanratty[8] speculated that finite-amplitude Kelvin-Helmholtz waves undergo an internal second harmonic resonance which would result in the doubling of the wavelength of the unstable wave. It was believed that this could be part of the initial mechanism which would lead to the formation of slugs.

Recently Romanova & Annenkov[67] studied three-wave resonant interactions in a multi-layer stratified flow using a Hamiltonian formulation. They derived a set of coupled nonlinear interaction equations for the evolution of a resonant triad with one interacting wave component being linearly unstable. They found that the resonant interaction with stable waves can stabilize the growth of the linearly unstable wave. Similar work was also carried out in the study of baroclinic wave dynamics based on a quasi-geostrophic two-layer model by Mansbridge & Smith[50], Loesch[47], and Pedlosky[64]. All of these studies did not focus on the growth of stable waves in the resonance. In addition, the influence of the zero-th harmonic (resulted from
quadratic self-interactions in finite depth) on the evolution of the interacting waves was not accounted for.

In this work, we study theoretically and computationally the effects of nonlinear resonant wave interactions coupled with interfacial instability upon the development of long waves on the interface of a two-fluid stratified flow. We consider a two-dimensional canonical problem of triad interfacial wave resonance involving one unstable short wave, which is linearly unstable due to the Kelvin-Helmholtz mechanism, and two stable waves in a two-layer stratified horizontal channel flow. Based on the observation of slug flow experiments, it is of interest to have one of the stable waves in the resonant triad with a wavelength much larger than that of the unstable wave. Since our focus is on the understanding of the nonlinear mechanism for energy transfer from short unstable waves to long stable waves, we assume simple uniform base flows for the two fluids and formulate the problem in the context of potential flow (§2.1). We derive the evolution equations for the amplitudes of the interacting waves, including both interfacial instability and resonant wave interaction effects, by the use of the method of multiple scales (§2.2.4). Based on the evolution equations, we analyze the characteristic features of triad resonance and nonlinear interfacial instability. Of particular interest is that under certain flow conditions, there exists a strong mechanism for effectively transferring energy from (unstable) short waves to (stable) long waves (§2.2.5). For validation of the theory and application to realistic situations involving multiple resonances, an effective numerical method based on the high-order pseudo-spectral approach is developed (§2.3). The theory and numerical simulation are cross validated for the characteristic cases presented. Moreover, the trends observed in the direct simulation agrees qualitatively with the experimental measurement of initial slug time/length for a two-layer flow entering into a horizontal pipe (§2.4). This work provides an insight into the basic nonlinear physics that may play a significant role in the initial development of slugs in stratified channel/pipe flows.
2.2 Theoretical Analysis

This analysis considers the nonlinear evolution of interfacial waves propagating through a stratified two-fluid horizontal channel. It is of fundamental interest to understand the characteristic features of wave energy transfer associated with triad resonant interaction, particularly when one of the wave components in the triad is linearly unstable to the Kelvin-Helmholtz mechanism.

2.2.1 Fully Nonlinear Governing Equations

A fixed Cartesian coordinate system is established with the origin located at the undisturbed interface between the two fluids with the $x$-axis extending horizontally to the right and the $y$-axis being directed vertically upwards. The fluids have equilibrium depths of $h_u$ and $h_l$ with the upper and lower fluids being denoted by the subscripts $u$ and $l$ respectively. The vertical displacement of the interface away from its undisturbed position is defined by the function $y = \eta(x,t)$. The two fluids, which are assumed to be immiscible, are of density $\rho_u$ and $\rho_l$, with $\rho_u < \rho_l$. The effects of gravity $g$ and surface tension $\gamma$ are also taken into account. A sketch of the problem is illustrated in figure 2-1.

The flow in each domain is decomposed into a constant uniform current ($U_u$ and $U_l$) and a disturbance flow. It is assumed that both flows are incompressible and irrotational such that the velocity of each fluid is defined by the gradient of its potential function, $\varphi_u(x,y,t) = U_u x + \phi_u(x,y,t)$ and $\varphi_l(x,y,t) = U_l x + \phi_l(x,y,t)$. The disturbance potentials ($\phi_u$ and $\phi_l$) must satisfy Laplace's equation in the fluid.
domain:

\[ \nabla^2 \phi_u = 0, \quad \eta < y < h_u \quad (2.1) \]
\[ \nabla^2 \phi_l = 0, \quad -h_l < y < \eta. \quad (2.2) \]

At the channel walls, the no flux conditions are enforced as

\[ \phi_{u,y} = 0, \quad y = h_u \quad (2.3) \]
\[ \phi_{l,y} = 0, \quad y = -h_l. \quad (2.4) \]

Requiring that the interface remain material produces

\[ \eta_t + (U_u + \phi_{u,x}) \eta_x = \phi_{u,y}, \quad y = \eta \quad (2.5) \]
\[ \eta_t + (U_l + \phi_{l,x}) \eta_x = \phi_{l,y}, \quad y = \eta. \quad (2.6) \]

while the balance of normal stresses at the interface between the two fluids gives

\[
\mathcal{R} \left[ \phi_{u,t} + \frac{1}{2} (\nabla \phi_u)^2 + U_u \phi_{u,x} + \eta \right] - \left[ \phi_{l,t} + \frac{1}{2} (\nabla \phi_l)^2 + U_l \phi_{l,x} + \eta \right] \\
= -\frac{\eta_{xx}}{\mathcal{W}} (1 + \eta_x^2)^{-\frac{3}{2}}, \quad y = \eta \quad (2.7)
\]

where \( \mathcal{R} \equiv \rho_u/\rho_l \) is the density ratio and \( \mathcal{W} \equiv \mathcal{L}^2 g \rho_l / \gamma \) is the Weber number. In the above equations, the quantities are non-dimensionalized in term of the characteristic length \( \mathcal{L} \) and time \( T = (\mathcal{L}/g)^{1/2} \). This problem is complete with the specification of an appropriate set of initial conditions for \( \phi_u, \phi_l \) and \( \eta \).

\subsection*{2.2.2 Linear Theory And The Kelvin-Helmholtz Instability}

For the purpose of better understanding the nonlinear analysis in the following sections, it is beneficial to review the key findings from the classical linear theory. The
linearization of (3.1-3.7) in terms of the small interfacial wave steepness \( \epsilon \) produces

\[
\begin{align*}
\nabla^2 \phi &= 0, \quad 0 < y < h_u \\
\nabla^2 \phi &= 0, \quad -h_t < y < 0 \\
\phi_{u,y} &= 0, \quad y = h_u \\
\phi_{u,y} &= 0, \quad y = -h_t \\
\eta_t + U_u \eta_x - \phi_{u,y} &= 0, \quad y = 0 \\
\eta_t + U_t \eta_x - \phi_{t,y} &= 0, \quad y = 0 \\
\mathcal{R} (\phi_{u,t} + U_u \phi_{u,x} + \eta) - (\phi_{t,t} + U_t \phi_{t,x} + \eta) + \frac{\eta_{xx}}{\mathcal{W}} &= 0, \quad y = 0.
\end{align*}
\]

A traveling wave solution of (2.8) takes the form:

\[
\begin{align*}
\eta &= \frac{\eta_0}{2} e^{i(kx - \omega t)} + \text{c.c.} \\
\phi_u &= \frac{-i (U_u k - \omega)}{2k \tanh kh_u} \cosh k(y - h_u) e^{i(kx - \omega t)} + \text{c.c.} \\
\phi_t &= \frac{i (U_t k - \omega)}{2k \tanh kh_t} \cosh k(y + h_t) e^{i(kx - \omega t)} + \text{c.c.}
\end{align*}
\]

where \( \eta_0 \) is the amplitude of the (initial) wave disturbance, \( k \) is the wavenumber, and \( \omega \) is the frequency. The symbol “c.c.” represents the complex conjugate of the preceding term(s). The frequency \( \omega \) is related to the wavenumber \( k \) by the dispersion relation:

\[
\omega = \frac{k (U_u \mathcal{R} T_l + U_l T_u)}{\mathcal{R} T_l + T_u} \pm k \left[ \frac{1}{k} \left( \frac{T_u T_l}{\mathcal{R} T_l + T_u} \right) \left( 1 - \mathcal{R} + \frac{k^2}{\mathcal{W}} \right) - \mathcal{R} \frac{(U_u - U_l)^2 T_u T_l}{(\mathcal{R} T_l + T_u)^2} \right]^{\frac{1}{2}}
\]

(2.10)

where \( T_{u/l} \equiv \tanh kh_{u/l} \). From (2.10), it is clear that \( \omega \) is a complex number if \( |U_u - U_l| > U_c \) with the critical velocity \( U_c \) defined as

\[
U_c(k) = \left[ \frac{\mathcal{R} T_l + T_u}{\mathcal{R} k} \left( 1 - \mathcal{R} + \frac{k^2}{\mathcal{W}} \right) \right]^{\frac{1}{2}}.
\]

(2.11)
Under this condition, the wave (of wavenumber $k$) is unstable with its amplitude growing exponentially with time by drawing energy from base flows.

Without loss of generality, we assume $U_u > U_i$ in the following analysis and we consider the case with $U_u - U_i$ slightly exceeding $U_c$, i.e., $U_u - U_i = U_c (1 + \Delta)$ where $0 < \Delta \ll 1$. In this case, the frequency can be written as

$$\omega \equiv \omega_R + i\omega_I = \frac{k(U_uRT_i + U_iT_u)}{RT_i + T_u} \pm i \left[ \frac{2kT_uT_i}{RT_i + T_u} \left( 1 - \frac{k^2}{\mathcal{W}} \right) \right]^{1/2} \Delta^{1/2} + O(\Delta^{3/2}) \quad (2.12)$$

Clearly, the growth rate $\omega_I = O(\Delta^{1/2})$ while $U_u - U_i - U_c = O(\Delta)$.

2.2.3 Triad Resonant Wave-Wave Interaction

In the context of linear theory, waves of different wavelengths (or frequencies) travel independently in time/space. When nonlinear interactions among them are accounted for, locked waves are generated. The amplitudes of the locked waves are generally of higher order compared to the primary waves. If the frequency and wavenumber of the locked wave satisfy the dispersion relation (2.10), the locked wave becomes a free wave. In this case, the interaction becomes resonant. As a result, the generated free wave can grow significantly with its amplitude being comparable to that of the primary waves. Resonant interactions are known to play a critical role in the evolution of nonlinear ocean surface waves as they cause energy transfer among different wave components in the wave spectrum (e.g. Phillips[65]).

In this study, we consider a triad resonant interaction in which one of the primary waves is unstable due to the Kelvin-Helmholtz effect. The focus is on the mechanism of energy transfer from the unstable wave to the stable waves in the triad. For definiteness, we consider a triad consisting of three free waves with wavenumbers $k_1$, $k_2$, and $k_3$. Without loss of generality, we let $k_3 < k_1 < k_2$ with the $k_2$ wave being unstable (and $k_1$ and $k_3$ waves being stable). Unlike in the conservative wave system in which the frequencies of interacting waves involved are all real, the frequency of the unstable $k_2$ wave is complex in the present case. The multiple-scale analysis
commonly used in the conservative wave system cannot be directly applied here. The evolution of the amplitudes of interacting waves in the triad is now affected not only by the resonant interaction but also by the interfacial instability. The time scales of these two processes need to be properly considered in the analysis. To obtain a basic understanding of the interaction mechanism of these two processes, we consider the $k_2$ wave to be slightly unstable with $U_u - U_l = U_c(1 + \Delta)$, $\Delta \ll 1$. In the present analysis, we choose to expand the interaction problem at $U_u - U_l = U_c(1 + \Delta)$ with respect to the marginally stable state at $U_u - U_l = U_c$ in terms of $\Delta$ (Loesch[47]).

A sketch of the interacting waves in the plane formed by wavenumber and velocity jump is illustrated in figure 2-2(a).

At the marginally stable state, the $k_2$ wave is neutrally stable. Following the analysis of Phillips[65] for conservative resonances, a resonant triad involving $k_1$, $k_2$ and $k_3$ waves are formed under the condition: $k_2 - k_1 = k_3$ and $\omega_2 - \omega_1 = \omega_3$, where $\omega_1(k_1)$ and $\omega_3(k_3)$ are given by (2.10) while for a given $k_2$, the frequency, $\omega_2(k_2)$, is given by the real part of (2.12)). Figure 2-2(b) shows a typical result of $k_1$ and $k_3$ (normalized by $k_2$) as a function of $U_c(k_2)$. The result shown corresponds to the left branch of the neutral stability curve in 2-2(a). For lower $U_c$, the triad resonance involving long and short waves exist. There also exists a subharmonic resonance between $k_2$ and its subharmonic $k_1 = k_3 = 1/2 k_2$. For larger $U_c$, the triad resonance converges to its special case of sub-harmonic resonance. In this work, the analysis is focused on the case of general triad resonance with $k_1 \neq k_3$ (at relatively lower $U_c$). The sub-harmonic resonance is to be analyzed in a separate study.

In the following analysis, for generality, we consider a near-resonance triad in which the wavenumbers and frequencies of the interacting waves have the relations

$$\begin{cases}
    k_2 - k_1 = k_3 \\
    \omega_2 - \omega_1 = \omega_3 + \sigma
\end{cases}$$

(2.13)

where $\sigma$ (with $\sigma \ll \omega_j, j = 1, 2, 3$) represents the frequency detuning. Note that the triad resonance condition is satisfied exactly when $\sigma = 0$. In addition, for both simplicity and clarity, the second harmonic of the $k_2$ wave is assumed to be stable. When
this constraint is relieved, the analysis procedure is similar, but a different combination in the order of magnitudes of the interacting waves needs to be considered.

2.2.4 Multiple-Scale Analysis

In this section we shall derive the analytic equations governing the time evolution of the amplitudes of the interacting waves in the triad by use of the method of multiple scales which captures the combined effect of the nonlinear Kelvin-Helmholtz instability and resonant wave-wave interaction.

Perturbation Expansions

As (2.12) indicates, the growth rate of the \( k_2 \) wave is of \( O(\Delta^{\frac{1}{2}}) \). Thus, there exists two distinct time scales. One is the fast time \( t \) associated with the rapid change of the phases of the waves. The other is the slow time \( \tau \) associated with the time variation of the amplitudes of the waves. Clearly, we have \( \tau = \Delta^{\frac{1}{2}} t \) for the present problem. For the perturbation analysis, we assume that the steepness of the interface is small with \( O(\epsilon) = O(\Delta^{\frac{1}{2}}) \). For convenience, we expand the velocity potentials \( \phi_u \) and \( \phi_l \) as well as the interfacial displacement \( \eta \) in a regular perturbation expansion of the form

\[
\phi_u (x, y, t, \tau) = \sum_{m=1}^{5} \Delta^{\frac{m+1}{4}} \phi_u^{(m)} (x, y, t, \tau) + O\left(\Delta^{\frac{3}{4}}\right) \tag{2.14a}
\]

\[
\phi_l (x, y, t, \tau) = \sum_{m=1}^{5} \Delta^{\frac{m+1}{4}} \phi_l^{(m)} (x, y, t, \tau) + O\left(\Delta^{\frac{3}{4}}\right) \tag{2.14b}
\]

\[
\eta (x, t, \tau) = \sum_{m=1}^{5} \Delta^{\frac{m+1}{4}} \eta^{(m)} (x, t, \tau) + O\left(\Delta^{\frac{3}{4}}\right) \tag{2.14c}
\]

where \( \phi_u^{(m)}, \phi_l^{(m)} \) and \( \eta^{(m)}, m = 1, \ldots, 5, \) are \( O(1) \). This expansion is established under the assumption that the amplitude of the unstable wave \( \langle k_2 \rangle \) is \( O(\Delta^{\frac{1}{2}}) \) while that of the other two stable waves \( \langle k_1, k_3 \rangle \) are \( O(\Delta^{\frac{3}{4}}) \). With this order arrangement, the effect of the cubic self-interaction on the evolution of the \( k_2 \) wave could be comparable to that of the quadratic (resonant) interactions of the \( k_1 \) and \( k_3 \) waves. Following the standard procedure of the multiple-scale perturbation analysis, the nonlinear
Figure 2-2: (a) Sketch of the neutral stability curve (—) and wavenumbers of the primary waves forming a resonant triad \((k_1, k_2, k_3)\) along with the second harmonic \((2k_2)\) of the \(k_2\) wave. (b) Normalized wavenumbers \(k_1/k_2\) (· · · ) and \(k_3/k_2\) (— — ) in a resonant triad as a function of the critical velocity \(U_c(k_2)\). The curve (— — ) in (b) represents the sub-harmonic resonance with \(k_1/k_2 = k_3/k_2 = 0.5\). The results are obtained with \(\mathcal{R} = 1.23 \times 10^{-3}, \mathcal{W} \approx 845.5, H = 1.25, \alpha \equiv h_u/H = 0.5,\) and \(U_l = 1.13\). (With \(L \approx 0.08\) m and \(T \approx 0.09\) s, as an example, these parameters correspond to an air-water flow in a horizontal channel with fixed \(h_u = h_l = 0.05\) m, \(U_l = 1.0\) m/s, and \(\gamma = 0.073\) N/m.)
boundary-value problem for $\phi_u$ and $\phi_t$ is decomposed into a sequence of linearized boundary-value problems for $\phi_u^{(m)}$ and $\phi_t^{(m)}$, $m=1,2,\ldots,5$, which are presented in Appendix A. These problems are then solved sequentially starting from $m=1$.

The $O\left(\Delta^1\right)$ Solution

At this order, the boundary-value problem is identical to that outlined in §2.2.2. The solution for the linearly unstable $k_2$ wave is

$$\phi_u^{(1)}(\tau) = \frac{\cosh k_2(y-h_u)}{\cosh k_2 h_u} \left( \mathcal{P}_2(\tau) e^{i\psi_2} + c.c. \right) + \phi_w^{(1)}(\tau)$$

$$\phi_t^{(1)}(\tau) = \frac{\cosh k_2(y+h_t)}{\cosh k_2 h_t} \left[ \mathcal{Q}_2(\tau) e^{i\psi_2} + c.c. \right] + \phi_t^{(1)}(\tau)$$

$$\eta^{(1)}(\tau) = \frac{A_2(\tau)}{2} e^{i\psi_2} + c.c.$$  \hspace{1cm} (2.15c)

where $\mathcal{P}_2$, $\mathcal{Q}_2$ and $\psi_2$ are given from the general expressions:

$$\mathcal{P}_j(\tau) = \frac{-i\mathbb{D}_{uj} A_j(\tau)}{2k_j \tanh k_j h_u}, \quad \mathcal{Q}_j(\tau) = \frac{i\mathbb{D}_{ij} A_j(\tau)}{2k_j \tanh k_j h_t}, \quad \psi_j = k_j x - \omega_j t$$  \hspace{1cm} (2.16)

and $\mathbb{D}_{uj} \equiv (U_l + U_c) k_j - \omega_j$ and $\mathbb{D}_{ij} \equiv U_l k_j - \omega_j$ with the subscript $j=2$. Note that the boundary-value problem for $\phi_u^{(1)}$ and $\phi_t^{(1)}$ admits space-independent solutions $\phi_w^{(1)}(\tau)$ and $\phi_t^{(1)}(\tau)$ which will be shown to be important in the higher order solutions. The slow-time dependent amplitude, $A_2(\tau)$, and potentials, $\phi_w^{(1)}(\tau)$ and $\phi_t^{(1)}(\tau)$, are governed by the evolution equations to be developed below.
The $O(\Delta^3)$ Solution

At this order, the solution represents the (stable) $k_1$ and $k_3$ waves:

\begin{align*}
\phi_u^{(2)} &= \frac{\cosh k_1(y - h_u)}{\cosh k_1 h_u} \left[ \mathcal{P}_1(\tau)e^{i\psi_1} + c.c. \right] + \frac{\cosh k_3(y - h_u)}{\cosh k_3 h_u} \left[ \mathcal{P}_3(\tau)e^{i\psi_3} + c.c. \right] + \phi_{u0}^{(2)}(\tau) \\
\phi_i^{(2)} &= \frac{\cosh k_1(y + h_i)}{\cosh k_1 h_i} \left[ \mathcal{Q}_1(\tau)e^{i\psi_1} + c.c. \right] + \frac{\cosh k_3(y + h_i)}{\cosh k_3 h_i} \left[ \mathcal{Q}_3(\tau)e^{i\psi_3} + c.c. \right] + \phi_{i0}^{(2)}(\tau) \\
\eta^{(2)} &= \frac{A_1(\tau)}{2} e^{i\psi_1} + \frac{A_3(\tau)}{2} e^{i\psi_3} + c.c.,
\end{align*}

(2.17a)

(2.17b)

(2.17c)

where $\mathcal{P}_{1,3}$, $\mathcal{Q}_{1,3}$ and $\psi_{1,3}$ are given from the general expressions (2.16) with the subscript $j=1$ and 3, respectively. Like the $m=1$ problem, the boundary-value problem at $m=2$ also admits space-independent potentials, $\phi_{u0}^{(2)}(\tau)$ and $\phi_{i0}^{(2)}(\tau)$, which are functions of slow time. The slow-time dependent amplitudes, $A_1(\tau)$ and $A_3(\tau)$, are governed by the evolution equations to be developed below.

The $O(\Delta)$ Solution

The inhomogeneous forcing terms at this order are:

\begin{align*}
f_1^{(3)} &= -\frac{1}{2} \dot{A}_2 e^{i\psi_2} + p_4 A_2^2 e^{2i\psi_2} + c.c. \\
f_2^{(3)} &= -\frac{1}{2} \dot{A}_2 e^{i\psi_2} + d_4 A_2^2 e^{2i\psi_2} + c.c. \\
f_3^{(3)} &= f_2 \dot{A}_2 e^{i\psi_2} + f_7 A_2^2 e^{2i\psi_2} + c.c. + f_8 |A_2|^2 + \phi_{i0}^{(1)} - \mathcal{R}\phi_{v0}^{(1)}
\end{align*}

(2.18a)

(2.18b)

(2.18c)

with the coefficients given by

\begin{align*}
p_4 &= -\frac{1}{2} i k_2 \mathcal{D}_{u2} C_{u2}, & d_4 &= \frac{1}{2} i k_2 \mathcal{D}_{u2} C_{u2}, \\
f_2 &= \frac{1}{2k_3} \left( \mathcal{D}_{u2} C_{i2} + \mathcal{R} \mathcal{D}_{u2} C_{u2} \right), & f_7 &= -\frac{1}{8} \left( (3 - C_{u2}^2) \mathcal{D}_{i2}^2 - \mathcal{R} (3 - C_{u2}^2) \mathcal{D}_{u2}^2 \right), \\
f_8 &= \frac{1}{4} \left( (C_{u2} - 1) \mathcal{D}_{i2}^2 - \mathcal{R} (C_{u2}^2 - 1) \mathcal{D}_{u2}^2 \right)
\end{align*}

50
where $C_{u/j} \equiv \coth \left( k_j h_{u/j} \right)$ and the symbol $\cdot'$ denotes the derivative with respect to slow time $\tau$. There are zero-th, first, and second harmonics in the forcing terms.

The boundary-value solution associated with the zero-th and second harmonic forcing can be obtained directly. Since the first harmonic is the homogeneous solution of the boundary-value problem, the first-harmonic forcing must satisfy the solvability condition, specified by the Fredholm alternative, in order to obtain a nontrivial inhomogeneous solution. By applying Green’s theorem between the $O(\Delta^{\frac{1}{2}})$ and $O(\Delta)$ solutions (for the first harmonic), the solvability condition is found to take the form:

$$\frac{i \dot{A}_2}{k_2} \left[ \mathcal{D}_{12} + \frac{\mathcal{D}_{T_2}}{T_{12}} \right] = G_2 \left[ \frac{\mathcal{D}^2_{12}}{k_2 T_{12}} + \frac{\mathcal{D}^2_{T_2}}{k_2 T_{12}} - 1 + \mathcal{R} - \frac{k_2^2}{\mathcal{W}} \right]$$

(2.19)

where $T_{u/j} \equiv \tanh \left( k_j h_{u/j} \right)$ and $G_2$ denotes the half amplitude of the first-harmonic (propagating wave) of $\eta^{(3)}$. Since $k_2$ and $\omega_2$ satisfy the dispersion relation, it is straightforward to show that the terms in the brackets on both sides of (2.19) are identically zero. Since all of the structure of the propagating $k_2$ wave is contained in the $O(\Delta^{\frac{1}{2}})$ solution, $G_2$ can be set to zero without any loss of generality. The total solution for $\phi_u^{(3)}$, $\phi_i^{(3)}$, and $\eta^{(3)}$ are then written as:

$$\phi_u^{(3)} = \frac{\cosh k_2 (y - h_u)}{\cosh k_2 h_u} \left( \alpha_2 A_2 e^{i \omega_2} + \text{c.c.} \right) + \frac{\cosh 2k_2 (y - h_u)}{\cosh 2k_2 h_u} \left( \alpha_{22} A_2^2 e^{2i \omega_2} + \text{c.c.} \right)$$

(2.20a)

$$\phi_i^{(3)} = \frac{\cosh k_2 (y + h_l)}{\cosh k_2 h_l} \left( \beta_2 A_2 e^{i \omega_2} + \text{c.c.} \right) + \frac{\cosh 2k_2 (y + h_l)}{\cosh 2k_2 h_l} \left( \beta_{22} A_2^2 e^{2i \omega_2} + \text{c.c.} \right)$$

(2.20b)

$$\eta^{(3)} = \nu_{22} \left( A_2^2 e^{2i \omega_2} + \text{c.c.} \right) + \nu_0 |A_2|^2 + \frac{\dot{\phi}_{l0} - \mathcal{R} \dot{\phi}_{u0}}{\mathcal{R} - 1}$$

(2.20c)
where

$$\nu_{22} = \frac{k_2 D_{12}^2 \left[ 2C_4 C_{12} - \frac{1}{3} (3 - C_{12}^2) \right] - \mathcal{R} D_{u2}^2 \left[ 2C_{u2} C_{u2} - \frac{1}{2} (3 - C_{u2}^2) \right]}{8 \mathcal{R} C_4 D_{12}^2 + 8 C_{12} D_{u2}^2 + 4 k_2 \left( \mathcal{R} - 1 - \frac{4k_2^2}{\mathcal{W}} \right)}$$

$$\nu_0 = \frac{(C_{12}^2 - 1) D_{12}^2 - \mathcal{R} (C_{u2}^2 - 1) D_{u2}^2}{4 (\mathcal{R} - 1)}$$

$$\alpha_2 = -\frac{1}{2k_2 T_{u2}}, \quad \alpha_{22} = -i D_{u2} \left( \frac{k_2 C_{u2} + 4 \nu_{22}}{4k_2 T_{u2}} \right)$$

$$\beta_2 = \frac{1}{2k_2 T_{12}}, \quad \beta_{22} = -i D_{12} \left( \frac{k_2 C_{12} - 4 \nu_{22}}{4k_2 T_{14}} \right)$$

with $C_{u/1} = \coth \left( 2k_2 h_{u/1} \right)$ and $T_{u/1} = \tanh(2k_2 h_{u/1})$.

One notes that if the two stable waves ($k_1$ and $k_3$) are set to be of $O(\Delta^{1/2})$ like the unstable wave ($k_2$), an unbalanced resonant interaction term would be present in (2.19). By letting the two stable waves be $O(\Delta^{1/2})$ smaller than the unstable $k_2$ wave, the effects of triad resonant interaction and nonlinear Kelvin-Helmholtz instability will converge together correctly in the $O(\Delta^{3/2})$ problem without the presence of singularities in the lower order solution.

The $O \left( \Delta^{3/2} \right)$ Solution

At this order, the only terms in $f^{(j)}_j$ ($j = 1, 2, 3$) which produce secular terms are those with phase $\psi_1$ and $\psi_3$. Upon neglecting the non-secular terms, $f^{(j)}_j$, $j=1, 2,$ and $3$, takes the form:

$$f^{(4)}_1 = -\frac{1}{2} \hat{A}_1 e^{i\psi_1} - \frac{1}{2} \hat{A}_3 e^{i\psi_3} + p_1 A_1^* A_2 e^{i(\psi_2 - \psi_1)} + p_3 A_2 A_3^* e^{i(\psi_2 - \psi_3)} + c.c. \quad (2.21a)$$

$$f^{(4)}_2 = -\frac{1}{2} \hat{A}_1 e^{i\psi_1} - \frac{1}{2} \hat{A}_3 e^{i\psi_3} + d_1 A_1^* A_2 e^{i(\psi_2 - \psi_1)} + d_3 A_2 A_3^* e^{i(\psi_2 - \psi_3)} + c.c. \quad (2.21b)$$

$$f^{(4)}_3 = f_1 \hat{A}_1 e^{i\psi_1} + f_3 \hat{A}_3 e^{i\psi_3} + f_4 A_1^* A_2 e^{i(\psi_2 - \psi_1)} + f_6 A_2 A_3^* e^{i(\psi_2 - \psi_3)} + c.c. + \dot{\phi}_{00}^{(2)} - \mathcal{R} \dot{\psi}_{00}^{(2)} \quad (2.21c)$$
where \( \psi_2 - \psi_1 = \psi_3 - \omega \tau \) and \( \psi_2 - \psi_3 = \psi_1 - \omega \tau \) with \( \omega = \sigma \Delta^{-1/2} \), and the coefficients are given by

\[
p_1 = -\frac{1}{4} i k_3 \left( D_{u1} C_{u1} + D_{u2} C_{u2} \right), \quad p_3 = -\frac{1}{4} i k_1 \left( D_{u2} C_{u2} + D_{u3} C_{u3} \right)
\]

\[
d_1 = \frac{1}{4} i k_3 \left( D_{i1} C_{i1} + D_{i2} C_{i2} \right), \quad d_3 = \frac{1}{4} i k_1 \left( D_{i2} C_{i2} + D_{i3} C_{i3} \right)
\]

\[
f_1 = \frac{1}{2 k_1} \left( D_{i1} C_{i1} + R D_{u1} C_{u1} \right), \quad f_3 = \frac{1}{2 k_3} \left( D_{i3} C_{i3} + R D_{u3} C_{u3} \right)
\]

\[
f_4 = \frac{1}{4} \left\{ R \left[ D_{u1}^2 + D_{u2}^2 - (1 + C_{u1} C_{u2}) D_{u1} D_{u2} \right] - D_{i1}^2 - D_{i2}^2 + (1 + C_{i1} C_{i2}) D_{i1} D_{i2} \right\}
\]

\[
f_6 = \frac{1}{4} \left\{ R \left[ D_{u1}^2 + D_{u3}^2 - (1 + C_{u1} C_{u3}) D_{u1} D_{u3} \right] - D_{i2}^2 - D_{i3}^2 + (1 + C_{i2} C_{i3}) D_{i2} D_{i3} \right\}
\]

Upon the use of the solvability conditions which can be realized by applying Green’s theorem to the \( O(\Delta^{3/4}) \) and \( O(\Delta^{5/4}) \) solutions, we obtain the evolution equations for \( A_1 \) and \( A_3 \):

\[
\dot{A}_1 = i B_{23} A_2 A_3^* e^{-i \gamma \tau} \quad \text{(2.22a)}
\]

\[
\dot{A}_3 = i B_{12} A_1^* A_2 e^{-i \gamma \tau} \quad \text{(2.22b)}
\]

where

\[
B_{23} = \frac{k_1}{4} \left[ \frac{D_{i1} C_{i1} \left( D_{i2} C_{i2} + D_{i3} C_{i3} \right) - D_{i2}^2 - D_{i3}^2 + (1 + C_{i2} C_{i3}) D_{i2} D_{i3}}{R D_{u1} C_{u1} + D_{i1} C_{i1}} \right] - \frac{R k_1}{4} \left[ \frac{D_{u1} C_{u1} \left( D_{u2} C_{u2} + D_{u3} C_{u3} \right) - D_{u2}^2 - D_{u3}^2 + (1 + C_{u2} C_{u3}) D_{u2} D_{u3}}{R D_{u1} C_{u1} + D_{i1} C_{i1}} \right]
\]

\[
B_{12} = \frac{k_3}{4} \left[ \frac{D_{i3} C_{i3} \left( D_{i1} C_{i1} + D_{i2} C_{i2} \right) - D_{i1}^2 - D_{i2}^2 + (1 + C_{i1} C_{i2}) D_{i1} D_{i2}}{R D_{u3} C_{u3} + D_{i3} C_{i3}} \right] - \frac{R k_3}{4} \left[ \frac{D_{u3} C_{u3} \left( D_{u1} C_{u1} + D_{u2} C_{u2} \right) - D_{u1}^2 - D_{u2}^2 + (1 + C_{u1} C_{u2}) D_{u1} D_{u2}}{R D_{u3} C_{u3} + D_{i3} C_{i3}} \right]
\]

These two interaction coefficients, \( B_{23} \) and \( B_{12} \), are the same as those in the case where all three waves in the resonant triad are linearly stable to the Kelvin-Helmholtz effect, as shown in Campbell[13].
The $O(\Delta^3)$ Solution

The forcing terms $f_j^{(5)}$, $j=1, 2 \text{ and } 3$, are:

\begin{align*}
 f_1^{(5)} &= \left( p_2 A_1 A_3 e^{i\omega t} + p_5 |A_2|^2 A_2 + p_6 A_2 \left( R \phi_w^{(1)} - \phi_0^{(1)} \right) + p_7 A_2 + c.c. \right) e^{i\psi_2} \\
 & \quad + \frac{R \phi_w^{(1)} - \phi_0^{(1)}}{R - 1} - \nu_0 \frac{d}{dt} (|A_2|^2) \tag{2.24a} \\
 f_2^{(5)} &= \left( d_2 A_1 A_3 e^{i\omega t} + d_5 |A_2|^2 A_2 + d_6 A_2 \left( R \phi_w^{(1)} - \phi_0^{(1)} \right) + c.c. \right) e^{i\psi_2} \\
 & \quad + \frac{R \phi_w^{(1)} - \phi_0^{(1)}}{R - 1} - \nu_0 \frac{d}{dt} (|A_2|^2) \tag{2.24b} \\
 f_3^{(5)} &= \left[ f_5 A_1 A_3 + f_9 A_2 + f_{10} A_2 + f_{11} |A_2|^2 A_2 + f_{13} A_2 \left( R \phi_w^{(1)} - \phi_0^{(1)} \right) + c.c. \right] e^{i\psi_2} \\
 & \quad + f_{12} A_2 A_2^* + f_{12} A_2^* A_2 + f_{14} |A_1|^2 + f_{15} |A_3|^2 \tag{2.24c}
\end{align*}
where * denotes the complex conjugate and the coefficients are given by

\[ p_2 = -\frac{1}{4} ik_2 (D_{u1}C_{u1} + D_{u3}C_{u3}) \]
\[ p_5 = \alpha_{22} k_2^2 + \frac{1}{2} ik_2 D_{u2} \left[ \frac{3}{8} k_2 - C_{u2} (\nu_{22} + \nu_0) \right] \]
\[ p_6 = -\frac{ik_2 D_{u2} C_{u2}}{2 (R - 1)} \]
\[ p_7 = -\frac{iU_c k_2}{2} \]
\[ d_2 = \frac{1}{4} ik_2 (D_{l1} C_{l1} + D_{l3} C_{l3}) \]
\[ d_5 = \beta_{22} k_2^2 + \frac{1}{2} ik_2 D_{l2} \left[ \frac{3}{8} k_2 + C_{l2} (\nu_{22} + \nu_0) \right] \]
\[ d_6 = \frac{ik_2 D_{l2} C_{l2}}{2 (R - 1)} \]
\[ d_7 = \frac{1}{4} \left\{ (C_{l1} C_{l3} - 1) D_{l1} D_{l3} - D_{l1}^2 - D_{l3}^2 - R \left[ (C_{u1} C_{u3} - 1) D_{u1} D_{u3} - D_{u1}^2 - D_{u3}^2 \right] \right\} \]
\[ f_9 = \beta_2 - R \alpha_2 \]
\[ f_{10} = -\frac{1}{2} R U_c D_{u2} C_{u2} \]
\[ f_{11} = R \left( ik_2 D_{u2} \alpha_{22} [T_{u4} - C_{u2}] + \frac{1}{2} D_{u2} \left[ \nu_0 + \nu_{22} + \frac{5}{8} k_2 C_{u2} \right] \right) - \frac{3k_4^2}{16 W} + \frac{1}{2} D_{l2}^2 \left[ \frac{5}{8} k_2 C_{l2} - \nu_0 - \nu_{22} \right] + ik_2 D_{l2} \beta_{22} [T_{l4} - C_{l2}] \]
\[ f_{12} = \frac{i}{4} D_{l2} - \frac{i}{2} k_2 \beta_{22} D_{l2} C_{l2} - R \left[ \frac{i}{4} D_{u2} + \frac{i}{2} k_2 \alpha_{22} D_{u2} C_{u2} \right] \]
\[ f_{13} = \frac{D_{l2}^2 - RD_{l2}^2}{2 (R - 1)} \]
\[ f_{14} = \frac{1}{4} \left[ D_{l1}^2 (C_{l1}^2 - 1) - RD_{u1}^2 (C_{u1}^2 - 1) \right] \]
\[ f_{15} = \frac{1}{4} \left[ D_{l3}^2 (C_{l3}^2 - 1) - RD_{u3}^2 (C_{u3}^2 - 1) \right] \]

In (2.24), the non-relevant terms are not included for clarity.

Imposing the solvability condition for the forcing with phase \( \psi_2 \), we obtain the evolution equation for \( A_2 \):

\[ \dot{A}_2 = \Omega A_2 + \mathcal{N} |A_2|^2 A_2 + \mathcal{M} A_2 \left( R \dot{\phi}_u^{(1)} - \dot{\phi}_u^{(1)} \right) + B_{13} A_1 A_3 e^{i \pi r} \]  
(2.25)
with

\[ \Omega = \frac{-RU_2D_{u2}C_{u2}}{R\alpha_2 - \beta_2} \]  
\[ \mathcal{N} = \frac{i k_2}{R\alpha_2 D_{u2}} \left[ T_{u4} - 2C_{u2} \right] + \frac{\beta_2 D_{t2} \left[ T_{t4} - 2C_{t2} \right]}{R\alpha_2 - \beta_2} \]
\[ + \frac{RD_{u2} \left[ (\nu_0 + \nu_{22}) (1 - C_{u2}) + k_2 C_{u2} \right] + \mathcal{D}_{u2} \left[ (\nu_0 + \nu_{22}) (C_{t2}^2 - 1) + k_2 C_{t2} \right]}{2 (R\alpha_2 - \beta_2)} \]
\[ - \frac{3k_2^2}{16W(R\alpha_2 - \beta_2)} \]
\[ \mathcal{M} = \frac{\mathcal{D}_{t2} \left( 1 - C_{t2}^2 \right) + \mathcal{D}_{u2} \left( C_{u2}^2 - 1 \right)}{2 (R - 1) (R\alpha_2 - \beta_2)} \]
\[ B_{t3} = \frac{(C_{t1} C_{t3} - 1) \mathcal{D}_{t1} \mathcal{D}_{t3} - \mathcal{D}_{t1}^2 - \mathcal{D}_{t3}^2 - \mathcal{R} \left[ (C_{u1} C_{u3} - 1) \mathcal{D}_{u1} \mathcal{D}_{u3} - \mathcal{D}_{u1}^2 - \mathcal{D}_{u3}^2 \right]}{4 (R\alpha_2 - \beta_2)} \]
\[ - \frac{RD_{u2} C_{u2} \left( \mathcal{D}_{u1} C_{u1} + \mathcal{D}_{u3} C_{u3} \right) - \mathcal{D}_{t2} C_{t2} \left( \mathcal{D}_{t1} C_{t1} + \mathcal{D}_{t3} C_{t3} \right)}{4 (R\alpha_2 - \beta_2)} \]

Equation (2.25) requires that \( R\phi_{u0}^{(1)} - \phi_{t0}^{(1)} \) be solved for. For the zero-th harmonic, the boundary-value problem for \( \phi_{u0}^{(m)} \) and \( \phi_{t}^{(m)} \) in (A.2a-A.2f) allows for a solution that can be a function of slow time \( \tau \) only. Thus, the zeroth-harmonic forcing in \( f_1^{(5)} \) and \( f_2^{(5)} \) must vanish, which leads to:

\[ R\phi_{u0}^{(1)} - \phi_{t0}^{(1)} = (R - 1)\nu_0 \frac{d}{d\tau} |A_2|^2 \]  
\[ (2.30) \]

Integration of this equation with respect to \( \tau \) gives:

\[ R\phi_{u0}^{(1)} - \phi_{t0}^{(1)} = (R - 1)\nu_0 |A_2|^2 + C \]  
\[ (2.31) \]

where the integration constant is determined by the initial condition, \( C = R\phi_{u0}^{(1)} - \phi_{t0}^{(1)} - (R - 1)\nu_0 |A_2|^2 \) evaluated at \( \tau = 0 \).

The leading-order solution of the mean interface elevation in (2.20c) then becomes

\[ \bar{\eta}^{(a)} = \nu_0 |A_2|^2 - \frac{R\phi_{u0}^{(1)} - \phi_{t0}^{(1)}}{R - 1} = \frac{C}{1 - \bar{R}} \]  
\[ (2.32) \]

The zero-th harmonic forcing in \( j_3^{(4)} \) and \( j_3^{(5)} \) gives the higher-order solution: \( \bar{\eta}^{(4)} + \)
\[ \eta^{(5)} = [\dot{\phi}_{10}^{(2)} - \mathcal{R} \phi_{u0}^{(2)} + f_{12} \dot{A}_2 A_2^* + f_{13} \dot{A}_2^* A_2 + f_{14} |A_1|^2 + f_{15} |A_3|^2]/(\mathcal{R} - 1) \]

where the quantity \( \dot{\phi}_{10}^{(2)} - \mathcal{R} \phi_{u0}^{(2)} \) needs to be determined in the \( O(\Delta^4) \) boundary-value problem.

Upon substitution of \( t = \Delta^{-\frac{1}{2}} \tau \), \( a_2 = \Delta^\frac{1}{2} A_2 \), \( a_1 = \Delta^\frac{3}{2} A_1 \), \( a_3 = \Delta^\frac{3}{2} A_3 \), and \( \sigma = \omega \Delta^{-\frac{1}{2}} \), we rewrite the evolution equations for the amplitudes of the interacting waves in the triad in the form:

\[
\frac{d^2 a_2}{dt^2} = \tilde{\Omega} a_2 + \tilde{N}|a_2|^2 a_2 + B_{13} a_1 a_3 e^{i\omega t} \quad (2.33a)
\]

\[
\frac{da_1}{dt} = i B_{23} a_2 a_3^* e^{-i\omega t} \quad (2.33b)
\]

\[
\frac{da_3}{dt} = i B_{12} a_2 a_1^* e^{-i\omega t} \quad (2.33c)
\]

where \( \tilde{\Omega} = (\Omega + \mathcal{M}C) \Delta \) and \( \tilde{N} = N + \mathcal{M}(\mathcal{R} - 1)\nu_0 \). In the right-hand side of (2.33a), the first term represents the linear Kelvin-Helmholtz instability effect, the second-term the nonlinear correction (associated with cubic self-interactions) to the Kelvin-Helmholtz instability effect, and the third term the triad resonant interaction effect.

The effect of the mean interface elevation upon the Kelvin-Helmholtz instability is also considered with the inclusion of \( C \) in \( \tilde{\Omega} \).

### 2.2.5 Properties of The Nonlinear Interaction Equations

The analytic solution to (2.33) is, in general, not known and can only be found through numerical integration. In the case of a perfect resonance \( (\sigma = 0) \) or for small time, analytical solutions or integral properties can be derived.

**Resonant Interactions With \( \sigma = 0 \)**

For the perfect resonance case, \( \sigma = 0 \). With the decomposition \( a_j(t) = R_j(t) e^{i\phi_j(t)} \), \( j = 1, 2 \) and \( 3 \), equations (2.33b) and (2.33c) are separated into real and imaginary
parts:

\[
\begin{align*}
\dot{R}_1 &= -B_{23} R_2 R_3 \sin \Phi \quad (2.34a) \\
R_1 \dot{\theta}_1 &= B_{23} R_2 R_3 \cos \Phi \quad (2.34b) \\
\dot{R}_3 &= -B_{12} R_1 R_2 \sin \Phi \quad (2.34c) \\
R_3 \dot{\theta}_3 &= B_{12} R_1 R_2 \cos \Phi \quad (2.34d)
\end{align*}
\]

where \( \Phi = \theta_2 - \theta_1 - \theta_3 \). Using (2.34a) and (2.34c) and integrating with respect to time, we obtain

\[
B \left[ R_1^2 (\tau) - R_1^2 (0) \right] = \left[ R_3^2 (\tau) - R_3^2 (0) \right] \quad (2.35)
\]

where \( B = B_{12} / B_{23} \). For \( B < 0 \), we have from (2.35) that \( |B| R_1^2(t) + R_3^2(t) = |B| R_1^2(0) + R_3^2(0) \). It indicates that for \( B < 0 \), the growth of \( R_1 \) and \( R_3 \) remains bounded with their maximum amplitudes being limited by their initial conditions.

For \( B > 0 \), (2.35) shows that there is no restriction on how large the two waves can grow due to the resonant interaction with the unstable \( k_2 \) wave.

**Solution At Small Time**

When the amplitudes of the interacting modes are small (with \( k_2 a_2 \ll 1 \) and \( a_1/a_2, a_3/a_2 \ll 1 \)), the nonlinear terms in (2.33a) have a weak secondary effect. This leaves the evolution equation of \( a_2 \) to be dominated by the linear instability. With this simplification, the coupled evolution equations (2.33) can be solved for the (initial) growth rates of \( a_2, a_1, \) and \( a_3 \) due to the linear instability and the triad resonant interaction. Evolution at small initial time (with \( a_1/a_2, a_3/a_2 \ll 1 \)) is an example of this situation.

In this case, it is straightforward to obtain \( a_2(t) = \hat{a}_2 \exp\{\hat{\Omega}_2 t\} \) where \( \hat{a}_2 \) is the initial value of \( a_2 \) at \( t=0 \). By taking the time derivative of (2.33c), we have

\[
\ddot{a}_3 = i B_{12} \left[ a_1 \dot{a}_2 + \dot{a}_1 a_2 - i \sigma a_1^* a_2 \right] e^{-i\sigma t}
\]

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Upon substitution of \(a_2(t)\) and use of (2.33b), we rewrite the above equation as:

\[
\ddot{a}_3 = \left(\sqrt{\Omega} - i\sigma\right) \dot{a}_3 + |\dot{a}_2|^2 e^{2\sqrt{\Omega}t} B_{12} B_{23} a_3
\]  

(2.36)

Introducing the change of variable \(\xi = |\dot{a}_2| (B_{12} B_{23})^{\frac{1}{2}} \exp\{\Omega^{\frac{1}{2}} t\}\), we write (2.36) in the form:

\[
\xi^2 \frac{d^2 a_3}{d\xi^2} + \frac{i\sigma}{\sqrt{\Omega}} \frac{d a_3}{d\xi} - \frac{\xi^2}{\Omega} a_3 = 0
\]  

(2.37)

which is a transformed version of the Bessel differential equation. The general solution of (2.37) takes the form:

\[
a_3(\xi) = \xi^{-\nu} \left[ C_1^{(3)} J_\nu \left( \frac{-i\xi}{\sqrt{\Omega}} \right) + C_2^{(3)} Y_\nu \left( \frac{-i\xi}{\sqrt{\Omega}} \right) \right]
\]  

(2.38)

where \(\nu = (i\sigma - \sqrt{\Omega})/(2\sqrt{\Omega})\), and \(J_\nu\) and \(Y_\nu\) represent the Bessel functions of the first and second kinds, respectively. The constants \(C_1^{(3)}\) and \(C_2^{(3)}\) are determined from the initial condition to be:

\[
C_1^{(3)} = \frac{\xi_0 \left[ \ddot{a}_3 J_{\nu+1} \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right) + i\sqrt{\Omega} \ddot{\dot{a}}_2 B_{12} Y_{\nu+1} \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right) \right]}{J_\nu \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right) Y_{\nu+1} \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right) - J_{\nu+1} \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right) Y_\nu \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right)}
\]

\[
C_2^{(3)} = -\frac{\xi_0 \left[ \ddot{a}_3 J_{\nu+1} \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right) + i\sqrt{\Omega} \ddot{\dot{a}}_2 B_{12} Y_{\nu+1} \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right) \right]}{J_\nu \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right) Y_{\nu+1} \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right) - J_{\nu+1} \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right) Y_\nu \left( \frac{-i\delta_0}{\sqrt{\Omega}} \right)}
\]

where \(\xi_0 = \xi(t = 0)\) and \(\dot{a}_{1,3} = a_{1,3}(t = 0)\). Owing to the symmetry, the solution to \(a_1\) can be obtained from that of \(a_3\) by switching \(a_3\) with \(a_1\).

To assist in understanding the basic characteristics of the solution for the growth of \(a_3\) (and \(a_1\)), we consider the nearly perfect resonance case. With \(\sigma/\omega_3 \ll 1\), the middle term on the left hand side of (2.37) can be ignored. The general solution of
the resulting equation is given:

\[ a_3(\xi) = D_1 e^{\xi/\sqrt{n}} + D_2 e^{-\xi/\sqrt{n}} \]  

(2.39)

where the constants \( D_1 \) and \( D_2 \) are determined from the initial condition. Depending on the signs of the interaction coefficients \( B_{12} \) and \( B_{23} \), the solution in (2.39) shows different characteristic behaviors. When \( B_{12} \) and \( B_{23} \) have the same sign (i.e. \( B > 0 \)), \( \xi \) is purely real and is proportional to \( \exp(\hat{\Omega}^{1/2} t) \). As a result, \( a_3 \sim \exp[\exp(\hat{\Omega}^{1/2} t)] \) which shows a bi-exponential growth with time. This suggests a highly efficient mechanism for transferring energy to linearly-stable wave modes from a linearly-unstable wave mode through triad resonant interaction. In the study of (wind) wave generation by a sheared current, Janssen[36] predicted a similar energy transfer mechanism due to the nonlinear coupling between a linearly unstable mode and its second harmonic. When \( B_{12} \) and \( B_{23} \) have different signs (i.e \( B < 0 \)), \( \xi \) is purely imaginary. In this case, \( a_3 \sim \exp[i \exp(\hat{\Omega}^{1/2} t)] \) which shows an oscillatory feature in time with a constant amplitude but an exponentially growing frequency. (Note that this result is consistent with the finding in the preceding section for the perfect resonance case with \( B < 0 \ ).

2.3 Numerical Method

The theoretical analysis in §2.2 provides valuable insights into the dynamics of resonant triad wave interaction coupled with the Kelvin-Helmholtz instability effect for a two-fluid flow in a horizontal channel. To verify the theoretical prediction and deal with the practical situation involving multiple resonant interactions, we develop an effective numerical method that enables direct time simulation of the nonlinear initial boundary-value problem, (3.1-3.7).
2.3.1 Mathematical Formulation of A High-Order Spectral Method

An efficient high-order pseudo-spectral (HOS) method, originally developed by Dommermuth & Yue [19] for the study of nonlinear surface gravity waves, is modified to simulate the nonlinear interfacial evolution of stratified channel flows. The extension to nonlinear interactions of internal waves and surface waves over variable bottom topography was achieved by Alam, Liu & Yue [1]. This method solves the primitive equations of the problem by following the evolution of a large number of spectral interfacial wave modes and accounts for their interactions up to an arbitrarily high order of nonlinearity using a pseudo-spectral approach.

Time Evolution Equations

This process begins with the definition of the potentials at the interface within each fluid domain

\[ \phi^I_u (x, t) = \phi_u (x, \eta (x, t), t) \]  
\[ \phi^I_I (x, t) = \phi_I (x, \eta (x, t), t). \]

Applying chain rule to \( y = \eta \) allows for the standard derivatives on the interface \((y = \eta)\) to be written as

\[ \frac{\partial \phi}{\partial t} = \frac{\partial \phi^I}{\partial t} - \frac{\partial \phi}{\partial y} \frac{\partial \eta}{\partial t}, \quad y = \eta \]
\[ \frac{\partial \phi}{\partial x} = \frac{\partial \phi^I}{\partial x} - \frac{\partial \phi}{\partial y} \frac{\partial \eta}{\partial x}, \quad y = \eta. \]

With these new definitions of the potential derivatives, the boundary conditions maybe written as functions of the interface potentials. The kinematic boundary
conditions, (3.5) and (3.6), take the form

\[ \frac{\partial \eta}{\partial t} = - \left[ U_u + \frac{\partial \phi_u}{\partial x} \right] \frac{\partial \eta}{\partial x} + \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right] \frac{\partial \phi_u}{\partial y}, \quad y = \eta \]  

(2.42)

\[ \frac{\partial \eta}{\partial t} = - \left[ U_l + \frac{\partial \phi_l}{\partial x} \right] \frac{\partial \eta}{\partial x} + \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right] \frac{\partial \phi_l}{\partial y}, \quad y = \eta. \]  

(2.43)

while the dynamic boundary condition, (3.7), is expressed as

\[ \frac{\partial \Psi^l}{\partial t} = \frac{1}{2} \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right] \left[ \left( \frac{\partial \phi_l}{\partial y} \right)^2 - \mathcal{R} \left( \frac{\partial \phi_u}{\partial y} \right)^2 \right] + \frac{1}{2} \left[ \mathcal{R} \left( \frac{\partial \phi_u}{\partial x} \right)^2 - \left( \frac{\partial \phi_l}{\partial x} \right)^2 \right] \]

\[ + \mathcal{R} U_u \frac{\partial \phi_u^l}{\partial x} - U_l \frac{\partial \phi_l}{\partial x} - (1 - \mathcal{R}) \eta + \frac{1}{\mathcal{W}} \frac{\eta_{xx}}{(1 + \eta_y^2)^2}, \quad y = \eta \]  

(2.44)

where \( \Psi^l(x, t) \equiv \phi^l_l(x, t) - \mathcal{R} \phi_u(x, t) \). Together, (2.42) (or (2.43)) and (2.44) form a set of interfacial evolution equations that can be integrated in time to obtain the dynamic behavior of the interface between the two fluids, provided that the interface velocities \( \partial \phi_u / \partial y \) and \( \partial \phi_l / \partial y \) on \( y = \eta \) and potentials \( \phi^l_u \) and \( \phi^l_l \) can be solved from the boundary-value problem.

**Perturbation Expansions**

If it is assumed that \( \phi_u, \phi_l, \) and \( \eta \) are \( O(\epsilon) \ll 1 \), where \( \epsilon \) is a measure of the wave steepness, then these terms can be expanded in a perturbation series up to order \( M \) in terms of the small variable \( \epsilon \)

\[ \phi_{u/l}(x,y,t) = \sum_{m=1}^{M} \phi_{u/l}^{(m)}(x,y,t) \]  

(2.45)

where \( \phi_{u/l}^{(m)} \) and \( \phi_{l}^{(m)} \) are \( O(\epsilon^m) \). Since this is a free boundary problem, with \( \phi_u, \phi_l \) and \( \eta \) being unknown, we expand \( \phi_u \) and \( \phi_l \) on the interface \( (y = \eta) \) in Taylor series about the mean interface \( (y=0) \):

\[ \phi_{u/l}(x,\eta,t) = \sum_{m=1}^{M} \phi_{u/l}^{(m)}(x,\eta,t) = \sum_{m=1}^{M} \sum_{k=0}^{M-m} \frac{\eta^k}{k!} \frac{\partial^k \phi_{u/l}^{(m)}}{\partial y^k} \bigg|_{\eta=0}. \]  

(2.46)
It should be noted that the second summation is evaluated up to \( M - m \) in order to maintain a consistent expansion up to \( O(e^M) \).

Defining \( \Psi(x,y,t) \equiv \phi(x,y,t) - \mathcal{R}\phi_u(x,y,t) \) and using (2.46) produces

\[
\Psi'(x,t) = \sum_{m=1}^{M} \Psi^{(m)}(x,\eta,t) = \sum_{m=1}^{M} \sum_{k=0}^{M-m} \frac{\eta^k}{k!} \left. \frac{\partial^k \Psi^{(m)}}{\partial y^k} \right|_{y=0} .
\] (2.47)

If all terms of common order in (2.47) are collected, a set of Dirichlet boundary conditions for each \( \Psi^{(m)} = \phi^{(m)}(x,y,t) - \mathcal{R}\phi_u^{(m)}(x,y,t) \) on \( y=0 \) can be obtained as

\[
\Psi^{(1)}(x,0,t) = \psi'
\] (2.48)

\[
\Psi^{(m)}(x,0,t) = -\sum_{k=1}^{m-1} \frac{\eta^k}{k!} \left. \frac{\partial^k \Psi^{(m-k)}}{\partial y^k} \right|_{y=0} , \quad m = 2, 3, \ldots, M
\] (2.49)

with \( \Psi'(x,t) \) being obtained from the evolution equation (2.44) at any time \( t \).

Taking the difference between (3.5) and (3.6) gives a new form of the kinematic interfacial boundary condition:

\[
\Phi_y = \eta_x \left[ \Phi_x + (U_u - U_l) \right] , \quad y = \eta
\] (2.50)

where \( \Phi \equiv \phi_u - \phi_l \). Applying the perturbation expansions for \( \phi_u, \phi_l \) and then expanding them in Taylor series about \( y = 0 \) yields

\[
\Phi(x,\eta,t) = \sum_{m=1}^{M} \Phi^{(m)}(x,\eta,t) = \sum_{m=1}^{M} \sum_{k=0}^{M-m} \frac{\eta^k}{k!} \left. \frac{\partial^k \Phi^{(m)}}{\partial y^k} \right|_{y=0} .
\] (2.51)

Substituting this expansion into (2.50), a set of Neumann boundary conditions can be obtained for each \( \Phi^{(m)}_y \) on \( y=0 \):

\[
\Phi^{(1)}_y(x,0,t) = \eta_x (U_u - U_l)
\] (2.52)
\[ \Phi^{(m)}_y(x, 0, t) = \eta_x \sum_{k=1}^{m-1} \frac{\eta^{(k-1)}}{(k-1)!} \frac{\partial^{(k-1)}}{\partial y^{(k-1)}} \Phi^{(m-k)}_x \bigg|_{y=0} \eta^k \frac{\partial^{(k)}}{\partial y^{(k)}} \Phi^{(m-k)}_x \bigg|_{y=0} \]  

for \( m = 2, 3, \ldots, M \).

With these expansions, the nonlinear boundary-value problem for \( \phi_u, \phi_l \) is decomposed into a sequence of linear boundary-value problems for the perturbed potentials \( \phi_u^{(m)}, \phi_l^{(m)}, m=1,2,\ldots,M \), which can be solved sequentially starting from \( m=1 \) up to an arbitrary order \( M \).

**Solution of \( \phi_u^{(m)} \) And \( \phi_l^{(m)} \)**

Assuming a periodic boundary condition in the \( x \)-direction and choosing a normalization that makes the length of the computational domain \( L=2\pi \), the boundary-value solution at each order \( m \), which satisfies both Laplace’s equation and the zero-flux condition at the walls, can be written as a truncated Fourier Series of the form

\[ \phi_u^{(m)}(x, y, t) = \sum_{n=-N}^{N} A_n^{(m)}(t) \frac{\cosh k_n (y - h_u)}{\cosh k_n h_u} e^{i k_n x} \]  
\[ \phi_l^{(m)}(x, y, t) = \sum_{n=-N}^{N} B_n^{(m)}(t) \frac{\cosh k_n (y + h_l)}{\cosh k_n h_l} e^{i k_n x} \]

where the wavenumber of the \( n \)-th Fourier mode \( k_n = n \). From the Dirichlet boundary condition for \( \Psi^{(m)} \) and the Neumann boundary condition for \( \Phi^{(m)}_y \), the unknown Fourier modal amplitudes \( (A_n^{(m)} \text{ and } B_n^{(m)}) \) are determined to be:

\[ A_n^{(m)} = -\frac{\Phi_y^{(m)} + \Psi_u^{(m)} k_n \tanh k_n h_u}{k_n (\tanh k_n h_u + \mathcal{R} \tanh k_n h_l)} \]  
\[ B_n^{(m)} = \frac{\Psi_u^{(m)} k_n \tanh k_n h_u - \mathcal{R} \Phi_y^{(m)}}{k_n (\tanh k_n h_u + \mathcal{R} \tanh k_n h_l)} \]

for \( n = -N, -N+1, \ldots, N \) but \( n \neq 0 \), where \( \Psi_u^{(m)} \) and \( \Phi_y^{(m)} \) are the Fourier modal amplitudes of \( \Psi^{(m)}(x, 0, t) \) and \( \Phi_y^{(m)}(x, 0, t) \), respectively. For the mode of \( n = 0 \), the Dirichlet boundary condition requires that \( B_0^{(m)} - \mathcal{R} A_0^{(m)} = \Psi_0^{(m)} \) while the Neumann boundary condition is automatically satisfied. For the complete solution of the
problem, the specific values of $A_0^{(m)}$ and $B_0^{(m)}$ are not needed. For convenience in computation, we let $A_0^{(m)} = 0$ and $B_0^{(m)} = \Psi_0^{(m)}$.

After the boundary-value solutions of $\phi_u^{(m)}$ and $\phi_l^{(m)}$ are determined up to order $M$, the interface potentials $\phi_u^l$ and $\phi_l^l$ can be evaluated from (2.46). The interface velocities, $\partial \phi_u / \partial y$ and $\partial \phi_l / \partial y$ on $y = \eta$, can be evaluated similarly by making Taylor series expansions about $y=0$ and then substituting the solutions of $\phi_u^{(m)}$ and $\phi_l^{(m)}$. The evolution equations (2.42) (or (2.43)) and (2.44) can be integrated in time (with properly defined initial conditions) by the use of any high-resolution integration method, such as the fourth-order Runge-Kutta scheme.

One notes that owing to the use of perturbation expansion (2.45) and spectral expansion (2.54), the boundary-value solution converges exponentially fast with increasing the order $M$ and the number of spectral modes $N$ for moderately steep interfaces. However, as the interface steepness increases, the convergence rate with $M$ becomes slower. For steep waves, the present method is invalid because (2.45) will no longer converge with $M$. In this case, a fully-nonlinear scheme needs to be applied.

This numerical method is designed for efficiently simulating the nonlinear evolution of an interface composed of broadbanded wave components. It is also equally capable of simulating the nonlinear evolution of a specific discrete set of wave modes. For these problems, a simple bandpass filter is applied at each time step to remove all non-relevant spectral components.

2.3.2 Validation of The Numerical Method

As a verification of the numerical method, a relatively simple case of a single triad resonance is considered in which the three interacting waves are all linearly stable. For this problem, a closed-form analytic solution is available. The evolution equations
for the amplitudes of the interacting waves take the form (Campbell[13]):

\[
\begin{align*}
\frac{da_1}{dt} &= iB_{23}a_2a_3^*e^{-i\sigma t} \quad \text{(2.56a)} \\
\frac{da_2}{dt} &= iB_{13}a_1a_3e^{i\sigma t} \quad \text{(2.56b)} \\
\frac{da_3}{dt} &= iB_{12}a_1^*a_2e^{-i\sigma t} \quad \text{(2.56c)}
\end{align*}
\]

where the interaction coefficient \(b_{13}\) is

\[
b_{13} = \frac{k_2}{4(\mathcal{R}\mathbb{D}_{a_2}C_{a_2} + \mathbb{D}_{a_2}C_{a_2})} \left[ \mathbb{D}_{a_2}C_{a_2} \left( \mathbb{D}_{a_1}C_{a_1} + \mathbb{D}_{a_3}C_{a_3} \right) - \mathcal{R}\mathbb{D}_{a_2}C_{a_2} \left( \mathbb{D}_{a_1}C_{a_1} + \mathbb{D}_{a_3}C_{a_3} \right) \right. \\
+ \mathcal{R} \left( \mathbb{D}_{a_1}^2 + \mathbb{D}_{a_3}^2 \right) - \left( \mathbb{D}_{a_1}^2 + \mathbb{D}_{a_3}^2 \right) - \mathbb{D}_{a_1}\mathbb{D}_{a_3} \left( 1 - C_{a_1}C_{a_3} \right) + \mathcal{R}\mathbb{D}_{a_1}\mathbb{D}_{a_3} \left( 1 - C_{a_1}C_{a_3} \right) \right]
\]

and \(B_{12}\) and \(B_{23}\) are given by (2.23) in §2.2.4. For perfect resonance \((\sigma = 0)\), (2.56) can be solved analytically with the solution given in terms of Jacobian elliptic functions (e.g. McGoldrick[53]). With initial amplitudes \(a_1(0) = \hat{a}_1\), \(a_2(0) = 0\), and \(a_3(0) = \hat{a}_3\), as an example, the solution takes the form:

\[
\begin{align*}
a_1 &= \hat{a}_1 \text{dn}(\Xi|m) \quad \text{(2.57a)} \\
\hat{a}_2 &= \hat{a}_3 \sqrt{\frac{b_{13}}{B_{12}}} \text{sn}(\Xi|m) \quad \text{(2.57b)} \\
\hat{a}_3 &= \hat{a}_3 \text{cn}(\Xi|m) \quad \text{(2.57c)}
\end{align*}
\]

where \(\text{dn}, \text{sn}, \text{and cn}\) are the Jacobian elliptic functions with arguments

\[\Xi = \hat{a}_1 \left( B_{13}B_{12} \right)^{\frac{1}{2}} t, \quad m = \frac{B_{23}\hat{a}_3^2}{B_{12}\hat{a}_1^2} \leq 1\]

Figure 3-2 compares the numerical simulation result (with order \(M=2\)) with the analytic solution by (2.57) for the time variation of the amplitudes of interacting waves that form a resonant triad. As a numerical example, we use \(\mathcal{R} = 1.23 \times 10^{-3}\), \(\mathcal{W} \cong 8455.5\), \(H \equiv h_u + h_l \cong 1.25\), \(\alpha \equiv h_u/H \cong 0.5\), \(U_u \cong 7.94\), \(U_l \cong 1.1\), \(\hat{a}_1 \cong 6.28 \times 10^{-5}\), \(\hat{a}_2 = 0\), \(\hat{a}_3 = \frac{1}{2}\hat{a}_1\), and \(L=2\pi\). The interacting waves have wavenumbers \(k_1 = 11\), \(k_2 = 46\) and \(k_3=35\). (With normalization length \(L \cong 0.08\) m and time \(T \cong 0.09\).
Figure 2-3: Time evolution of the amplitudes of primary waves in a resonant triad. The plotted curves represent the direct numerical simulation result with $M=2$ (o) and the theoretical prediction (—) by (2.57).

$s$, as an example, these dimensionless parameters correspond to an air-water flow in a horizontal channel in a laboratory scale with $H=0.1$ m, $L=0.5$ m, $U_u \approx 7.0$ m/s, $U_l = 1.0$ m/s, $\gamma = 0.0734$ N/m, and $a_1 = 10^{-6}$ m). Excellent agreements between the numerical simulation result (of the leading order) and the analytic solution are shown, and thus validate the numerical method. The numerical simulation result with $M=3$ is graphically indistinguishable from that with $M=2$. Thus it is not shown in the figure. Note that the fluid properties used in this example correspond to those of air and water. They are chosen because these two fluids are commonly used in laboratory experiments; however, the theory and direct computation are applicable to general two-phase flows.

### 2.4 Results

In this section, we describe the characteristic features of triad resonant interfacial wave interactions, which are influenced by the Kelvin-Helmholtz instability, in a two-fluid channel flow. To assist in understanding the solution, the nonlinear self-interaction
effects on the Kelvin-Helmholtz instability are first investigated. Both theoretical solutions, based on the analysis in §2.2, and direct numerical simulation results using the method outlined in §2.3 are presented and discussed.

For numerical illustration, we use the same flow conditions as described in §2.3.2 with the exception that \( U_1 = U + U_c(1 + \Delta) \), where \( U_c \) is given by (2.11), and \( \Delta \) is chosen to be \( 10^{-4} \). The initial condition for the primary waves is chosen to be \( a_2(0) = 1.25 \times 10^{-5} \), \( a_1(0) = \frac{1}{2}a_2(0) \), \( a_3(0) = 0 \), and \( \phi_{u0}(0) = \phi_{v0}(0) = 0 \). The initial amplitudes of the primary waves are intentionally chosen to be small so that the associated velocity potentials at \( t=0 \) are properly given by the linear solution. This also allows for a longer period of nonlinear growth before the waves become too steep for the perturbation-based analysis and simulation. The initial condition for the second harmonic is given in terms of the primary wave by (2.20). This set of initial conditions are sufficient for direct numerical simulation since the evolution equations (2.42 or 2.43) and (2.44) only contain the first-order time derivatives. The theoretical model also requires \( \dot{a}_2(0) \) and \( \dot{\phi}_{u0}(0) - R\dot{\phi}_{u0}(0) \) since the second-order time derivative is present in (2.33a). We use \( \dot{a}_2(0) = a_2(0)\sqrt{\Omega} \) based on the linear growth rate of \( k_2 \) wave, and let \( \dot{\phi}_{u0}(0) - R\dot{\phi}_{u0}(0) \) equal to the initial value obtained in the numerical simulation (in order for direct comparison between the theoretical solution and the numerical simulation to be consistent).

This set of flow properties and initial conditions are used for the results presented in this section unless stated otherwise.

### 2.4.1 Self Interactions

By removing the term associated with the interaction of \( k_1 \) and \( k_3 \) waves in (2.33a), we obtain an evolution equation for the amplitude \( a_2 \) of the slightly unstable \( k_2 \) wave including the effect of third-order self interactions. The equation clearly indicates that depending on the sign of the parameter \( \mathcal{N} \), the inclusion of the third-order self-interaction effect can accelerate or stabilize the growth of \( k_2 \) wave as its amplitude becomes larger due to the linear instability effect.

Figure 2-4 shows the time evolution of the amplitude \( a_2 \) of an unstable wave with
Figure 2-4: Time evolution of the amplitudes of the primary wave ($a_2$), second harmonic ($a_4$) and zero-th harmonic ($a_0$). The plotted curves represent the theoretical solution (---) and numerical simulations with order $M=2$ (· · ·) and $M=3$ (— · —).

wavenumber $k_2=25$ for which $\hat{N} \simeq 4 \times 10^4$. The results for the second and zero-th harmonics are also shown. The evolution equation (c.f. (2.33)a) clearly shows that initially the growth of $a_2$ is dominated by the linear instability. This growth of $a_2$ continues according to linear theory until the third-order self-interaction term becomes significant. Since $\hat{N} > 0$, the third-order self-interaction increases the growth rate of $a_2$ above that of the linear solution. This nonlinear effect is stronger as $a_2$ becomes larger in the evolution. As time increases, the growth rate continues to increase until the amplitude of $a_2$ rapidly becomes unbounded, as shown in figure 2-4. The second harmonic is a locked wave that is stable in this case. Thus it grows at twice the rate of $a_2$. For the zero-th harmonic, the leading order theoretical solution is a near-zero constant. The next order solution is not pursued in this study.

The direct numerical simulation results ($M=2,3$) compare excellently with the theoretical prediction for both the primary wave and its second harmonic until the very late stage of evolution when the solution quickly blows up. The numerical solution is convergent with order $M$ since the solution of $M=2$ agrees well with that
of $M=3$ except at the late stage where the third order effects become apparent. The numerical simulation also provides a prediction for the zero-th harmonic that is generally about one order smaller than the second harmonic except at the late stage of evolution when the solution quickly becomes singular. One notes that as the wave amplitude $a_2$ becomes considerably large at late stage of evolution, higher order effects are important, which are not considered in the theoretical analysis. While the simulation can include such higher order effects, it will eventually fail to converge with $M$ as wave steepness continues to increase since the numerical simulation is based on a perturbation approach.

A numerical search over a wide range of flow parameters suggests that the case of $\hat{N} < 0$ occurs only when the second harmonic of the $k_2$ wave is also linearly unstable. The theoretical analysis in §2.2 does not apply to this case since the analysis assumes a linearly stable second harmonic. In this case, strong interactions between the primary wave and its second harmonic, called an overtone resonance, can occur. This itself is an interesting topic in nonlinear interfacial wave dynamics and is pursued in a separate study.

### 2.4.2 Triad Resonant Interactions

We now turn to the study of triad resonant interaction that involves one unstable and two stable interacting wave components. The discussion in §2.2.5 indicates that different characteristic solutions exist depending on the sign of the parameter $B \equiv B_{12}/B_{23}$. Both cases are considered here.

**$B < 0$**

For this example the triad resonance consists of the interacting waves with wavenumbers $k_1 = 24$, $k_2 = 25$, and $k_3 = 1$ which produces $\hat{N} \approx 4 \times 10^4$ and $B \approx -0.017$. The $k_2$ wave is slightly unstable. The resulting evolution of the amplitudes of these waves, as well as the second harmonic of $k_2$ wave and the zero-th harmonic, is shown in figure 2-5.
The analytic solution demonstrates that the amplitudes of the $k_2$ wave, its second harmonic, and the zero-th harmonic behave similarly to the case without the involvement of the triad resonance interaction, discussed in §2.4.1. The growth of $a_2$ is dominated by the linear instability while the second harmonic, which is locked to $a_2$, grows with twice the linear growth rate of $a_2$. As $a_2$ becomes large, the positive nonlinear self interaction term causes the growth rate to continuously increase until the solution becomes singular. This singularity causes the second harmonic to become singular as well. The theoretical solution predicts that the zero-th harmonic maintains a near-zero constant amplitude during the entire evolution. In this case, the amplitude of $k_1$ wave ($a_{k_1}$) remains almost unchanged. For the $k_3$ wave, $a_3$ grows rapidly from its much smaller initial value by taking energy from the other two primary waves through the triad resonance. During the later stage, as $a_3$ becomes about two orders of smaller than $a_1$, the growth rate of $a_3$ significantly decreases. Despite the continuous growth, $a_3$ remains about two orders of smaller than $a_1$ until the solution of the system becomes singular. During the entire evolution, the sum of energy of the $k_1$ and $k_3$ wave remains unchanged as predicted by the analysis in §2.2.5 (even though the $k_2$ wave significantly grows).

The direct simulation results show good agreements with the analytic solution except at the very late stage of evolution when the higher order effects become important and the unstable $k_2$ wave is significantly developed. At this stage of evolution, the numerical solution clearly indicates that the growth of $a_3$ is obtained due to the decrease of $a_1$ as predicted by the theory with $\mathcal{B} < 0$ in §2.2.5.

One notes that in this case, the growth of the $k_3$ wave is resulted with the energy transfer from the stable $k_1$ wave, but not from the unstable $k_2$ wave. Thus, this type of triad resonance (with $\mathcal{B} < 0$) does not provide a significant energy build up for long waves.

$\mathcal{B} > 0$

The analysis in §2.2.5 indicates that for $\mathcal{B} > 0$ it is possible for $a_1$ and $a_3$ to grow without bound. Figure 2-6 shows such a sample solution. This solution is obtained
Figure 2-5: Time evolution of the amplitudes of (a) primary waves ($a_2$, $a_1$, and $a_3$) and (b) second harmonic of $k_2$ wave ($a_4$) and zero-th harmonic ($a_0$). The plotted curves represent the theoretical solution ($-$) and numerical simulations with order $M=2$ ($\cdots$) and $M=3$ ($\cdots\cdots$).
with the same air-water flow as before, but with $\alpha = 0.05$, $k_1 = 11$, $k_2 = 12$, and $k_3 = 1$, which produces $\mathcal{N} \simeq 1.414 \times 10^5$ and $B \simeq 0.053$. At the initial stage of evolution, as the analysis in §2.2.5 shows, $a_2$ is dominated by the linear instability effect, and the initial growth rate of $a_3$ is bi-exponential as shown in figure 2-7. During the later stage of the evolution, the growth of $a_2$ is accelerated by the third-order self-interactions. As a result, both $a_1$ and $a_3$ obtain a growth rate even faster than bi-exponential as shown in figure 2-8. The interacting wave system then quickly blows up. The prediction by direct numerical simulations compares very well with the theoretical solution till the late stage of evolution when the interacting waves grow rapidly and become singular.

For the second-harmonic of the $k_2$ wave, the analytical solution also agrees well with the numerical solution. Both solutions predict that the second-harmonic eventually becomes unbounded like the primary waves. For the zero-th harmonic, the analytic solution predicts a near-zero constant amplitude while the numerical solution shows that the amplitude of the zero-th harmonic increases together with the other wave modes, and also becomes unbounded eventually.

We remark that in this case, stable waves are significantly developed by the energy transfer from the unstable wave through triad resonant wave interactions. This type of triad resonance (with $B > 0$) provides an effective mechanism for transferring energy from unstable short waves to stable long waves, leading to a fast development of large-amplitude long waves in a two-fluid channel flow.

### 2.4.3 Multiple Resonant Interactions

Realistic two-fluid flow problems involve a broadbanded spectrum of interfacial waves that can form multiple coupled triad resonant (and near resonant) interactions. Moreover, as the interface steepens, higher-order resonant interactions may also play a role. Large-amplitude long waves can be developed due to the combined effects of the linear instability and multiple resonant interactions. The extension of the theoretical analysis in §2.2 to include multiple coupled resonant interactions is in principle possible, but not straightforward. The direct numerical simulation developed in §2.3 is more
Figure 2-6: Time evolution of the amplitudes of (a) primary waves ($a_2$, $a_1$, and $a_3$) and (b) second harmonic of $k_2$ wave ($a_4$) and zero-th harmonic ($a_0$). The plotted curves represent the theoretical solution (-----) and numerical simulations with order $M=2$ (···) and $M=3$ (— — —).
Figure 2-7: Closeup of initial growth of the $k_3$ wave mode in figure 2-6a. The plotted curves represent the theoretical solution (-----) and the approximate solution (2.39) (- - -).

Figure 2-8: Closeup of the later bi-exponential growth of the $k_3$ wave mode in figure 2-6a. The plotted curves represent the theoretical solution (-----) and numerical simulations with order $M=2$ (- - -) and $M=3$ (--- - -).
appropriate for these practical cases.

As a numerical example, we consider the growth and time evolution of a large-amplitude long wave that is developed from a smooth interface in a two-phase channel flow. The spectrum shall be broadbanded and contain multiple resonant triads. This problem will approximate the spatial evolution of the first slug developed near the inlet of the two-phase channel flow. While there is a rich collection of laboratory experiments in the literature, the vast majority of these tests are carried out using cylindrical pipe geometries (e.g. Fan et al. [24]). Ideally, comparisons between numerical simulations and measurements should be made with identical geometry. However, to obtain an initial estimate, the numerical simulations presented here are performed using the same flow properties as those in pipe flow experiments.

We choose to compare the direct simulations against the results of Ujang et al. [76] & Ujang [75] who carried out experiments of air-water flows through a horizontal 78 millimeter pipe with different gas/liquid velocities and liquid holdups. We specifically focus on the case of figure 5c of Ujang et al. [76] or figure 5.3a of Ujang [75] for which the superficial gas and liquid velocities $U_{SG} \equiv U_u A_u / A_{pipe} = 4.64 \text{ m/s}$ and $U_{SL} \equiv U_l A_l / A_{pipe} = 0.61 \text{ m/s}$ with $A_{pipe}$, $A_u$ and $A_l$ being the cross sectional area of the pipe and the areas of the pipe occupied by the upper and lower fluids, respectively. The gas fraction for the pipe is $\alpha = 0.4$ and the surface tension coefficient is $0.037 \text{ N/m}$. In the simulation, the channel depth is set to be equal to the pipe diameter ($H = 78 \text{ mm}$) and the void fraction is fixed as $\alpha = 0.4$. The uniform gas and liquid velocities are $U_u = U_{SG} A_{pipe} / A_G = 12.53 \text{ m/s}$ and $U_l = U_{SL} A_{pipe} / A_L = 0.97 \text{ m/s}$.

In the numerical computation, we use $L = 0.318 \text{ m}$ and $T = 0.180 \text{ s}$ for length and time normalization. The length of the (periodic) computational domain corresponds to a laboratory channel length of $2.0 \text{ m}$. The density ratio is $\mathcal{R} = 1.18 \times 10^{-3}$ and the Weber number is $\mathcal{W} = 2.67 \times 10^4$. The simulations are performed with $N=32$ and different orders of nonlinearity ($M=1$, 2, and 3). The initial disturbance on the interface is given by the white noise with a near machine-zero amplitude of $10^{-15}$. We note that in reality, the growth of very short waves are limited by viscous damping and small-scale wave breaking which are not considered in the simulation based on
the potential flow formulation. Since our purpose is to demonstrate the growth of long waves through multiple/coupled resonant interactions with short waves, a spectral filtering is applied in the nonlinear simulations (with $M=2$ and 3) to limit the maximum steepness of each of the unstable short wave component (with $k > 17$) at $0.1 sech(0.05k)$. This approach was developed by Higgins & Cokelet[49] and Dommermuth & Yue[19] in the simulation of nonlinear breaking waves in the ocean.

Figures 2-9(a), 2-9(c), & 2-9(e) show the amplitude spectra of the interface at $t=0.844s$, $1.410s$, and $1.834s$ during the evolution of the two-phase channel flow. The simulation results obtained with $M=1$, 2 and 3 are compared. The corresponding interfacial shape is shown in figures 2-9(b), 2-9(d), & 2-9(f). The results in these figures indicate that the same energy transfer mechanism described in §2.4.2 is observed in the presence of multiple resonant and near-resonant interactions. The linearly unstable spectral modes (with $k > 17$) initially grow due to the Kelvin-Helmholtz mechanism as shown in figure 2-9(a) & 2-9(b). As their amplitudes increase and the nonlinearity becomes stronger, some of the energy supplied by the linear instability is transferred to the long wave components through multiple resonant interactions, as evidenced by the presence of two apparent peaks in the spectrum shown in 2-9(c). As this nonlinear process continues, the wave spectrum of the interface becomes broadbanded, resulting in the formation of a large-amplitude wave whose crest eventually touches the top boundary of the channel, as depicted in 2-9(e) & 2-9(f).

For this case, the interface bridges the channel diameter after $t \sim 1.83s$ of evolution. The width of the large wave disturbance is $\sim 1.8H$. By multiplying this time by the group velocity of the wave with wavelength of $\sim 1.8H$, where $C_g \approx 1.0$ m/s, we obtain the slug-initiation distance of approximately $1.85$ m from the inlet (Gaster[29]). The difference in the nonlinear solutions with $M=2$ and 3 are small until times just prior to the instant when the long wave crest touches the top boundary. The simulation with $M=1$ bridges the channel at much earlier time than the $M = 2$ & 3 cases, thus it is not shown in figures 2-9(c) & 2-9(f). This result compares qualitatively well with the experimental observation of Ujang[75] who reported that first slugging occurred in the region of $1.46 - 2.86$ meters from the inlet. Even though
Figure 2-9: The nonlinear simulation of the time evolution of broadbanded interfacial waves in a air-water channel flow. The left column shows the distribution of spectral amplitude (normalized by $C$) of interfacial wave components with $M = 1$ ($--$), $M = 2$ (---), and $M = 3$ (· · ·). The right column shows the corresponding interface shape, $(\eta + h_t)/H$. 

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this comparison is only a first order estimate, this method appears to consistently re-
produce results which have comparable length and times scales to the cases in figure
5 of Ujang[75].

2.5 Conclusions

This paper considers, both theoretically and computationally, the triad resonant inter-
actions of interfacial waves, which are influenced by the Kelvin-Helmholtz interfacial
instability, in an inviscid two-fluid incompressible flow through a horizontal channel.
The focus is on the mechanism of energy transfer from unstable short waves to stable
long waves through nonlinear resonant wave interactions. Based on a multiple-scale
analysis, in the context of a potential flow formulation, nonlinear interaction equa-
tions are derived which govern the amplitude evolution of the interacting waves in a
resonant triad, including the effects of interfacial instability. An effective numerical
method for direct simulations of nonlinear interfacial wave interactions is also devel-
oped based on a high-order pseudo-spectral approach. It is used for verification of
the theoretical analysis and the examination of practical applications involving mul-
tiple resonances with a broadbanded spectrum of waves. Cross-validations between
the theoretical solutions and direct numerical simulations are obtained for various
characteristic flows considered.

It is found that depending on the flow conditions, there exists an extremely efficient
energy transfer from the base flow to the stable long wave due to the coupled effects
of nonlinear wave resonance and interfacial instability. The growth rate of the long
wave can reach up to bi-exponential (or faster). Moreover, in this case, the (linearly)
unstable wave can grow unboundedly even when the nonlinear self-interaction effects
are accounted for, as do the stable waves in the resonant triad.

This work shows that the nonlinear coupling of an interfacial instability and non-
linear resonant wave interactions can cause a rapid development of long waves (which
are themselves linearly stable). Such a behavior of long wave growth bears simi-
larities to that of slug formation in stratified channel/pipe flows, as observed in
experiments. This suggests that the nonlinear mechanism found in this study may play an important role in slug formation and may improve slug-transition criteria. As a demonstration, we performed direct numerical simulations of nonlinear two-phase channel flow evolution involving multiple resonant and near-resonant wave interactions. The predicted slug initiation length compares qualitatively well with the laboratory measurement. The application of the present work to the general multi-phase flow problem for development of improved slug flow transition criteria, prediction of slug frequency/length, and direct comparisons to experimental measurements is the focus of ongoing research.
Chapter 3

Sub-harmonic Resonant Interaction Theory

In §2, it was shown that by coupling the Kelvin-Helmholtz instability to a resonant triad of interfacial wave modes, it is possible to see the rapid exchange of energy from the linearly unstable short wave to the linearly stable long wavelength components. A detailed investigation was carried out to examine the nonlinear behavior of the members of the triad; however, the analysis required the three resonant wavenumbers be distinct. A special case, referred to as the subharmonic resonance, is identified when two of the interacting modes in the triad share the same wavenumber. The analysis developed in §2 ignored this case, but the common observation of this subharmonic resonances in experiments justifies it’s investigation.

3.1 Introduction

In this chapter, a unique nonlinear mechanism in which two interfacial wave modes are nonlinearly coupled through a sub-harmonic resonance while at least one of the modes is linearly unstable to the Kelvin-Helmholtz instability is investigated.

This class of problem has been examined due to a wide range of problems which experience period-doubling behavior along with the formation of sub-harmonics in experimentally measured wave spectra. For instance, in the problem of wind-generated
gravity-capillary waves, experimental measurements have clearly shown the appearance of period-doubling in the wave spectra[14, 37]. Similarly, a unique wave type of disturbances has been observed in two-phase flows through pipes and channels. For the case of a gas blowing over a liquid, it is possible for small amplitude short wavelength waves to form at the interface and grow into large amplitude long wave disturbances which eventually fill the cross-section of the pipe or channel. This type of disturbance is referred to as slug flow. Experimental measurements on the initiation of slugging have shown that for certain flow conditions, strong energy transfer occurs between the short waves and long waves. This behavior is often accompanied by strong energy transfer to sub-harmonic modes [24, 40].

The prediction of the formation of slugs in pipes is of critical importance in many practical applications such as in the transport of complex multi-component fluids through oil pipelines. The presence of slugs in the pipeline can lead to pipe fatigue, complications in the separation of the flow into its constitutive components, and reduction in pipeline efficiency. This has motivated the development of various flow transition criteria for the prediction of the formation of slug flow. Traditionally these models are based on the Kelvin-Helmholtz instability and are supplemented with additional models to account for more complex physics such as interfacial and wall friction[43, 5], normal viscous stresses[27], and heuristic corrections which are presumed to account for the effects of nonlinearity[73]. A survey of several different transition methods and models was presented by Mata et al. [52].

By basing the slug transition criteria on linear instability theory, the possibility of modal energy transfer across time/length scales is prohibited and requires the use of nonlinear methods. The idea of resonant interaction theory[65] provides a simple nonlinear framework for accounting for important nonlinear wave-wave interactions in surface/interfacial wave-field development. Through the use of regular perturbation analysis, the initial growth rate and rate of energy transfer between different wave components can be analytically determined[45]. For long-time and/or large-distance interactions, the analysis based on the use of multiple scales is usually applied.

The resonant interaction between a primary wave and its second harmonic is a well
studied phenomenon, referred to as a second-harmonic resonance. The case where both modes are linearly stable has been well documented along with certain special asymptotic solutions\[54, 61\]. Attempts have been made to account for the effect of a linear instability in the resonant interaction, such as the damping due to molecular viscosity, by the simple addition of a linear term to the nonlinear interaction equations. For the case of damped oscillations, this model may capture the leading order effects\[55\]. However, for problems involving unstable modal growth, this approach becomes invalid since it neglects the nonlinear self-interactions that play a significant role in the long-term growth and overall nonlinear wave-field evolution.

The work in §2 examined the case of marginally unstable modes in conjunction with resonantly interacting triads of wave modes. It was found that the combination of these two mechanisms produced an efficient mechanism for the transfer of energy from linearly unstable short waves to linearly stable long waves. It was shown that for certain flow conditions, the resonant energy exchange in the presence of linearly unstable short waves could create bi-exponential growth for the stable long waves. Sub-harmonic (and second-harmonic) resonances should be of similar importance, but were not addressed at that time. Moreover, they considered the resonant interaction in which only one (primary) wave component is linearly unstable. Resonant interactions involving multiple linearly unstable wave components were not studied.

Therefore, in this paper, we theoretically and numerically examine the nonlinear evolution of a pair of interfacial wave modes undergoing a sub-harmonic resonance while under the influence of a Kelvin-Helmholtz instability. It is shown that depending on the location of the modes with respect to the neutral-stability curve, several different classes of resonances can occur. The method of multiple scales is employed to derive a pair of nonlinear interaction equations that govern the time evolution of the amplitudes of the resonantly interacting waves. Within each case, each type of resonance produces a set of interaction equations with a unique algebraic form. Additionally, it is shown that none of the cases can be obtained by supplementing the linearly stable interaction equations with a linear source term. All of these resonance cases are validated against the direct numerical simulations that are obtained based
on a separate robust high-order numerical scheme. This numerical scheme is also employed to examine more general cases, which are too cumbersome to be addressed by analytical methods, such as resonance chains where a single mode is involved in two different simultaneous resonances as well as practical situations where broadbanded resonant wave interactions are present.

### 3.2 Fully Nonlinear Governing Equations, Linear Solution, And Nonlinear Numerical Simulation

This analysis deals with the interfacial evolution of a stratified flow composed of two immiscible fluids in a horizontal channel. A Cartesian coordinate system is fixed at the location of the interface when the two fluids are in equilibrium. The $x$–axis points to the right and the $y$–axis points upwards. The upper and lower fluids, which are denoted by the subscripts $u$ and $l$ respectively, have densities $\rho_u$ and $\rho_l$ (with $\rho_u < \rho_l$) along with equilibrium fluid depths of $h_u$ and $h_l$. The instantaneous interface between the two fluids is described by the function $y = \eta(x, t)$. The effects of gravity and surface tension are also considered.

The velocity in each fluid domain is decomposed into a constant uniform current $(U_u/l)$ and a disturbance velocity. It is assumed that both the upper and lower fluids are incompressible and inviscid and the associated flows are irrotational. This allows for the velocity of each fluid to be defined by the gradient of its potential function, $\varphi_u(x, y, t) = U_u x + \phi_u(x, y, t)$ and $\varphi_l(x, y, t) = U_l x + \phi_l(x, y, t)$. Satisfaction of the continuity equations leads to:

\[
\nabla^2 \phi_u = 0, \quad \eta < y < h_u \tag{3.1}
\]

\[
\nabla^2 \phi_l = 0, \quad -h_l < y < \eta \tag{3.2}
\]
At the channel walls, the no-flux conditions are enforced as

\[
\phi_{u,y} = 0, \quad y = h_u \quad (3.3)
\]
\[
\phi_{l,y} = 0, \quad y = -h_l. \quad (3.4)
\]

Requiring that the interface remain material produces the kinematic boundary conditions

\[
\eta_t + (U_u + \phi_{u,x}) \eta_x = \phi_{u,y}, \quad y = \eta \quad (3.5)
\]
\[
\eta_t + (U_l + \phi_{l,x}) \eta_x = \phi_{l,y}, \quad y = \eta \quad (3.6)
\]

while the balance of normal stresses at the interface between the two fluids gives

\[
\mathcal{R} \left[ \phi_{u,t} + \frac{1}{2} (\nabla \phi_u)^2 + U_u \phi_{u,x} + \eta \right] - \left[ \phi_{l,t} + \frac{1}{2} (\nabla \phi_l)^2 + U_l \phi_{l,x} + \eta \right] =
\]
\[
-\frac{1}{\mathcal{W}} \eta_{xx} (1 + \eta_x^2)^{-3/2}, \quad y = \eta \quad (3.7)
\]

where \( \mathcal{R} \equiv \rho_u/\rho_l \) is the density ratio, \( \mathcal{W} \equiv L^2 g \rho_l/\gamma \) the Weber number, \( g \) the gravitational acceleration, and \( \gamma \) the surface tension coefficient. This problem is made dimensionless with the characteristic length and time scales \( \mathcal{L} \) and \( \mathcal{T} = \sqrt{\mathcal{L}/g} \). (For numerical examples presented in this work, \( \mathcal{L} \) is chosen such that the spatial domain has a length of \( 2\pi \).) This problem is complete with the specification of an appropriate set of initial conditions for the interface elevation and potentials.

The linear formulation of this problem yields the classical solutions for the Kelvin-Helmholtz instability. For clarity and to introduce the notation used in this paper, the key linear results are re-stated here. By linearizing the governing equations, eqn.
(3.1-3.7), a traveling wave solution is obtained of the form

\[ \eta = \frac{1}{2} \eta_0 e^{i(kx - \omega t)} + c.c. \quad (3.8a) \]
\[ \phi_u = -i\eta_0 \frac{(U_u k - \omega) \cosh k(y - h_u)}{2k \tanh kh_u} e^{i(kx - \omega t)} + c.c. \quad (3.8b) \]
\[ \phi_l = i\eta_0 \frac{2k \tanh kh_l}{2k \tanh kh_l} \cosh k h_u e^{i(kx - \omega t)} + c.c. \quad (3.8c) \]

where 'c.c' denotes the complex conjugate, \( i = \sqrt{-1} \), \( \eta_0 \) is the initial wave disturbance amplitude, \( k \) is the wave number and \( \omega \) is the wave frequency given by the dispersion relationship

\[ \omega = \frac{k (U_u R T_l + U_l T_u)}{R T_l + T_u} \pm \frac{1}{k} \left[ \frac{T_u T_l}{R T_l + T_u} \left( 1 - \frac{k^2}{W} \right) - \frac{R (U_u - U_l)^2 T_u T_l}{(R T_l + T_u)^2} \right]^{1/2} \quad (3.9) \]

where \( T_{u/l} \equiv \tanh kh_{u/l} \). From (3.9), it is evident that \( \omega \) is complex if \( |U_u - U_l| > U_c \) with the critical velocity being defined as

\[ U_c(k) = \sqrt{\frac{R T_l + T_u}{R k} \left( 1 - \frac{k^2}{W} \right)} \quad (3.10) \]

In the limit of marginally unstable modes, where \( |U_u - U_l| \) slightly exceeds \( U_c \), a modified solution can be obtained. Without loss of generality, we let \( U_u - U_l = U_c (1 + \Delta) \) with \( U_u > U_l \) and \( 0 < \Delta \ll 1 \). Under these conditions, the frequency can be written as

\[ \omega = \frac{k (U_u R T_l + U_l T_u)}{R T_l + T_u} \pm \frac{2k T_u T_l}{R T_l + T_u} \left( 1 - \frac{k^2}{W} \right)^{1/2} \Delta^{1/2} + O \left( \Delta^{3/2} \right) \quad (3.11) \]

This demonstrates that for marginally unstable modes, the growth rate \( \omega \sim O \left( \Delta^{1/2} \right) \) while \( |U_u - U_l - U_c| \sim O (\Delta) \).

The stated nonlinear problem can in principle be solved by use of the methods of regular perturbation expansion and/or multiple-scale analysis. In practice, however, the theoretical solution of the nonlinear problem is achievable only for relatively
simple cases when a small number of wave components and low-order nonlinearity are concerned. In practical applications involving steep broadbanded interfacial wave-field development, numerical tools are usually employed to find the nonlinear solution. In §2.3, an effective numerical method has been developed for efficiently simulating the nonlinear evolution of an interface in a two-phase channel flow, including the effects of both nonlinear wave resonances and linear instability. This method solves the stated nonlinear initial boundary-value problem, eqn. (3.1 - 3.7), by following the time evolution of a large number ($N$) of spectral interfacial wave modes and accounts for their interactions up to an arbitrarily high order ($M$) of nonlinearity in the wave steepness using a pseudo-spectral approach. The numerical simulation was validated by comparisons with the theoretical prediction of various discrete triad wave resonances and the experimental observation of slug development near the entrance of two-phase flow into a pipe. In this work, the direct simulation with this numerical method is used to verify the theoretical analysis for sub-harmonic wave resonances and study the effects of multiple chained sub-harmonic (and triad) wave resonances in the broadbanded interfacial wave-field development.

3.3 Resonant Interfacial Wave Interactions

Following the analysis of Phillips\cite{Phillips65} for conservative resonances, a triad resonance involving waves with wavenumber $k_1$, $k_2$ and $k_3$ is formed when they satisfy the relationships $k_2 - k_1 = k_3$ and $\omega_2 - \omega_1 = \omega_3$ where $\omega_j$ is related to $k_j$ by (3.9). For the special case where $k_1 \rightarrow k_3$, a second-harmonic resonance is formed between two unique wave components when $k_2 = 2k_1$. Similarly, sub-harmonic resonances are possible when $k_2 = \frac{1}{2}k_1$. The sub-harmonic resonance (as opposed to the second-harmonic resonances) is of interest because there are many practical problems which exhibit the formation of large amplitude long waves which evolve from shorter waves. In order to account for those wave modes that do not perfectly satisfy the resonance condition, a more general (near) sub-harmonic resonance condition is utilized for
which the wavenumbers and frequencies satisfy

\[
\begin{align*}
    k_1 &= 2k_2 \\
    \omega_{R1} &= 2\omega_{R2} + \sigma
\end{align*}
\]  

(3.12)

where \( \sigma \) (with \( \sigma \ll \omega_{Rj}, j = 1, 2 \)) represents the frequency detuning. Note, the resonance condition is satisfied exactly when \( \sigma = 0 \).

Depending on the wavenumber values of the interacting waves, there are several classes of sub-harmonic interactions which can result in very different modal evolution equations. A summary of four classes of sub-harmonic resonances are shown in figure 3-1. The first case, shown in figure 3-1(a), is the stable sub-harmonic resonance which is analogous to the classic (conservative) second-harmonic resonance. The second class of sub-harmonic resonances, shown in figure 3-1(b), involves the case where one of the resonant modes is marginally unstable while the second is linearly stable. The third case is illustrated in figure 3-1(c), in which both modes are marginally unstable. These three cases are analyzed theoretically by use of the method of multiple scales in §3.4, where the theoretical results are also compared with the direct numerical simulation. Figure 3-1(d) shows a case which involves strongly unstable waves. This case is of less theoretical interest since the wave growth is dominated by the linear instability rather than the sub-harmonic resonance. This case is included in the general situation of broadbanded wave-field evolution which is examined by direct numerical simulations in §3.5.2.

Additionally, practical applications are rarely limited to flows with discrete (isolated) resonant interactions, but may involve multiple and chained resonances. To demonstrate the presence of these types of sub-harmonic resonant interactions in more general flows, direct numerical simulations of broadbanded flows is carried out and examined in §3.5. The role of these subharmonic resonances is identified and the strength of these interactions is compared against the more general triadic interactions.
Figure 3-1: Summary of the four different classes of sub-harmonic resonances and their wavenumber distribution with respect to the neutral Kelvin-Helmholtz stability curve (solid line). (a) Both modes are linearly stable; (b) one mode is marginally unstable and one mode is linearly stable; (c) both modes are marginally unstable; and (d) both modes are strongly (linearly) unstable.
3.4 Modal Evolution Equations by Multiple-Scale Analysis

In this section, the nonlinear modal equations governing the time evolution of the amplitudes of the interacting waves are derived using the method of multiple scales. These interaction equations incorporate the effect of nonlinear resonant wave-wave interactions along with a possible linear interfacial instability. All of the interaction equations derived within this section are cross-validated against a more general (time-domain) perturbation solution obtained by direct numerical simulation based on a high-order pseudo-spectral method (for two-phase flows) developed in §2.3.

3.4.1 The $k_1$ And $k_2$ Modes Are Both Linearly Stable

The case of nonlinear second-harmonic interactions amongst linearly stable wave modes was examined in works dating back to the 1970’s. McGoldrick[54] derived a pair of nonlinear resonant interaction equations for the case of second-harmonic interactions among capillary-gravity waves. It was found for the case where the amplitudes do not contain spatial variations, the interaction equations could be solved in a closed form in terms of Jacobi elliptic functions and an incomplete elliptic integral of the third kind. Similarly, Nayfeh and Saric[61] examined the case of second-harmonic resonances at the interface between two superposed fluids of infinite depth. In the limit of a perfect resonance ($\sigma \to 0$), exact solutions to the second-order interaction equations were found to be of the form of hyperbolic functions (tanh and sech) provided that there was no spatial variations to the amplitude.

This class of subharmonic resonance is the simplest case which can be examined. For completeness, an analogous set of expressions to those derived by Nayfeh and Saric[61] are derived for the case of sub-harmonic resonances at the interface of a two-layer wall bounded flow. This introduces the notation used in this paper, provides a clean validation case to compare against the numerical method, and provides a basis for comparison in the unstable cases.
This analysis begins with the standard regular perturbation expansion of the velocity potentials \( \phi_u, \phi_l \) and the interfacial displacement \( \eta \) of the form

\[
\phi_u(t,x,y) = \sum_{m=1}^{2} \epsilon^m \phi_u^{(m)}(t,x,y) + O(\epsilon^3)
\]

\[
\eta(t,x,y) = \sum_{m=1}^{2} \epsilon^m \eta^{(m)}(t,x,y) + O(\epsilon^3)
\]

with \( \epsilon \ll 1 \) being a measure of the wave steepness and \( \tau = ct \) being the slow time scale. Following the standard procedure for the method of multiple-scale perturbation analysis, substitution of eqn. (3.13) into eqns. (3.1-3.7) decomposes the nonlinear boundary value problem into a series of linearized boundary value problems for \( \{ \phi_u^{(m)}, \phi_l^{(m)}, \eta^{(m)} \} \) given by (B.1) with the non-zero nonlinear forcing functions being listed in Appendix B.1. These linearized boundary-value problems can then be solved sequentially. The \( O(\epsilon) \) boundary-value solution is of the form of the general linear solution given by (3.8)

\[
\phi_u^{(1)} = \frac{\cosh k_i (y - h_u)}{\cosh k_i h_u} [P_1(\tau) e^{i\psi_1} + c.c] + \frac{\cosh k_2 (y - h_u)}{\cosh k_2 h_u} [P_2(\tau) e^{i\psi_2} + c.c]
\]

\[
\phi_l^{(1)} = \frac{\cosh k_i (y + h_l)}{\cosh k_i h_i} [Q_1(\tau) e^{i\psi_1} + c.c] + \frac{\cosh k_2 (y + h_l)}{\cosh k_2 h_i} [Q_2(\tau) e^{i\psi_2} + c.c]
\]

\[
\eta^{(1)} = [iA_1(\tau)e^{i\psi_1} + c.c] + [iA_2(\tau)e^{i\psi_2} + c.c]
\]

where \( P_j, Q_j, \) and \( \psi_j \) are given by the expressions

\[
P_j = -\frac{iG_{uj}A_j(\tau)}{2k_j \tanh k_j h_u}, \quad Q_j = \frac{iG_{lj}A_j(\tau)}{2k_j \tanh k_j h_l}, \quad \psi_j = k_j x - \omega_j t
\]

and \( G_{uj} = U_u k_j - \omega_j \) and \( G_{lj} = U_l k_j - \omega_j \) with the subscripts spanning \( j = 1,2 \).

With this \( O(\epsilon) \) solution, the second-order solution can be obtained by invoking the proper solvability condition, based on Green’s Theorem, to terms having phase \( \psi_1 \) or \( \psi_2 \). This produces a set of nonlinear evolution equations for the wave amplitudes.
\( \frac{da_1}{dt} = iB_{22} a_2^* e^{i\omega t} \)  \hfill (3.15a)

\( \frac{da_2}{dt} = iB_{12} a_1 a_2^* e^{-i\omega t} \)  \hfill (3.15b)

where

\[
B_{12} = k_2 \frac{G_{12} G_{12} (G_{11} C_{11} + G_{12} C_{12}) + G_{11} G_{12} (1 + C_{11} C_{12}) - G_{11}^2 - G_{12}^2}{4 (R G_{u2} C_{u2} + G_{12} C_{12})}
- \mathcal{R} k_2 \mathcal{G}_{u2} C_{u2} (G_{u1} C_{u1} + G_{u2} C_{u2}) + \mathcal{G}_{u1} \mathcal{G}_{u2} (1 + C_{u1} C_{u2}) - G_{u1}^2 - G_{u2}^2
\]

\[
B_{22} = k_2 \frac{1}{2} \left( \mathcal{R} G_{u2}^2 (3 - C_{u2}^2) - G_{12}^2 (3 - C_{12}^2) \right) - \mathcal{R} G_{u1} C_{u1} G_{u2} C_{u2} + \mathcal{G}_{u1} C_{u1} G_{12} C_{12}
\]

\[\frac{4 (R G_{u1} C_{u1} + G_{11} C_{11})}{4 (R G_{u1} C_{u1} + G_{11} C_{11})}\]

The characteristic behaviors of this class of coupled ordinary differential equations are well documented\[54, 61\] making them appropriate for the validation of the high-order spectral method for two-phase flows documented in §2.3. A sample case is presented in figure 3-2 which is generated using the flow conditions \( U_u \approx 2.58, U_l \approx 0.53, h_u \approx 0.014, h_l \approx 0.260, k_1 = 174, k_2 = 87, \mathcal{R} \approx 0.00123, \) and \( W \approx 17891. \) Given these flow conditions, the interaction coefficients are found to be \( B_{12} \approx -107.17 \) and \( B_{22} \approx -53.35. \) At \( t = 0, \) the \( a_1 \) mode is given an initial amplitude of \( a_1(0) = 3 \times 10^{-5} \) while the \( a_2 \) mode's amplitude is set to \( a_2(0) = 0.1 a_1(0). \) As expected, the subharmonic resonant coupling between the two modes allows for the nonlinear transfer of energy between the \( a_1 \) and the \( a_2 \) modes. Clearly, good agreement is observed between the theoretical and numerical solutions.

### 3.4.2 The \( k_1 \) Mode Is Marginally Unstable And The \( k_2 \) Mode Is Linearly Stable

The second class of resonant interactions to be examined consists of one mode, of wavenumber \( k_1, \) being marginally unstable to the Kelvin-Helmholtz mechanism while simultaneously being involved in a sub-harmonic resonance with a wave mode of wavenumber \( k_2. \) This class of resonant interactions is a special case of the more
Figure 3-2: Time evolution of the amplitudes of the interfacial waves in a stable subharmonic resonance. The plotted curves represent the direct numerical simulation result with the nonlinearity order set to $M=2$ (---) and the theoretical prediction (—) by eqn. (3.15).

As in the case for the marginally unstable triad resonance, the time scale of the linear instability, shown in eq. (3.11), is $O(\Delta^{1/2}t)$ while the time scale of the rapid phase oscillations is $O(t)$. This suggests that the proper scaling for the perturbation expansion is of the form

$$
\phi_u (x, y, t, \tau) = \sum_{m=1}^{5} \Delta^{(m+1)/4} \phi_u^{(m)} (x, y, t, \tau) + O(\Delta^{7/4}) \quad (3.17a)
$$

$$
\phi_l (x, y, t, \tau) = \sum_{m=1}^{5} \Delta^{(m+1)/4} \phi_l^{(m)} (x, y, t, \tau) + O(\Delta^{7/4}) \quad (3.17b)
$$

$$
\eta (x, t, \tau) = \sum_{m=1}^{5} \Delta^{(m+1)/4} \eta^{(m)} (x, t, \tau) + O(\Delta^{7/4}) \quad (3.17c)
$$

where it is assumed that the steepness of the interface is small such that $O(\epsilon) \sim O(\Delta^{1/2})$. With this choice of scaling and this definition of the perturbation expansion, nonlinear governing equations (3.1-3.7) can be expanded in the form as (B.1) with the nonlinear forcing functions (at each order) being listed in Appendix B.2.
This expansion is established under the assumption that the amplitude of the unstable wave \((k_1)\) is \(O(\Delta^{1/2})\) while the stable wave mode \((k_2)\) is \(O(\Delta^{3/4})\). This choice of scaling causes the effect of the cubic self interaction of the unstable \((k_1)\) wave to be comparable to that of the quadratic interaction of the stable \((k_2)\) wave. This causes the solution of the first two orders of the perturbation to have the same general form which is consistent with the linear solution

\[
\phi_u = \Delta \frac{1}{2} \left[ \phi_{u0}^{(1)}(\tau) + \left\{ S_1(\tau) \frac{\cosh k_1(y - h_u)}{\cosh k_1 h_u} e^{i\psi_1} + c.c \right\} \right] + \Delta \frac{3}{4} \left[ \phi_{u0}^{(2)}(\tau) + \left\{ S_2(\tau) \frac{\cosh k_2(y - h_u)}{\cosh k_2 h_u} e^{i\psi_2} + c.c \right\} \right] \tag{3.18a}
\]

\[
\phi_l = \Delta \frac{1}{2} \left[ \phi_{l0}^{(1)}(\tau) + \left\{ T_1(\tau) \frac{\cosh k_1(y + h_l)}{\cosh k_1 h_l} e^{i\psi_1} + c.c \right\} \right] + \Delta \frac{3}{4} \left[ \phi_{l0}^{(2)}(\tau) + \left\{ T_2(\tau) \frac{\cosh k_2(y + h_l)}{\cosh k_2 h_l} e^{i\psi_2} + c.c \right\} \right] \tag{3.18b}
\]

\[
\eta = \Delta \frac{1}{2} \left[ \frac{i}{2} A_1(\tau) e^{i\psi_1} + c.c \right] + \Delta \frac{3}{4} \left[ \frac{i}{2} A_2(\tau) e^{i\psi_2} + c.c \right] \tag{3.18c}
\]

and \(S, T,\) and \(\psi_j\) are given by

\[
S_j = -i \frac{D_{wj} A_j(\tau)}{2 k_j \tanh k_j h_u}, \quad T_j = \frac{i D_{lj} A_j(\tau)}{2 k_j \tanh k_j h_l}, \quad \psi_j = k_j x - \omega_j t \tag{3.19}
\]

and \(D_{wj} = (U_l + U_c) k_j - \omega_j\) and \(D_{lj} = U_l k_j - \omega_j\) where the subscripts \(j = 1, 2,\) Following the standard multiple-scale analysis procedure documented in §2.2.4, the nonlinear interaction equations are found to be

\[
\frac{d^2 A_1}{d\tau^2} = \Omega A_1 + \mathcal{N}|A_1|^2 A_1 + \mathcal{M} A_1 \frac{d}{d\tau} \left[ \mathcal{R} \phi_{u0}^{(1)} - \phi_{l0}^{(1)} \right] + B_{22} A_2^2 e^{i\sigma \tau} \tag{3.20a}
\]

\[
\frac{d A_2}{d\tau} = i B_{12} A_1 A_2^* e^{-i\sigma \tau} \tag{3.20b}
\]

\[
\frac{d^2}{d\tau^2} \left[ \mathcal{R} \phi_{u0}^{(1)} - \phi_{l0}^{(1)} \right] = \nu_0 (\mathcal{R} - 1) \frac{d}{d\tau} \left[ |A_1|^2 \right] \tag{3.20c}
\]
where

\[
\Omega = 2k_1RU_{u1}D_{u1}C_{u1} + C_{u1} + RC_{u1}
\]

\[
\mathcal{N} = ik_1\frac{R\alpha_{u1}D_{u1}(T_{u12} - 2C_{u1}) + \beta_{u1}D_{u1}(T_{u12} - 2C_{u2}) - \frac{3k_1}{16W(R\alpha_1 - \beta_1)}}{2(R\alpha_1 - \beta_1)} + \frac{RD_{u1}^2((\upsilon_0 + \upsilon_1)(1 - C_{u1}^2) + k_1C_{u1}) + DD_{u1}^2((\upsilon_0 + \upsilon_1)(C_{u1}^2 - 1) + k_1C_{u1})}{2(R\alpha_1 - \beta_1)}
\]

\[
\mathcal{M} = \frac{RD_{u1}^2(1 - C_{u1}^2) + DD_{u1}^2(C_{u1}^2 - 1)}{2(R - 1)(\beta_1 - R\alpha_1)}
\]

\[
B_{12} = k_2\frac{G_{u2}C_{u2}(G_{u2}C_{u1} + G_{u1}C_{u2}) + G_{u1}G_{u2}(1 + C_{u1}C_{u2}) - G_{u1}^2 - G_{u2}^2}{4(RG_{u2}G_{u2} + G_{u2}C_{u2})} - \frac{G_{u2}C_{u2}(G_{u2}C_{u1} + G_{u1}C_{u2}) + G_{u1}G_{u2}(1 + C_{u1}C_{u2}) - G_{u1}^2 - G_{u2}^2}{4(RG_{u2}G_{u2} + G_{u2}C_{u2})}
\]

\[
B_{22} = k_1\frac{RD_{u1}D_{u2}C_{u1}C_{u2} - DD_{u1}D_{u2}C_{u1}C_{u2} - \frac{1}{2}DD_{u1}^2(C_{u2}^2 - 3) + \frac{1}{2}RD_{u2}^2(C_{u2}^2 - 3)}{2(C_{u1} + RC_{u1})}
\]

\[
\nu_0 = \frac{(C_{u1}^2 - 1)D_{u1}^2 - R(C_{u1}^2 - 1)D_{u1}^2}{4(R - 1)}
\]

\[
\nu_{11} = k_1\frac{[2c_{u2}C_{u1} - \frac{1}{2}(3 - C_{u1}^2)] - \frac{1}{2}(3 - C_{u1}^2)}{8RC_{u2}D_{u1}^2 + 8C_{u2}D_{u1}^2 + 4k_1(R - 1 - 4k_1^2/W) + \frac{1}{2k_1T_{u1}}}
\]

along with \(C_{u1/2} = \coth(2k_1h_{u1/2})\), \(T_{u1/2} = \tanh(2k_1h_{u1/2})\) and \(x = \Delta^{\frac{1}{4}}\sigma\). Integrating (3.20c) with respect to \(\tau\) gives

\[
\frac{d}{d\tau} \left[ R\phi_{u0}^{(1)} - \phi_{u0}^{(1)} \right] = \nu_0(R - 1)|A_1|^2 + C
\]

with \(C\) being a constant of integration. Substituting (3.22) along with the scaling \(a_1 = \Delta^{\frac{1}{2}}A_1\), \(a_2 = \Delta^{\frac{1}{2}}A_2\), \(\tau = \Delta^{\frac{1}{4}}t\) and \(\sigma = \Delta^{\frac{1}{4}}x\) into (3.20a) and (3.20b) produces a modified set of nonlinear interaction equations of the form

\[
\frac{d^2a_1}{dt^2} = \hat{\Omega}a_1 + \hat{\mathcal{N}}|a_1|^2a_1 + B_{22}a_2^2e^{i\omega t}
\]

\[
\frac{da_2}{dt} = iB_{12}a_1a_2^*e^{-i\omega t}
\]

with \(\hat{\Omega} = (\Omega + \mathcal{M}\Delta)\Delta\) and \(\hat{\mathcal{N}} = \mathcal{N} + \mathcal{M}(R - 1)\nu_0\).
The nonlinear system of resonant interaction equations, eqns. (3.23), shows that there are different classes of competing physics. In eqn. (3.23a), the first term on the right hand side represents the effect of the linear instability while the second term represents the effect of the third-order self interaction. For positive values of $N$, the growth rate of the instability increases generating faster than exponential growth (for large wave amplitude at later stage of evolution). For negative values of $N$, the nonlinearity will suppress the linear instability eventually generating a bounded solution. (Over the course of this study, the authors have not seen cases involving negative values of $N$ for a linearly unstable $k_1$ mode whose second harmonic is linearly stable. This comment is made observationally and stated without proof.) The final term represents the sub-harmonic resonant coupling to the $k_2$ mode and permits the intermodal exchange of energy. Because of the second-order time derivative, the resonant interaction coefficient $(B_{22})$ is of a different algebraic form than that of the linearly stable case presented in §3.4.1.

The evolution of the $a_2$ mode is governed purely by the sub-harmonic resonant interaction term, as (3.23b) shows. For the $a_2$ mode, the forms of the evolution equation and the interaction coefficient $B_{12}$ are identical to those of the linearly stable case. But the growth rate is completely different since the $a_1$ mode grows rapidly by the instability effect. To estimate and understand the growth rate of the $a_2$ mode, we consider the initial phase of evolution where $a_1$ is dominated by the linear instability.

The nonlinear interaction equations can be reduced to

\[
\begin{align*}
\frac{d^2 a_1}{dt^2} & \approx \hat{\Omega} a_1 \\
\frac{da_2}{dt} & = iB_{12}a_1a_2^*e^{-i\sigma t}.
\end{align*}
\]

By making the change of variables $\xi \equiv |a_1(0)|B_{12}\exp\{\sqrt{\hat{\Omega}}t\}$, eqn. (3.24) can be combined to produce a single differential equation for $a_2$ which takes on the form of a transformed version of the Bessel differential equation

\[
\xi^2 \frac{d^2 a_2}{d\xi^2} + i\sigma \frac{da_2}{d\xi} - \frac{\xi^2}{\hat{\Omega}} a_2 = 0.
\]
In the case of nearly perfect resonance \((\sigma/\sqrt{\hat{\Omega}} \approx 0)\), this equation has a solution of the form

\[
a_2 = C_1 e^{\xi/\sqrt{\hat{\Omega}}} + C_2 e^{-\xi/\sqrt{\hat{\Omega}}}. \tag{3.25}
\]

Since \(B_{12}\) is real and \(\hat{\Omega} > 0\), the behavior of the \(a_2\) mode is dominated by bi-exponential growth in time. At the later stage of evolution, the \(a_1\) mode grows faster than exponentially (for \(\hat{\mathcal{N}} > 0\)), the \(a_2\) mode should at least continue its bi-exponential growth.

Figure 3-3 compares the theoretical prediction and direct numerical simulation of the evolution of two sample cases with different flow conditions. For figure 3-3(a), we use \(U_u = U_l + U_c \approx 6.80\), \(U_l = 0.10\), \(h_u = 0.09\), \(h_l = 0.01\), \(k_1 = 4\), \(k_2 = \frac{1}{2} k_1\), \(\mathcal{R} = 0.002\), and \(\mathcal{W} = 380\). At these flow conditions, we have \(\hat{\Omega} \approx 0.33 \Delta\), \(\hat{\mathcal{N}} \approx 10^4\), \(B_{12} \approx -85.70\), \(B_{22} \approx 1.34\), and \(\Delta = 10^{-5}\). At the initial time \(t=0\), we set \(a_1(0) = 10^{-7}\) and \(a_2(0) = 10^{-3} a_1(0)\). For relatively small time, the \(a_1\) mode grows exponentially with time under the influence of the linear Kelvin-Helmholtz instability. The second harmonic of the \(a_1\) component (denoted as \(a_{12}\)) is also shown, which is a linearly stable locked wave in this case. Its evolution is directly coupled with the \(a_1\) wave and is predicted to grow with twice the growth rate of the \(a_1\) mode. The nonlinear sub-harmonic resonance couples the linearly stable \(a_2\) mode with \(a_1\) mode allowing for the direct exchange of energy between them. This allows the (linearly stable) sub-harmonic wave mode to achieve faster than exponential growth. As the amplitude of the \(a_1\) mode becomes large at later stage of the evolution, the nonlinearity of the system causes the growth rate of \(a_1\) to increase above exponential until it eventually becomes strongly nonlinear. The nonlinear coupling then forces the \(a_2\) and the \(a_{12}\) to follow. Good agreement between the analytical and numerical solutions is observed. The third order \((M = 3)\) and fourth order \((M = 4)\) numerical solutions are compared well in figure 3-3(a) demonstrating the overall convergence of the numerical solution. The difference between the \(M = 3 \& M = 4\) solution is only discernable near the very end of the simulations when all of the modal components blow up.

The growth rate of the sub-harmonic \(a_2\) depends on the flow conditions, ultimately
Figure 3-3: Time evolution of the amplitudes of the unstable primary waves ($a_1$) and the subharmonic ($a_2$) along with the second harmonic of the unstable primary mode ($a_{12}$). The plotted curves represent the theoretical solution (-----), the third order numerical solution, $M = 3, (---)$, and the fourth order numerical solution, $M = 4, (\cdots)$. Different flow conditions are used for results in (a) and (b).
the values of $B_{12}$ and $\tilde{\Omega}$ (and the magnitude of the primary $a_1$ mode). For the case shown in figure 3-3(a), the bi-exponential growth behavior of $a_2$ is seen to be weak except at the very late stage of the evolution. However, for the case shown in figure 3-3(b), the $a_2$ mode shows a clear and strong bi-exponential growth behavior. For this case, we use $U_u = U_l + U_c \approx 7.09$, $U_l = 0.10$, $h_u = 0.06$, $h_l = 0.14$, $k_1 = 4$, $k_2 = \frac{1}{2} k_1$, $\mathcal{R} \approx 0.00123$, and $\mathcal{W} = 830$. The interaction coefficients are found to be $\hat{\Omega} \approx 4.13 \Delta$, $\hat{\dot{\mathcal{N}}} \approx 8.1 \times 10^5$, $B_{12} \approx -2301$, $B_{22} \approx 25.33$, and $\Delta = 10^{-5}$. The initial values of $a_1$ and $a_2$ are the same as in the case of figure 3-3(a). In this case, the values of $B_{12}$ and $\hat{\Omega}$ are one order of magnitude larger than those in figure 3-3(a), producing a more visible bi-exponential growth of $a_2$ in the evolution. The growth features predicted by the theoretical analysis compare reasonably well with those from the direct numerical simulations. We note that as $a_1$ and $a_2$ grow in the evolution, the frequency detuning slightly increases due to nonlinear effects on frequencies ($\omega_{R1}$ and $\omega_{R2}$). This effect is accounted for in the direct numerical simulation, but not in the theoretical analysis. The growth of $a_1$ and $a_2$ by the theoretical prediction at the very late stage of the evolution is thus slightly faster than that by the numerical simulation, as seen in both figures 3-3(a) and 3-3(b).

3.4.3 The $k_1$ And $k_2$ Modes Are Both Marginally Unstable

The next case involves the two interacting interfacial modes, both of which are marginally unstable and nonlinearly coupled through a subharmonic resonance. Since both modes are unstable, they both experience the same fast and slow time scales which are $O(t)$ and $O(\Delta^{1/2}t)$ respectively. Therefore, unlike in §3.4.2, both of the resonance modes are scaled to be of the same order. A perturbation expansion of the
\[ \phi_u (x, y, t, \tau) = \sum_{m=1}^{3} \Delta^{(m+1)/2} \phi_u^{(m)} (x, y, t, \tau) + O (\Delta^{5/2}) \]  
\[ \phi_l (x, y, t, \tau) = \sum_{m=1}^{3} \Delta^{(m+1)/2} \phi_l^{(m)} (x, y, t, \tau) + O (\Delta^{5/2}) \]  
\[ \eta (x, t, \tau) = \sum_{m=1}^{3} \Delta^{(m+1)/2} \eta^{(m)} (x, t, \tau) + O (\Delta^{5/2}) \]

is defined where it is assumed that the steepness of the interface is small such that \( O (\epsilon) \sim O (\Delta^{1/2}) \). Substituting this expansion into (3.1-3.7) produces a leading order solution of the form

\[ \phi_u = \Delta \left[ \left\{ S_1 (\tau) \frac{\cosh k_1 (y - h_u)}{\cosh k_1 h_u} e^{i\psi_1} + S_2 (\tau) \frac{\cosh k_2 (y - h_u)}{\cosh k_2 h_u} e^{i\psi_2} + c.c. \right\} + \phi_{u0} (\tau) \right] \]  
\[ \phi_l = \Delta \left[ \left\{ T_1 (\tau) \frac{\cosh k_1 (y + h_l)}{\cosh k_1 h_l} e^{i\psi_1} + T_2 (\tau) \frac{\cosh k_2 (y + h_l)}{\cosh k_2 h_l} e^{i\psi_2} + c.c. \right\} + \phi_{l0} (\tau) \right] \]  
\[ \eta = \Delta \left[ \frac{1}{2} A_1 (\tau) e^{i\psi_1} + \frac{1}{2} A_2 (\tau) e^{i\psi_2} + c.c. \right] \]

with \( S_j \) and \( T_j \) being defined by eqns. (3.19). The nonlinear wave-wave interactions produce nonlinear forcing terms which are displayed in Appendix B.3. As in the previous cases, the solvability condition is enforced at each nonlinear order and generates the third order nonlinear interaction equations for wave amplitudes \( a_1 = \Delta A_1 \) and \( a_2 = \Delta A_2 \)

\[ \frac{d^2 a_1}{dt^2} = \hat{\Omega}_1 a_1 + B_{22} a_2^* e^{i\sigma t} \]  
\[ \frac{d^2 a_2}{dt^2} = \hat{\Omega}_2 a_2 + B_{12} a_2 a_1^* e^{-i\sigma t} \]
where

\[
\hat{\Omega}_1 = \Omega_1 \Delta = \frac{2\mathcal{R} k_1 \mathcal{D}_{u_1} C_{u_1} U_c}{C_{l_1} + \mathcal{R} C_{u_1}} \Delta \\
\hat{\Omega}_2 = \Omega_2 \Delta = \frac{2\mathcal{R} k_2 \mathcal{D}_{u_2} C_{u_2} U_c}{C_{l_2} + \mathcal{R} C_{u_2}} \Delta \\
B_{22} = k_1 \left( \frac{3}{4} \mathcal{D}_{l_2}^2 - \frac{1}{4} \mathcal{D}_{l_2} C_{l_2} - \mathcal{R} \left( \frac{3}{4} \mathcal{D}_{u_2}^2 - \frac{1}{4} \mathcal{D}_{u_2}^2 C_{u_2} \right) + \frac{1}{4} \mathcal{R} \mathcal{D}_{u_1} C_{u_1} \mathcal{D}_{u_2} C_{u_2} \right) \\
B_{12} = k_2 \left( \frac{3}{4} \mathcal{D}_{l_2}^2 - \frac{1}{4} \mathcal{D}_{l_2} C_{l_2} - \mathcal{R} \left( \mathcal{D}_{l_1}^2 - \mathcal{D}_{l_1} D_{l_2} (1 + 2C_{l_1} C_{l_2}) \right) \right) \\
\frac{2 \left( C_{l_2} + \mathcal{R} C_{u_2} \right)}{2 \left( C_{l_2} + \mathcal{R} C_{u_2} \right)} - \mathcal{R} \left[ \mathcal{D}_{u_2}^2 (1 - C_{u_2}^2) + \mathcal{D}_{u_1}^2 - \mathcal{D}_{u_1} \mathcal{D}_{u_2}^2 (1 + 2C_{u_1} C_{u_2}) \right] \right]
\]

Each interaction equation in (3.28) is composed of a term responsible for the linear instability and a sub-harmonic resonant interaction term. While the resonant terms appear to be of a similar form to that of the linearly stable case, shown in §3.4.1, the second order time derivative produces interaction coefficients with different algebraic forms. These interaction equations contain up to third-order effects as opposed to the linearly stable equations which only contain second-order interactions. Also, because of the choice of scaling in the perturbation expansion, the nonlinear self interaction term is no longer present.

A sample case is shown in figure 3-4 for which the flow conditions are $U_w \approx 12.60$, $U_l \approx 1.60$, $h_u \approx 1.26$, $h_l \approx 1.26$, $k_1 = 20$, $k_2 = 10$, $\Delta = 10^{-5}$, $\mathcal{R} \approx 0.00123$, and $W \approx 200$. At these flow conditions, we have $\hat{\Omega}_1 \approx 119.7\Delta$, $\hat{\Omega}_2 \approx 29.9\Delta$, and $B_{22} \approx B_{12} \approx 149.3$. At the initial time ($t=0$), we set $a_1(0) = 10^{-7}$ and $a_2(0) = 0.1a_1(0)$. Clearly, both modes grow initially at an exponential rate due to driving influence of the linear instability. As their amplitude becomes large, both of their growth rates increase to a rate which is faster than exponential. The numerical solutions, with $M = 3, 4$, are in very good agreement up with the theoretical prediction until the amplitude becomes very large until the very end stage of the evolution, as shown in figure 3-4(b).
Figure 3-4: (a) The time evolution of the wave amplitudes $a_1$ and $a_2$ in a sub-harmonic resonant interaction with both modes being marginally unstable. The plotted curves denote the theoretical prediction (---), the third order numerical solution with $M = 3$ (---), and the fourth order numerical solution with $M = 4$ (. . .). (b) Close-up of the faster than exponential growth of $a_1$ and $a_2$ at later stage of evolution.
3.5 Generalized Wave-Field Evolution

The analysis presented in §3.4.1-3.4.3 describes several classes of sub-harmonic resonant wave-wave interactions. When one or more of these resonant modes are marginally unstable, a mechanism is found which permits the rapid transfer of energy from unstable short wavelength disturbances to stable long wavelength disturbances. While these mechanisms have been described for the case of discrete interactions, the more physically realistic problems of interest involve a wider spectrum of interacting waves modes. Under these conditions, there can be a large range of unstable wavenumbers which can satisfy the resonant interaction conditions. Furthermore, it is also evident that a single wave mode may be an active component in several simultaneous resonant (sub-harmonic, triadic, quartets, etc.) exchanges. These more complicated interactions can further accelerate the transfer of energy across the spectrum.

To understand how the additional interactions change the energy exchange process from the discrete cases which have already been analyzed, two additional problems are presented. In the first case, a sub-harmonic resonance between two modes is examined with one mode being marginally unstable. The sub-harmonic mode is also simultaneously coupled to a separate triad resonance. This problem demonstrates how the effects of the linearly unstable short waves can propagate across the spectrum to even longer wave components compared to the sub-harmonic exchange alone. In the second case, the nonlinear evolution of a broadbanded spectrum is simulated whereby all of the linearly unstable modes in the spectrum are resolved and the resulting energy cascade is captured. This allows for the most realistic comparison of the differences between the discrete mechanisms and the generalized problem.

3.5.1 Discrete Resonance Cascade

The resonance mechanisms shown in §3.4.2 & 3.4.3 demonstrate that when two modes are in a sub-harmonic resonance, the rate of energy transfer and the maximum modal amplitudes rapidly increase when one or more of the modes is marginally unstable. For the case of a single marginally unstable wave, the \( k_1 \)-mode grows due to the linear
instability and transfers a portion of its energy to the resonantly coupled \( k_2 \)-mode. Because there are only two modes in the resonance, the energy cascade stops at the \( k_2 \) mode resulting in rapid, faster than exponential, growth.

If we now allow the interface to contain more wave modes, the energy cascade can continue further across the spectrum. For this study, we carry out the numerical simulation with the flow conditions \( U_0 = 9.1, U_1 = 0.1, h_u = 0.1, h_l = 0.9, W = 320 \), and \( R \approx 0.00123 \). At these conditions, there is a sub-harmonic resonance between \( k_1 = 8 \) and \( k_2 = 4 \) (with \( \omega_1 - 2\omega_2 \sim 0.09 \)). By examining the dispersion relationship, it is also apparent that the \( k_2 \) mode satisfies the triad resonance conditions with the \( k_3 = 3 \) and \( k_4 = 1 \) modes (with \( \omega_2 - \omega_3 - \omega_4 \sim 0.02 \)). Figure 3-5(a) shows the results of nonlinear simulations. Clearly, the \( a_1 \) and \( a_2 \)-modes behave in a similar fashion to that described in §3.4.2. The \( a_1 \) mode initially grows exponentially due to the Kelvin-Helmholtz instability and the linearly stable \( a_2 \) mode grows at a faster than exponential rate through the sub-harmonic resonant energy exchange. However, now that the \( a_2 \) mode is in a second resonance, with the \( a_3 \) and \( a_4 \) modes, a portion of the energy in the \( a_2 \) mode is passed on to the two longer wave modes by the triad resonance. An energy map of this resonance cascade is sketched in figure 3-5(b). In general, the amount of energy which is passed on to the \( a_3 \) and \( a_4 \) modes depends on the magnitude of the interaction coefficients, the amplitude of the wave components in the triad, and the growth rate of the \( a_1 \) mode. It is fair to assume that in a more general system containing additional stable wave modes, the energy would continue to be spread out across the spectrum.

To further verify that this growth in the \( a_3 \) and \( a_4 \) modes is a consequence of the resonant coupling to the \( a_2 \), and in turn the \( a_1 \) mode, a simulation is carried out in which the \( a_1 \) mode is removed. This leaves only the three linearly stable modes \( a_2, a_3, \) and \( a_4 \). In this case, the three wave system experiences a small-amplitude oscillatory behavior, which is typical of stable resonant triads, without any sizable long term growth being observed.
Figure 3-5: Discrete resonance cascade involving a sub-harmonic resonance between the $k_1$ and $k_2$ modes and a simultaneous triad resonance among the $k_2$, $k_3$ and $k_4$ modes. (a) The time evolution of the amplitudes of the $k_1$, $k_2$, $k_3$ and $k_4$ waves obtained from direct numerical simulations with nonlinearity order $M = 3 \ (-\ -\ -)$ and $M = 4 \ (-----)$; and (b) sketch of energy transfer in the resonance cascade (with dash line representing the neutral instability curve).
3.5.2 Broadbanded Wave-Field Evolution

The final case which is considered is the direct extension of the problem examined in §3.5.1 to the case which permits broadbanded wave interactions. Until now, all of the cases considered only permit a few discrete modes to interact, which allows for the specific resonant mechanisms to be examined and quantified. However, it is not clear how these interactions play a role under more complicated flow conditions. The flow conditions in these cases are also chosen such that the modes are only marginally unstable. In a broadbanded spectrum, it is possible to observe resonances among strongly unstable modes such as those depicted in figure 3-1(d).

A numerical simulation is carried out which accounts for up to forth-order nonlinear interactions among modes with $k \in [0, 32]$. The flow conditions are $U_u = 9.05$, $U_l = 0.10$, $h_u = 0.1$, $h_l = 0.9$, $W = 360$, and $R \approx 0.00123$. The unstable band of wavenumbers are in the range of $k \in (6, 16)$. The initial amplitude of all of the spectral components is semi-arbitrarily set to be $O(10^{-8})$ with each mode having a unique random phase. This amplitude is chosen to be small so that the initial condition could be given by the linearized modal solutions given by eqn. (3.8). The small initial amplitude would also allow for an adequate period of linear interfacial evolution before the nonlinear effects become strong. Figure 3-6 shows the results of the nonlinear simulation.

Initially, only those modes which are linearly unstable to the Kelvin-Helmholtz mechanism grow with their amplitudes increasing exponentially with time, as shown at $t = 10.73$ in figure 3-6(a). As the unstable modal amplitudes increase, the nonlinear interactions begin transferring energy across the spectrum as shown at $t = 20.28$. A strong spectral peak is observed among the short waves, $k \sim [20, 28]$. These are components which are generally non-resonant, linearly stable, locked waves whose amplitudes are directly coupled to the growth of the unstable Kelvin-Helmholtz modes. A second spectral peak is observed between $k \in [0, 6]$. These modes evolve due to the multiple nonlinear resonant interactions. Examination of the dispersion relationship shows that there is a sub-harmonic resonance formed by the $k_1 = 12$ and
Figure 3-6: The nonlinear time evolution of a broadbanded interfacial wave spectrum in an air-water channel flow: (a) distribution of the interfacial wave spectral amplitude at time $t=10.73$, 20.28, and 21.65; and (b) the liquid hold-up/interface profile $(\eta + h_t)/H$ (with $H = h_u + h_l$) inside the channel at $t=21.65$. The plotted curves represent the nonlinear numerical simulation results with $M = 3$ (---) and $M = 4$ (···).
$k_2 = 6$ modes and a large number of triadic near resonances formed by waves with wavenumbers in the range of $k \in (0, 16)$. As the flow continues to evolve, it eventually becomes broadbanded, as shown at $t = 21.65$ in figure 3-6(a), with a slight peak around the strongest linearly unstable modes. The interface profile corresponding to the broadbanded spectrum at $t = 21.65$ is displayed in figure 3-6(b).

We note that though the sub-harmonic mode has grown by nearly three orders of magnitude by time $t=20.28$, it is not the dominant wave mode in the spectrum in this case. Examination of figure 3-6(a) indicates that the most rapidly growing long waves are around $k = 2$. This suggests that the more general triadic resonant interactions play a strong role in the interfacial evolution in this case. Upon examining the dispersion relationship, it is apparent that the $k_2$-mode alone can participate in several near resonant interactions. Similarly, most of the other linearly stable modes satisfy near-resonance conditions (often with linearly unstable modes) showing that there is a large volume of resonant triads in the linearly stable portion of the spectrum. This simulation result suggests that even though the sub-harmonic resonances can produce rapid wave growth, it is possible for the sheer volume of resonant triads in the spectrum to have a stronger effect than the single sub-harmonic resonance. Therefore, examining the dispersion relationship in search of sub-harmonic resonances appears to be an insufficient condition to predict the formation of a dominant sub-harmonic spectral peak in physical systems. Analysis of the resonant triads or numerical simulations need to be carried out to determine which resonant mechanism will be the dominant feature.

3.6 Conclusions

The role of nonlinear sub-harmonic resonant interactions among interfacial waves is investigated when one or more of the modes is linearly unstable to the Kelvin-Helmholtz instability. Three classes of discrete sub-harmonic resonances (with both modes linearly stable, one mode linearly unstable, and both modes linearly unstable) are examined analytically. By the method of multiple scales, nonlinear interaction
equations are developed which govern the time evolution of the amplitudes of the interacting waves under the combined influence of the linear instability and nonlinear sub-harmonic resonant interactions. It is shown that by resonantly coupling a linearly stable wave with a linearly unstable component, faster than exponential (or even bi-exponential) growth could be achieved for both interacting (short and long) waves. More general cases are examined by direct numerical simulations, in which interfacial wave modes are simultaneously engaged in sub-harmonic and triad resonances. It is shown by nonlinear numerical simulations that this chain of resonances could draw energy from the shorter linearly unstable wave components and rapidly transfer that energy across the spectrum to the longest wave components. For practical applications, this suggests that problems like slug flow, which can begin as small amplitude short waves and grow into large amplitude long waves, could be formed through similar resonant mechanisms.
Chapter 4

Viscous Stability and Nonlinear Resonance Analysis

The analysis presented in the preceding chapters have examined the role of nonlinear resonant wave interactions in the development of large amplitude nonlinear waves and slugs in horizontal channel flows. Detailed comparisons against experimental measurements demonstrated strong similarities to the behavior predicted by the analytic theory. However, the analysis was built on the assumption that each of the fluids were inviscid and irrotational. This allowed for a large range of powerful analytical and fast numerical methods to be utilized. Due to the high Reynolds numbers reported in experiments and industrial applications, this assumption seemed reasonable.

Strength would be given to that resonant interaction theory if a more comprehensive viscous analysis still showed the presence of strong resonant interactions. Therefore, a linear viscous analysis is carried out to determine the importance of viscosity and turbulence on the interfacial development. This theoretical analysis will provide an eigenvalue problem for the wave phase velocities and stream function using a Chebyshev spectral method. Knowing the calculated wave spectrum then allows for possible resonant interactions to be identified and validated against experimentally measured wave spectra. The resulting analysis will provide support to the physical legitimacy of the resonant cascade mechanism for the nonlinear wave growth in two-phase channel flows.
4.1 Introduction

Hydrodynamic stability analysis has always been a rich academic research area dating back to the days of Kelvin, Reynolds, Taylor, and Rayleigh. Investigations into the theory of single phase channel flow have been well documented yielding a wide range of theoretical and numerical techniques. This has caused a shift towards the more complicated cases involving the stability of two-fluid flows. The classic work of Yih in 1967 [84] examined the linear stability of two superposed fluids in plane Couette and Poiseuille flow. The analysis presented linearized governing equations for the two-fluid flow which gave way to a pair of Orr-Sommerfeld equations along with the corresponding wall and interfacial boundary conditions. Asymptotic expansions were utilized to determine the leading order behavior of long wavelength disturbances and identified a new interfacial instability that is due to the presence of viscosity stratification. This work showed that there may be a number of instability mechanisms in stratified flows. In this section, the analysis of Yih is repeated in order to introduce the nomenclature used within this chapter. The analysis is then generalized to the case of a turbulent fluid flowing over a separate laminar (or turbulent) fluid.

4.1.1 Laminar Linear Stability Analysis

In this section, the equations governing the viscous linear stability of a two-phase laminar stratified flow are derived in detail. The Squire’s transformation is invoked allowing for the two-dimensional form of the governing equations to be examined. The velocity fields are decomposed into a mean flow component $U_j$ and a small perturbation velocity $(u_j, v_j)$ field. The upper and lower fluids are denoted by the subscripts $j = 1, 2$ respectively. A similar decomposition is applied to the pressure field to account for a hydrostatic pressure $P_j$ and a small dynamic pressure perturbation $p_j$. 
For $\epsilon \ll 1$, the velocity and pressure fields may be written as

\begin{align}
\tilde{u}_j(x, y, t) &= u_j(y) + \epsilon u_j(x, y, t) \\
\tilde{v}_j(x, y, t) &= v_j(x, y, t) \\
\tilde{p}_j(x, y, t) &= p_j(y) + \epsilon p_j(x, y, t).
\end{align}

Substituting this decomposition into the Navier-Stokes equations produces the dimensionless $O(\epsilon)$ continuity and momentum equations. For the upper fluid ($0 < y < h_1$), they are

\begin{align}
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} &= 0 \\
\frac{\partial u_1}{\partial t} + U_1 \frac{\partial u_1}{\partial x} + U'_1 v_1 &= -\frac{dp_1}{dx} + \frac{1}{Re} \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right) \\
\frac{\partial v_1}{\partial t} + U_1 \frac{\partial v_1}{\partial x} &= -\frac{dp_1}{dy} + \frac{1}{Re} \left( \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} \right)
\end{align}

while the lower fluid ($-h_2 < y < 0$), governing equations are

\begin{align}
\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} &= 0 \\
\frac{\partial u_2}{\partial t} + U_2 \frac{\partial u_2}{\partial x} + U'_2 v_2 &= -\frac{1}{r} \frac{dp_2}{dx} + \frac{n}{r Re} \left( \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right) \\
\frac{\partial v_2}{\partial t} + U_2 \frac{\partial v_2}{\partial x} &= -\frac{1}{r} \frac{dp_2}{dy} + \frac{n}{r Re} \left( \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} \right)
\end{align}

with the primes denoting differentiation with respect to the $y$-variable. Within these equations, the density and viscosity ratios are defined as $r \equiv \frac{\rho_2}{\rho_1}$ and $n \equiv \frac{\mu_2}{\mu_1}$, and $Re \equiv \frac{\nu U h_1}{\mu_1}$ is the Reynolds number. For laminar problems, the scaling velocity $U$ is the interfacial velocity; while for the turbulent problem, it is an effective shear velocity which is defined in §4.2.4. A stream function $\Psi$ formulation is utilized such that the velocity perturbations may be written as

\[ u' = \Psi_y \quad \text{and} \quad v' = -\Psi_x. \]
Assuming a traveling wave solution for all perturbation quantities allows for the
perturbations to be written as
\[
\{ \Psi_1, \Psi_2, p_1, p_2 \} = \{ \phi_1(y), \phi_2(y), f_1(y), f_2(y) \} e^{i\alpha(x-ct)} \tag{4.5}
\]
where \( \alpha \) is the wavenumber and \( c \) is the phase velocity of the perturbation. Substituting
the stream function into linearized Navier-Stokes equations, eqns. (4.2 & 4.3), and
removing the pressure perturbation term \( f_j \) produces the well-known Orr-Sommerfeld
equation for each fluid
\[
(D^2 - \alpha^2)^2 \phi_j(y) = i\alpha \text{Re} \left\{ (U_j - c) (D^2 - \alpha^2) - U_j'' \right\} \phi_j(y) \quad 0 < y < h_1
\]
\[
(D^2 - \alpha^2)^2 \phi_j(y) = i\alpha \text{Re} \left\{ (U_j - c) (D^2 - \alpha^2) - U_j'' \right\} \phi_j(y) \quad -h_2 < y < 0
\tag{4.6}
\]
with \( D \equiv d/dy \) and both \( U_j \) and \( U_j'' \) being functions of \( y \). At the walls, the no-flux
and no-slip boundary conditions are enforced through
\[
\phi_1(h_1) = 0 \quad \phi_2(-h_2) = 0
\]
\[
\frac{d}{dy} \phi_1(h_1) = 0 \quad \frac{d}{dy} \phi_2(-h_2) = 0. \tag{4.7}
\]
At the interface, the continuity of \( v' \) requires that
\[
\phi_1(0) = \phi_2(0) = \phi(0). \tag{4.8}
\]
Since \( U'(y) \) is not continuous across the interface, the variation of the total \( \tilde{u} \)-velocity
must be continuous at the interface. The interfacial kinematic boundary condition
may be written as
\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \eta = v' = -i\alpha \phi(0) e^{i\alpha(x-ct)}
\]
which permits the interfacial amplitude may be expressed as
\[
\eta = \frac{\phi(0)}{c - U(0)} e^{i\alpha(x-ct)}.
\]
This forces the continuity of $u'$ to be satisfied through

$$(c - U_1(0)) \phi_1'(0) + U_1'(0) \phi_1(0) = (c - U_2(0)) \phi_2'(0) + U_2'(0) \phi_2(0).$$  \hspace{1cm} (4.9)

The continuity of shear stress at the interface is expressed as

$$\phi_1''(0) + \alpha^2 \phi_1(0) = n \left( \phi_2''(0) + \alpha^2 \phi_2(0) \right)$$  \hspace{1cm} (4.10)

while the normal stresses are balanced by

$$\frac{i \alpha \text{Re} \phi(0)}{c - U_1(0)} \left[ \alpha^2 S + \mathcal{F} \right] = n \left( D^3 \phi_2 - 3 \alpha^2 \phi_2' \right) + i n \text{Re} \left[ (c - U_2(0)) \phi_2' + U_2'(0) \phi_2 \right]$$  

$$- \left( D^3 \phi_1 - 3 \alpha^2 \phi_1' \right) - i \alpha \text{Re} \left[ (c - U_1(0)) \phi_1' + U_1'(0) \phi_1 \right]$$  \hspace{1cm} (4.11)

where

$$\mathcal{F} = \frac{g H_1 (r - 1)}{U^2} \quad S = \frac{\gamma}{\rho_1 H_1 U^2}.$$

In this analysis, $\mathcal{F}$ denotes the density weighted reciprocal of the Froude number squared while $S$ is the reciprocal of the Weber number. Together, equations (4.6-4.11) define an eigenvalue problem for the eigenfunctions $\phi_j(y)$ and eigenvalue $c = c_r + ic_i$. In §4.2, a pseudo-spectral method is derived for the numerical solution of this eigenvalue problem.

### 4.1.2 Turbulent Stability Analysis

Náraigh et al. [60] carried out a similar analysis for a turbulent gas flowing over a laminar (or turbulent) liquid layer. It was found that the linearized governing equations closely resembled the laminar Orr-Sommerfeld equations with only an additional term representing the perturbation of the Reynolds stress. Similarly, the only modification to the boundary conditions occurred in the balance of normal stresses. Their paper implemented several closure models for the different Reynolds stress perturbations; however, it was found that the dominant effects of the turbulence are conveyed though the time averaged turbulent velocity profile. This observation constitutes what they
referred to as the quasi-laminar hypothesis. Using several different closure models, they demonstrated that this proposition captured the dominant turbulent physics allowing for the perturbed Reynolds stresses to be ignored. Numerical studies showed that the deviations between the stability predictions made with turbulent closure models and the predictions made using the quasi-laminar hypothesis agreed to within less than 10%.

Boomkamp et al. [10] commented that this conclusion is supported by experiments for liquid films that can have a perfectly smooth surface in the presence of turbulent gas flows if the mean air velocity is below a certain critical gas velocity. They suggested that time scale associated with the turbulent fluctuations is in general much smaller than the time scale for the growth of disturbances of the time averaged flow.

Based on these observations, the analysis carried in this work utilized the quasi-laminar hypothesis which resulted in the turbulent stability analysis having governing equations which were of identical structure to the laminar Orr-Sommerfeld equations and boundary conditions, given by eqns. (4.6-4.11). The only modification is that the original parabolic velocity profile is replaced with a time-averaged turbulent velocity profile.

In the following section, a numerical method is developed for the solution of the two-fluid eigenvalue problem. A convenient consequence of the quasi-laminar hypothesis assumption is that rigorously validating the numerical method for the laminar Orr-Sommerfeld problem validates the method for the corresponding turbulent problem. No additional validation or code development was necessary for the turbulent stability analysis.

### 4.2 Solution of Orr-Sommerfeld Equations

In this section, the eigenvalue problem given by eqns. (4.6-4.11) is solved numerically. The Orr-Sommerfeld problem is known to be a stiff eigenvalue problem due to the large range of eigenvalues. Historically, this caused the numerical solutions to
converge slowly. Several different numerical schemes have been developed specifically to address stiff eigenvalue problems.

Yiantsios & Higgins [82] developed an algorithm based on the compound matrix method for the numerical solution of two-point boundary and eigenvalue problems involving stiff differential operators. This method converts the boundary value problem into an initial value problem that can then be solved with standard shooting methods; however, since this is an iterative technique, the method needs to have an initial guess for the eigenvalue be supplied as input. The method also requires a knowledge of whether the modes are linearly stable or unstable. A common way to deal with this was to supplement the compound matrix method with a method that calculates all of the eigenvalues for the discretized problem using a simple finite difference or finite element method. This provided an initial guess which would then be passed to the compound matrix method that would then refine the solution for a particular eigenmode.

Orszag [63] examined the stability of plane Poiseulle flow using the Chebyshev Tau method to approximate the solution to the Orr-Sommerfeld equation. The Chebyshev spectral methods permit simulations of very high accuracy with a very low computational expense. This method was shown to be significantly more accurate than previous methods. The accuracy and efficiency of spectral methods has led to their continued development and application to a range of problems. Boomkamp et al. [9] utilized a Chebyshev collocation method for solving laminar parallel two-phase flow stability problems. The solutions were found to be very accurate and demonstrated rapid convergence.

More recently, such as in the work by Valluri et al. [77] for the solution of the Orr-Sommerfeld stability problem for parallel laminar two-fluid flows, there has been a resurgence of the classic finite difference methods for stiff eigenvalue problems. Due to the increase in computational power and storage, the concerns over grid resolution requirements and convergence rates has become a lesser concerns.

After surveying the available numerical methods for eigenvalue problems, the (pseudo-spectral) Chebyshev collocation scheme was selected. The rapid convergence
(at a very low computational cost) along with the ease of implementation make this method well suited for the laminar and turbulent stability problems which are examined in this work. In the following section, a detailed derivation of the numerical scheme is provided along with a rigorous validation procedure.

### 4.2.1 Chebyshev Solution Method

To solve the eigenvalue problem given by eqns. (4.6-4.11), the Chebyshev collocation (pseudo-spectral) method is utilized. Since Chebyshev functions are defined on the domain \( z \in [-1,1] \), the \( y \)-space for each fluid is transformed through the linear mapping

\[
z_1 = \frac{2y}{h_1} - 1, \quad z_2 = -\frac{2y}{h_2} - 1
\]

(4.12)

where in both cases \( z = -1 \) represents the interface between the two fluids. Application of chain-rule redefines the vertical derivatives to be of the form

\[
\frac{d^n}{dy^n} = \begin{cases} \left( \frac{n}{h_1} \right)^n \frac{d^n}{dz_1}, & y \in [0, h_1] \\ \left( -\frac{n}{h_2} \right)^n \frac{d^n}{dz_2}, & y \in [-h_2, 0] \end{cases}
\]

Denoting \( \phi_1(z_1) = \phi_1(y) \), \( \psi(z_2) = \phi_2(y) \), \( \dot{U}_1(z_1) = U_1(y) \), and \( \dot{U}_2(z_2) = U_2(y) \), the linear Orr-Sommerfeld operators can be expressed in terms of the \( z \)-variable. The eigenfunctions, \( \phi(z_1) \) and \( \psi(z_2) \), are approximated by Chebyshev expansions of the form

\[
\phi(z_1) = \sum_{n=0}^{N_1} \phi_n T_n(z_1) \quad (4.13a)
\]

\[
\psi(z_2) = \sum_{n=0}^{N_2} \psi_n T_n(z_2) \quad (4.13b)
\]

where \( T_n(z) \) denotes the \( n^{th} \) Chebyshev polynomial of the first kind and \( \phi_n \) and \( \psi_n \) are the expansion coefficients. The derivative of the eigenfunctions are obtained by
differentiating eqn. (4.13) directly producing

\[ D^{(k)} \phi (z_1) = \sum_{n=0}^{N_1} \phi_n T_{n}^{(k)} (z_1) \quad (4.14a) \]

\[ D^{(k)} \psi (z_2) = \sum_{n=0}^{N_2} \psi_n T_{n}^{(k)} (z_2) \quad (4.14b) \]

where \( k \geq 1 \) denotes the order of the derivative. These expressions are evaluated by utilizing the following recurrence relationships between Chebyshev polynomials and their derivatives

\[ T_0^{(k)} (z) = 0 \]
\[ T_1^{(k)} (z) = T_0^{(k-1)} (z) \]
\[ T_2^{(k)} (z) = 4T_1^{(k-1)} (z) \]
\[ T_n^{(k)} (z) = 2nT_{n-1}^{(k-1)} (z) + \frac{n}{n-1} T_{n-2}^{(k-1)} (z) \quad n = 3, 4, \ldots \]

Applying eqns. (4.14) to eqns. (4.6) produces

\[ \sum_{n=0}^{N_1} \left\{ \frac{16}{h_1^4} T_{n}^{(4)} - \frac{4}{h_1^2} \left( 2\alpha^2 - i\alpha Re U_1 \right) T_{n}^{(2)} \right\} \]
\[ + \left( \alpha^4 + i\alpha Re \left[ \alpha^2 U_1 - \frac{4}{h_1^2} U_1'' \right] \right) T_{n} \]
\[ = -\frac{i\alpha Rec}{n} \sum_{n=0}^{N_1} \phi_n \left\{ \frac{4}{h_1^2} T_{n}^{(2)} - \alpha^2 T_{n} \right\} \quad (4.16a) \]

\[ \sum_{n=0}^{N_2} \left\{ \frac{16}{h_2^4} T_{n}^{(4)} - \frac{4}{h_2^2} \left( 2\alpha^2 - \frac{ir\alpha Re}{n} U_2 \right) T_{n}^{(2)} \right\} \]
\[ + \left( \alpha^4 + \frac{ir\alpha Re}{n} \left[ \alpha^2 U_2 - \frac{4}{h_2^2} U_2'' \right] \right) T_{n} \]
\[ = -i\alpha Re \sum_{n=0}^{N_2} \phi_n \left\{ \frac{4}{h_2^2} T_{n}^{(2)} - \alpha^2 T_{n} \right\} \quad (4.16b) \]
Equations (4.16) are required to be satisfied on the standard Chebyshev grid that has points located at the Gauss-Lobatto grid points defined by

\[ z_{mj} = \cos \left( \frac{j\pi}{N} \right), \quad j = 1, \ldots, M - 1 \text{ and } m = 1, 2 \quad (4.17) \]

where \( M = N_m - 2 \). This yields \( N_1 - 3 \) equations in the upper fluid and \( N_2 - 3 \) equations in the lower fluid for the expansion coefficients \( \phi_n \) and \( \psi_n \) respectively. The remaining equations come from the four interfacial and four wall boundary conditions (4.7-4.11) which are re-written in the transformed coordinates as

\[
\phi(1) = 0 \quad (4.18a)
\]

\[
\frac{d\psi}{dz_2} (1) = 0 \quad (4.18b)
\]

\[
\psi(1) = 0 \quad (4.18c)
\]

\[
\frac{d\psi}{dz_2} (1) = 0 \quad (4.18d)
\]

\[
\phi(-1) - \psi(-1) = 0 \quad (4.18e)
\]

\[
U(0) \left[ \frac{1}{h_1} \phi' + \frac{1}{h_2} \psi' \right] + \frac{1}{2} [U'_2(0)\psi - U'_1(0)\phi] = c \left( \frac{1}{h_1} \phi' + \frac{1}{h_2} \psi' \right) \quad (4.18f)
\]

\[
\alpha^2 \phi + \frac{4}{h_1^2} \phi'' = n \left( \alpha^2 \psi + \frac{4}{h_2^2} \psi'' \right) \quad (4.18g)
\]

\[
\frac{i\alpha Re (F + \alpha^2 S)}{U'_2(0) - U'_1(0)} \left[ \frac{\phi'}{h_1} + \frac{\psi'}{h_2} \right] - n \left( \frac{3\alpha^2}{h_2} \psi' - \frac{4}{h_2^2} \psi'' \right) + \left( \frac{4}{h_1^2} \phi'' - \frac{3\alpha^2}{h_1} \phi' \right) - i\alpha Re H \left[ \frac{1}{2} U'_2 \psi + \frac{1}{2} U_2 \psi' \right] - i\alpha Re H \left[ \frac{1}{2} U'_1 \phi - \frac{1}{h_1} U_1 \phi' \right] = -i\alpha Re \left[ \frac{r}{h_2} \psi' + \frac{1}{h_1} \phi' \right] \quad (4.18h)
\]

where (4.18e-4.18h) are evaluated at \( z_1 = z_2 = -1 \) and the primes denoted differentiation with respect to their variable \( (z_1 \text{ or } z_2) \). The boundary conditions are numerically evaluated by using the Chebyshev identities

\[
T_n(1) = 1, \quad T_n(-1)^n, \quad \frac{d^p T_n}{dz^p} \bigg|_{z = \pm 1} = (\pm 1)^{n+p} \prod_{k=0}^{p-1} \frac{n^2 - k^2}{2k + 1}.
\]
With the enforcement of the boundary conditions given by eqns. (4.16-4.18), the discretized expansions are written as a generalized eigenvalue problem

\[ [A] \{ \Phi \} = c [B] \{ \Phi \} \]  (4.19)

where \( \Phi = \{ \phi_0, \ldots, \phi_{N_1+2}, \psi_0, \ldots, \psi_{N_2+2} \}^T \) and \( A, B \) are \( ((N_1 + 2) + (N_2 + 2)) \times ((N_1 + 2) + (N_2 + 2)) \). Since only eqns. (4.18a-4.18d, 4.18e, & 4.18g) are independent of the eigenvalue \( c \), the resulting \( B \)-matrix is singular and can be solved using the QZ-algorithm developed by [58]. For laminar flows, the QZ-algorithm is an efficient and accurate algorithm; however, for problems with turbulent mean flows the singular \( B \)-matrix resulted in poorly conditioned eigenvector matrices. This poor conditioning often exhibited poor eigenvalue convergence or for a given eigenvalue \( (c_j) \) and eigenvector \( (\Psi_j) \), the \( L_\infty | A \Psi_j - c_j B \Psi_j | \) could be as large as \( O(1) \). Following the work of Gardner et al. [28, eq.(2.24-2.26)], the issue of matrix conditioning was alleviated by utilizing a matrix transformation which removes the homogeneous boundary condition rows making \( B \) non-singular. As a result, the transformed eigen-matrices were recast in standard eigenvalue form:

\[ \left[ \hat{B} \right]^{-1} \left[ \hat{A} \right] \{ \Phi \} = \left[ \hat{A} \right] \{ \Phi \} = c \{ \Phi \} . \]

This matrix transformation significantly improves the matrix conditioning resulting in rapid convergence of both the eigenvalue and eigenvector solutions.

It should be noted that prior to utilizing the Chebyshev collocation method, the \( D^2 \) and \( D^4 \) Chebyshev Tau methods described by Dongarra et al.[22] were used. For laminar flows, the method was found to be computationally efficient and rapid convergence of the eigenvalues was observed. However, for problems with turbulent mean flows, the eigenvalue problem was found to be poorly conditioned with the resulting eigenvector matrix having condition numbers: \( \text{cond}[\Psi] \equiv ||\Psi|| ||\Psi^{-1}|| \sim O(10^{30}) \). From numerical linear algebra, the condition number provides the general guidance that if the coefficients of \( [A] \) are known to \( t \)-digits of precision and the \( \text{cond} [A] = 10^c \), then the solution is only valid to \( t - c \) digits (or has rounding errors \( \approx 10^{c-t} \)).
<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$N$</td>
<td>$c$</td>
</tr>
<tr>
<td>30</td>
<td>$0.2375268224 + 0.0037388759i$</td>
</tr>
<tr>
<td>40</td>
<td>$0.2375264810 + 0.0037396712i$</td>
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<tr>
<td>50</td>
<td>$0.2375264888 + 0.0037396707i$</td>
</tr>
<tr>
<td>60</td>
<td>$0.2375264888 + 0.0037396706i$</td>
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<tr>
<td>70</td>
<td>$0.2375264889 + 0.0037396710i$</td>
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<tr>
<td>80</td>
<td>$0.2375264899 + 0.0037396706i$</td>
</tr>
<tr>
<td>90</td>
<td>$0.2375264863 + 0.0037396736i$</td>
</tr>
<tr>
<td>100</td>
<td>$0.2375265003 + 0.0037396581i$</td>
</tr>
</tbody>
</table>

Table 4.1: Chebyshev approximation of the most unstable mode in plane Poiseuille flow for $\alpha = 1$, $Re = 10^4$.

Therefore, for the condition numbers found using the Chebyshev Tau method, there would be zero accurate significant digits using current double precision platforms. Several modifications to the method were considered, yet accurate results were never achieved. This motivated the selection of the Chebyshev collocation method which had significantly lower eigenvector condition numbers.

4.2.2 Validation for Laminar Problems

A detailed validation of this eigen solver is presented against various laminar single and two-phase flow problems which are known to be governed by the Orr-Sommerfeld equations. The first validation test examines the classic problem of the linear stability of laminar plane Poiseuille flow. Spectral methods were used to examine this problem in the classic work of Orszag [63]. In this problem, a flow was considered which had a perturbation wavenumber $\alpha = 1$ and $Re = 10^4$. Using Orszag’s Chebyshev-Tau method, the unstable eigenvalue was identified as $c = 0.23752649 + 0.00373967i$. Due to the symmetric nature of the mean flow, the Chebyshev method developed in §4.2.1 calculated the solution with $N_1 = N_2 = N$ resulting in the eigen spectrum shown in figure 4-1. Table 4.1 shows the convergence of the most unstable eigenvalue. With only $N = 50$ modes, the eigenvalue is in complete agreement (to within round-off precision) with the value reported by Orszag. The solution has converged such that the variations of eigenvalue are $O(10^{-9})$. 

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The second validation tests recovered the leading eigenvalues for a two-layer Poiseuille flow calculated by Dongarra et al. [22]. In that work, $\tilde{Re} = 10^4$, $\tilde{\alpha} = 1$, $\mu_1/\mu_2 = 2$, $\rho_1/\rho_2 = 1$, $g = 0$ and $d_1/d_2 = 1.2$. The choice of normalization used by Dongarra et al. is different than this work; therefore, the equivalent flow conditions wavenumber and Reynolds number are $\alpha = 1.2$ and $Re = 6000$ respectively. With $N = 50$, the calculated eigenvalues, shown in Table 4.2, are in good agreement with those shown in [22, Table 5] with the deviation in the shear and the interfacial modes being $O(10^{-9})$. Additional validation tests were carried out by comparing the numerical solution against the asymptotic solutions of Yih [84] along with the asymptotic and numerical solutions reported in the work by Yiantsios & Higgins [81]. Similar convergence and accuracy was observed in all cases.

### 4.2.3 Instability Classification Through Energy Methods

The linear stability analysis, derived in §4.1, provides the eigenvalues and eigenfunctions which together are capable of producing unstable growth within the flow. Such an unstable solution is known to extract energy from the mean flow allowing for unstable growth to occur; however, the large parameter space of the problem
Table 4.2: Chebyshev approximation of the leading unstable modes in a two-phase plane Poiseuille flow with $\mu_1/\mu_2 = 2$, $\rho_1/\rho_2 = 1$, $\mathcal{F} = 0$, $d_1/d_2 = 1.2$, $\alpha = 1.2$, $Re = 6000$.

\[
\begin{array}{|c|c|c|}
\hline
N & c(\text{shear mode}) & c(\text{interface mode}) \\
\hline
30 & 0.25789462 + 0.00087713i & 1.00390746 + 0.00179187i \\
40 & 0.25789420 + 0.00087790i & 1.00390743 + 0.00179189i \\
50 & 0.25789420 + 0.00087789i & 1.00390743 + 0.00179189i \\
60 & 0.25789420 + 0.00087789i & 1.00390745 + 0.00179188i \\
70 & 0.25789420 + 0.00087789i & 1.00390743 + 0.00179189i \\
80 & 0.25789420 + 0.00087789i & 1.00390691 + 0.00179167i \\
90 & 0.25789420 + 0.00087789i & 1.00390786 + 0.00179184i \\
100 & 0.25789423 + 0.00087786i & 1.00390786 + 0.00179184i \\
\hline
\end{array}
\]

\{Re, Fr, n, r, h_1/h_2, S, \alpha\} allows for there to be a number of different possible physical mechanisms which may be responsible for causing the unstable growth. From the linear Orr-Sommerfeld analysis alone, it is not easy to identify the source of the instability.

Following the analysis of Boomkamp & Miesen \[10\], energy equations which describe the rate of change of kinetic energy are developed which isolate the physical dependence of each variable through the calculation of the dissipation, Reynolds stresses, along with the normal and tangential interfacial stresses. Multiplying the linearized momentum, eqn. (4.2 & 4.3), by $u_i$ produces

\[
\left[ \frac{\partial}{\partial t} + U_i^{(m)} \frac{\partial}{\partial x} \right] KE^{(m)} + u_i^{(m)} u_2^{(m)} \frac{dU_i^{(m)}}{dy} = u_i^{(m)} \frac{\partial \tau_{ij}^{(m)}}{\partial x_j} \quad (4.20)
\]

where $KE = \frac{1}{2} u_i^{(m)} u_i^{(m)}$, and the superscript $m = 1, 2$ denotes the upper and lower fluid respectively. Using the tensor identity

\[
\frac{\partial}{\partial x_j} [u_i \tau_{ij}] = \tau_{ij} \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial \tau_{ij}}{\partial x_j} \quad (4.21)
\]

and invoking the Newtonian constitutive equation for the total fluid stress

\[
\tau_{ij}^{(m)} = -\frac{1}{r^{(m)}} \rho^{(m)} \delta_{ij} + 2 \frac{n^{(m)}}{r^{(m)}} e_{ij}^{(m)} \quad (4.22)
\]
with

\[ n^{(m)}, r^{(m)} = \begin{cases} 
1 & m = 1 \\
0, r & m = 2 
\end{cases} \quad (4.23) \]

and where \( e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \) is the strain rate tensor, yields the final energy equation

\[
\left[ \frac{\partial}{\partial t} + U_i^{(m)} \frac{\partial}{\partial x_i} \right] KE^{(m)} = -u_1^{(m)} u_2^{(m)} \frac{dU_1^{(m)}}{dy} + \frac{\partial}{\partial x_j} \left[ u_i^{(m)} \tau_{ij}^{(m)} \right] - \frac{2n^{(m)}}{r^{(m)} Re} e_{ij} e_{ij}.
\]

(4.24a)

Integrating over the depth of each phase, averaging over a wavelength, and summing over both phases produces the final energy balance equation for the system

\[
\sum_{m=1}^{2} KIN^{(m)} = \sum_{m=1}^{2} DIS^{(m)} + \sum_{m=1}^{2} REY^{(m)} + INT
\]

(4.25)

where

\[
KIN^{(m)} = \frac{r^{(m)}}{\lambda} \int_{a_m}^{b_m} \int_0^\lambda \frac{1}{2} \left( u_1^{(m)} u_2^{(m)} \right) \, dx \, dy
\]

(4.26a)

\[
DIS^{(m)} = -\frac{n^{(m)}}{Re} \lambda \int_{a_m}^{b_m} \int_0^\lambda \left[ 2 \left( u_1^{(m)} u_1^{(m)} \right)^2 + \left( u_1^{(m)} u_2^{(m)} \right)^2 + 2 \left( u_1^{(m)} u_2^{(m)} \right)^2 \right] \, dx \, dy
\]

(4.26b)

\[
REY^{(m)} = \frac{r^{(m)}}{\lambda} \int_{a_m}^{b_m} \int_0^\lambda \left[ -u_1^{(m)} u_2^{(m)} \frac{dU_1^{(m)}}{dy} \right] \, dx \, dy
\]

(4.26c)

with

\[ [a_m, b_m] = \begin{cases} 
[0, 1] & m = 1 \\
[-h_2, 0] & m = 2 
\end{cases} \]

and \( \lambda = \frac{2\pi}{\alpha} \). The \( KIN \)-term denotes the spatially averaged time rate of change of kinetic energy. For linearly unstable modes, the exponential growth of the squared velocity components allows for \( \frac{d}{dt} \to 2 \Im \{ \alpha c \} \). The \( DIS \)-term represents the viscous dissipation of the perturbation. As expected, this quantity is negative definite which causes it to oppose the growth of the perturbed velocity field. The \( REY \)-term denotes the interaction between the Reynolds stresses (the product of the perturbation velocities) and the mean flow. Depending on the nature of the flow, this quantity can
be stabilizing or destabilizing to the system.

The \( INT \)-term denotes the energy contributions due to the presence of an interface. It can be decomposed into its normal and tangential components \( INT = NORM + TAN \) where

\[
NORM = \frac{1}{\lambda} \int_{0}^{\lambda} \left[ v_2 T_2^{(yy)} - v_1 T_1^{(yy)} \right]_{y=0} dx
\]

\[
TAN = \frac{1}{\lambda} \int_{0}^{\lambda} \left[ u_2 T_2^{(xy)} - u_1 T_1^{(xy)} \right]_{y=0} dx
\]

and \( T_2^{yy} \) and \( T_1^{yy} \) are the components of the stress tensor based on the perturbed velocity field. The \( NORM \)-term can be simplified by invoking the normal stress boundary condition at the interface

\[
T_2^{yy} - T_1^{yy} = S \eta_{xx} - F \eta
\]

along with the continuity of the vertical velocity component at the interface, \( v_1(y = 0) = v_2(y = 0) \) producing

\[
NORM = \frac{1}{\lambda} \int_{0}^{\lambda} [v S \eta_{xx}]_{y=0} dx - \frac{1}{\lambda} \int_{0}^{\lambda} [v \eta F]_{y=0} dx.
\]

The \( TAN \)-term is then evaluated directly through the shear stress components of the viscous stress tensor

\[
T_j^{xy} = \frac{1}{Re} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right).
\]

For this analysis, the pseudo-spectral method derived in §4.2.1 was used to supply the eigenvalue and eigenvector solutions. The integrals of eigenvectors in the \( y \)-direction were evaluated using Chebyshev polynomial quadrature and integrals in the \( x \)-direction were evaluated analytically. This leads to spectral convergence of the energy integrals.

As an example, consider the case of a laminar flow with \( r = 100, \ n = 1000, \ S = 0.1, \ F = 0.25, \ h_1 = 1, \ h_2 = 2.33, \) and \( Re = 300 \). The Orr-Sommerfeld solver finds that there is an unstable eigenvalue with wave speed \( c = 1.4261 + 0.9957i \).
Table 4.3: Energy distribution for an unstable Orr-Sommerfeld mode for a flow with $r = 100$, $n = 1000$, $S = 0.1$, $F = 0.25$, $h_1 = 1$, $h_2 = 2.33$, and $Re = 300$. All terms have been normalized by the total kinetic energy term $KIN$ such that $KIN^{(1)} + KIN^{(2)} = 1$.

Carrying out an energy analysis for this flow, shown in table 4.3, finds that the energy balance equation is satisfied to within machine precision which shows the high level of accuracy of both the numerical quadrature scheme and eigen solution. To convey the relative importance of each term, all of the energy terms have been normalized by the total kinetic energy term ($KIN$) such that $KIN^{(1)} + KIN^{(2)} = 1$. As expected the value of the dissipation terms are negative, which is consistent with the operator being negative definite. The largest positive term on the right hand side of the energy balance expression is $TAN$ which shows that this strong instability is generated by the work done between the interaction of the tangential velocity and shear stress interaction at the interface. Additionally, the positive energy associated with upper fluid’s Reynolds stress shows that it had a weakly destabilizing contribution.

### 4.2.4 Model Equations For Turbulent Analysis

While the examination of the stability of two-phase parallel laminar flows has lead to the development of sophisticated analytical/numerical techniques as well as provided insight into the complex dynamics of two-fluid flows, the majority of the applied problems of interest involve one or both of the fluids being turbulent. The corresponding turbulent Orr-Sommerfeld analysis requires as an input the time-averaged turbulent velocity profile. One possible solution to this problem is to carry out direct numerical simulations to develop exact velocity profile; however such a method is prohibitively expensive. A second approach is to utilize the classic asymptotic boundary layer solution for single phase turbulent channel flow as an approximate solution. However, errors in this approximation may produce significant error in the resulting calculated eigen spectra.
Another approach is to utilize the approximate turbulent mean flow solution developed by Náraigh et al. [60]. In their work, a time-averaged, spatially-uniform mean velocity solution was constructed for a fully developed turbulent gas flowing over a laminar liquid layer in a horizontal channel. Their analysis also showed how to modify the theory for the case of a turbulent liquid layer.

In the lower laminar fluid, the Navier-Stokes equations reduce to the standard balance between the pressure and viscous stress (Poiseuille flow)

$$\mu_2 \frac{\partial^2 U_2}{\partial y^2} - \frac{\partial P}{\partial x} = 0$$

(4.30)

where $U_2$, $P$, and $\mu_2$ denote the mean velocity, pressure, and dynamic viscosity of the lower fluid. Integrating this solution and invoking the continuity of shear stress at the interface $\mu_2 \frac{\partial U_2}{\partial y} |_{z=0} = \tau_i$ and no-slip at the lower wall $U_2(y = -d_2) = 0$ produces a laminar velocity profile.

In the upper fluid, the Reynolds averaged Navier-Stokes equations reduced to

$$\frac{\mu_1}{\rho_1} \frac{\partial U_1}{\partial y} - \rho_1 \overline{u'u'} = \tau_i + \frac{\partial \rho}{\partial x} y$$

(4.31)

where $u'$ and $v'$ are the turbulent fluctuating velocity components. The closure of the turbulent Reynolds stress was achieved by introducing an interpolation function for the eddy viscosity[59] which mimics the traditional mixing-length theory near the interface and walls. Van Driest-type wall functions were utilized to damp the effects of the turbulence to zero near the interface and wall.

Scaling these governing equations by the upper fluid depth $d_1$ and the effective shear velocity $U = \sqrt{\frac{d_1}{\rho_1} \left| \frac{\partial P}{\partial x} \right|}$ produced the dimensionless velocity for the upper fluid

$$U_1(y) = \frac{1}{n} \left( \frac{1}{2} \delta^2 Re + \frac{Re^2}{Re} \right) + \frac{Re^2}{Re} \int_0^z \frac{1 - \frac{Re^2}{Re} s}{1 + \frac{Re^2}{Re} \sqrt{|s|} G(s) \psi_i(s) \psi_w \left( 1 - s \right)} ds$$

(4.32)

and lower fluid

$$U_2(y) = \frac{1}{n} \left[ -\frac{1}{2} Re (z^2 - \delta^2) + \frac{Re^2}{Re} (z + \delta) \right]$$

(4.33)
where \( Re = \frac{\bar{U}d_{1}}{\mu_{1}} \), \( Re_{*} = \frac{\bar{U}d_{1}}{\mu_{1}} \), \( n = \frac{\bar{U}}{\mu_{*}} \), and \( \delta = \frac{d_{1}}{\mu} \). In eqn. (4.32), \( G(s) \) is an interpolation function designed to reproduce the “law of the wall” near the interface and the upper wall, \( R = \left[ 1 - \frac{(Re/Re_{*})^{2}}{1 - 1/(Re_{*})^{2}} \right]^{-1} \), \( \psi_{i/w} \) are interfacial and wall functions respectively (defined by [60, eq.(13)]), and \( \kappa = 0.41 \) is the von Kármán constant.

An example solution is shown in figure 4-2 for a flow with \( r = 1000 \), \( n = 100 \), \( \delta = 0.2 \), and \( Re = 300 \). Clearly, the parabolic velocity profile for a laminar velocity field is observed in the lower fluid. Similarly, the upper fluid’s profile shows strong similarity to the traditional turbulent velocity profile for single phase channel flow. Figure 4-3 shows the upper fluid velocity profile scaled by wall units. Figure 4-3(a) shows the velocity profile on the interfacial side while figure 4-3(b) shows the solution in the vicinity of the wall. Good agreement is observed between the velocity profile calculated by eqns. (4.32 & 4.33) and the classic asymptotic solutions for the viscous and logarithmic sublayers. Náraigh et al. [60] compared this new model for the turbulent velocity profile against solutions computed from direct numerical simulations of the Navier-Stokes equations and excellent agreement is observed. Therefore, unless stated otherwise, this method supplied the turbulent velocity profiles for the
Figure 4-3: Turbulent mean velocity profile calculated by [60, eq.(14)] for \( r = 1000, \) \( n = 100, \) \( \delta = 0.2, \) and \( Re = 300 \) plotted in wall units from the interfacial and wall boundaries.

remaining turbulent stability analysis.

### 4.3 Comparisons of Linear Stability Predictions With Experimental Observation

The previous sections of this chapter developed the necessary theoretical and numerical methods to investigate the interfacial stability of either laminar or turbulent flows. With this capability, the calculated wave spectra provide a means of determining the bandwidth of unstable modes and the frequency spectra for identifying possible resonant interactions in viscous shear flows. To demonstrate that these predictions are accurate, tests were carried out using the flow conditions reported in the experiments by Jurman et al. [40] which utilized glycerine-water solutions (with the liquid viscosity ranging between 8-20 cP) flowing through a 30 cm wide, 2.54 cm high and 9 m long horizontal rectangular channel. In the experiments, wave probes stationed along the channel were used to measure the wave height and determine the frequency spectrum as a function of fetch. From these measurements, the wave modes that were initially unstable were identified along with the resulting nonlinear spectral evolution.

For each experiment, the investigators reported a liquid and gas Reynolds number,
Table 4.4: Selected case conditions from Jurman et al. [40] for a 2.54 cm deep channel with $\mu_L = 10 \text{ cP}$, $\mu_g/\mu_l = 0.0018$, $\rho_g/\rho_l = 0.0012$, and $\frac{d\rho}{dx}$ being found using the analytic models developed by [60]. Experimental and numerically defined values are denoted by superscript (E) and (N).

<table>
<thead>
<tr>
<th>Case</th>
<th>$Re_l^{(E)}$</th>
<th>$Re_g^{(E)}$</th>
<th>$h_l$ [cm]</th>
<th>$U_{air}$ [m/s]</th>
<th>$\frac{d\rho}{dx}^{(N)}$ [N/m]</th>
<th>$Re_l^{(N)}$</th>
<th>$F^{(N)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>5150</td>
<td>0.45</td>
<td>3.7</td>
<td>-6.73</td>
<td>9.13</td>
<td>1453.38</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>6300</td>
<td>0.33</td>
<td>4.3</td>
<td>-8.03</td>
<td>5.73</td>
<td>1214.02</td>
</tr>
</tbody>
</table>

equilibrium depth, average gas velocity, and the liquid’s dynamic viscosity. In order to generate a numerical initial condition, the pressure gradient across the channel would need to be determined. Utilizing the turbulent velocity profile described in §4.2.4 and specifying the mean velocity of the gas as an input, a unique pressure gradient was calculated using [60, eq.(14)]. The resulting pressure gradient, numerical liquid Reynolds number based on the mean velocity, and density-weighted reciprocal of the Froude number are included in the last three columns of table 4.4. The value of surface tension was set to the value of 0.06 N/m. Clearly this theoretical approach accurately produces flow conditions that are in close agreement with the experientially reported values.

Consider Case 1 from table 4.4 that corresponds to the test shown in Figure 7 of Jurman et al. [40]. The experiments show that during the initial phase of evolution, a fundamental mode with a wavelength of approximately 2.3 cm forms due to a linear interfacial instability. As the interface continues to evolve, a subharmonic mode ($\lambda_{sub} = 2\lambda_{fund}$) develops into a strong component within the wave spectrum. Using these flow conditions, the Orr-Sommerfeld frequency spectrum was calculated and an analysis of possible resonant modes was carried out.

Using the exact flow conditions from the experiments and the numerical procedure described above, the calculated turbulent Orr-Sommerfeld spectrum shown in figure 4-4 demonstrates that there are no linearly unstable modes. The peak of the growth rate spectrum has a decay rate (negative growth rate) of $O(-10^{-2})$. The peak component in the higher wavenumber portion of the spectrum corresponds to a wave mode with a wavelength $\lambda_{peak} \approx 2.79$ cm that is slightly higher than the experimentally
measured value of the fundamental mode. The lack of linearly unstable modes and the value of $\lambda_{peak}$ being larger than the experimentally measured fundamental mode can be attributed to a few inconsistencies between the numerical solution and the experiments. First, the experiments used wave probe measurements to generate time traces of the interfacial displacement. These measurements show strong fluctuations of average liquid depth as a function of fetch. For marginally unstable flow conditions, this displacement of the mean liquid depth can have a significant effect on the interfacial stability. Second, the experiments report fluid Reynolds numbers based on a mean velocity which was likely obtained by normalizing the inlet fan/pumps by the cross sectional area of each phase rather than calculating it from an experimentally measured velocity profile. Therefore, it is possible and likely, that the mean velocity profile is not fully developed and in perfect agreement with the theoretical profile given by eqn. (14) of Naraigh et al.[60]. Additionally, the linear stability analysis utilized the quasi-laminar hypothesis that neglected the role of linear perturbations to the Reynolds stress components. It is presumed that these perturbed Reynolds stresses are small, but may have a weak effect on the location of the neutral curve.
These inconsistencies make accurately estimating the growth rate of marginally unstable modes challenging.

To examine the interfacial stability, while acknowledging the effect of these weak inconsistencies between the numerical and experimental base flow conditions, the values of the gas Reynolds number and equilibrium liquid depth were perturbed until a marginally unstable linear mode was identified. One set of marginal flow conditions was found to be at $Re_G = 5,900$, $Re_L = 16.1$, $h_L = 0.51 cm$. Using these conditions, the presence of resonant interactions could be identified and a variable sensitivity analysis was carried out to confirm the robustness of these predictions.

The first set of tests shown in figure 4-5 examined the how the marginally stable flow conditions changed due to small changes in the equilibrium liquid depth. Figure 4-5(a) show the dimensional wave frequencies while the corresponding growth rate is shown in figure 4-5(b). Clearly, as the liquid depth increases there is a regular increase in the growth rate. The frequency exhibits only a weak dependence on the liquid depth. As the liquid depth changes from $h_L = (0.47, 49, 51, 53) cm$, the wavelength corresponding to the peak of the growth rate holds approximately constant at $\lambda \approx 2.3 cm$ which is consistent with the wavelength of the fundamental mode reported in the experiment.

Examination of the frequency spectrum permits the identification of sub-harmonic
resonant pairs which may satisfy the resonance conditions: \( k_2 = \frac{1}{2} k_1 \) and \( |\frac{1}{2} \omega_1 - \omega_2| < \sigma \). As in chapters 2 & 3, \( \sigma \) denotes the resonance detuning parameter. Due to the large volume of frequencies associated with each wavenumber in the spectrum, several filters were applied in order to remove the non-relevant frequencies and wavenumbers.

First, in this analysis only those interactions for which the value of \( |\sigma| \leq \sigma_{\text{tol}} = 0.025 \) were examined such that only the strongest resonant pairs would be considered. This choice of \( \sigma_{\text{tol}} \) was semi-arbitrarily chosen to be small enough to focus attention on the strong resonance while also being large enough to generate a sufficiently large collection of resonances. Sensitivity studies were carried out to demonstrate the robustness of the resulting trends.

Second, the numerically calculated eigenspectrum returns many strongly damped modes with decay rates that are \( O(-10^3) \). Realistically, due to the high damping rates, it is not expected that these modes play a significant role in energy transfer and long wave growth, so an additional filter was imposed in the resonance analysis which only allows modes where \( \omega_i > -0.1 \) be considered.

Using these two conditions to filter the eigenspectrum, the resulting resonance analysis is shown in figure 4-6. For all of the values of the equilibrium liquid depth, there are a large number of sub-harmonic resonance modes. However, for each cases there is a narrow band of wavenumbers for which \( \sigma < 10^{-5} \) indicating a strong resonance. For the \( h_L = 0.47 \text{ cm} \) case, a narrow band is concentrated near \( \lambda \approx 4.25 \text{ cm} \). However, as the liquid depth increases, this wavelength shifts to smaller wavelengths. For the marginally unstable case, with \( h_L = 0.51 \text{ cm} \), the sub-harmonic mode is identified with \( \lambda \approx 2.4 \text{ cm} \). Using the dispersion curve from figure 4-5(a), the primary (peak unstable wave) mode is identified with wavelength \( \lambda_1 = 2.4 \text{ cm} \) and a wave frequency \( f_1 = 9.5 \text{ Hz} \). The resonance curve shown in figure 4-6 shows that this mode would generate a sub-harmonic with wavelength \( \lambda_2 = 4.8 \text{ cm} \) and wave frequency \( f_2 = 4.9 \text{ Hz} \). Figure 4-6 also predicts a second strong sub-harmonic resonance for \( (\lambda_1, f_1) = (6.4 \text{ cm}, 3.4 \text{ Hz}) \) which would yield a subharmonic mode with \( (\lambda_2, f_2) = (12.8 \text{ cm}, 1.8 \text{ Hz}) \). The locations of the resonances and the values of the modal frequencies are in complete agreement with the results from figure 7 in
Figure 4-6: Distribution of strong ($|\sigma| \leq \sigma_{tot}$) resonant interactions for (o) $h_L = 0.47$, ($\times$) $h_L = 0.49$, (□) $h_L = 0.51$, (*) $h_L = 0.53$ and $Re = 5900$.

the Jurman et al. [40] manuscript. This strongly supports the conclusion that energy from the linearly unstable mode is transferred through a sub-harmonic resonance to the linearly stable mode and is the dominant mechanism responsible for the observed nonlinear spectral evolution.

A similar sensitivity analysis shown in figure 4-7 was carried out that examines the effects of the gas Reynolds number on the stability of the interface around the original marginally unstable flow conditions. Similar trends are observed as in the previous case. As the gas Reynolds number was increased over the range $Re = (5200, 5900, 6500, 7200)$, the wavelength corresponding to the peak in the growth rate spectrum experienced a small, but consistent, down shift taking on the values $\lambda = (2.59, 2.34, 2.19, 2.10) \text{ cm}$ as shown in figure 4-7(b). The corresponding wave frequencies demonstrate a weak dependence on the gas Reynolds number as shown in figure 4-7(a). Examination of the resonant detuning parameter shown in figure 4-8 is again consistent with the results in figure 4-6. For each gas Reynolds number, there are two clear sub-harmonic resonances within the spectrum. The first involves a mode near the peak of the growth rate spectrum and its corresponding sub-harmonic.
Figure 4-7: Dependence of (a) wave frequency and (b) linear growth rate on gas Reynolds number where \((-\cdots) Re = 5,200, (-\cdots) Re = 5,900, (\cdots) Re = 6,500, (-\cdots) Re = 7,200\) and \(h_L = 0.51\ cm\).

Figure 4-8: Distribution of strong (\(|\sigma| \leq \sigma_{tot}\)) resonant interactions for (\(\circ\) \(Re = 5,200\), (\(\times\) \(Re = 5,900\), (\(\Box\) \(Re = 6,500\), (\(\ast\) \(Re = 7,200\) with \(h_L = 0.51\ cm\).
The second resonant pair involves two long wavelength modes. As expected, all of these perturbed conditions generate resonances that involve wave modes that have frequencies and wavelengths that are consistent with the experimental wave spectra.

The results of this sensitivity analysis demonstrate that, while the initial numerically calculated parameters identified in table 4.4 were found to be marginally stable and free of strong sub-harmonic resonances, small perturbations to the flow conditions were capable of generating instabilities and resonances that were consistent with the experimental findings. The magnitude of the perturbations were small enough that it is reasonable to assume that they fall within the range of experimental error or higher order corrections to the numerical procedure (i.e. determination of the effective pressure gradient).

Given these findings, a similar analysis was carried out using the flow conditions of Case 2 from table 4.4 which correspond to figure 8 from Jurman et al.[40]. As in the previous case, the flow conditions for Case 2 were found to be marginally stable with growth rates being $O(-10^{-2})$. Perturbing the experimental conditions to $Re_G = 7200$ and $h_L = 0.38 \text{ cm}$ identified a marginally unstable set of wave modes. The resulting frequency and growth rate spectrum are shown in figure 4-9. Figure 4-9(b) shows a peak growth rate occurring for a wavelength of $\lambda_{peak} \approx 2.05 \text{ cm}$ that corresponds to a wave frequency of $f_1 = 10.6 \text{ Hz}$. The location of $\lambda_{peak}$ had a weak
Analysis of the wave frequencies allowed for the presence of subharmonic resonances to be identified. The sub-harmonic resonance distribution is shown in figure 4-10. There are a large number of wave modes which approximately satisfy the resonance condition to within $|\sigma| < 0.1$ As the liquid depth increases, $\sigma \to 0$ at $\lambda_{\text{peak}} \approx 5 \text{ cm.}$; however, this corresponds to a portion of the the spectrum which is was shown to be linearly stable. For wave modes in the vicinity of the peak of the growth rate spectrum $(\lambda_1, f_1) = (2.05 \text{ cm.}, 10.6 \text{ Hz.})$, $|\sigma (\lambda_{\text{peak}})| \approx 0.1$ that shows that there is a near resonant interaction with the sub-harmonic mode $(\lambda_2, f_2) = (4.1 \text{ cm.}, 5.1 \text{ Hz.})$. This near sub-harmonic resonant interaction permits the exchange of energy from the linearly unstable $\lambda_1$-mode to the linearly stable $\lambda_2$-mode; however, for $|\sigma| \approx 0.1$, the rate of energy transfer and modal amplitude will be reduced from the strong "pure" resonance case shown in the previous example. These findings are consistent with the experimental results. Both the frequencies of the fundamental $(\lambda_{\text{peak}})$ and sub-harmonic modes agree with the experimentally measured wavenumber spectrum. Additionally, the power spectral density functions of the wave height show clear but
weak sub-harmonic growth.

4.4 Conclusions

This chapter carried out viscous stability analysis of parallel two-fluid channel flows for the purpose of identifying possible interfacial instabilities and strong resonant interactions. For laminar flows, the linearized momentum and continuity equations resulted in two of the classical Orr-Sommerfeld equations and along with the corresponding interfacial and wall boundary conditions. For turbulent flows, the quasi-laminar hypothesis resulted in stability equations which were of identical form to those of the laminar problem. A Chebyshev spectral method was implemented for the accurate and efficient solution of the governing eigenvalue problem.

Using the flow conditions provided in several of the experiments by Jurman et al. [40], it was demonstrated that the viscous stability analysis described within this chapter produced stability predictions which are in close agreement with experimental measurements. Using the experimental flow conditions, the model for the turbulent base flow described in §4.2.4 produced marginally stable wave spectra. To account for variations in the experimental flow conditions, weak perturbations to the flow conditions demonstrated that the numerically calculated wave frequencies of the most unstable modes are in agreement with the peak frequencies in the experimentally measured wave spectra. The growth rate spectrum was found to be sensitive to the variations in the liquid depth observed in the experiments along with the amount of ambient turbulence in the gas layer. The calculated growth rate spectrum may improve with accurate closure models for the perturbed Reynolds stresses in the turbulent stability equations; however, the sensitivity analysis suggested that the location of the peak of the growth rate spectrum would not change significantly. The stability analysis also demonstrated that there are clear sub-harmonic resonances. The sensitivity analysis demonstrated that these resonances were present over a wide range of flow conditions within the vicinity of the experimental parameter space. The predicted resonant wavelengths and frequencies were also found to be in close
agreement with the experimental measurements demonstrating the cascade of energy from linearly unstable short waves to linearly stable long waves is the dominant mechanism for the interfacial evolution. These predictions are consistent with those made using potential flow analysis of chapters 2 & 3.

The accuracy of this stability analysis, compared to experiments, allows for this method to be an effective tool for identifying useful flow conditions for future direct numerical simulations of turbulent two-fluid flows. Additionally, this method has potential applications in the development of future slug transition conditions.
Chapter 5

Numerical Methods for Viscous Two-Phase Flows

5.1 Introduction

Chapter 4 carried out a detailed analysis of the viscous stability of laminar and turbulent two-phase flows. Theoretical and numerical methods were developed for the prediction of the behavior of small amplitude linear modes and a nonlinear analysis was carried out to identify possible resonant triads and subharmonics. In order to validate that predictive capability and observe the resulting nonlinear evolution, numerical methods capable of solving two-phase Navier-Stokes equations are developed.

Within this chapter, the fully nonlinear viscous governing equations for a two-phase channel flow are defined along with a description of the numerical algorithms which are used to solve them. Detailed validation tests are carried out demonstrating the accuracy and second order convergence of the numerical scheme. Additional numerical tools are developed that are used for the initiation and growth of nonlinear interfacial waves.
5.2 Governing Equation

Within the context of this work, it is assumed that each phase consists of an incompressible Newtonian fluid. The governing equations are expressed with a single-fluid formulation with the Navier-Stokes equations being defined on a Cartesian coordinate system with the domain spanning \((x, y, z) \in [0, L_x] \times [0, L_y] \times [0, L_z]\). This results in the following set of nonlinear governing equations.

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \bar{u} = 0 \tag{5.1a}
\]
\[
\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho Re} \nabla \cdot (2\mu \bar{D}) + \frac{k}{Fr^2} \tag{5.1b}
\]

with the deformation tensor being defined as \(\bar{D} = \frac{1}{2} (u_{i,j} + u_{j,i})\). The dimensionless parameters are defined as \(Re = \frac{UL}{\mu}, Fr^2 = \frac{U}{L} \) with \(U\) and \(L\) being characteristic length and time scales that are uniquely defined for each problem under investigation. At the walls, the velocity satisfies the standard no-flux and no-slip boundary conditions.

Since this work does not account for changes in phase and the fluids are assumed to be incompressible, the velocity satisfies the interfacial jump condition \([\bar{u}]_{int} = 0\) with \([x]_{int} = x_2 - x_1\) being the standard interfacial jump notation. Additionally, momentum conservation at the interface requires that the normal and tangential stresses satisfy the interfacial stress boundary conditions

\[
[-p + 2\mu \hat{n} \cdot \bar{D} \cdot \hat{n}]_{int} = -\sigma \kappa \tag{5.2a}
\]
\[
[2\mu \hat{l}^{(1)} \cdot \bar{D} \cdot \hat{n}]_{int} = 0 \tag{5.2b}
\]
\[
[2\mu \hat{l}^{(2)} \cdot \bar{D} \cdot \hat{n}]_{int} = 0 \tag{5.2c}
\]

where \(\bar{D}\) is the deformation tensor, \(\hat{n}\) is the interfacial normal unit vector, and \(\hat{l}^{(k)}\) \((k = 1, 2)\) are the tangential unit vectors which are directed along the \((x, z)\)- and \((y, z)\)-planes respectively.
5.3 Flow Solver Formulation

In this section, the velocity field, governed by the Navier stokes equations and incompressible continuity equation, is calculated as an initial-boundary-value problem using a second-order finite-volume numerical algorithm. The scheme utilizes the marker-and-cell (MAC) method developed by Harlow & Welch[30] that puts vector valued quantities at the cell-face centers and the scalar quantities at the cell center. The velocity field is integrated in time through a two stage projection method where in the first step, the continuity equation is coupled to the Navier-Stokes equation, creating a variable density Poisson equation. The second step corrects the velocity, yielding a divergence free velocity field. The density and viscosity fields are determined through a second-order volume conserving interface capturing scheme.

It should be noted that the code developed for this work is based off of a legacy code[32] which implements a different coordinate system than the one used in the prior chapters of this thesis. Therefore, to avoid confusion it is important to point out that in this chapter and in §6, the x-y axes denote the horizontal plane (with the x-direction extending horizontally to the right) and the z-direction being oriented vertically upwards.

5.3.1 Numerical Method For Tracking Fluid Interface

A large range of numerical methods have become available for tracking the interface between two-fluids. Methods such as front tracking, level set, volume-of-fluid (VOF) and phase field (summarized nicely by Tryggvason et al.[74]) are capable of producing high-order approximation for the interfacial dynamics. In order to accurately simulate multiphase flows, it is required that the method be implementable for both two- and three-dimensional flows, be capable of capturing complex topological changes of the interface, maintain global volume conservation across the domain in the presence of violent turbulent flows, and ideally be (relatively) easy to implement.

In front tacking methods, the boundary between the two fluids is represented by connected points on the interface that are moved by the fluid. This method has
been used extensively for direct numerical simulations of bubble fields, but the algorithms appears to be complicated and expensive to implement. It requires constant re-meshing (interface point redistribution), have special handling of any topological changes, and requires special consideration of the interaction between the moving marker points with the solution on a fixed Cartesian grid.

The level set and phase field methods have similar formulations[69]. Both methods can easily compute complex geometric quantities, be implemented in both two- and three-dimensional problems, and can easily handle topological changes. Initially in this work, a level set method[31] was implemented for capturing the interface. For laminar and weakly turbulent flows, mass conservation was observed; however, as the flow became more violent the level set method suffered from extreme mass loss. For strongly turbulent channel flows or flows with drop formation, the numerical domain would quickly drain. More recently, volume-conserving level set methods were developed[62], but their algorithms appeared complicated to implement compared to some of the simple VOF schemes.

The VOF method utilizes a marker function which takes on different values in the different fluids. As the fluid moves, the location of the interface between the two fluids changes and the marker function is updated. Updating the location of the marker function accurately is difficult, but there are a large number of robust numerical algorithms in the literature which have addressed this issue. While the VOF method is easily able to account for topological changes, run two- or three-dimensional problems, and is generally easy to implement numerically, one considerable advantage is that recent numerical algorithms have been developed which can carry out three-dimensional advection while still maintaining global mass/volume conservation. It is for this reason that the VOF method is utilized in this thesis.
5.3.2 Finite-Volume Solution of Navier-Stokes Equations

In this work, each fluid is identified through the use of a color function which is defined as

\[
    c(x) = \begin{cases} 
        1 & z \geq \eta \\ 
        0 & z < \eta 
    \end{cases} 
\]  

(5.3)

Following the standard procedure of volume-of-fluid (VOF) algorithms on Cartesian meshes, a discrete volume fraction is obtained by integrating the color function over a discrete cell volume

\[
    f = \frac{\int_{\Omega} c(x) \, dV}{\Delta \Omega}, \quad 0 \leq f \leq 1
\]  

(5.4)

with \( \Delta \Omega \) denoting the volume of a grid cell. Since the fluid property does not change along a particle path, the color function is passively advected with the flow and governed by the transport equation

\[
    \frac{Dc}{Dt} = \frac{\partial c}{\partial t} + \bar{u} \cdot \nabla c = 0
\]  

(5.5)

where \( \frac{D}{Dt} \) denotes the material derivative. Due the discontinuous nature of the color function, the gradients of \( c \) are not defined across the interface. Casting (5.5) in its integral form alleviates this as an issue and produces an alternative form of the transport equation

\[
    \frac{\partial f}{\partial t} \Delta \Omega + \oint_{\partial \Omega} cu_n \, dS = \oint_{\Omega} c \nabla \cdot \bar{u} \, dV.
\]  

(5.6)

The numerical solution of (5.6) has been an actively researched area for more than 30 years [33] with a plethora of different numerical algorithms being available in the literature. Many of these methods employ complex (and expensive) strategies for identifying the location and orientation of the interface or numerically integrating the advection equation forward in time. For instance, some of the early methods[39] utilized least-squares curve fitting algorithms for reconstructing the interface shape but the formulation required an expensive nonlinear minimization technique. Algorithms, such as those of [39, 66], developed unsplit advection algorithms based on
geometric flux integration routines. While being second-order accurate, they are algorithmically complex to implement (particularly for three-dimensional flows). Some developers coupled the best features of the VOF together with level set methods in order to accurately incorporate surface tension effects[72]; however, solving two interface tracking schemes can become computationally expensive. For a long time, despite the sophistication of these VOF algorithms, volume conservation was a severe shortcoming of the method. Several methods could enforce conservation for simple two-dimensional flows, but suffered strong volume loss for violent or three-dimensional flows.

In this work, the conservative VOF scheme developed by Weymouth & Yue[80] is implemented. This method is capable of efficiently carrying out the nonlinear evolution of an interface while conserving mass to within machine precisions for both two- and three-dimensional flows. At the start of this work, to the author’s knowledge, no other VOF algorithm available in literature was able to achieve three-dimensional mass conservation to such a high degree of numerical accuracy making this algorithm appropriate for the three-dimensional turbulent simulations which are carried out in this work. More recently, additional conservative VOF methods have been developed [3, 4, 16, 79], but their methods are not as simple to implement and did not have any additional capabilities which motivated switching from the current conservative VOF scheme developed by Weymouth & Yue[80].

Following the numerical algorithm developed by Dommermuth et al. [21], a second-order two-stage Runge-Kutta scheme is used to integrate the Navier-Stokes equations and the VOF advection equation in time. During the first stage of the Runge-Kutta algorithm, the velocity at the current time step, \( u_i^{(k)} \) (\( i = 1, 2, 3 \)), is used to make a discrete Poisson equation for the pressure of the form

\[
\frac{\partial}{\partial x_i} \left( \frac{1}{\rho (f^{(k)})} \frac{\partial p}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left( \frac{u_i^{(k)}}{\Delta t} - A_i^{(k)} + \frac{D_i^{(k)} + F_i^{(k)}}{\rho^{(k)}} \right)
\]  

(5.7)

where the cell averaged values of the advection, diffusion, and body force operators
being defined as

\[ A_i = \frac{1}{\Delta\Omega} \oint_S u_i (\tilde{u} \cdot \hat{n}) \, dS, \quad D_i = \frac{1}{\Delta\Omega} \oint_S \mu \left[ \nabla_h \tilde{u} + (\nabla_h \tilde{u})^T \right] \cdot \hat{n} \, dS, \quad F_i = \frac{1}{\Delta\Omega} \oint_V f_i (\tilde{x}) \, dV \]

with \( \nabla_h \) being the discrete gradient operator and \( \hat{n} \) being the unit normal vector. This pressure field is then used to project the velocity onto a divergence free velocity field by

\[ u_i^{(k+\frac{1}{2})} = u_i^{(k)} + \Delta t \left( -A_i^{(k)} + \frac{D_i^{(k)} + f_i^{(k)}}{\rho^{(k)}} - \frac{1}{\rho^{(k)}} \frac{\partial p^{(k+\frac{1}{2})}}{\partial x_i} \right). \quad (5.8) \]

Similarly, the volume fraction is updated through a forward Euler step

\[ f^{(k+\frac{1}{2})} = f^{(k)} + VOF \left( u_i^{(k)}, f^{(k)}, \Delta t \right). \quad (5.9) \]

For brevity, the details of the time integration of the volume fraction (i.e. VOF reconstruction and advection algorithm) are simply denoted by VOF; however, a summary is provided in §5.3.3 and a complete description is described by Weymouth & Yue[80]. This process is repeated for the second step of the Runge-Kutta algorithm, beginning with a second Poisson equation for the pressure of the form

\[ \frac{\partial}{\partial x_i} \left[ \frac{1}{\rho^{(k+\frac{1}{2})}} \frac{\partial p^{(k+1)}}{\partial x_i} \right] = \frac{\partial}{\partial x_i} \left[ \frac{u_i^{(k+\frac{1}{2})} + u_i^{(k)}}{\Delta t} - A_i^{(k+\frac{1}{2})} + \frac{D_i^{(k+\frac{1}{2})} + f_i^{(k+\frac{1}{2})}}{\rho^{(k+\frac{1}{2})}} \right]. \quad (5.10) \]

The new pressure field is then used to update the final velocity field onto a divergence free velocity field by

\[ u_i^{(k+1)} = \frac{u_i^{(k+\frac{1}{2})} + u_i^{(k)}}{2} + \frac{\Delta t}{2} \left[ -A_i^{(k+\frac{1}{2})} + \frac{D_i^{(k+\frac{1}{2})} + f_i^{(k+\frac{1}{2})}}{\rho^{(k+\frac{1}{2})}} - \frac{1}{\rho^{(k+\frac{1}{2})}} \frac{\partial p^{(k+1)}}{\partial x_i} \right]. \quad (5.11) \]

The volume fraction is similarly updated by the second step of the Runge-Kutta algorithm through

\[ f^{(k+1)} = f^{(k)} + VOF \left( u_i^{(k)} + u_i^{(k+\frac{1}{2})}, f^{(k)}, \Delta t \right). \quad (5.12) \]
In this work, these governing equations are generally discretized through second-order approximations. The $x$-component of the advection operator is discretized by the midpoint rule

\[
(A_x)_{i+1/2,j} = \frac{1}{\Delta x \Delta z} \left\{ \left[ (uu)_{i+1,j} - (uu)_{i,j} \right] \Delta z + \left[ (uw)_{i+1/2,j+1/2} - (uw)_{i+1/2,j-1/2} \right] \Delta x \right\}
\] (5.13)

while the $z$-component is approximated by

\[
(A_z)_{i,j+1/2} = \frac{1}{\Delta x \Delta z} \left\{ \left[ (uw)_{i+1/2,j+1/2} - (uw)_{i-1/2,j+1/2} \right] \Delta z + \left[ (ww)_{i,j+1} - (ww)_{i,j} \right] \Delta x \right\}
\] (5.14)

Using the MAC scheme, the velocities are located at the center of the momentum control volumes, but the momentum fluxes need to be specified at the control volume boundaries. For the majority of the cases presented in this thesis, the momentum fluxes are approximated through a second-order linear interpolation scheme forcing eqn. (5.13) to be of the form

\[
(A_x)_{i+1/2,j}^{(n)} = \frac{1}{\Delta x} \left[ \frac{(u_{i+3/2,j}^{(n)} + u_{i+1/2,j}^{(n)})^2}{2} - \frac{(u_{i+1/2,j}^{(n)} + u_{i-1/2,j}^{(n)})^2}{2} \right]
+ \frac{1}{\Delta y} \left[ \frac{(u_{i+1/2,j+1}^{(n)} + u_{i+1/2,j}^{(n)})}{2} \left( \frac{(w_{i+1,j+1/2}^{(n)} + w_{i,j+1/2}^{(n)})}{2} \right) - \frac{(u_{i+1/2,j+1}^{(n)} + u_{i+1/2,j}^{(n)})}{2} \right]
- \frac{1}{\Delta y} \left[ \frac{(u_{i+1/2,j}^{(n)} + u_{i+1/2,j-1}^{(n)})}{2} \left( \frac{(w_{i+1,j-1/2}^{(n)} + w_{i,j-1/2}^{(n)})}{2} \right) - \frac{(u_{i+1/2,j}^{(n)} + u_{i+1/2,j-1}^{(n)})}{2} \right]
\]
while eqn. (5.14) becomes

\[
(A_x)^{(n)}_{i,j+1/2} = \frac{1}{\Delta x} \left[ \left( \frac{u^{(n)}_{i+1/2,j} + u^{(n)}_{i+1/2,j+1}}{2} \right) \left( \frac{w^{(n)}_{i,j+1/2} + w^{(n)}_{i+1,j+1/2}}{2} \right) - \left( \frac{u^{(n)}_{i-1/2,j+1} + u^{(n)}_{i-1/2,j}}{2} \right) \left( \frac{w^{(n)}_{i,j+1/2} + w^{(n)}_{i-1,j+1/2}}{2} \right) \right] + \frac{1}{\Delta z} \left[ \left( \frac{w^{(n)}_{i,j+3/2} + w^{(n)}_{i,j+1/2}}{2} \right)^2 - \left( \frac{w^{(n)}_{i,j+1/2} + w^{(n)}_{i,j-1/2}}{2} \right)^2 \right].
\]

This second-order (centered) scheme is more accurate than non-centered second-order schemes, but was occasionally found to be unstable resulting in numerical instabilities. In those cases, the velocity at the boundaries of the control volume was approximated using a third-order upwind-biased polynomial approximation, refereed to as QUICK (quadratic upstream interpolation for convective kinematics)\[42\] having the form

\[
u_{i,j} = \begin{cases} 
\frac{1}{8} \left( 3u_{i+1/2,j} + 6u_{i-1/2,j} - u_{i-3/2,j} \right) & \text{if} \ \frac{1}{8} \left( u_{i-1/2,j} + u_{i+1/2,j} \right) > 0 \\
\frac{1}{8} \left( 3u_{i-1/2,j} + 6u_{i+1/2,j} - u_{i+3/2,j} \right) & \text{if} \ \frac{1}{8} \left( u_{i-1/2,j} + u_{i+1/2,j} \right) \leq 0.
\end{cases} \tag{5.15}
\]

The viscous fluxes are also approximated by the midpoint rule resulting in

\[
(D_x)^{(n)}_{i+1/2,j} = \frac{T^{(v,x)}_{i+1,j} - T^{(v,x)}_{i,j}}{\Delta x} + \frac{T^{(v,x)}_{i+1/2,j+1/2} - T^{(v,x)}_{i+1/2,j-1/2}}{\Delta z}
\tag{5.16}
\]

and

\[
(D_y)^{(n)}_{i,j+1/2} = \frac{T^{(v,x)}_{i+1/2,j+1/2} - T^{(v,x)}_{i-1/2,j+1/2}}{\Delta x} + \frac{T^{(v,x)}_{i,j+1} - T^{(v,x)}_{i,j}}{\Delta z}
\tag{5.17}
\]

Using standard second-order centered differences, the viscous stress \(T^{(v,x)}\) are approx-
imated by

\[ T_{i,j}^{(v,xx)} = 2\mu_{i,j} \frac{u_{i+1/2,j}^{(n)} - u_{i-1/2,j}^{(n)}}{\Delta x} \]

\[ T_{i+1/2,j+1/2}^{(v,xx)} = \mu_{i+1/2,j+1/2}^{(n)} \left( \frac{u_{i+1/2,j+1}^{(n)} - u_{i+1/2,j}^{(n)}}{\Delta z} + \frac{w_{i+1,j+1/2}^{(n)} - w_{i,j+1/2}^{(n)}}{\Delta x} \right) \]

\[ T_{i,j}^{(v,xz)} = 2\mu_{i,j} \frac{w_{i,j+1/2}^{(n)} - w_{i,j-1/2}^{(n)}}{\Delta z}. \]

In these expressions, the normal viscous stresses are approximated using the standard linear interpolation method

\[ \mu_{i,j} = \mu_1 f_{i,j} + \mu_2 (1 - f_{i,j}) \]

while for shear stress components utilize a harmonic average approximation for the viscosity having the form

\[ \mu_{i+1/2,j+1/2} = \begin{cases} \left( \frac{\phi}{\mu_1} + \frac{1-\phi}{\mu_2} \right)^{-1} & \text{if } \frac{1}{2} \leq f_j + f_{j+1} \leq \frac{3}{2} \\ \mu_1 f_{i,j} + \mu_2 (1 - f_{i,j}) & \text{otherwise} \end{cases} \]

where \( \phi = f_j + f_{j+1} - \frac{1}{2} \). The theoretical support for this approximation is well documented in §3.4 of Tryggvason et al. [74].

The Poisson equation given by eqn. (5.7) is discretized through standard centered differences resulting in

\[ \frac{1}{\Delta x^2} \left( \frac{P_{i+1,j}^{(n)} - P_{i,j}^{(n)}}{\rho_{i+1,j}^{(n)} + \rho_{i,j}^{(n)}} - \frac{P_{i,j}^{(n)} - P_{i-1,j}^{(n)}}{\rho_{i,j}^{(n)} + \rho_{i-1,j}^{(n)}} \right) + \frac{1}{\Delta z^2} \left( \frac{P_{i,j+1}^{(n)} - P_{i,j}^{(n)}}{\rho_{i,j+1}^{(n)} + \rho_{i,j}^{(n)}} - \frac{P_{i,j}^{(n)} - P_{i,j-1}^{(n)}}{\rho_{i,j}^{(n)} + \rho_{i,j-1}^{(n)}} \right) \]

\[ = \frac{1}{2\Delta t} \left( \frac{u_{i+1/2,j}^{(*)} - u_{i-1/2,j}^{(*)}}{\Delta x} + \frac{u_{i,j+1/2}^{(*)} - u_{i,j-1/2}^{(*)}}{\Delta z} \right) \]

(5.18)

where \( \bar{\eta}^{(*)} \) is defined as the source term inside of the divergence term in eqn. (5.7) scaled by \( \Delta t \). This Poisson equation is solved subject to the appropriate boundary conditions. Consider the vertical boundary on the left side of the domain. The
continuity equation for the pressure node at \((i, j)\) next to the boundary denoted by

\[
\frac{u_{i+1/2,j}^{(n+1)} - U_{b,j}}{\Delta x} + \frac{u_{i,j+1/2}^{(n+1)} - u_{i,j-1/2}^{(n+1)}}{\Delta z} = 0
\]

provides an expression for the boundary velocity \(u_{i+1/2,j}^{(n+1)}\). Using the correction velocity given by eqn. (5.8), the discrete Poisson eqn. (5.18) next the boundary becomes

\[
\frac{1}{\Delta x^2} \left( \frac{p_{i+1,j}^{(n)} - p_{i,j}^{(n)}}{\rho_{i+1,j}^{(n)} + \rho_{i,j}^{(n)}} \right) + \frac{1}{\Delta z^2} \left( \frac{p_{i,j+1}^{(n)} - p_{i,j}^{(n)}}{\rho_{i,j+1}^{(n)} + \rho_{i,j}^{(n)}} - \frac{p_{i,j}^{(n)} - p_{i,j-1}^{(n)}}{\rho_{i,j}^{(n)} + \rho_{i,j-1}^{(n)}} \right) = \frac{1}{2\Delta t} \left( \frac{u_{i+1/2,j}^{(*)} - U_{b,j}}{\Delta x} + \frac{u_{i,j+1/2}^{(*)} - u_{i,j-1/2}^{(*)}}{\Delta z} \right)
\]

(5.19)

Similar equations are derived for the pressure next to the other boundaries and for each corner point. In most problems, the pressure Poisson equation is solved subject to periodic boundary conditions in the \(x\)- and \(y\)-directions and this zero Neumann boundary conditions at the walls shown in eqn. (5.19).

### 5.3.3 Volume-of-Fluid Algorithm

The conservative volume-of-fluid method (cVOF) developed by Weymouth & Yue[80] solves eqn. (5.6) in two steps. In the first step, the explicit interface location is locally approximated from the volume fraction field by second-order reconstruction algorithm. In the second step, integrates the advection equation forward in time generating solving for a new volume fraction field.

During the reconstruction step, the interface is approximated as being piecewise linear across the grid cell. This local linear reconstruction is expressed as \(\vec{m} \cdot \vec{x} = \alpha\) where \(\vec{m}\) is the surface normal vector and \(\alpha\) is the intercept. In general, analytical solutions for \(\alpha = \alpha(f, \vec{m})\) and \(f = f(\alpha, \vec{m})\), are known, but an approximation scheme is necessary for the determination of the surface normal vector. In this method, central differences of \(f\) are used to determine the interface orientation given by \(\max\{|m_x|, |m_y|, |m_z|\}\). For discussion, assume that this is the \(z\)-axis. The next step is to use a height function base method is implemented which approximates the
interface height (in say the z-direction) by

\[ \bar{z}_{i,j,k} = \sum_{l=-1}^{1} f_{i,j,k+l} \Delta z_{i,j,k+l} \] (5.20)

Finally, the surface normals are approximated by

\[ m_x = \frac{\partial \bar{z}}{\partial x}, \quad m_y = \frac{\partial \bar{z}}{\partial y} \] (5.21)

where if \( f_{i,j,k} \) is more than half full, a backward finite difference method is utilized; otherwise a forward finite difference method is used. This method allows for the exact reconstruction of a linear interface in 2D or 3D using only the local 3 x 3 (x3) block of cells.

After the interface reconstruction step is complete, eqn. (5.6) is used to update the volume fractions. Using an operator-split method, the advection equation is decomposed as

\[ \Delta f'_{i,j,k} \frac{\Delta \Omega}{\Delta t} = F_{i+1/2} - F_{i-1/2} + \int_{\Omega} c_c \frac{\partial u}{\partial x} dV \] (5.22a)

\[ \Delta f''_{i,j,k} \frac{\Delta \Omega}{\Delta t} = G_{j+1/2} - G_{j-1/2} + \int_{\Omega} c_c \frac{\partial v}{\partial y} dV \] (5.22b)

\[ \Delta f'''_{i,j,k} \frac{\Delta \Omega}{\Delta t} = H_{k+1/2} - H_{k-1/2} + \int_{\Omega} c_c \frac{\partial w}{\partial z} dV \] (5.22c)

where \( F, G, \) and \( H \) are the fluxed volume of fluid in the \( x-, y-, \) and \( z- \)directions respectively. The gradients in each of the 1D divergence terms are calculated through standard second order central differences. This method is guaranteed to be volume conserving as long as the following conditions are satisfied:

1. The flux terms are conservative, and
2. the divergence terms sum to zero, and
3. no over- or under-filling of a cell which violates \( 0 \leq f \leq 1. \)

By implementing an operator-split, the fluxed volume can be analytically determined.
using the relations in [68] satisfying condition (1). Weymouth and Yue proved that if

\[ c_c = \begin{cases} 
1 & \text{if } f^{(n)} > 1/2 \\
0 & \text{else}
\end{cases} \]

and the time step is restricted by

\[ \Delta t \sum_{d=1}^{N} \left| \frac{u_d}{\Delta x_d} \right| < \frac{1}{2} \tag{5.23} \]

conditions (2) and (3) would be satisfied exactly.

### 5.3.4 Dynamic Time Stepping For Time Integration

The numerical Navier-Stokes algorithm described in §5.3.2 utilized an explicit Runge-Kutta scheme which requires a time step restriction in order to guarantee numerical stability. A von Neumann stability analysis can be used to determine the stability bounds for each term in the Navier-Stokes equations. The resulting time step restrictions based on the viscous diffusion, hydrostatic pressure field, convection (CFL), and the VOF time step necessary to maintain volume conservation given by eqn. (5.23) are denoted as

\[ \Delta t_{Re} < \min \left( \frac{3}{14} Re \frac{\rho(f)}{\mu(f)} \Delta x^2 \right) \tag{5.24a} \]

\[ \Delta t_{Fr} < \min (\Delta \bar{x}) Fr^2 \tag{5.24b} \]

\[ \Delta t_{CFL} < \min \left( \frac{\Delta \bar{x}}{\| \bar{u} \|} \right) \tag{5.24c} \]

\[ \Delta t_{VOF} \sum_{d=1}^{N} \left| \frac{u_d}{\Delta x_d} \right| < \frac{1}{2} \tag{5.24d} \]

where in eqn. (5.24d), the \( N \) denotes the number of spatial dimensions in the simulation. These different time constraints are calculated for each time step, and the
minimum value in the set is implemented into the Runge-Kutta algorithm

\[ \Delta t_{\text{simulation}} = 0.8 \min \{ \Delta t_{Re}, \Delta t_{CFL}, \Delta t_{Fr}, \Delta t_{VOF} \}. \quad (5.25) \]

Since these time step constraints were derived from a linear von Neumann analysis, a small scaler value (0.8) is used to ensure that the minimum time step is well within the numerically stable range.

### 5.4 Flow Solver Validation Scheme

A suite of test cases are developed to demonstrate the consistently second order accuracy of the numerical Navier-Stokes flow solver and proper coupling between the flow solver and VOF algorithm.

#### 5.4.1 VOF Reconstruction And Transport Method Verification

The VOF algorithm has two primary components, an interface reconstruction algorithm and an advection algorithm, which needs to be properly validated in order for second-order convergence to be achieved.

To compute the fluxes in the transport algorithm, the local interface orientation needs to be reconstructed from the volume fraction. For a reconstruction algorithm to achieve second-order convergence rates, it must be capable of calculating the interfacial normals of a linear interface exactly. Therefore, a simple validation test case was used in which a unit cube is intersected by a linear plane. This produces an interface with constant normal vector in all of the intersected cells, those grid cells for which \( f \neq 0, 1 \). Tests were carried out with variations of the global volume fractions ranging from \( f_{\text{global}} \in (0, 1) \) and the normal vectors for the interface being restricted to only those in first quadrant. This first quadrant restriction is acceptable because the advection routines in the VOF algorithm always re-orient any interface to the first quadrant. Using the second order reconstruction routines from Weymouth and
Yue [80], more than one thousand interface orientations and global volume fractions were considered and for each case, the error in the calculated normal vectors were always found to be of the order of machine precision.

The second routine to be validated was the VOF advection algorithm. The simplest standard advection test involves the transport of a circular geometry over a unit distance, with unit velocity, in a periodic unit domain. To demonstrate convergence with grid refinement, the time step, $\Delta t$ was fixed by enforcing the user specified CFL-condition of $\Delta t/\Delta x = 1/4$. Additionally, the radius was set to be $R = 16\Delta x$.

Two different error metrics were used to quantify the behavior of the solution. The first error norm defines the advection error using the metric

$$E \{f\} = \Delta x \Delta y \Delta z \sum_{i,j} |f_{i,j,k} - \tilde{f}_{i,j,k}|$$

(5.26)

where $\tilde{f}_{i,j,k}$ is a reference solution. In order to ensure that this error metric isolates only the transport error, all of these cases were initialized in a way to avoid reconstruction errors. A high resolution initial condition was constructed from $N = 2048$ mesh points. This fine grid volume fraction solution was then used to build each of the $\tilde{f}_{i,j,k}$ reference solutions on the $N = 64, 128, 256, 512$ mesh cases. Due to the periodicity of the computational domain, the error was then calculated as the difference between the final advected solution and initial condition, $\tilde{f}_{i,j,k}$. The second error metric measures the cumulative volume loss in the system after each time step of the transport algorithm. Defining the volume as $V = \Delta x \Delta y \Delta z \sum_{i,j,k} f(i,j,k)$, an $L_1$-error metric on the volume advection was defined as

$$L_1 \{V\} = \frac{1}{T} \sum_{t=1}^{T} |V_t - V_{t-1}|$$

(5.27)

Table 5.2 shows the second order convergence of the $E$-norm and the volume conservation of the VOF algorithm on the $L_1$-norm. Similar convergence results were observed for translation tests in any of the coordinate directions.

All of the remaining validations tests presented in the work by Weymouth & Yue
Unit Distance Translation: $\Delta t/\Delta x = 1/4$ and $R = 16\Delta x$

<table>
<thead>
<tr>
<th>Mesh Points</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>2.04E-03</td>
<td>5.34E-04</td>
<td>1.48E-04</td>
<td>4.19E-05</td>
</tr>
</tbody>
</table>

Table 5.1: Convergence of the VOF algorithm for translation test for a circular geometry in uniform velocity field.

were carried out to validate the current numerical algorithm. In all cases, second order convergence of the $E$-norm and machine accurate volume conservation was observed.

5.4.2 Two-Fluid Couette Flows

The simplest validation cases that was considered was the lid driven two-phase Couette flows through a horizontal channel. This test contains a well known steady-state mean flow solution which allows for the effectiveness of the viscosity interpolation methods employed by the VOF algorithm to be demonstrated.

The mean velocity field is derived from the linearized steady-state Navier-Stokes equations. A fixed Cartesian coordinate system is established with the origin located at the undisturbed interface between the two fluids. The upper and lower fluids have viscosities $(\mu_1, \mu_2)$, densities $(\rho_1, \rho_2)$, and equilibrium depths $(h_1, h_2)$ respectively. The problem has been made dimensionless by the equilibrium depth, density, and viscosity of the upper phase along with the interfacial velocity. For the Couette flow problem, the resulting velocity solution is given by

$$U_{\text{theory}}(z) = \begin{cases} 
C_{11} z + C_{12} & z \geq 0 \\
C_{21} z^2 + C_{22} & z < 0 
\end{cases}$$

$$C_{11} = -n(U_b - U_t) \frac{h_2}{h_2 + nh_1}, \quad C_{12} = \frac{U_t h_2 + U_b h_1 n}{h_2 + nh_1}$$

$$C_{21} = -\frac{(U_b - U_t)}{h_2 + nh_1}, \quad C_{22} = \frac{U_t h_2 + U_b h_1 n}{h_2 + nh_1}$$

and $U_t$ and $U_b$ represent the velocity of the top and bottom walls respectively. The linear velocity profile allows for the velocity gradients in the viscous term to be cal-
Mesh Points \((N)\) | 64  | 128  | 256  | 512  \\
---|---|---|---|---
\(L_\infty(u)\) | 0. | 0. | 0. | 0. \\

Table 5.2: Grid convergence of two-fluid Couette velocity field for a flow with \(h_1 = 0.3\), \(h_2 = 0.7\), \(r = 1\), \(n = 10\), and \(Re = 1000\).

calculated exactly while the viscosity interpolation should cause the shear stress at the interface be satisfied exactly.

A numerical case was considered with gas and liquid depths of \(h_1 = 0.3\) and \(h_2 = 0.7\) along with density and viscosity ratios of \(r \equiv \frac{\rho_2}{\rho_1} = 1\) and \(n \equiv \frac{\mu_2}{\mu_1} = 10\) respectively. The Reynolds number was set to be \(Re = 1000\). The \(L_\infty = \max \{|U - U_{\text{theory}}|\}\) norm of the velocity field was used measure the errors in the numerically calculated Couette velocity field and is summarized in table 5.2. As expected, the Couette velocity field is recovered exactly showing that all of the wall boundary conditions have been implemented correctly and the viscosity interpolation scheme is functioning as expected.

5.4.3 Two-Fluid Poiseuille Flows

The second validation case that was considered was the two-phase Poiseuille flows through a horizontal channel. Like the Couette flow, this problem has a well known analytic steady-state mean flow solution making it an ideal validation problem for the accurate determination of the space/time convergence rate of the numerical algorithm.

The same scaling that was used in the two-fluid Couette flow was implemented into this problem resulting in an analytic velocity profile of the form

\[
U^{(\text{P})}(z) = \begin{cases} 
A_{11}z^2 + A_{12}z + 1 & z \geq 0 \\
A_{21}z^2 + A_{22}z + 1 & z < 0 
\end{cases} \tag{5.29}
\]

where

\[
A_{11} = -\frac{n+h_2}{h_2(1+h_2)}, \quad A_{12} = \frac{n-h_2}{h_2(1+h_2)} \\
A_{21} = -\frac{n+h_2}{nh_2(1+h_2)}, \quad A_{22} = \frac{n-h_2}{nh_2(1+h_2)}
\]
Figure 5-1: Grid convergence of two-fluid poiseuille velocity field where (- - -) represents the numerical error and (- - -) denotes a second order reference convergence rate.

For this particular numerical example, \( n = 10, \ Re = 500, \ h_2 = 1.0, \ \frac{dh}{dx} = -0.022 \). In order to show the second-order convergence of the numerical method, the numerical velocity field was calculated on a grid composed of \( N_x = N_y = N = 64, 128, 256, 512 \) grid points. The time step, \( \Delta t \), was fixed at value satisfying the CFL-condition for the finest grid used in the convergence study \( (N = 512) \). For this case, the values \( \Delta t = 10^{-4} \) was used. Figure 5-1 shows the convergence of the \( L_{\infty} \) norm of the velocity field as a function of grid resolution. Clean second-order convergence is observed.

### 5.4.4 Two-Dimensional Standing Wave

The second test case that was used to validate the Navier-Stokes flow solver and demonstrate its effective coupling with the VOF algorithm was the case of a linear standing wave on the interface between two fluids. From linear theory, the time evolution of the interface satisfies

\[
\eta (x,t) = A \cos (2\pi kx) \cos(\omega t)
\]  

(5.30)

where \( A \) is the wave amplitude, \( k \) is the wavenumber, and the linear dispersion relationship provides the wave frequency \( \omega_{\text{theory}} = \sqrt{k/Fr^2} \) in deep-water with \( Fr \)
Figure 5-2: Grid convergence of two-dimensional standing wave period where (---) represents the numerical error and (----) denotes a second order reference convergence rate

being the Froude number. Since this analytical frequency is derived from linear potential flow theory, it is not expected to satisfy the Navier-Stokes equations exactly. To minimize the error in this approximation, inviscid simulations were carried out for a flow with a large density ratio within a unit domain. For this simulation \( \rho_{\text{air}}/\rho_{\text{water}} = 10^{-3}, Fr = 1, A/\lambda = 0.05 \) and \( N = \lambda/\Delta x = 64, 128, 256 \). In order to get converged statistics for the wave frequency/period, the numerical simulations were run for ten consecutive wave periods. All quantities were made dimensionless by the liquid properties. With this choice of scaling, the VOF volume fraction is equal to unity in the liquid domain and zero in the gas domain. A second order interface profile was reconstructed from the VOF volume fraction as \( \eta(x_i) = \sum_j f(x_i, z_j) \Delta z \) where \( f \) is the volume fraction and \( \Delta z \) is the cell height in the z-direction. A spectral decomposition of the interface was then computed using discrete Fourier transforms. The resulting time evolution of the modal amplitude was computed and discrete estimates of the wave periods were calculated from the wave crests for each of the ten wave periods. The error in the wave frequency was measured with respect to the frequency calculated from a 512\(^2\)-grid. The convergence of the wave frequency with grid refinement is shown in figure 5-2. Clean second-order convergence of the wave period is observed validating the coupling between the VOF algorithm and the effectiveness
5.5 Simulation of Laminar Orr-Sommerfeld Flows

This section examines this numerical methods ability to capture the evolution of an interfacial Orr-Sommerfeld instability. A series of simulations are carried out showing that, for sufficiently fine grid resolution, the proper linear growth rate and flow energetics can be accurately captured.

5.5.1 Initial Condition

The viscous linear instability analysis and Chebyshev spectral solution derived in §4.1.1-4.2.1 provided highly accurate solutions for the stream functions and unstable wave speeds. These solutions are used to develop the initial conditions for the direct numerical simulations of the Navier-Stokes equations. Experience has shown that the way this initial condition is numerically implemented into the Navier-Stokes solver can lead to error in the interfacial growth rate which may be larger than $O(10)$%.

The Orr-Sommerfeld analysis was derived by utilizing Taylor series expansions of the solution around the free surface that allowed the functions to be evaluated at $z = 0$ instead of the actual free surface position $z = \eta(x, t)$. Therefore, the upper and lower eigenfunctions were calculated on the domain $z \in [0, h_1]$ and $z \in [-h_2, 0]$ respectively. The initial condition used by the Navier-Stokes solver contains a deformed interface corresponding to $z = \eta(x, t)$. Therefore, the eigen solution must be modified to produce an initial velocity field that accurately conforms to the deformed interfacial position. This can be achieved through a domain mapping that transforms a rectangular domain to one with a wavy interface.

Consider the lower fluid in the Orr-Sommerfeld problem which has a Chebyshev solution defined on the domain of $(\tilde{x}, \tilde{z}) \in [0, L_x] \times [-h_2, 0]$ while the physical wavy solution has a domain defined on $(x, z) \in [0, L_x] \times [-h_2, \eta(x, z)]$. The coordinate
transformation

\[
\hat{x} = x
\]
\[
\hat{z} = h_2 \frac{z - \eta(x,t)}{\eta(x,t) + h_2}
\]

provides a unique mapping between the two domains.

The initial perturbed velocity field is obtained through calculating derivatives of the stream function

\[
u_1(x, z) = \frac{\partial \psi}{\partial y} \quad u_3(x, z) = -\frac{\partial \psi}{\partial x}
\]

For a given Chebyshev eigenfunction, \( \hat{\psi}(\hat{x}, \hat{z}) \), the transformation given by eqn. (5.31) allows the eigenfunctions within the wavy domain to obtained through the derivative transformations resulting in the perturbed velocity fields of the form

\[
u_1(x, z) = \frac{h_2}{h_2 + \eta} \frac{\partial \hat{\psi}}{\partial \hat{z}}
\]
\[
u_3(x, z) = -\left( \frac{\partial \hat{\psi}}{\partial \hat{x}} + \frac{h_2 (h_2 + z)}{(\eta + h_2)^2} \frac{\partial \eta}{\partial \hat{x}} \frac{\partial \hat{\psi}}{\partial \hat{z}} \right).
\]

By analogy, a similar mapping was implemented for the upper fluid having the form

\[
\hat{x} = x
\]
\[
\hat{z} = h_1 \frac{z - \eta(x,t)}{h_1 - \eta(x,t)}
\]

which produces the mapped perturbed velocity fields

\[
u_1(x, z) = \frac{h_1}{h_1 - \eta} \frac{\partial \hat{\psi}}{\partial \hat{z}}
\]
\[
u_3(x, z) = -\left( \frac{\partial \hat{\psi}}{\partial \hat{x}} - \frac{h_1 (h_1 - z)}{(\eta - h_2)^2} \frac{\partial \eta}{\partial \hat{x}} \frac{\partial \hat{\psi}}{\partial \hat{z}} \right).
\]

Through this method, it was found that provided the grid resolution was fine enough,
the initial velocity field would produce results that were consistent with the theoretical Orr-Sommerfeld theory. This domain mapping routine was found to be consistent with the one used by Coward et al. [17].

The initial pressure field was given by the theoretical hydrostatic pressure profile. Due to the discretization used in the Poisson equation, the numerical hydrostatic profile, is smoothed across the density discontinuity resulting in differences from the analytic pressure profile. To minimize errors over the first few time steps, the initial pressure field was given by this numerical hydrostatic pressure profile.

5.5.2 Validation of Orr-Sommerfeld Theory

In this section, direct numerical simulations of the Navier-Stokes equations are carried out for the purpose of accurately capturing a laminar interfacial Orr-Sommerfeld instability. Consider a two-phase flow with equilibrium fluid depths of $h_1 = 1$ and $h_2 \equiv \frac{dL}{dG} = 2.33$ with density and viscosity ratios $r \equiv \frac{\rho G}{\rho C} = 100$ and $n \equiv \frac{\mu G}{\mu C} = 500$. The Reynolds number, density-weighted Froude numbers, and dimensionless pressure gradient have values of $Re = 150$, $\mathcal{F} = 99$, and $\frac{d\phi}{dx} = -0.861$. For simplicity, the effects of surface tension are neglected in this simulation. Carrying out a laminar linear stability analysis with these flow conditions results in an eigen-spectrum shown in figure 5-3 and contains an unstable band of wavenumbers ranging from approximately $\alpha \in [0, 4.9]$. This analysis shall examine a mode near the peak of the spectrum at $\alpha = 2$ for which $c \approx 1.4176 + 0.1874i$. The horizontal domain is scaled such that it can contain one wavelength resulting in the computational domain $(x, z) \in [0, \pi] \times [-2.333, 1]$. Since the kinematic interface condition is of the form

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \eta = \nu'$$

it can be shown that

$$\eta(x, t) = \frac{\phi(0)}{c - U(0)} e^{i\alpha(x - ct)}$$

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Figure 5-3: Computed laminar Orr-Sommerfeld growth rate spectrum for $h_1 = 1$, $h_2 = 2.33$, $r = 100$, $n = 500$, $Re = 150$, $\mathcal{F} = 99$, and $\frac{\partial P}{\partial x} = -0.861$.

Therefore the initial interfacial amplitude is given by the eigen-solution and interface velocity. To make the initial condition smaller, allowing for a longer period of initial linear growth, a scalar value of $\epsilon = 10^{-3}$ was applied to the interfacial amplitude, velocity, and pressure perturbations.

Direct numerical simulations of this two-fluid channel flow were carried and the resulting convergence of the growth rate is shown in figure 5-4. Clearly, the simulation shown in figure 5-4(a) shows stable interfacial growth while figure 5-4(b) demonstrates the convergence of the growth rate towards the expected value from Orr-Sommerfeld theory. Small deviations are observed between the numerical and analytical solutions that result in variations in the numerical growth rate. It should be noted that the method that was used to generate the initial volume fraction of the wavy interface utilizes a first-order reconstruction method. The error in this method can generate error in the phase between the wave interface profile and velocity components that in turn leads to an error in the growth rate. Using a higher-order method, such as a Romberg integration scheme, can reduce this initial phase error.
Figure 5-4: Recovery of linearly unstable Orr-Sommerfeld mode from direct numerical simulation with \( N_x = N_z = (\text{---}) 512, (\text{---}) 1024 \) grid points, and (---) theoretical. (a) Evolution of Fourier spectrum for most unstable mode; (b) Evolution of linear growth rate of primary mode.

5.6 Pressure Forcing Schemes

In §5.5.2, it was shown that the current numerical algorithm was able to accurately recover the behavior of linear interfacial instabilities. For studies where the long time nonlinear interfacial evolution is to be examined, the cost of simulating marginally unstable modes (where \( 0 < \omega_i \ll 1 \)) can be prohibitively expensive. A numerical technique which accelerates the initial linear interfacial growth phase without compromising the resulting nonlinear interfacial evolution can provide significant value for the simulation and modeling of nonlinear waves.

5.6.1 Numerical Formulation

One numerical strategy that can achieve this goal is an interfacial pressure forcing scheme which applies a concentrated stress at the interface to generate a user defined perturbation. Such a method was utilized by Dommermuth et al. [20] for the purpose of generating realistic ocean wave fields. Starting from the Navier-Stokes equations,
an interfacial surface stress \( (p_s) \) is applied

\[
\frac{\partial u_i}{\partial x_i} = 0
\]

\[
\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\delta_{i3}}{Fr^2} + \frac{1}{\rho Re} \frac{\partial}{\partial x_i} (2\mu e_{ij}) - \frac{p_s}{\rho} \frac{\partial H(f)}{\partial x_i}
\]

(5.36a)

(5.36b)

where \( e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \) and \( \frac{\partial H(f)}{\partial x_i} \) is a Dirac delta function defined by the gradient of the VOF Heaviside function.

Several different formulations of the surface stress are available. In the work by Dommermuth et al. [20], the surface stress was defined by the linear superposition of Fourier modes

\[
p_s = G(t) \left[ \sum_{n=1}^{N} A_n \cos (k_n x - \omega_n t) \right]
\]

(5.37)

where \( A_n, k_n \) and \( \omega_n \) are the user defined Fourier forcing amplitudes, wavenumber, and frequency respectively. This method is referred to as the traveling wave pressure forcing scheme. The original method was developed for flows without a mean flow where the frequency can be approximated by the deep-water linear dispersion relationship \( \omega_n^2 = k_n/Fr^2 \). For pressure driven channel flows, the frequency is supplied from the solution of the Orr-Sommerfeld problem (described in §4.2.1). The pressure forcing is activated smoothly by the function

\[
G(t) = \frac{1}{2} \left[ 1 - \cos \left( \frac{2\pi t}{T_f} \right) \right]
\]

(5.38)

over \( 0 \leq t \leq T_f \) and \( G(t) = 0 \) for \( t > T_f \). This activation function, \( G(t) \), is non-unique, but is intended to prevent impulsively stressing the interface and smoothly deactivates the forcing. This careful transition reduces the formation of interfacial noise and unintended standing waves.

An alternative formulation of the surface stress utilizes slope coherent pressure forcing. By decomposing the interface into its spectral components

\[
\eta(x) = \sum_{n=1}^{N} \eta_n e^{i k_n x} + c.c
\]

(5.39)
the surface pressure is defined by the derivative of the spectral components

\[ p_s = G(t) \sum_{n=1}^{N} A_n \left[ i k_n \hat{n} e^{ik_n x} + c.c \right] \]  

(5.40)

where \( c.c \) denotes the complex conjugate and \( A_n \) are the user defined Fourier forcing coefficients. The benefit of this forcing strategy comes from not having to force at a particular wave frequency, as was the case for the traveling wave method. As the waves begin to grow out of the linear regime, the nonlinearity of the system may result in a nonlinear adjustment of the dispersion relationship. As a result, continued forcing with the linear frequency can hinder the growth of the forced wave, excite additional unintended wave modes, or generate significant noise at the interface. The slope coherent method would remain effective since it does not require a description of the wave’s dispersive properties.

In order to implement this numerical forcing method, eqn. (5.36) requires a numerical treatment of the Dirac delta function \( \delta(\bar{x}) \equiv \frac{\partial H(f)}{\partial x_i} \). Several rounds of numerical experiments proved that special attention must be paid to the selection of the functional form of this term. In most continuous surface force methods, \( \nabla H(f) = -\delta_h \hat{n} \), with the delta function \( \delta \) being approximated by \( |\nabla_h f| \) (where the subscript ‘h’ denotes a numerical approximation of grid size \( h \)); however, it was found that this approximation resulted in the rapid formation and growth of interfacial noise. A smoother delta function prevented the formation of interfacial noise. To smooth the spatial distribution of the \( \delta \)-function, the volume fraction \( f \) was convolved with a smoothing kernel. The methods used in this work is thoroughly documented in §7.1.4 of Tryggvason et al. [74]. The key points of their description shall be briefly highlighted. A smoothed heaviside function is formed through

\[ H(\bar{x}) = \int_{\Omega} H(x') K(\bar{x} - \bar{x}', \varepsilon) d\bar{x}' \]  

(5.41)
with $\varepsilon$ being the width of the filter and the kernel being defined by

$$K = \begin{cases} A(\varepsilon) \left[ 1 - \left( \frac{|\vec{x}|^2}{\varepsilon^2} \right) \right] & |\vec{x}| < \varepsilon \\ 0 & |\vec{x}| > \varepsilon \end{cases}$$  \hspace{1cm} (5.42)

Using the convolution kernel, a discrete smoothed volume fraction is written as

$$\hat{f}_{ijk} = A(\varepsilon) \sum_l \sum_m \sum_n f_{lmn} \left[ 1 - \frac{x_{il}^2 + y_{jm}^2 + z_{kn}^2}{\varepsilon^2} \right]^4 h^2$$  \hspace{1cm} (5.43)

where $x_{il} = x_i - x_l$, $y_{jm} = y_j - y_m$, and $z_{kn} = z_k - z_n$, and the normalization constant, $A(\varepsilon)$, satisfies

$$A(\varepsilon) \sum_l \sum_m \sum_n \left[ 1 - \frac{x_{il}^2 + y_{jm}^2 + z_{kn}^2}{\varepsilon^2} \right]^4 h^2 = 1$$

From this, gradients of the volume fraction are found through direct differentiation of (5.42), followed by direct convolution with the volume fraction allowing for the an approximated $\delta$-function to be expressed as

$$\delta_{S,ijk} = 6 \hat{f}_{ijk} \left( 1 - \hat{f}_{ijk} \right) \left\| \nabla_h \hat{f}_{ijk} \right\|$$  \hspace{1cm} (5.44)

This $\delta$-function is centered over the smoothed region and satisfies the normalization condition

$$\int_{-\varepsilon}^{\varepsilon} \delta_{S}(\xi) \, d\xi = \int_0^1 6 \hat{f}_{ijk} \left( 1 - \hat{f}_{ijk} \right) \, d\tilde{f}$$

with $\xi$ being an interface coordinate which is normal to the interface.

### 5.6.2 Sample Pressure Forcing Cases

In this section, a few canonical tests are carried out which demonstrate the effectiveness of the two wave forcing schemes (traveling wave and slope coherent). These tests are designed to demonstrate the effectiveness of each of the forcing schemes while also providing a means of identifying how quickly each method can generate monochromatic waves. The results also suggest under what circumstances each forcing scheme is appropriate.
Demonstration of Wave Forcing Methods

The first validation test examines the interfacial evolution of a two-dimensional progressive wave under the influence of interfacial pressure forcing. The tests are carried out on a unit square domain with grid resolution $N_x \times N_z = 256^2$. The fluids are chosen to be an inviscid air-water flow with density ratio $\rho_u/\rho_l = 0.00123$ with equilibrium liquid depths $h_u = h_l = 0.5$. The initial wave perturbation is initialized with a wave steepness $\epsilon \equiv ak = 0.1$ and Froude number $Fr^2 = 1$. The numerical scheme invokes free-slip boundary conditions because of the inviscid flow assumption.

The numerical forcing scheme is applied to grow a primary wave mode with $k = 2\pi$ with forcing parameters $A_1 = 0.1$ and $T_f = 5$. The smoothed interfacial delta function was developed using a smoothing kernel width $\varepsilon = 2\Delta x$. For traveling wave forcing, the linear forcing frequency is approximated by $\omega = \sqrt{k/Fr^2}$. Three tests are considered.

The first simulation considered the natural evolution of a plane progressive wave without any interfacial forcing. Figure 5-5(a) shows the time evolution of the modal amplitude. As expected, the amplitude time history is nearly constant with only small amplitude oscillations and a slight drift being observed. These deviations from the theoretical constant amplitude profile are primarily attributed to weakly nonlinear dispersive effects which result from the wave having a wave steepness of $\epsilon = 0.1$. Additionally, the initial volume fraction distribution of the wave was calculated with a first-order reconstruction method. It is possible that error in this reconstruction resulted in small phase shifts in the interface which caused the interface to undergo small amplitude oscillations as the code adjusts to the correct profile.

The second test utilizes the traveling wave pressure forcing scheme. The resulting time evolution of the modal amplitude is presented in figure 5-5(b). The modal solution exhibits clear growth due to the pressure forcing scheme. Due to the transition function $G(t)$ given by eqn. (5.38), strong growth doesn’t occur until $t \sim O(3)$ at which point weak exponential growth is observed. With the forcing variables $(A_1, T_n) = (0.1, 5)$ the modal amplitude nearly doubles within the simulation time.
The final test utilizes the slope coherent wave forcing scheme. The time evolution of the modal amplitude is shown in figure 5-5(c). The modal amplitude exhibits strong exponential growth throughout the entire simulation and increases in amplitude by a factor of six over the course of the simulation. Additionally, the strong interfacial forcing has created modal growth without the presence of the weak oscillations that were observed in the traveling wave forcing method shown in figure 5-5(b).

Comparing the forced solutions shown in figure 5-5(b) & 5-5(c) shows that both methods are effective at generating wave growth; however, given the same forcing parameters, the slope coherent pressure forcing scheme is capable of producing significantly faster growth. This is a logical result because as the wave steepness increases, the wave forcing becomes stronger resulting in a feedback forcing mechanism. This method is highly efficient in the event that there is a pre-existing wave disturbance or if the natural mechanics of the system are expected to generate waves. However, starting from a flat interface in a stable system (i.e. stratified two-fluid poiseuille flow without any interfacial perturbation), the slope coherent forcing would be inefficient because the computed wave slope is nearly zero. In such a case, the traveling wave method is the more appropriate scheme.

It was also found that when developing large amplitude waves, the slope coherent method was less prone to generating interfacial noise than the traveling wave method. This is attributed to the traveling wave method requiring a linear forcing frequency. As the wave becomes large and nonlinear, the wave frequency experiences a nonlinear correction. Continued forcing at the linear wave frequency results in the generation of sidebands, a reduction in the growth rate of the primary mode, and in some cases strong interfacial noise. Not having to specify a wave frequency in the slope coherent scheme makes it a more reliable forcing scheme.

**Forced Growth of Unstable Laminar Orr-Sommerfeld Mode**

The examples shown in the previous section demonstrated the efficacy of the interfacial pressure for wave generation; however, they did not demonstrate the effect of the forcing on the resulting dynamics of the flow. This section examines the the growth
Figure 5-5: Time-evolution of the primary interfacial mode of a plane progress wave: (a) without pressure forcing; (b) with traveling wave pressure forcing; (c) with slope coherent pressure forcing.
of a linearly unstable laminar Orr-Sommerfeld interfacial mode under the influence of interfacial pressure forcing. From this analysis, it is shown that the pressure forcing can accelerate the wave growth above the theoretical growth rate and then return back to its natural state of growth with minimal adverse effects.

Consider a flow with the same operating conditions as those described in §5.5.2 for which direct numerical simulations, without any external forcing, were shown to accurately recover the wave growth predicted by analytic theory. Two sets of simulations are carried out. The first set of tests document the effects of the interfacial smoother, given by eqns. (5.43-5.45), which are used to generate the interfacial delta function. It is necessary to understand how smoothing the interfacial discontinuities (density and viscosity) effect the resulting modal evolution without any external pressure forcing. The second set of tests examine how the magnitude of the forcing coefficients \( A_n \) in eqn. (5.40) effects the wave growth and growth rate of the interfacial mode. Such a test can help quantify how strongly the interface needs to be forced, determine if the resulting nonlinear behavior is strongly effected, and observe how the resulting physics are changed if the interface is forced too strongly.

The effects of the interfacial smoother were examined by comparing the modal evolution with different kernel smoother widths (\( \varepsilon \)). Figure 5-6 shows the modal amplitude and linear growth rate of the unstable wave component for a case without any smoothing filter against two trials where interfacial smoothing has been applied with filter kernel widths of \( \varepsilon = \Delta \) and \( 2\Delta \) where \( \Delta \equiv \sqrt{\Delta x_i^2} \). For the case of the small kernel width, the modal evolution follows closely with that of the unfiltered direct numerical simulation case. The linear growth rate is found to be marginally smaller with the minimum growth rates differing by only \( \sim 12\% \). The evolution of the modal amplitude exhibits similar behavior going through exponential growth followed by the nonlinear steepening and then bounding of the solution. The maximum amplitudes are found to differ by \( \sim 17\% \) at the peak. The modal evolution obtained with the wider smoothing kernel presents with significant deviations from the unfiltered simulation results. The maximum linear growth rate is nearly twice that of the unfiltered case and exhibits stronger oscillations during the recovery phase that never settle.
Figure 5-6: Effects of interfacial smoother on unstable Orr-Sommerfeld (a) modal amplitude and (b) growth rate. The plotted curves represent the growth of the unstable mode determined from: (-----) Orr-Sommerfeld theory, (---) direct numerical simulation, (- - -) numerical simulation using $K_8$-smoother with $\varepsilon = \Delta$ and (···) numerical simulation using $K_8$-smoother with $\varepsilon = 2\Delta$ where $\Delta \equiv \sqrt{\Delta x_i^2}$. 
around the theoretical growth rate. Additionally, the resulting nonlinear evolution does not exhibit the same increase in the growth rate prior to reaching the maximum modal amplitude as was exhibited in the previous two cases.

The departure of the smoothed cases from the unforced case may be due to the strong oscillations which are found in the Orr-Sommerfeld eigenfunctions in the vicinity of interface. Introducing strong smoothing of the interfacial discontinuities causes the eigenfunctions to no longer be accurate (or consistent with the theory) and can cause the numerical algorithm to go through an adjustment period in an attempt to converge to a consistent disturbance profile. This suggests that, unless it is known that for a particular flow the eigenfunctions have a simple profile, smaller kernel smoothing widths should be utilized. Unless otherwise stated, in this work the kernel width is set to $\varepsilon = \Delta$.

The next set of tests demonstrate the effect of the forcing coefficient on the modal evolution and provide a means of identifying if the interfacial forcing results in significantly altered solution behavior. For the following tests, the slope coherent forcing routines given by eqn. (5.39) are applied with a forcing interval given by eqn. (5.38) with $T_f = 1$ and the kernel smoother defined with $\varepsilon = \Delta$. Three forcing coefficients, $A_1 = 1, 10, 100$ were chosen to cover a large range and provide a clear depiction of the solution behavior under varying intensities of interfacial forcing. Figure 5-7 shows the modal amplitude of the unstable Orr-Sommerfeld mode and corresponding linear growth rate. The unforced case with kernel smoothing applied described in figure 5-6 is provided as a basis for comparison. With $T_f = 1$, interfacial forcing is only applied from $t \in [0, \pi]$. For the weakest forcing coefficient, the mode experiences only a weak departure from the unforced case. The growth rate slightly increases during the forcing interval but then recovers the unforced growth rate after the period of interfacial forcing is over. Clear exponential behavior is observed. As the forcing coefficient increases, the growth rate departs further from the unforced case; but once again, returns to the unforced behavior once the forcing has been removed. For the final case with $A_1 = 100$, the maximum growth rate increases to $O(6)$ times larger than the unforced case before returning back to the original unforced linear growth
Figure 5-7: The (a) modal amplitude and (b) growth rate of unstable Orr-Sommerfeld under the influence of interfacial pressure forcing using $K_{8}$-smoother with $\varepsilon = \Delta$ and $T_{f} = 1$. The plotted curves represent the growth of the unstable mode determined from: (——) unforced case, interfacial forcing with: (---) $A_{1} = 1$, (---) $A_{1} = 10$ and (· · ·) $A_{1} = 100$. 

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Being able to provide short intervals of interfacial forcing to accelerate the growth of interfacial modes can provide a powerful numerical tool. Often, the flow conditions may produce a narrow band of unstable modes with growth rates that are $O (0.001 - 0.01)$. Such modes require long simulation times in order to see sizable modal growth and is often too computationally expensive. Alternatively, interfacial pressure forcing can be applied to accelerate process. Interfacial pressure forcing can be applied and cause the initial perturbation to grow by several orders of magnitude and then removed giving the mode time to recover its initial unforced linear behavior and allowing for accurate examinations of the resulting nonlinear evolution. Such a procedure results in significant computational savings. Furthermore, there may be cases where the form of the initial perturbation is not known. In such an event, an initial interval of pressure forcing can be applied to generate a growing mode. Once the forcing is removed, the correct perturbation behavior can be recovered and studied.

It should be noted that while there is flexibility in choosing the magnitude of the forcing coefficient, care must be taken in its selection. If a sufficiently large value is chosen, the wave may grow so quickly that the nonlinearity becomes strong while the forcing function is still active. Consider the case shown in figure 5-8 which has a forcing coefficient of $A_1 = 1000$. As expected, the forcing causes rapid growth of the modal amplitude; however, the forcing has caused the interface to become so steep that it becomes bounded by nonlinearity (at $t = 2.4$) while the transition function, $G(t)$, is still active. The strong forcing has caused the peak amplitude to be $\sim 125\%$ larger than in the unforced case and because nonlinearity bounded the solution before the forcing was removed, there was never a period which allowed for the mode to recover the correct linear growth rate. As a result, it is not clear how closely the resulting nonlinear evolution will resembled the true unforced behavior. Therefore, excessively forcing the solution will generate growth but there is no guarantee that the resulting modal behavior will resemble that of the unforced case from figure 5-7.
Figure 5-8: The modal amplitude of unstable Orr-Sommerfeld under the influence of strong interfacial pressure forcing using $K_8$-smoother with $\varepsilon = \Delta$ and $T_f = 1$ with: (—) unforced case and (- - -) interfacial forcing with $A_1 = 1000$.

## 5.7 Conclusions

In this chapter, the fully nonlinear viscous governing equations for an incompressible, Newtonian, two-phase channel flow were expressed in terms of a single-fluid formulation of the Navier-Stokes equations. A second-order numerical discretization of the Navier-Stokes equations coupled with a conservative VOF method was discussed and detailed validation tests were presented demonstrating the convergence and consistency of the numerical scheme. A detailed description of the procedure for developing a proper initial condition for unstable interfacial (unstable Orr-Sommerfeld) waves was provided and highlighted the way the solution in the vicinity of the interface must be treated in order to recover accurate wave growth. Direct comparisons of the the numerical algorithm against theoretical growth rates was made and rapid convergence of the solution was observed.

To overcome the excessive computational cost associated with marginally unstable waves, an interfacial pressure forcing scheme was developed. It was shown that this method was capable of accelerating the initial phase of unstable growth while still providing reasonably accurate predictions of the later nonlinear interfacial evolution.
A forcing sensitivity study was presented which demonstrated the usefulness of the method while also highlighting the limitations of the forcing method.
Chapter 6

Direct Numerical Simulation of Turbulent Two-Phase Flows

Within this chapter, the physics of a combined linear instability and resonant interaction theory are further examined under more generalized flow conditions. Chapters 2 and 3 identified a general mechanism that was capable of generating large amplitude long waves from linearly unstable short-wave modes using nonlinear potential theory. This chapter, generalizes the results from Chapter 4, by carrying out turbulent of the interfacial evolution. Using the advanced numerical algorithm developed in Chapter 5, simulations are carried out that examine the initial interfacial instability and resulting nonlinear energy cascade from short- to long-wavelength wave modes. Such tests demonstrate that the initial nonlinear mechanism described in Chapter 2 under the assumption of potential theory still persists in more complex turbulent flows.

6.1 Introduction

The experiments carried out by Jurman et al.[40], described in detail in section 4.3, provide a parameter space that demonstrates strong nonlinear resonant interactions among interfacial waves. In this chapter, one of the experimental flow conditions described in figure 7 of Jurman et al.[40] is examined which is composed of a turbulent gas flowing over a laminar liquid-layer. Within this experiment, a strong
sub-harmonic resonant interaction is observed. By repeating this experimental test numerically, several concepts can be examined. First, direct numerical simulations carried out at this flow condition can be used to validate the predictions of the turbulent linear stability analysis from Chapter 4. This can confirm the accuracy of the model equations given by eqn. (4.2.4) used in the stability analysis to approximate the mean flow profiles while also demonstrating how well the turbulent linear dispersion relationship, calculated with the quasi-laminar hypothesis, matches the result from the direct numerical simulations. The numerical simulation also provides a more detailed history of the nonlinear spectral evolution of the interfacial waves allowing for the wave-interactions to be examined in detail and confirm the trends identified by the analytic resonant interaction theory. Furthermore, the mechanics of the interfacial evolution can be quantified allowing form quantities such as interfacial stress to be modeled. Such terms are of vital importance in improving the accuracy of industrial slug simulators and theoretical slug transition models.

A large body of work exists which is focused on examining the influence of near-interface turbulence on the coupled gas-liquid flows. Initial works carried out direct numerical simulations of turbulent air-water flows in channel (e.g. Couette) flows[48, 44]. In these simulations, interface is kept flat allowing for the continuity of velocity and shear stress to be enforced. In doing so, the air- and water-side boundary layers could be characterized and a statistical analysis of the velocity, vorticity, and turbulent kinetic energy budget confirmed features such as the similarity of the air-side interface boundary layer to the wall boundary layer.

The next layer of complexity to the problem was introduced by carrying out direct numerical simulations of a turbulent flow over a smooth wavy wall undergoing transverse motions in the form of a streamwise traveling wave[2, 70]. This study allowed for the effects of the wave boundary on the mean flow and the turbulent statistics to be quantified. Significant differences from the flat-wall case were observed. The mean flow was found to be significantly altered and a strong redistribution of energy in a layer close to the boundary (downstream of the wave troughs) was observed. Strong flow separation over the wave crests was identified and was found to be a function of
the ratio of the mean velocity to the wave speed. This effect was also found to have a strong influence on the form drag on the wave profile.

More recently, the problem was generalized to consider direct numerical simulations of turbulence in a counter-current sheared air-water flow with a deformable interface[26]. In this study, the deformation of the interface were primarily of the form of capillary waves with wave steepness of \( ak \sim O(0.01) \). It was found that the interfacial motion strongly influenced the time-averaged turbulent statistics in the near-interface region. The interfacial motion tended to damp the turbulent fluctuations and become less anisotropic in the vicinity of the interface which intern reduced the interfacial dissipation.

In all of these studies, spectral or coupled spectral finite difference methods were implemented such that accurate high order turbulent solutions could be obtained. For the simulations that incorporated propagating waves or freely deforming interfaces, domain mappings were implemented such that the mapped solutions could be solved spectrally on a rectangular domain. This technique prohibits the possibility of wave breaking or topological changes to the interface. As a result, all of the interfacial waves were limited to being of very small wave steepness. To overcome this strong limitation, in this work a volume-of-fluid interface capturing method is utilized which permits the formation of steep waves which are free to break, form sprays and entrain bubbles.

6.2 Identification of Flow Conditions

For each experimental case presented by Jurman et al. [40], the authors reported the upper and lower fluid Reynolds numbers \( (Re_u^{(e)} \) and \( Re_l^{(e)} \), average air velocity \( (\bar{U}_{air}) \), liquid depth \( (d_l) \), and the dynamic viscosity of the liquid \( \mu_l \). In order to generate an initial condition, the corresponding applied pressure gradient across the channel would need to be determined. In this work, an estimation of the pressure gradient was made using the same method that was described in §4.3. By specifying the mean velocity of the upper fluid as an input, a unique pressure gradient was
Table 6.1: Selected case conditions from [40] for a 2.54 cm deep channel where $d_1 = 0.45 \text{ cm}$, $\bar{U}_{air} = 3.7 \text{ m/s}$, $\mu_1 = 10 \text{ cP}$, $\mu_0/\mu_1 = 0.0018$ and $\rho_0/\rho_1 = 0.0012$ with $\frac{dp}{dx}$ being found using the analytic models developed by [60, eq.(14)]. Experimental and numerically defined Reynolds numbers denoted by superscript (e) and (n) with $Re_u = \rho_u \bar{U}_u h_u/\mu_u$, $Re_l = \rho_l \bar{U}_l h_l/\mu_l$, and $Fr^2 = gh_u/\bar{U}^2$.

<table>
<thead>
<tr>
<th>$Re_l^{(e)}$</th>
<th>$Re_u^{(e)}$</th>
<th>$\frac{dp}{dx}$</th>
<th>$\left(\frac{N}{m^2}\right)$</th>
<th>$Re_l^{(n)}$</th>
<th>$Re_u^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>5150</td>
<td>-6.73</td>
<td>9.1</td>
<td>5150</td>
<td></td>
</tr>
</tbody>
</table>

calculated using [60, eq.(14)].

In this chapter, flow conditions corresponding to the results shown in figure 7 of Jurman et al. [40] were chosen because it has a moderately high gas Reynolds number which could be adequately resolved by direct numerical simulations of the Navier-Stokes equations. Using these flow conditions, the resulting pressure gradient and corresponding numerical upper and lower fluid Reynolds numbers, based on the mean velocity within each phase are shown in the last three columns of table 6.1. Fair agreement is observed between the experimental Reynolds number and the Reynolds numbers from the model equations for the turbulent mean flow. It should be noted that the model equations for the turbulent mean flow profile developed by Naraigh et al.[60] are based on a simple eddy-viscosity model for the turbulent Reynolds stresses. For lower Reynolds number flows (near the transition Reynolds number), the eddy-viscosity model may become less accurate and introduce error. Therefore, one may expect some deviation between the DNS/experimental profile and the mean flow profile computed with the Naraigh[60] model.

In order to capture the nonlinear interfacial evolution under the influence of a turbulent gas blowing over a laminar liquid layer, an accurate turbulent gas field must be generated as the initial condition to this problem. Therefore, the simulations carried out in this chapter are broken into two separate phases. The first set of simulations are dedicated to developing accurate initial conditions of a statistically steady (fully developed) turbulent gas field which blows over a laminar liquid layer. The second set of simulations take this initial condition and examines the fully nonlinear evolution of the interface between the two fluids. Each step contains significant technical
challenges which need to be addressed.

6.3 Generation of Initial Turbulent Flow

The first set of simulations are dedicated to developing an accurate turbulent initial condition for the later nonlinear evolving interface trials. The strategy for developing this initial condition was constrained by several requirements. First, given that one of the primary objectives of this work is to observe which wavelengths become linearly unstable under the influence of turbulent forcing, it is necessary for the interface remain flat, \( \eta(\tilde{x}, t) = 0 \), during the initial turbulent development phase. Second, since the liquid Reynolds number shown in tables 6.1 is laminar, with \( Re_L \sim O(10) \), the generation of the initial turbulent gas field can not corrupt the liquid layer resulting in its transition to turbulence. Special attention must be paid to ensuring that the liquid layer remains laminar.

Therefore, to generate a fully developed turbulent gas field while preserving the interface and laminar liquid properties, the following numerical strategy was developed. The two-fluid laminar Poiseuille velocity and pressure profiles are given by

\[
\begin{align*}
    u &= \begin{cases}
        \frac{Re}{2N} \frac{dp}{dx} z^2 + C_1 z + C_2, & z \in [-h_l, 0] \\
        \frac{Re}{2} \frac{dp}{dx} z^2 + C_3 z + C_4, & z \in [0, 1]
    \end{cases} \\
    p &= \begin{cases}
        -\frac{p}{\rho g} (z + h_l), & z \in [-h_l, 0] \\
        -\frac{1}{\rho g} (z + Re h_l), & z \in [0, 1]
    \end{cases}
\end{align*}
\]

with

\[
\begin{align*}
    C_1 &= -\frac{dp}{dx} \frac{Re (N - h_l^2)}{2N (N + h_l)}, & C_3 = NC_1 \\
    C_2 &= -\frac{dp}{dx} \frac{Re h_l (1 + h_l)}{2 (N + h_l)}, & C_4 = C_2
\end{align*}
\]

where \( h_l \equiv d_l/d_u \), \( Re \equiv \rho_l/\rho_u \), and \( N \equiv \mu_l/\mu_u \). Comparing this laminar liquid profile the the modeled turbulent velocity profile given by [60, eq.(14)], it was found that the
interface velocity and the laminar liquid velocity profile are in excellent agreement. To simplify the process of generating a turbulent gas field, the problem was adjusted to that of a single phase pressure driven Couette flow where the upper wall would enforce the zero-velocity no-slip condition while the lower wall would move at the interface velocity of the two fluids given by eqn. (6.1a) or [60, eq.(14)]. Reformulating the problem from a two-phase to a single-phase flow reduces the computation cost, increases the effective numerical resolution of the turbulent gas field, and ensures that the interface remains flat. By requiring that the lower wall move at the interface velocity, this method implicitly enforces that the liquid layer is laminar and will therefore be well conditioned for the next stage of two-phase simulations. Converting the initial problem to that of a single phase flow also has the added benefit of not requiring that the initial round of simulations comply with the strict VOF CFL constraint,[80, eq.(21)]. This allows the simulations run faster with a larger stable time step.

This single phase problem is initialized with the laminar velocity profile given by eqn. (6.1a). In order to force the velocity field to transition to turbulence, a random divergence free velocity field whose magnitude is $O(10\%)$ of the maximum velocity of the laminar mean flow is superposed on the initial laminar velocity field. Initially, this random velocity field is not representative of the proper turbulent statistics. However, allowing the numerical algorithm to have a ‘warm-up’ period to evolve the velocity field forces the transition to turbulence and allows the turbulent fluctuations to recover physically realistic statistical distributions.

The length of the computational domain in the x-direction was chosen to be of the order of the wavelength of the dominant wave component observed in Jurman et al.[40] while the domain length in the y-direction was chosen to be large enough such that the two-point correlation of the energy spectra would decay to zero for large separation values. For all cases, the grid spacing in the z-direction was selected such that there would be a few points within the viscous sub-layer. For the structured uniform grid used by this numerical method, the first point is located a distance of $z^+ = \frac{1}{2} \Delta z^+$ from the interface and wall.
Table 6.2: Computational parameters used in the direct simulation of the flow conditions detailed in Table 6.1 where the \((\Delta x, \Delta y, \Delta z)^\dagger = (\Delta x, \Delta y, \Delta z) u_{ii}/u_u\).

<table>
<thead>
<tr>
<th>(Re)</th>
<th>(Fr^2)</th>
<th>(\mathcal{R} \equiv \frac{\rho_l}{\rho_u})</th>
<th>(\mathcal{N} \equiv \frac{\mu_l}{\mu_u})</th>
<th>(L_x)</th>
<th>(L_y)</th>
<th>(L_z)</th>
<th>Mesh</th>
<th>((\Delta x_i^+, \Delta y_i^+, \Delta z_i^+))</th>
</tr>
</thead>
<tbody>
<tr>
<td>476.95</td>
<td>0.57</td>
<td>555.5</td>
<td>833.3</td>
<td>6</td>
<td>4</td>
<td>1.0</td>
<td>256(^2) \times 128</td>
<td>(7.88, 5.25, 2.63)</td>
</tr>
</tbody>
</table>

To generate an accurate initial turbulent field, the initial solution is integrated forward in time until the velocity field reaches a statistically steady state. Steady state is identified by a linear profile of the total stress as predicted by eqn. (6.7) in section 6.4.3. Once this condition is satisfied, the velocity field is integrated further in time in order for a running time average of various statistical quantities to be determined. Based on the viscous length scale \(\delta_\ast = u_u / u_{\ast i}\) and the interfacial friction velocity \(u_{\ast i} = \sqrt{t_i / \rho_u}\), a dimensionless time scale \(t_\ast = \frac{\delta_\ast u}{u_{\ast i}}\) was used to guide the selection of the sample rate for the time averaging. In this paper, the sampling rate of \(\Delta t = 0.01\) was used. To improve the volume of statistical samples, all quantities are averaged over the horizontal planes \((x, y)\). Therefore, for the remainder of this section, all \(<\) refer to quantities which have been averaged over \(x, y,\) and \(t\) while \((\ )'\) refer to turbulent fluctuations.

### 6.4 Statistical Analysis of Initial Turbulent Flow

Within this, and the following sections, the traditional statistical analysis of turbulent flow is carried out to validate that the velocity and pressure fields have converged to a steady-state solution and are sufficiently accurate for the resulting investigations involving the nonlinear interfacial evolution.

#### 6.4.1 Mean Velocity Profile

The profile of the turbulent mean flow, made dimensionless by the interfacial- and wall-shear velocities, are shown in figures 6-1(a) and 6-1(b) respectively. The dashed line represents the law of the wall and the log law profiles. Within the interfacial and wall (denoted by the subscripts \(i\) and \(w\) respectively) viscous sublayers, \(z_i^+ < \) and \(z_w^+ < \)
5, the numerical results follow the linear law of the wall profiles given by \( u_i^* = \bar{U}(0)U/\upsilon_{\text{w}} + z^+ \) and \( u_w^* = z^+ \) where \( \upsilon_w = (\tau_1/\rho_u)^{1/2} \). Excellent agreement is observed even though there are only a few grid points present within this region. In the log-law region, the computational results are in agreement with the logarithmic velocity profile with \( u_{i/w}^* = \frac{1}{\kappa} \log z^+ + B_{i/w} \) with \( \kappa = 0.41 \) being the von Kármán constant and the constants \( B_i = 6.5 \) and \( B_w = 6 \) respectively. The mean of the \( v^- \) and \( w^- \) velocity components, denoted as \( V \) and \( W \) respectively, were calculated and averaged resulting in max \( |V| \sim O(0.01) \) and max \( |W| \sim O(10^{-14}) \). While \( V \ll 1 \), as expected, its non-zero magnitude is most likely a numerical artifact due to the sampling frequency used for the time averaging.

Additional mean flow parameters are computed to assess the quality of the mean velocity profile. Integrating the RANS momentum equations once across the vertical direction produces the total stress distribution across the channel

\[
\tau(z) \equiv \mu_u \frac{\partial \bar{U}}{\partial z} - \rho_u \bar{w} \bar{w}' = \tau_i + \frac{\partial \bar{P}}{\partial x} z. \tag{6.3}
\]

Evaluating \( \tau(z) \) at the wall and normalizing by the interfacial stress produces

\[
\frac{-\tau_w}{\tau_i} = 1 + \frac{\partial \bar{P}}{\partial x} \frac{d_u}{\tau_i} = 1 - \left( \frac{Re}{Re_{\text{w}}} \right)^2 \tag{6.4}
\]
Numerically evaluating the interfacial and wall stress along with the Reynolds number based on the interfacial friction velocity produces the effective Reynolds number $Re_e = 479.57$, which is in close agreement with the prescribed Reynolds number of $Re = 476.95$. The small error which is present in the computed interfacial and wall stresses results in approximately 0.5% error in the Reynolds number. Additionally, the bulk mean velocity, defined as

$$U_{mean} = \frac{1}{h_u} \int_0^{h_u} u \, dz$$  \hspace{1cm} (6.5)$$

is calculated as well as the maximum velocity ($U_{max}$). A summary of the resulting computed parameters are reported in Table 6.3.

### 6.4.2 Turbulence Intensities

The turbulence intensities, normalized by the scaling velocity $U$, are shown in figure 6-2(a). As expected, they exhibit a nearly symmetric behavior about the center-line ($z = 0.5$). The relative order of magnitudes among the turbulent fluctuations is consistent with that of single phase turbulent channel flow where it is found that $w_{rms} < v_{rms} < u_{rms}$. The pressure fluctuations, normalized by $p_u U^2$, shown in figure 6-2(b), also exhibit nearly symmetric behavior. The magnitude of the fluctuating quantities is smaller than those in some of the previously reported results for single phase turbulent channel flow (i.e. Kim, Moin, & Moser [41]); however, the Reynolds numbers used in this work are lower resulting in weaker turbulence intensities.

### 6.4.3 Reynolds Shear Stress

The total shear stress distribution across the flow is shown in figure 6-3. The theoretical distribution of the total stress is obtained from eqn. (6.3). Normalizing this

<table>
<thead>
<tr>
<th>$Re_e$</th>
<th>$Re_{mean}$</th>
<th>$Re_{max}$</th>
<th>$U_{mean}/u_{r,i}$</th>
<th>$U_{max}/u_{r,i}$</th>
<th>$U_{max}/U_{mean}$</th>
<th>$u_{r,i}/u_{r,w}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>479.57</td>
<td>5391.52</td>
<td>6248.38</td>
<td>15.93</td>
<td>18.47</td>
<td>1.16</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 6.3: Characterization of the the time-averaged streamwise velocity profile
Figure 6-2: Root-mean square (a) velocity fluctuations \( u_{\text{rms}} \), \( v_{\text{rms}} \), and (b) pressure fluctuations. All quantities are normalized by wall shear velocity, \( u_\tau \), and shown in global coordinates normalized by upper layer equilibrium depth, \( h_u \).

The relationship with the scaling \( \tau = \tau_w \tilde{\tau} \) and \( z = h_u \tilde{z} \) allows the dimensionless stress to be written as

\[
\tilde{\tau}_u(z) = \frac{\tau_i}{\tau_w} + \left( \frac{d_u}{\tau_w \frac{d\tau}{dx}} \right) \tilde{z}
\]  

(6.6)

With this choice of scaling, the magnitude of the dimensionless pressure gradient \( \frac{dp}{dz} \) is equal to unity. Noting that

\[
\left( \frac{Re}{Re_*} \right)^2 = \left( \frac{U_p}{u_\ast} \right)^2 = \left( \frac{d_u}{\rho_u \frac{d\tau}{dx}} \right) \left( \frac{\rho_u}{\tau_i} \right)
\]

along with eqn. (6.4) allows for the dimensionless total stress, given by eqn. (6.6), to be written as

\[
\tilde{\tau}(\tilde{z}) = \frac{Re^2_{si}}{Re^2_{*si} - Re^2} \left[ 1 - \frac{Re^2_{si}}{Re^2_{*si}} \tilde{z} \right].
\]  

(6.7)

This theoretical linear stress distribution is confirmed by the numerical solution as shown in figure 6-3(a). The agreement between the two solutions confirms that the numerical simulation has reached a fully developed equilibrium state. Figure 6-3(b) shows the decomposition of the total stress into the viscous and Reynolds stress components. As expected, there is a rapid transition from the dominant viscous stress in the vicinity of the wall/interface to a Reynolds stress dominated field away.
Figure 6-3: Shear stress distribution normalized by the upper wall stress: (a) total shear stress compared with (6.7) (b) \( \frac{1}{\tau_w} \frac{d\tau}{dz} \), \( -\frac{\bar{w}'w'}{\tau_w} \), and \( \frac{\bar{w}'w'}{\tau_w} + \frac{1}{\tau_w} \frac{d\tau}{dz} \) from the interface/wall.

### 6.4.4 Two-Point Correlation Functions

The instantaneous two-point correlation functions are given by

\[
R_{ii}(z, r_i, t) = \frac{\langle u_i(x, y, z, t) u_i(x + r_i y, z, t) \rangle}{\langle u_i(x, y, z, t)^2 \rangle} \quad (6.8a)
\]

\[
R_{ii}(z, r_3, t) = \frac{\langle u_i(x, y, z, t) u_i(x, y + r_2 z, t) \rangle}{\langle u_i(x, y, z, t)^2 \rangle} \quad (6.8b)
\]

for \( i = 1, 2, 3 \) (no summation convention implied). The samplings of the x- and y-correlations are shown in figures 6-4 at three different vertical locations. As expected, for small separation distances, the velocity components exhibit correlation with values close to unity; however, as the separation increases, the correlation decays rapidly towards zero. Additionally, the \( u_1 \)-component remains correlated over larger streamwise separation distances when compared against the vertical and spanwise velocity fluctuations.

The rapid decay of the correlation coefficients signifies that the large scale coherent structures within the flow are captured within the computational domain. Previous simulation attempts carried out on a domain of size \( \frac{1}{2}L_x \times \frac{1}{2}L_y \) produced two-point correlations functions with strong peaks at separation distances corresponding to half
Figure 6-4: Two-point (one time) correlations functions: (——) $R_{uu}$, (-----) $R_{vv}$, (---) $R_{ww}$ as a function of (a,c,e) streamwise ($x$-direction) separation and (b,d,f) spanwise ($y$-direction) separations.
domain lengths implying that the largest vortex structures were being strongly impacted by the domain size and the artificial periodic boundary conditions. Increasing the domain size to its current configuration captured the dominant scales, removed that sub-peak, and resulted in the rapidly decaying two-point correlation coefficients shown in the current studies.

6.4.5 Turbulent Kinetic Energy

The general equation governing the transport of the turbulent kinetic energy, \( k \equiv \frac{1}{2} u_i^2 \), is given by

\[
\frac{\partial k}{\partial t} + \nabla \cdot T = P - \varepsilon
\]

where

\[
T_i = \frac{1}{2} u_i u_j u_j + \frac{\bar{u}_i \bar{p}' / \rho - 2 \nu \bar{u}_i \bar{s}_{ij} }{2}
\]

\[
\varepsilon = \nu \frac{\partial u_i' \partial u_i'}{\partial x_j} \partial x_j
\]

and with \( s_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \) denoting the rate-of-strain tensor for the turbulent velocity fluctuations. For statistically steady flows, eqn. (6.9) reduces to

\[
\frac{\partial u_i' \partial u_i'}{\partial z} - \frac{\partial}{\partial z} \left( \frac{p' w'}{\rho} \right) - \frac{1}{2} \frac{\partial u_i' \partial w_i}{\partial z} + \frac{1}{2} \nu \frac{\partial^2 u_i' \partial u_i'}{\partial x_j \partial x_j} = 0
\]

whereby the turbulent kinetic energy that is produced by the interaction of the Reynolds stress and the mean flow is dissipated by viscous dissipation and transported by the combined effects of the pressure and velocity fluctuations. The balance of these terms in eqn. (6.11) across the channel is shown in figure 6-5.

Clearly, the production and dissipation provide the contributions to the turbulent kinetic energy over the majority of the domain. In the vicinity of the wall, the production decreases and the viscous dissipation becomes balanced by the viscous diffusion. Additionally, the turbulent transport terms become strong around \( z_f \sim \)
Figure 6-5: Balance of the turbulent kinetic energy distribution: (——) dissipation, (- - -) turbulent transport, (- - -) viscous diffusion, (· · ·) Production

10 – 15 where the production is large, indicating that the energy is being advected away from the wall. These findings are in good agreement with the trends identified from other high resolution direct numerical simulation studies of turbulent channel flow (i.e. Moin & Kim [57]).

6.4.6 Comparison Against Naraigh et al. Model For Turbulent Velocity Profile

Section 4.3 utilized turbulent linear stability analysis to determine the bandwidth of linearly unstable modes and identify sub-harmonic resonances. The analysis relied upon linear stability theory which is governed by the Orr-Sommerfeld equations. In that analysis, profiles for $\bar{U}(z)$ and $d^2\bar{U}(z)/dz^2$ are supplied by the model equations developed by Naraigh et al. [60] that implemented a turbulent eddy-viscosity model as closure for the turbulent Reynolds stresses. In this section, the accuracy of that model is examined.

Figure 6-6 shows a comparison between the time-averaged turbulent mean velocity profile computed by direct numerical simulations and the model equation for
the mean velocity profile given by [60, eqn. (14)]. Within the interfacial and wall viscous-sublayers excellent agreement is observed. Similar slopes are observed within the log-layer; however, the maximum velocity of the numerical simulation is found to be approximately 3.36% larger \( \{ U - \bar{U}_{\text{naraigh}} \} = 0.76 \) than the model equation prediction. The source of the discrepancy can be identified through examination of the accuracy of the modeled closure relationship for the turbulent Reynolds stresses by an eddy-viscosity model given by eqn. (8) in Naraigh et al. Figure 6-7 compares the modeled shear Reynolds stress against the results computed through direct numerical simulations. The characteristic behavior of the two solutions are in close agreement; however, the modeled Reynolds stresses have a maximum value which is approximately -9.89% larger than the numerical simulation. This results in a slower, on average, mean velocity profile. The deviation between these two solutions is likely attributed to the moderate friction Reynolds number of the simulation being \( Re = 476.95 \). As the Reynolds number increases (resulting in stronger turbulence) the eddy-viscosity model is likely to exhibit improved performance.

The Orr-Sommerfeld equations given by eqn. (4.6) used in the turbulent linear stability analysis also require an accurate representation of \( d^2 \bar{U}(z)/dz^2 \). Figure 6-8 shows a comparison between the direct simulation result and the profile obtained through
Figure 6-7: Comparison of Reynolds shear stress distributions given by: (---) model equations developed by Naraigh et al. [60, eqn. (8)], (-- ---) direct numerical simulation.

differentiation of the modeled mean velocity profile given by [60, eqn. (14)]. Both curves exhibit similar characteristic behavior. The modeled profile exhibits stronger curvature with the maximum value in the profile being approximately 16.48% larger in magnitude ($U_{\infty} \{ \max |U'_{DNS}(z)| - \max |U''_{Naraigh}(z)| \} = 771$). The computed error in this quantity is likely a combination of the error in the modeling process used by Naraigh et al. and the moderate grid resolution in the interfacial and wall sublayer region of the profile where the curvature is highest. As in the case of the previous quantities, it is expected that the agreement between the two solutions will improve at higher Reynolds numbers with increased grid resolution.

### 6.5 Linear Stability Analysis

In §6.4, a detailed statistical analysis of the turbulent solution was carried out for the purpose of demonstrating its quality as an initial condition. The second stage of numerical tests involve the simulation of the nonlinear evolution of the interface between a turbulent gas blowing over a laminar liquid layer.
Figure 6-8: Comparison of the second derivative of the mean velocity profile given by: (——) model equations developed by Naraigh et al. [60, eqn. (14)], (o) direct numerical simulation.

In order to understand the mechanics of the initial time evolution of the interface, a linear stability analysis is carried out to identify the dispersion relationship for the interfacial waves along with the possible bandwidth and growth rate of any linearly unstable modes. This analysis utilizes the numerical tools developed in §4.1 for linear stability analysis for two-fluid flow. In light of the discrepancies with the modeled mean velocity profile (given by [60, eqn. (14)]) and the DNS solution, this linear stability analysis is carried out using the time-averaged mean velocity profile obtained from the direct numerical simulations described in §6.4.1.

The linear stability analysis was carried out using approximately forty Chebyshev modes to resolve the liquid perturbation while approximately ninety Chebyshev modes were used for the gas layer. To confirm the accuracy of the solution, convergence studies were carried out to ensure the accuracy of the most unstable eigenvalues. Additionally, each eigenvalue-vector pair were back substituted into the eigenvalue problem to confirm that the accuracy of the solution. For each solution, the maximum error in the eigen-solution never exceed $L_{\infty} \{c_j, \phi_j\} = \max \{ |A| \phi_j - c_j \phi_j \} \sim O(10^{-6})$. 

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Figure 6-9: Calculated wave frequency and growth rate spectra where the mean velocity profile was obtained through (——) direct numerical simulation, (— — —) [60, eqn. (14)].

The resulting eigenspectra of the most unstable mode as a function of wavenumber are shown in figure 6-9. Figure 6-9(a) shows how the phase velocity varies as a function of wavenumber while figure 6-9(b) shows the bandwidth of unstable modes. The growth rate spectrum is found to be weakly stable with max \( \omega'_1 \sim O(-0.02) \). As was explained in §4.3, the stability of this case, according to linear theory, is most likely due to small deviations between these numerical flow conditions and the experimental operating conditions. It is unlikely that the mean velocity profile of the gas was in a fully developed state. Similarly, the experimentally measured wave spectra reported large oscillations in the mean liquid depth as a function of fetch. Therefore the numerically determined stability of this case is not surprising.

A linear energy analysis was also carried out to determine this dominant class of physics, viscous dissipation/Reynold/normal/interfacial stress, which governs the behavior of the perturbation. The resulting energetics are documented in table 6.4. In all three cases, the energy attributed to the viscous dissipation and Reynolds stresses are stronger in the gas than in the liquid. For each energy source, the magnitude of the energy in the gas phase is usually at least an order of magnitude larger than within the liquid layer. While the highest destabilizing force is due to the tangential stress, it is overcome by the stabilizing influence of the viscous dissipation within the body of the fluid.
Table 6.4: Perturbation energetics for the most linearly unstable mode wave mode.

<table>
<thead>
<tr>
<th>$k_{peak}$</th>
<th>$\omega_{l,peak}$</th>
<th>DIS(1)</th>
<th>DIS(2)</th>
<th>REY(1)</th>
<th>REY(2)</th>
<th>NORM</th>
<th>TAN</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>-0.020</td>
<td>-6660.79</td>
<td>-689.69</td>
<td>-1828.29</td>
<td>42.15</td>
<td>67.52</td>
<td>8998.96</td>
</tr>
</tbody>
</table>

As an additional comparison, the stability predictions based on the model equation for the mean flow [60, eqn. (14)] are also reported in figure 6-9. While, the statistical analysis of the turbulent gas flow described in §6.4 showed that for lower Reynolds numbers the model equation becomes less accurate, a comparison with the DNS solution can provide insight into the sensitivity of the eigenspectrum. Figure 6-9(a) shows a comparison of the phase velocity computed with the DNS and [60, eqn. (14)] mean flow profiles. There is nearly perfect agreement between the two methods. The differences are only observable in the growth rate spectra shown in figure 6-9(b). For these marginally unstable flow conditions, the growth rate is a higher order quantity which is sensitive to the properties of the mean flow. However, despite the sensitivities, both predictions produce growth rates which are of comparable orders of magnitude and exhibit similar distributions across the spectrum.

6.6 Nonlinear Evolution of a Turbulent Wave Field

The turbulent velocity field developed in §6.3, and statistically analyzed in §6.4, is used as an initial condition to the nonlinear evolving interface problem. For this stage of simulation, fully nonlinear two-phase simulations were carried out with a domain spanning $(x, y, z) \in [0, L_x] \times [0, L_y] \times [-h_2, 1]$ with grid spacing $(\Delta x, \Delta y, \Delta z) = \left( \frac{L_x}{N_x}, \frac{L_y}{N_y}, \frac{1+h_2}{N_z} \right)$ with $(L_x, L_y, h_2)$ being defined in table 6.2. The initial velocity and pressure fields are defined such that the liquid layer is governed by the laminar fields given by eqn. (6.1) while the gas layer is given by the direct numerical simulations’s turbulent velocity and pressure field. Trilinear interpolation was used to transfer the turbulent solution from the single phase domain to this new two-phase domain. The manner in which the initial single phase turbulent gas field was generated ensured that the current interpolated initial condition would reasonably satisfy the proper
velocity and pressure continuities at the interface.

Using these flow conditions, the nonlinear time evolution of the interface was examined. The simulation was carried out with the interfacial position being reconstructed every \( \Delta t = 0.01 \) from the volume fraction distribution with a second order approximation given by \( \eta(x_i, y_j, t) = \sum_k f(x_i, y_j, z_k) \Delta z \). A spectral analysis of the interfacial evolution was carried out through a standard spatial Fourier decomposition method

\[
\eta(x, y, t) = \sum_{-N_x/2+1}^{N_x/2} \sum_{-N_y/2+1}^{N_y/2} \hat{\eta}(k_x, k_y, t) e^{2\pi i(k_x x + k_y y)}
\]

(6.12)

that allows for direct comparisons to be made against the theoretical linear stability analysis carried out in §4.3.

**Direct Numerical Simulation of Turbulent Wave Evolution**

Figure 6-10 shows the initial evolution of the interface under the influence of the turbulent gas blowing over a laminar liquid layer. At \( t = 0 \), the interface is perfectly flat with \( \eta(x, y, t = 0) = 0 \). As the turbulent fluctuations interact with the interface, small amplitude high-wavenumber streamwise and cross-stream perturbations are generated as shown in figure 6-10(a). Figure 6-10(b) shows that as the laminar liquid layer and the turbulent gas flow interact at the interface, long streaks occur on the interface with strong cross stream variations which propagate in the streamwise direction. For \( t \in [0, 10] \) the maximum interfacial elevation never grows larger than \( O(10^{-5}) \). The linear stability analysis assumes that the dominant flow features are two-dimensional. Therefore, analysis of the two-dimensional Fourier interfacial modes, denoted by \( a_j(t) = \hat{\eta}(k_x = j, k_y = 0, t) \), is carried out to confirm the predictions made by the theoretical linear stability analysis. Figure 6-11 shows \( |a_j(t)| \) for the first six two-dimensional Fourier modes. Clearly, none of the modes present with observable long-term growth. This finding is consistent with the stable linear spectrum provided by the theoretical analysis results shown in figure 6-9(b).

Phase resolved modal amplitudes, defined by \( 2\Re\{a_j(t)\} \) and shown in figure 6-12, permit the identification of the dominant interfacial wave frequencies. Clear har-
Figure 6-10: Interfacial elevation at (a) $t = 0.10668$ and (b) $t = 8.6504$. 
Figure 6-11: Time evolution of the first six two-dimensional Fourier interfacial modes obtained from direct numerical simulation of turbulent wave growth.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\omega (k_1)$</th>
<th>$\omega (k_2)$</th>
<th>$\omega (k_3)$</th>
<th>$\omega (k_4)$</th>
<th>$\omega (k_5)$</th>
<th>$\omega (k_6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical</td>
<td>0.6103</td>
<td>1.2323</td>
<td>1.8618</td>
<td>2.5126</td>
<td>3.191</td>
<td>3.899</td>
</tr>
<tr>
<td>Fourier</td>
<td>—</td>
<td>1.211</td>
<td>1.816</td>
<td>1.816</td>
<td>2.421</td>
<td>3.027</td>
</tr>
<tr>
<td>Wave Period</td>
<td>—</td>
<td>1.281</td>
<td>1.654</td>
<td>1.827</td>
<td>2.614</td>
<td>3.075</td>
</tr>
</tbody>
</table>

Table 6.5: Modal frequencies identified through Fourier analysis and estimation half-wave period of $|2\Re \{a_j(t)\}|$ obtained through surface reconstruction from direct numerical simulations of turbulent wave growth.

Monomeric oscillations are observed in the time history permitting a temporal Fourier transform to be calculated to determine the dominant frequency content of each interfacial modes. The relatively short duration of the numerical simulation yields a temporal Fourier transform which exhibits poor resolution among the low frequency modes with a frequency spacing of $\Delta \omega = 0.605$. An alternative estimate of the wave frequency is obtained through identifying the half periods (or local minima) of the signal $|2\Re \{a_j(t)\}|$. Table 6.5 summarizes the comparison of the theoretical linear predictions against the temporal Fourier analysis and half-period wave frequency estimation. Figure 6-12(a) shows that the $a_1$ mode has not yet converged to a regular oscillatory wave behavior. Therefore, an accurate estimation of the wave frequency could not be obtained. For the remaining two-dimensional wave modes, both the Fourier and half-wave period method agree to within $O(10\%)$ for all reported cases.
Figure 6-12: Time evolution of the phase resolved modal amplitudes of the first six two-dimensional Fourier interfacial modes with the wavenumber corresponding to (a) $k = 1$, (b) $k = 2$, (c) $k = 3$, (d) $k = 4$, (e) $k = 5$, (f) $k = 6$. 
Additionally, fair agreement between the theoretical solutions and the direct numerical simulation results is observed. Allowing the simulations to run for a longer time interval should cause the waves to converge to a stable wave frequency and cause both measures to converge.

The experimental observations and the linear stability analysis carried out in §4.3 indicated the presence of a nonlinear subharmonic resonance which satisfies the resonance condition $k_2 = \frac{1}{2}k_1$ and $\omega_2 \approx \frac{1}{2}\omega_1$. Analysis of the wavenumber spectrum and experimental interfacial power spectrum measurements suggest that the a sub-harmonic resonance exists between $\{k_1, k_2\} = \{6, 3\}$. The phase resolved amplitudes of this alleged sub-harmonic pair are shown in figure 6-13. Both wave modes have settled and are undergoing clear harmonic oscillations with a regular wave frequency. Figure 6-14 shows the Fourier frequency spectrum for the sub-harmonic pair. The dominant frequencies are concentrated at $\omega_6 = 3.027$ and $\omega_3 = 1.816$. Due to the the resolution $\Delta \omega$ of the numerical Fourier spectrum, these two modes satisfy the resonance condition to within $|\sigma| = \Delta \omega$. Using the half-period estimate of the wave frequency, it is found that the resonance condition is satisfied to within
Figure 6-14: Temporal Fourier transform of $2\Re \{a_j(t)\}$ for (a) $k = 6$ with $\omega_{\text{peak}} = 3.027$ and (b) $k = 3$ with $\omega_{\text{peak}} = 1.816$.

$|\sigma| = |\frac{1}{2}\omega_6 - \omega_3| = 0.117$. This is in fair agreement with the value of $|\sigma| = 0.0877$ obtained from linear theory. Having $|\sigma| \ll 1$ indicates the presence of a sub-harmonic resonance that is capable of permitting the possibility of strong energy transfer from the $a_6$ mode to the $a_3$ modes. The stability of the growth rate spectrum implies that the $a_6$ mode has a negligible amount of energy to transfer to the $a_3$ mode which makes it difficult to observe and confirm resonant growth. However, if the $k = 6$ mode were to somehow receive energy, the resonance would allow it to directly pass energy across the spectrum to the linearly stable $a_3$ mode. This will be demonstrated in the following section, by using interfacial pressure forcing to supply energy to this $a_6$ mode and observing the growth of the corresponding $a_3$ mode.

**Direct Numerical Simulation of Forced Turbulent Wave Evolution**

While the wave frequencies determined through the direct numerical simulation is in good agreement with the theoretical stability analysis, the overall stability of the interface limits the interfacial growth and trivializes any resulting nonlinear analysis of the interfacial evolution. Ideally, the turbulent mean flow should naturally generate linearly unstable modes; however, section 4.3 showed that the linear growth rate spectrum exhibited a strong sensitivity to parameters such as the gas Reynolds number and the equilibrium liquid depth. A new turbulent simulation could be carried out
where the Reynolds number and/or liquid depth could be varied until an interfacial instability is observed. However, the computational expense associated with the generation of a new turbulent solution makes this solution prohibitively expensive. An alternative approach used in this work is to utilize weak interfacial pressure forcing to destabilize certain wave modes and generate interfacial wave growth.

In this section, the traveling wave forcing described in section 5.6 is implemented in order to generate interfacial waves. Since the initial condition to the simulation is a perfectly flat interface, the slope coherent forcing would be rendered largely ineffective without an excessively large forcing coefficient. The least stable mode, identified as $k = 6$ according to the linear stability analysis, is selected and forced with its linear wave frequency. The interfacial smoother described in §5.6.1 is implemented with $\Delta = \sqrt{\Delta x^2}$. The least stable mode in the growth rate spectrum was found to fall between the $k = [5,6]$ integer wavenumbers. The resonance analysis also identified the $k = 6$ wavenumber as possibly belonging to a strong subharmonic resonance. Therefore, interfacial pressure forcing is applied to the $a_6$-mode. In order to destabilize the interface, weak forcing is applied with forcing coefficient $A_6$ over a forcing time interval $T_f$. In order to demonstrate the robustness of the solution, several simulations are carried out over a range of forcing parameter values.

Figure 6-15 shows the time evolution of the first six two-dimensional Fourier interfacial modes over a range of pressure forcing coefficient values and forcing time intervals. Figure 6-15(a) uses $A_6 = 5$ and $T_f = 1$ while figures 6-15(b) & 6-15(c) use $A_6 = 1$ with $T_f = 1$ and $T_f = 2$ respectively. Clearly, the $k = 6$ mode experiences rapid growth under the influence of the traveling wave forcing. Comparing figures 6-15(a) and 6-15(b) show that increasing the forcing coefficient from $A_6 = 1$ to $A_6 = 5$ causes the maximum wave steepness to increase by a factor of $O(4)$. It is also apparent that the simulations carried out with interfacial pressure forcing present with additional components that also experience rapid growth. Comparing the simulations from figure 6-15 to the unforced case shown in figure 6-11 shows that the modal evolution of the $k \in [1,5]$ is due to the interfacial forcing since none of the unforced modal amplitudes reached amplitudes larger $O(10^{-5})$. Experience with
Figure 6-15: Time evolution of the two-dimensional modal amplitudes under the influence of traveling wave interfacial pressure forcing of the $k = 6$ mode.
interfacial pressure forcing has shown that using the traveling wave pressure forcing scheme with a forcing frequency that does not closely satisfy the flow's dispersion relationship can result in the generation of interfacial noise. The numerical simulations carried documented in table 6.5 indicated that there is a small amount of error between the theoretical linear frequency and the natural frequency of the system. Additionally, the frequency spectrum reported by Jurman et al.[40] indicated that as the wave amplitude increased, the frequency of the fundamental mode increased possibly indicating a nonlinear correction to the dispersion relationship. Such a frequency shift is not accounted for in the traveling wave forcing scheme. As a result, the generation of disturbance other than the \( a_6 \)-mode is expected. After the forcing is removed, the subsequent interfacial dynamics will help discern what behavior is due to forcing versus nonlinear resonant interactions.

For the \( A_6 = 5 \) case, the interfacial forcing is applied over the interval \( t \in [0, \pi] \). After the forcing is removed, the \( k = 6 \) mode continues to grow until it reaches a maximum wave steepness of \( \max \{ \epsilon \} = 0.0123 \). After the forcing is removed, the \( a_3 \) mode also experiences exponential growth. It initially stops growing prior to the bounding of the \( a_6 \)-mode. As the simulation continues, the \( a_6 \)-mode begins to slowly decrease in amplitude while the \( a_3 \) mode begins to gradually increase until eventually becomes larger than the \( a_6 \) mode. Two possible mechanisms may be at play. The \( a_6 \) mode could be decreasing due to the stability of the interface \( \omega(k = 6) < 0 \); however, the linear stability analysis predicted the decay rate to be \( O(0.01) \) that would suggest that over \( t \in [\pi, 14] \) the amplitude should decrease by \( O(10\%) \). The \( a_6 \) time history, shown in figure 6-15(a), decreases by nearly an order of magnitude during that time. Such a rapid energy exchange is more consistent with the coupled instability-resonant interaction mechanism since the \( a_3 \) mode has also grown by an order of magnitude during that time. This trend suggests that after the forcing is removed at \( t = \pi \), the \( a_6 \) mode no longer received the destabilizing energy and reached a limiting wave steepness. The presence of the sub-harmonic resonance allowed for the energy supplied to the \( a_6 \) mode to be transferred to sub-harmonic \( a_3 \) mode resulting in its rapid growth. Without a continued supply of energy, the siphoning of energy from the \( a_6 \)
mode by the $a_3$ mode causes the amplitude of the $a_6$ mode to decrease permitting the continued growth of $a_3$. The other forcing trials, shown in figures 6-15(b) & 6-15(c), support this conclusion with the only significant difference being the maximum wave steepness, $c_6 \equiv \max \{a_6(t)\}/\lambda$, of the $a_6$-mode being $c_6 (A_6 = 1, T_f = 1) = 0.0032$, $c_6 (A_6 = 1, T_f = 2) = 0.0026$ as compared to $c_6 (A_6 = 5, T_f = 1) = 0.0123$.

Figure 6-16 shows the reconstructed surface elevation at two different times. At $t = 5$, figure 6-16(a) shows that initially the interface is dominated by the $a_6$ mode. The forced wave perturbation spans the entire width, though cross-stream perturbations introduce some non-uniformity across the span of the domain. At a later time, $t = 14$, the results indicated in figure 6-15(a) show that the dominant two-dimensional mode corresponds to the $a_3$ subharmonic mode. Figure 6-16(b) confirms this by showing three clear wave crests propagating in the streamwise direction. As expected, throughout the simulation the dominant interfacial features are two-dimensional waves profiles.

As in the unforced case, a temporal Fourier analysis of the first six two-dimensional spatial Fourier modes can be carried out to determine how the interfacial forcing modifies the temporal frequencies and determine the level of agreement with the theoretical linear stability analysis. Table 6.6 shows the results of the frequency analysis calculated through Fourier analysis and by estimation from the half-periods of $|2R \{a_j(t)\}|$. Like the unforced case, presented in §6.6, it was found that for these forcing trials the $k = 1, 2$ modes had not yet developed into regular oscillatory

| $\omega (k_1)$ | $0.6103$ | $-$ | $-$ | $-$ | $-$ |
| $\omega (k_2)$ | $1.2323$ | $1.211 (1.281)$ | $-$ | $-$ | $-$ |
| $\omega (k_3)$ | $1.8618$ | $1.816 (1.654)$ | $1.817 (1.946)$ | $1.817 (1.965)$ | $2.124 (2.167)$ |
| $\omega (k_4)$ | $2.5126$ | $1.816 (1.827)$ | $2.428 (2.601)$ | $2.423 (2.561)$ | $2.731 (3.035)$ |
| $\omega (k_5)$ | $3.191$ | $2.421 (2.614)$ | $3.029 (3.105)$ | $3.332 (3.514)$ | $3.338 (3.450)$ |
| $\omega (k_6)$ | $3.899$ | $3.027 (3.075)$ | $3.635 (3.735)$ | $3.937 (3.782)$ | $3.945 (3.809)$ |

Table 6.6: Modal frequencies identified through Fourier analysis and ( - ) estimation half-wave period of $|2R \{a_j(t)\}|$ obtained through surface reconstruction from direct numerical simulations of forced wave growth
Figure 6-16: $f = 0.5$ isocontour of the volume fraction distribution across the channel at (a) $t = 5$ and (b) $t = 14$. 
Table 6.7: Satisfaction of the sub-harmonic resonance condition from the detuning parameter, $\sigma = \left| \frac{1}{2} \omega_6 - \omega_3 \right|$, calculated from the Fourier frequency and $(- - -)$ half period frequency estimations.

<table>
<thead>
<tr>
<th>$\frac{1}{2} \omega_6 - \omega_3$</th>
<th>Theory</th>
<th>$A = 0 \ T_f = 0$</th>
<th>$A = 1 \ T_f = 1$</th>
<th>$A = 1 \ T_f = 2$</th>
<th>$A = 5 \ T_f = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.088</td>
<td>0.308</td>
<td>0.088</td>
<td>5E-4 (0.079)</td>
<td>0.152 (0.074)</td>
<td>0.152 (0.263)</td>
</tr>
</tbody>
</table>

behavior. The high wave periods require longer simulation times in order for their behavior to converge to that of a regular wave. Therefore, for these modes, the Fourier transform did not show an obvious dominant spectral component. For the remaining $k \in [3,6]$. Both frequency estimation methods were found to be in close agreement with each other. The table clearly shows that modal frequency has a weak dependence on the forcing coefficient; however, within the simulated parameter space, the simulation results never depart significantly from the linear stability theory. For the case with $A_6 = 1$, the frequency analysis appears to be in closest agreement with the theoretical predictions.

Using the estimated wave frequencies from the surface reconstruction, a search was carried out to identify the alleged two-dimensional sub-harmonic resonance between the $k = 3$ and $k = 6$ wave mode that was observed in the experimental measurements. Using the different wave frequency estimates, obtained from the various forced wave growth trials, the resonance condition defined by $k_6 = k_3$ and $\sigma = \left| \frac{1}{2} \omega_6 - \omega_3 \right|$ was estimated. Table 6.7 shows that all of the weakly forced trials exhibit sub-harmonic resonant pairs which satisfy $|\sigma| < 1$ with the $A_6 = 1$ demonstrating the strongest resonant interaction. The coarseness of the $\Delta \omega$ used in the Fourier transform and the simple averaging used by the half-period method are capable of introducing a moderate amount of error in the values of $\sigma$; however, in each case, at least one of the frequency estimation methods yields an estimate of $|\sigma| < O(0.1)$.

Examination of the frequency spectra, shown in figure 6-17, of the modal amplitudes further supports the presence of interfacial subharmonic resonance. To simplify direct comparisons against the experimental measurements reported in figure 7 of Jurman et al.[40], the numerical frequency spectrum have been reported in dimensional units. The dominant energy of the $a_6$ mode is concentrated at the $f \approx 10 \ Hz$ fre-
quency while the \( a_3 \) mode is focused around \( f \approx 5 \) Hz. The energy content of the fundamental \( a_6 \) mode is just slightly less than twice that of the subharmonic \( a_3 \) mode. These findings are in nearly perfect agreement with the experimental findings. The level of agreement between the numerical simulations and the experimental measurements confirm the quality of the simulations and lend strong support to the observations and theories proposed in this chapter.

### 6.7 Development and Distribution of Interfacial Stress

From the direct numerical simulations, it is possible to calculate the detailed fluid stress components anywhere within either fluid domain. It is known that the fluid stress tensor for a Newtonian fluid is defined as

\[
\sigma_{ij} = -p\delta_{ij} + \frac{\mu(f)}{Re} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]  

(6.13)
where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

By examining a differential tetrahedron shown in figure 6-18 a local stress traction \( \vec{T} \) can be determined and used to calculate the normal and tangential interfacial stresses. Denoting \( \vec{F}_b \) as a general body force per unit mass, the summation of forces in the \( x_i \)-direction be be expressed as

$$\vec{F}_b^{(i)} \left( \frac{1}{3} \rho \Delta h \Delta S \right) + T_n^{(i)} \Delta S - \sigma_{i1} (n_1 \Delta S) - \sigma_{i2} (n_2 \Delta S) - \sigma_{i3} (n_3 \Delta S) = \left( \frac{1}{3} \rho \Delta h \Delta S \right) a_i$$

(6.14)

where \( a_i \) is the component of acceleration in the \( x_i \)-direction and \( n_i \) are the components of the interfacial unit vector \( \hat{n} \) which points out of the liquid layer. In the limit where \( (\Delta S, \Delta h) \rightarrow 0 \), the stress balance reduces to

$$T^{(i)} = \sigma_{i1} n_1 + \sigma_{i2} n_2 + \sigma_{i3} n_3.$$  

(6.15)

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The stress acting normal to the interface is obtained from the stress traction as
\[ \sigma_N = \mathbf{n} \cdot \mathbf{T} \] while the two components of the interfacial shear stresses are defined as
\[ \tau_i^{(1)} = \hat{i}_1 \cdot \mathbf{T} \] and \[ \tau_i^{(2)} = \hat{i}_2 \cdot \mathbf{T} \] where \( \hat{i}_1 \) and \( \hat{i}_2 \) are the unit vectors tangential to the interface in the \( xz \)- and \( yz \)-directions respectively. The interfacial direction vectors are defined as
\[
\hat{i}_1 = \frac{(1, 0, \eta_x)}{\sqrt{1 + \eta_x^2}}, \quad \hat{i}_2 = \frac{(0, 1, \eta_y)}{\sqrt{1 + \eta_y^2}}.
\] (6.16)

These direction vectors were numerically determined by calculating \( \mathbf{n} \) from the VOF normal vector routines and by calculating \( \eta_x \) and \( \eta_y \) from central differences of the quantity \( 1 - f(\mathbf{x}) \).

### 6.7.1 Turbulent Shear Stresses on a Flat Interface

Within slug transition models and slug simulators, the complicated interfacial and wall stresses are often approximated through the use of simple empirical models. While this study examines the complex, nonlinear evolution of the interfacial stresses, it is useful to begin the examination by seeing how well those reduced order models agree with the initial condition.

At \( t = 0 \), the interface is flat and the initial interfacial shear stress is dictated by the initial turbulent velocity profile. The interfacial and wall friction coefficients defined as \( C_{f_i/w} = \tau_{i/w} / \left( \frac{1}{2} \mu C U^2_{\text{mean}} \right) \) are found to take on values of \( C_{f_i} = 7.6205 \times 10^{-3} \) and \( C_{f_w} = -7.7033 \times 10^{-3} \) respectively. Over years of research in this field, a number of empirical correlations have been proposed for the friction coefficients. One of the simplest single phase methods, developed by Dean [18] for two-dimensional rectangular duct flow, estimates the skin friction as \( C_f = 0.073 Re_{\text{g,mean}}^{-0.25} = 8.5191 \times 10^{-3} \). Comparing this model against the direct simulation finds that an -9.59\% and -10.59\% percent error in the interfacial and wall skin friction coefficients respectively. This margin of error is reasonable given that the Dean skin friction coefficient is only valid for \( Re_{\text{g,mean}} \in [6.0 \times 10^3, 6.0 \times 10^5] \) that is slightly above the numerical simulations mean gas Reynolds number of \( Re_{\text{g,mean}} = 5391.5 \). Even the high resolution simu-
lations carried out by Kim, Moin, & Moser [41] found an error in the skin friction coefficient of -3.18%.

Therefore, the margin of error in the Dean model for this case seems reasonable.

Within the field of two-phase flows, more complicated empirical models have been proposed. The Taitel & Dukler slug transition criterion method [73] assumed the gas and liquid friction coefficients were of the form

$$C_{f,\text{liquid}} = C_L \left( \frac{D_L \bar{U}_L}{\nu_L} \right)^{-n} \quad C_{f,\text{gas}} = C_G \left( \frac{D_G \bar{U}_G}{\nu_G} \right)^{-m} \quad (6.17)$$

with $D_L$ and $D_G$ being the hydraulic diameters evaluated by

$$D_L = \frac{4A_L}{S_L} \quad D_G = \frac{4A_G}{S_G + S_i}$$

and where for a turbulent flow $C_G = C_L = 0.046$, $n = m = 0.2$ while for a laminar flow $C_G = C_L = 16$, $n = m = 1.0$. The pipe/channel geometry is accounted for with the cross-sectional area $A_{l/g}$ and wetted perimeters $S_{l/g}$ of each phase. Another common method was utilized by the Lin & Hanratty [43] model assumed that the interfacial and wall stresses coefficients could be approximated by Blasius equation for a smooth surface

$$C_{f,w/i} = 0.0665Re_G^{-0.25} \quad (6.18)$$

With these definitions of the friction coefficients, the shear stresses were then evaluated as

$$\tau_{w,l} = f_l \frac{\rho_I \bar{U}_l^2}{2} \quad \tau_{w,G} = f_G \frac{\rho_G \bar{U}_G^2}{2} \quad \tau_{w,G} = f_G \frac{\rho_G (\bar{U}_G - U_{int})^2}{2} \quad (6.19)$$

with $(\ldots)$ denoting the average value.

Table 6.8 compares the accuracy of these empirical models of the interfacial and wall friction coefficients to those obtained through direct numerical simulations. The percentage error term is reported first for the wall friction coefficient and then is indicated by the term in parenthesis for the interfacial friction coefficient. The percent-
age difference indicators demonstrate that all three of the methods exhibit adequate agreement with the numerical simulations; however, the two-phase flow models produce predictions with significantly smaller error than the simple single-phase model proposed by Dean. The Blasius equation used in the Lin & Hanratty model shows exceptional agreement with both the interfacial and wall friction coefficients agreeing to within less than once percent.

### 6.7.2 Turbulent Shear Stresses on an Evolving Interface

It was shown in the previous section that simple empirical models were capable of accurately representing the interfacial and wall stress coefficients of the initial time- (and horizontally-) averaged turbulent velocity field. However, it is unknown how well these model predictions will agree with the more complex case of a freely evolving linear or nonlinear interface.

The time and spatial evolution of the turbulent interfacial stress are depicted at different instances of time in figures 6-19(a), 6-20(a), and 6-21(a). By comparing figures 6-19(a), 6-20(a), and 6-21(a), it is apparent that the initial interfacial stress at \( t = 0 \) is dominated by the tangential shear stress along the \( x-z \) plane, denoted by \( \tau_{i}^{(1)} \). The initial stress distribution is generally uniform with localized stress concentrations generated by the turbulent velocity fluctuations in the interfacial boundary layer. The second interfacial shear stress component and the normal shear stress only exhibit a weak long wavelength streamwise fluctuation against its mean value.

As time progresses, the interface becomes deformed due to the interfacial pressure forcing of the \( k = 6 \) wavenumber. The pressure forcing scheme requires that an

<table>
<thead>
<tr>
<th></th>
<th>Dean</th>
<th>TD</th>
<th>LH</th>
<th>%{Dean}</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( C_f )</td>
<td>0.00852</td>
<td>0.00751</td>
<td>0.00776</td>
<td>-10.59(-9.59)</td>
<td>2.52(3.40)</td>
<td>-0.74(0.17)</td>
</tr>
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</table>

Table 6.8: Comparisons of the friction coefficients \( C_f \) computed for the Dean, Taitel & Dukler (TD), and Lin & Hanratty (LH) models. The percentage difference for the wall and \((-\rightarrow)\) interfacial friction coefficients being defined with the reference coefficients obtained from direct numerical simulations having values of \( C_{f,w}^{(DNS)} = 0.00770 \) and \( C_{f,s}^{(DNS)} = 0.00777 \).
interfacial smoother be used in order to reconstruct an interfacial delta function for
the identification of the portion of the domain for which the pressure forcing stress
should be applied. As a consequence of the smoothing, both the density and viscosity
fields become distorted over an additional grid cell on either side of the interface.
This smoothing, combined with high shear rates in the turbulent gas phase, allow for
a small amount of liquid to be torn from the wave crests and become dispersed within
the gas phase. As a result, stress concentrations are identified in the vicinity of the
wave crests and are of magnitudes that are significantly larger than anywhere else
within the remainder of the flow domain. The color bounds of the isocontour were
adjusted such that the high-intensity stress concentrations were removed showing the
remaining stress distributions over the wave surface. As a result, these figures may be
used to observed regions with strong stress fluctuations and identify possible coupling
with the interfacial elevation or wave slope.

Figures 6-19(b), 6-20(b), & 6-21(b) shows the resulting stress distributions at \( t = 5 \)
after the interfacial forcing has stopped. The normal interfacial stress maintains a
near uniform stress across the interface with the peak stress intensities occurring
near the wave crests. Both of the shear stress components exhibit strong fluctuations
across the domain with strong stress concentrations occurring at the wave crests due
to entrainment events and in the troughs due to the interaction of the interface with
turbulent fluctuations.

As the simulation is allowed to evolve and transition from the fundamental \( k = 6 \)
mode to a state dominated by the subharmonic \( k = 3 \) mode, the interfacial stresses
under go a regime change shown in figures 6-19(c), 6-20(c), & 6-21(c). The normal
stress remains nearly uniform across the domain with the dominant stresses being
concentrated at the wave crest of dominant the \( a_3 \) mode. The tangential shear stress
also show strong adjustmen due to the transition to the subharmonic mode. The \( \tau_i^{(1)} \)
stress exhibits strong fluctuations which which are predominantly in the wave troughs
while the \( \tau_i^{(2)} \) stress are very small throughout the domain except in the vicinity of
the wave crests at the drop entrainment sites.

Observing the stress distributions across the wave surface demonstrate that the in-
terfacial stresses experience strong cross-stream variation due to the interaction of the wave surface and the turbulent fluctuations. The stress distributions redistribute in accordance with the nonlinear evolution of the interface and remain strongly coupled to both the location of drop dispersal sites and dominant interfacial wave mode.

Due to the drop dispersal site’s association with strong stress concentrations, it was difficult to clearly observe the relationship between the interfacial stresses and the wave profile. To make any possible coupling more apparent, cross-correlation functions and coefficients were calculated

\[
\text{corr} \{f, g\} (\xi, y, t) = \frac{\int f (x, y, t) g (x + \xi, y, t) \, dx}{\text{mean} \{f\} \text{mean} \{g\}}.
\]  

This function was further simplified by averaging the cross-correlation across the span of the domain in the y-direction. The value associated with \(\xi = 0\) denotes the correlation coefficient between the two functions

Figure 6-22 examines the correlation between the \(\tau_1^{(1)}\) and either the interface profile or wave slope given by \(\eta_x (x, y, t)\). The y-averaged cross-correlation function and coefficients between the \(\tau_1^{(1)}\) and the interface profile are shown in figures 6-22(a) and 6-22(b). During the initial evolution, for which the pressure forcing is active with \(t \in [0, \pi]\), there is weak negative correlation between the interface profile and the interfacial shear stress. The y-averaged cross-correlation function shows a distinct phase of transition between \(t \in [\pi, 5]\). Examination of figure 6-15(a) shows that this time interval corresponds to the transition period after the pressure forcing is removed and extends until the \(a_6\) mode reaches its maximum value. During this time interval, the correlation coefficient shows that the shear stress and interface profile are nearly uncorrelated. After the \(a_6\) mode becomes bounded at \(t \approx 5\), figure 6-15(a) demonstrates that this is the time interval corresponding to strong subharmonic resonant interactions. The cross-correlation coefficient shows that during this time interval, the shear stress remains weakly correlated to the fundamental \(a_6\) mode with \(\text{corr} \left\{ \tau_1^{(1)}, \eta \right\} \sim O(0.1)\). At \(t \approx 10\), the subharmonic \(a_3\) mode becomes larger than the fundamental mode and the triggers a transition phase where the shear
Figure 6-19: Time evolution of the normal interfacial stress plotted over the \( f=0.5 \) isocontour of the volume fraction distribution across the channel plotted at (a) \( t=0 \), (b) \( t=5 \), and (c) 14.
Figure 6-20: Time evolution of the $\tau^{(1)}_t$ interfacial shear stress plotted over the $f=0.5$ isocontour of the volume fraction distribution across the channel plotted at (a) $t=0$, (b) $t=5$, and (c) 14.
Figure 6-21: Time evolution of the $\tau_{i}^{(2)}$ interfacial shear stress plotted over the $f=0.5$ isocontour of the volume fraction distribution across the channel plotted at (a) $t=0$, (b) $t=5$, and (c) 14.
Figure 6-22: Identification of interfacial shear stress, $\tau_{i1}$, coupling with (a,b) interfacial elevation $\eta(x,y,t)$ and (c,d) wave slope $\eta_{,x}(x,y,t)$. Cross-correlation functions shown in (a,c) and correlation coefficients shown (b,c).
stress redistributes and concentrates across the dominant $a_3$ mode. Similar regime transitions are observed between the $\tau_i^{(1)}$ stress component and the interfacial wave slope as shown in figures 6-22(c) & 6-22(d). Despite similar regime transitions being observable in the cross-correlation function, consistently smaller correlation value are reported signifying that the interfacial shear stress is more strongly associated with the interfacial elevation than the wave slope.

It is expected that the correlation values are small due to the strong isolated stress concentrations at the entrainment sights. It is speculated that if the simulation were to be re-run with the entrainment events being mitigated, similar trends in the cross-correlations would be observed but with higher absolute correlation values being observed. Several attempts were made to better condition the interfacial shear stress field, but the noise due to the entrainment remained significant and dominant. Through more advanced post-processing and conditional averaging of the interfacial stresses, it may be possible to remove these large problematic droplet induced stress concentrations and gain a clear understanding of the stress-interface correlations behavior.

Figure 6-23 examines the result of a similar study between the interface profile and wave slope with the normal interfacial stress. Figure 6-23(b) shows that during the initial stage of the simulation from $t \in [0, \pi]$ during which the interfacial pressure forcing is active, strong (negative) correlation is seen between the normal stress and the interface profile. This is expected given that the interfacial pressure forcing scheme is a concentrated interfacial stress which is responsible for the generation of the interfacial disturbance. As the forcing is removed, the correlation quickly changes from strong negative correlation to moderate positive correlation with $\text{corr} \{P_{normal}, \eta\} \sim O(0.5)$. Unlike the interfacial shear stress, there is no apparent transition region during which the $a_6$ mode becomes bounded. Instead, for the remainder of the simulation, the normal stress remains moderately correlated, with $\text{corr} \{P_{normal}, \eta\} \sim O(0.3 - 0.4)$, increasing slowly with time. Examining the relationship between the normal stress and the wave slope shows during the time interval with active interfacial pressure forcing, the correlation is weak with $|\text{corr} \{P_{normal}, \eta\}| \sim O(0.1 - 0.2)$. Once the
Figure 6-23: Identification of normal interfacial stress, $P_{\text{normal}}$, coupling with (a,b) interfacial elevation $\eta(x,y,t)$ and (c,d) wave slope $\eta_x(x,y,t)$. Cross-correlation functions shown in (a,c) and correlation coefficients shown (b,c).
forcing is removed, the normal stress and the wave slope become uncorrelated. It is not until the sub-harmonic mode become the dominant interface feature that the normal stress begins to become increasingly (though still weakly) correlated with the wave slope.

This analysis of interfacial stress demonstrated two important findings. First, the initial (time and spatially averaged) turbulent shear stresses may approximated accurately by simple empirical models. Several models were compared against direct numerical simulations all were found to make predictions to within less than ten percent error. The model formulated off of the Blasius equation for a smooth surface was found to show particularly excellent agreement with a percentage error well below one percent.

However, when the interface is allowed to deform under the influence of the ambient gas-turbulence, the resulting interfacial stress distributions were found to differ significantly from the empirical models. Traditional slug transition analysis carried out in the literature utilize simple interfacial shear stress equations, given by eqn. (6.19). Within the context of linear stability analysis, these model equations are simply modulated by a traveling wave solution corresponding to the most unstable wavenumber and frequency. Such a solution assumes that the stress is uniform in the cross-stream direction. The stress analysis presented in this section demonstrated that strong spatial-temporal cross-stream fluctuations in the shear and normal stress components exist. Additionally, the nonlinear interfacial evolution resulted in the significant redistribution of stress from the fundamental \((a_0)\) mode the the subharmonic \((a_3)\) mode. During this transition process, cross-correlation analysis presented the change in the coupling between the interfacial stresses and wave profile. In order to better improve the slug transition predictions, significantly more advanced interfacial friction models are required which account for strong cross-stream fluctuations and the nonlinear interfacial evolution.
6.8 Conclusions

This chapter carried out detailed direct numerical simulations of turbulent two-phase flows through horizontal channels capturing the fully nonlinear interfacial evolution. In order to accomplish this task, a numerical procedure was developed for which the initial turbulent gas phase was generated as a pressure driven Couette problem (with proper interfacial boundary conditions that match the laminar liquid layer) and then re-initialized as a two-phase problem with an evolving interface. In doing so, the turbulent gas was found to be in excellent agreement with theoretical asymptotic solutions (and various statistical methods) while the liquid layer remained laminar and flatness of the interface was preserved.

With this initial condition, fully nonlinear simulations of the interfacial evolution were carried out allowing for the initial instability and resulting resonant wave-wave interactions to be examined in detail. The wave frequencies were estimated through a Fourier analysis and half-period estimation scheme. When compared to linear theory, good agreement was observed. Further comparisons were made against experimental power spectrum measurements and excellent agreement was observed. The theoretical analysis and numerical simulations predicted, and confirmed, the existence of strong nonlinear sub-harmonic resonant interactions that transfer the linearly unstable energy to stable long-wave components.

Additional studies were carried out to examine the distribution and evolution of the turbulent interfacial stresses. For the initial case where the interface remains flat, detailed comparisons were made against simple empirical models showing that simple correlations were capable of describing the initial state of interfacial stress. As the interface is allowed to evolve under the influence of the ambient flow and gas turbulence, the interfacial stress quickly departs from the empirical models. Examination of the time evolution of the interfacial stress components demonstrated streamwise and cross-stream variation due to strong turbulent fluctuations. During the initial interfacial evolution, stress fluctuations remained strongly coupled to the dominant fundamental mode resulting in high wavenumber fluctuations. Strong nonlinear in-
terfacial resonances caused the eventual transition and redistribution of stresses to the sub-harmonic mode. Cross-correlation analysis demonstrated strong coupling between the normal stresses and the interfacial profile. The interfacial shear stresses underwent several regime transitions based on if the fundamental or sub-harmonic mode was the dominant interfacial mode which resulted in weaker correlation between the interfacial elevation and wave slope. Strong regime dependence in the solution was identified from the cross-correlation analysis, but liquid entrainment in the gas was accompanied by intense stress concentrations. Numerically mitigating the drop dispersal is expected to increase the stress correlations.

In section 6.7, the interfacial stress distribution over the evolving interface was calculated; however, the results were contaminated by large shear localized shear stresses associated with droplet formation. It was proposed that this droplet formation was due to the presence of interfacial smoothing. Another issue may be associated with the interfacial boundary layer not being sufficiently resolved in these high Reynolds number VOF simulations. For flows with large density jumps across the interface, a numerical breakdown may occur due to the jump that occurs in the tangential velocity across the interface. In VOF simulations, where this jump occurs right at the interface, physically unrealistic tearing may occur which would normally be prevented by the viscous boundary layer transition across the interface. Filtering, such as the scheme developed by Fu et al.[25], could be applied to reduce the jump in tangential velocity across the interface. While the work by Fu et al. demonstrated the effectiveness of the method, it is an expensive solution because it requires solving an extra variable density Poisson problem. Such a solution to the droplet formation problem may be worth further study.

To the author's knowledge, this investigation is among the first works to implement a second-order finite volume scheme with a volume-of-fluid interface tracking scheme for the purpose of examining the interfacial dynamics of turbulent two-phase flows. Traditionally, numerical investigations involving turbulent flow utilize higher order methods. In this work, it was found that, with a sufficiently high resolution, it was possible to demonstrate excellent agreement between the numerical simulation and
asymptotic theory. While it was relatively easy to converge the wave frequencies, it was found that high grid resolutions was necessary to recover accurate interfacial growth rates.
Chapter 7

Nonlinear Slug Transition Criterion

Chapters 2 and 3 demonstrated, through the use of potential flow, that a powerful nonlinear mechanism exists that is capable of coupling an interfacial instability with nonlinear resonances. It was shown that this process extracts energy from the interfacial instability and, through the resonant interactions, transfers the energy to linearly stable modes resulting in the rapid growth of long wavelength waves. Despite being formulated under ideal fluid assumptions, chapter 6 carried out direct numerical simulations of turbulent two-phase flows and show that is possible for this coupled instability-resonant mechanism to persist in more complex flows while still demonstrating the same efficient energy transfer process. Within this chapter, these concepts are generalized into a condition that is capable of predicting the onset of slugging in horizontal channels and pipes.

7.1 Introduction

Many industrial applications involve the transport of multiphase flows through pipes. For instance, the design and operation of oil transport and production facilities rely heavily on understanding the hydrodynamics of multiphase flows. Predicating the flow regime based off of the geometry, fluid properties, and flow conditions has proven
to be a significant technical challenge and remains an active area of research. One particularly complex flow regime is slug flow in which, for a horizontal pipe, waves grow on the interface between the two fluids until they bridge the pipe diameter trapping large gas bubbles within the liquid. The presence of slugs often results in a significant change in the pressure drop across the pipe reducing the overall transport efficiency of the system. The design of processing equipment relies heavily on understanding the expected level of slugging which may occur over the lifetime of the facility. This phenomena has been well studied experimentally, but a rigorous theoretical understanding of the underlying physics is still lacking due to the number of physical mechanisms which are at play.

Experiments have shown that there are a number of different physical processes that can result in the transition from a stratified to slug flow. For small superficial gas velocities (typically \( u_{SG} \leq 3 \text{m/s} \)), the experiments conducted by Fan et al. [24] found that initially, a linear instability would excite the growth of short waves on the interface.

As the waves evolved, there was an energy transfer from the unstable short waves resulting in the growth of large amplitude long waves that may eventually form a slug (assuming that the liquid depth was sufficiently deep). As the superficial gas velocity increases, experiments suggest that the slug formation is due to wave coalescence. Additional mechanisms exist which may lead to slug formation, such as pipe inclination; however, this paper will focus on horizontally oriented pipes and channels.

Early theoretical work on the subject of slug prediction was based on the classical Kelvin-Helmholtz (KH) instability theory for the prediction of the critical gas velocity for which long wavelength disturbances become unstable and presumably grow into slugs. While inviscid KH-theory by itself generally over predicts the critical velocity of the gas (compared to experiments) numerous works have attempted to improve the transition criteria by supplementing the theory with additional physics such as interfacial and wall friction models (Lin & Hanratty [43] and Barnea & Taitel [5]), normal viscous stresses at the interface (Funada & Joseph [27]) just to name a few.
The survey paper by Mata et al [52] demonstrated that the methods produce a wide range of predictions of the onset of slug formation.

One commonality of these methods is that they assume that the large amplitude long waves seen in experiments can be predicted by a linear analysis. However, numerous experiments, such as those by Fan et al. [24], demonstrated that (for a wide range of flow conditions) the modes which are linearly unstable are significantly shorter than the waves that transition into slugs. A bifurcation process is responsible for transferring the energy from short waves to the long waves. Such a process cannot be predicted by any linear stability theory and requires the use of nonlinear predictive analysis. Therefore, a new slug transition criterion is needed that incorporates the ideas from linear stability theory with the physics of the nonlinear wave interactions.

Phillip’s [65] developed analytic methods for examining the evolution of weak nonlinear wave-wave interactions in an ocean wave field. This work demonstrated that resonant interactions are responsible for transferring significant amounts of energy across the wavenumber spectrum and that analysis provided significant insight into the mechanisms for the nonlinear evolution of surface gravity-waves. The initial investigations in resonant interaction theory utilized perturbation expansions to determine the modal evolution, growth rates, and energy transfer among the resonantly interacting (linearly stable) waves. It was found that these modes can experience rapid energy exchange between wave modes; however, the energy of the system is bounded by the initial conditions.

The fact that for stable resonant triads the amplitude of the resonant modes is bounded by the initial conditions makes it an unlikely method for generating large amplitude waves that may then transition into slugs. However, the work from chapter 2 considered a modification to the traditional resonant interaction theory. That work analytically examined a resonant triad of interfacial modes, but allowed the smallest wavelength mode to be weakly unstable to the Kelvin-Helmholtz (KH) mechanism. Initially, the unstable mode followed tradition KH theory, but as the wave became steeper, the resonant coupling permitted the transfer of the energy, provided by the KH instability, to the other longer linearly stable wave modes. This energy cascade
resulted in rapid wave growth that could never be achieved by stable triad resonances. Additionally, under certain flow conditions, the long wave modes were found to grow bi-exponentially fast. They then carried out broadbanded numerical simulations which showed that such interactions could generate large amplitude disturbances that could grow until they touch the top of the channel. Similar observations were made for the case of linearly unstable subharmonic resonant interactions[12]. Such a mechanism seems consistent with the experimental observations seen in the low superficial velocity cases reported by Fan et al.[24].

In this chapter, a novel slug transition criterion is proposed which incorporates the coupled effects of a linear interfacial instability with nonlinear resonant interaction theory. The canonical problem of two-fluids flowing through a horizontally oriented two-dimensional channel is considered. In §7.2, an Orr-Sommerfeld analysis is carried out and an asymptotic solution for the wave speed and growth rate is obtained. This dispersion relationship is then used to determine the possible location of any resonant triads. In §7.4, a transition criterion is developed that seeks a critical gas velocity that minimizes the number of resonant modes that are linearly unstable. In §7.5, this new transition criterion is validated against experiments carried out in horizontal square channels. In order to make this method applicable to the more common pipe flow configurations, a heuristic method is proposed which finds the “equivalent” channel flow problem for which the theoretical solution is valid. Comparisons are made against experiments and consistently good agreement is observed. While this method may not capable of predicting all cases of slug formation, such as through wave coalescence events, it among the first methods incorporate the experimentally observed wave-wave interactions into slug transition predictions.

7.2 Viscous Linear Stability Analysis

This analysis carries out a viscous linear stability analysis for the case of waves propagating at the interface of a stratified two-phase flow through a horizontal channel. It is of interest to develop asymptotic solutions to the governing equations in order to
understand the physics of the unstable waves and properties of the dispersion relationship. The asymptotic approximations will prove to be valuable in significantly reducing the computational cost in §7.4 by not requiring a full numerical Orr-Sommerfeld eigen-solver be developed and implemented into the slug transition method.

### 7.2.1 Linearized Governing Equations

The experiments carried out by Cohen & Hanratty [15] examined the mechanism responsible for the initial generation of waves on the interface of a gas-liquid flow through a horizontal channel. Their work found that the wave motion of the liquid received its energy from pressure and shear stress variations at the interface resulting from fluctuations in the gas flow caused by the waves. The pressure variations transmit energy to the liquid through liquid velocity components in the vertical direction, while shear stress fluctuations transmit energy through liquid velocity components in the horizontal direction. Their work carried out a linear stability analysis for a liquid film that produced an Orr-Sommerfeld equation with modeled pressure and shear stress terms having the form

\[ p = \hat{P} e^{i(\alpha x + \beta z - \omega t)} \]  
\[ T = \hat{T} e^{i(\alpha x + \beta z - \omega t)} \]  

that are responsible for representing the energy transfer from the gas to the lower liquid. The coefficients are allowed to be complex having the form \( \hat{P} = P_R + i P_I \) and \( \hat{T} = T_R + i T_I \). From this work, they were able to derive an asymptotic approximation of the dispersion relationship for the linear phase velocities and growth rates. A summary of their analysis shall be repeated for clarity and to unambiguously introduce the nomenclature used in this work.

A fixed Cartesian coordinate system is established with the origin located at the undisturbed interface between the two fluids with the x-axis extending horizontally to the right and the y-axis being directed vertically upwards. The flow is decomposed into a steady-state mean solution \( U(y) \) and a fluctuating velocity field denoted
by \( \bar{u}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)) \) along with a similar decomposition for the steady-state mean pressure solution \( P(y) \) and fluctuating pressure field \( p(x, y, z, t) \). Substitution of the velocity and pressure decompositions into the Navier-Stokes equations allows for linearized momentum equations to be written as

\[
\begin{align*}
\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} + \frac{1}{Re} \nabla^2 u \quad (7.2a) \\
\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial y} + \frac{1}{Re} \nabla^2 v \quad (7.2b) \\
\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} &= -\frac{\partial p}{\partial z} + \frac{1}{Re} \nabla^2 w \quad (7.2c)
\end{align*}
\]

along with the continuity equation for the velocity perturbation

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (7.3)
\]

where the \((\cdot)'\) denotes differentiation with respect to \( y \), and \( Re = U_0 h_l / \nu_l \). This problem is scaled by the interface velocity \( (U_0) \), the liquid's depth \( (h_l) \) and viscosity \( (\nu_l) \). Substituting a normal mode solution of the form

\[
\{u, v, w, p\} = \{\hat{u}(y), \hat{v}(y), \hat{w}(y), \hat{p}(y)\} e^{i(\alpha x + \beta z - \omega t)}
\]

into eqns. (7.2 & 7.3) yields:

\[
\begin{align*}
\left[ D^2 - (\alpha^2 + \beta^2) - i\alpha Re (U - c) \right] \hat{u} &= i\alpha Re \hat{p} + Re \hat{v}U' \quad (7.4a) \\
\left[ D^2 - (\alpha^2 + \beta^2) - i\alpha Re (U - c) \right] \hat{v} &= i\alpha Re \hat{p} \quad (7.4b) \\
\left[ D^2 - (\alpha^2 + \beta^2) - i\alpha Re (U - c) \right] \hat{w} &= i\beta Re \hat{p} \quad (7.4c) \\
i\alpha \hat{u} + D \hat{v} + i\beta \hat{w} &= 0 \quad (7.4d)
\end{align*}
\]

where \( c \) is the complex wave speed and \( \alpha, \beta \) are the streamwise and spanwise wavenumbers. Taking the divergence of the linearized momentum equations and using the
continuity equation yields the governing equation for the pressure disturbance

$$\nabla^2 p = -2U' \frac{\partial v}{\partial x}$$  \hspace{1cm} (7.5)

Taking the divergence of (7.2b) and utilizing (7.5) results in an expression for the vertical velocity

$$\left[ \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - U'' \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^4 \right] v = 0$$ \hspace{1cm} (7.6)

In order to completely describe the three-dimensional flow, a second condition is needed which incorporates the $u$- and $w$-velocity components. By taking the curl of the momentum equations, the normal vorticity component, $\zeta_y = \frac{\partial w}{\partial x} - \frac{\partial v}{\partial z}$, is found to be governed by

$$\left[ \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^2 \right] \zeta_y = -U' \frac{\partial v}{\partial z}$$ \hspace{1cm} (7.7)

Substituting a normal mode solution into eqns. (7.6 & 7.7) yields

$$\left[ i\alpha (U - c) \left( D^2 - (\alpha^2 + \beta^2) \right) - i\alpha U'' - \frac{1}{Re} \left( D^2 - (\alpha^2 + \beta^2) \right)^2 \right] \hat{v} = 0$$ \hspace{1cm} (7.8a)

$$\left[ i\alpha (U - c) - \frac{1}{Re} \left( D^2 - (\alpha^2 + \beta^2) \right) \right] \hat{\zeta} = -i\beta U' \hat{v}$$ \hspace{1cm} (7.8b)

### 7.2.2 Two-Dimensional Disturbances

It was demonstrated by Squire [71] that for channel flow of a uniform fluid between rigid boundaries, and later by Yih [83] for stratified flows, that it is sufficient to consider two-dimensional disturbances. Therefore, this study will utilize the two-dimensional form of (7.8a) which reduces to

$$\left[ ik (U - c) \left( D^2 - k^2 \right) - ikU'' - \frac{1}{Re} \left( D^2 - k^2 \right)^2 \right] \hat{v} = 0.$$ \hspace{1cm} (7.9)

where $k$ is the two-dimensional wavenumber which results from Squire’s transformation. This problem is made complete with the specification of the boundary condi-
tions. At the lower walls, the flow must satisfy no-slip and no-flux conditions
\[ \hat{u} = \hat{v} = 0 \quad (y = -1) \] (7.10)

For waves of small steepness, the kinematic boundary condition is written as
\[ \frac{\partial \eta}{\partial t} + U_0 \frac{\partial \eta}{\partial x} = v, \quad (y = 0) \] (7.11)

which upon substitution of a normal mode solution \( \eta = \hat{\eta} e^{i(k(x-ct))} \) into (7.11) produces
\[ \hat{\eta} = \frac{i \hat{v}_0}{k(c-1)} \] (7.12)

where \( \hat{v}_0 = \hat{\nu} (y = 0) \) and \( U_0 = 1 \) due to the scaling of the problem by the interfacial velocity. The remaining boundary conditions are derived from the continuity of normal and shear stresses at the interface. In general, the shear stress balance is satisfied on the free surface which can be denoted by an \( x' - y' \) coordinate system where \( x' \)-coordinate is measured along the wave surface and \( y' \)-is measured normal to the wave surface. The linearization allows the stress balance to occur along the original undisturbed interface denoted by the \( x - y \) coordinate system. Since the shear stress fluctuations in the gas are denoted by the modal equation (7.1b), special treatment of the governing equations is needed. Defining the angle between the \( x \)-axis and the \( x' \)-axis as the angle \( \theta \), the small angle approximation, which is consistent with the linearization of the free surface, allows \( \theta \approx \frac{d\eta}{dx} \). The shear stress on the deformed surface \( (x' - y' \) coordinate system) can be projected onto the \( x - y \) coordinate system yielding

\[ \tau_{x'y'}^{(g)} = \tau_{x'y'} \cos (\theta) \]
\[ \tau_{y'y'}^{(g)} = \tau_{x'y'} \sin (\theta) + \tau_{x'x'} \sin (\theta). \]
which upon applying small angle approximation allows for the linear fluctuation terms to be denoted as

\[ \tau_{xy}^{(s)} \approx \hat{T} \quad (7.14a) \]
\[ \tau_{yy}^{(s)} \approx 2\hat{T} \frac{d\eta}{dx} \quad (7.14b) \]

where \( \hat{T} = \left. \frac{1}{Re} \frac{dU}{dy} \right|_{y=0} \). With these approximations, the shear stress in the liquid is balanced by the wave induced shear stress variation in the gas phase and is denoted by

\[ \tau_{yy,\text{liquid}} = \frac{1}{Re} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] y=0 = \hat{T} \text{e}^{ik(x-ct)} = \tau_{yy,\text{gas}} \quad (7.15) \]

Upon substitution of the normal modes decomposition of the velocity field along with (7.4d) and (7.12), the shear stress balance becomes

\[ D^2 \hat{v} + k^2 \hat{v} - \frac{Re \hat{v}}{\eta (c-1)} = 0 \quad (y = 0) \quad (7.16) \]

where \( D \equiv \frac{d}{dy} \). The final condition enforces the continuity of normal stress at the interface

\[ \tau_{yy,\text{gas}} - \tau_{yy,\text{liquid}} = -\frac{1}{W} \frac{d^2}{dy} \quad (7.17) \]

where \( W = \frac{\sigma}{\rho U_{\infty}} \) denotes the Weber number with \( \sigma \) being the value of the surface tension. The normal stress in the liquid is represented by \( \tau_{yy,\text{liquid}} = -p + Fr^{-2} \eta + \frac{2}{Re} \frac{\partial \eta}{\partial y} \), while in the gas, \( \tau_{yy,\text{gas}} = -p + 2\hat{T} \frac{d\eta}{dx} \) where the pressure is given by (7.1a).

The liquid’s pressure term \( p \) in \( \tau_{yy} \) is determined from the linearized horizontal momentum equation

\[ -p = -\frac{1}{k^2 Re} \left( D^2 \hat{v} - k^2 D \hat{v} \right) - \frac{1}{ik} (1 - c) D \hat{v} + \frac{1}{ik} U_{\infty} \hat{v}. \quad (7.18) \]
With this representation of $p$, along with (7.12), the normal stress condition is written as

$$
- \frac{2k^2 \bar{v} \bar{T} \text{Re}}{c - 1} + (D^3 - 3k^2 D) \bar{v} + ik \text{Re} [(c - 1) \bar{v}' + U_0 \bar{v}] =
$$

$$
+ ik \text{Re} \left( F r^{-2} + \frac{k^2}{\mathcal{W}} + \frac{\hat{P}}{\hat{\eta}} \right) \frac{\bar{v}}{(c - 1)} \quad (y = 0) \quad (7.19)
$$

Cohen & Hanratty [15] carried out an asymptotic analysis, summarized in $C$, which produced leading order estimates of the linear phase velocity

$$
c_r = 1 - \frac{U_0}{2k} \tanh (k) \pm \left[ \left( \frac{U_0}{2k} \tanh (k) \right)^2 + \tanh (k) \left( \frac{k}{\mathcal{W}} + \frac{1}{k F r^{-2}} + \frac{\hat{P}}{k \hat{\eta}} \right) \right]^{1/2} \quad (7.20)
$$

and growth rate

$$
\omega_i = k c_i = \frac{\frac{\hat{P}}{\hat{\eta}} + \frac{\hat{T}}{\hat{\eta}}} {2 (c_r - 1) \coth (k) + \frac{U_0}{k}} \left( \coth (k) + \frac{U_0}{k (c_r - 1)} \right) - \frac{4 k U_0'}{\text{Re}} - \frac{4 k^2 (c_r - 1) \coth (k)}{\text{Re}} - \frac{k^{5/2} (c_r - 1)^2 (\coth^2 (k) - 1)}{\sqrt{2 c_r \text{Re}}}
$$

$$
\quad 2 (c_r - 1) \coth (k) + \frac{U_0}{k} \quad (7.21)
$$

### 7.2.3 Models for Gas Pressure Effects

The derivation carried out in section 7.2.2 incorporated the effect of the gas phase through the simple models for the gas shear and pressure terms given by eqn. (7.1); however, the form of these functions has yet to be specified. In this section, several special cases are examined. The first case examines a models for the gas pressure for which the Kelvin-Helmholtz dispersion relationship (for stable waves) can be recovered from the asymptotic solutions. This provides a simple validation of the dispersion relationship derived in §7.2.2. For the second case, a model for the pressure is described which incorporates the classic Jeffreys' wave sheltering mechanism.
Recovery of Kelvin-Helmholtz solution

Consider the case of an ideal fluid, for which $\nu_l = 0$ and $\hat{T} = 0$, in a uniform current ($U_g$ in the gas and $U_l = U_0 = 1$ in the liquid) where the derivative of the velocity profile at the interface is zero, $U'_l = 0$. Under these conditions, if the a model pressure term is denoted by

$$\frac{\hat{P}_{r}}{\eta} = -rk \coth (kh_g) (U_g - c)^2 - \frac{r}{Fr^2} \quad \hat{P}_{l} = 0 \quad (7.22)$$

is substituted into eqn. (7.20), the Kelvin-Helmholtz dispersion relationship for the wave speed, $c_{KH}$, is obtained

$$c_{KH} = \frac{(rU_gT_l + U_lT_g)}{rT_l + T_g} \pm \left[ \frac{T_gT_l}{k(rT_l + T_g)} \left( 1 - \frac{r}{Fr^2} + \frac{k^2}{W} \right) - \frac{r(U_g - U_l)^2}{(rT_l + T_g)^2} \right]^{1/2} \quad (7.23)$$

where $T_g/l = \tanh (kh_g/l)$ and $r = \rho_g/\rho_l$. This example provides a simple check of the validity of the asymptotic dispersion relationship.

Incorporation of Jeffreys’ wave sheltering mechanism

In 1925, Jeffreys [38] suggested that the separation of the gas flow over the top of wave crests causes a pressure variation that is in phase with the wave slope. This can be incorporated into the surface pressure through the model equation

$$\hat{P}_{r} = 0 \quad \frac{\hat{P}_{l}}{\eta} = skr (U_g - c)^2 \quad (7.24)$$

where $s$ is empirically determined sheltering coefficient. From the problem of wind generated waves in deep water, Jeffreys proposed the value of $s = 0.3$. Even in the absence of gas separation over the top of the wave crests, this type of slope coherent model is appropriate. The work by Miles & Benjamin solved the Orr-Sommerfeld equation, along with an assumed turbulent velocity profile, and derived the solution for the pressure and shear stress variations over a wavy surface. In the limit of high Reynolds number (and assuming that there is one way coupling between the
fluctuations and the turbulent gas velocity profile) it was found that in the viscous layer, phase shifts between the velocity and pressure fluctuations produce a pressure component at the interface that is in-phase with the wave slope. In such a case, eqn. (7.24) would still be an appropriate approximation.

The use of this Jeffreys model in eqns. (7.20 & 7.21) can be used to predict a critical gas velocity for which two-dimensional waves can form at the interface. Using the value of $s = 0.3$, the experiments carried out by Cohen & Hanratty [15] found that this model accurately predicts the formation of small amplitude two-dimensional waves on the interface.

### 7.3 Resonant Interaction Theory

From linear stability theory, waves of different wave length (or frequency) travel independently in space/time. By accounting for the nonlinear wave-wave interactions between the different wave modes, it is possible for new locked waves to be generated. These locked waves are generally higher-order compared to the primary waves; however, if the frequency and wavenumber of the locked wave satisfy the dispersion relationship, given by eqn. (7.20), the locked wave becomes a free wave. In this case, it is possible for the free wave to grow significantly with its amplitude becoming comparable to that of the original primary waves. When a locked wave satisfies the dispersion relationship, it is said to be in a resonant interaction with the primary waves. Such resonant interactions are well documented and have been shown to play a significant role in the evolution of ocean surface waves resulting in strong energy transfer across the wave spectrum.

Traditional resonant interaction theory is derived under the assumption that all of the resonant wave modes are linearly stable; however, the modal behavior changes significantly when under the influence of a linear instability. To examine the coupled influence of resonant interactions and a linear instability Campbell & Liu [11, 12] examined the behavior of both triadic and subharmonic resonances. That work utilized both an analytical and numerical methods to quantify nonlinear modal evolution. It
was found that the nonlinear coupling between a linear instability and wave-wave resonances can significantly increase the rate of energy transfer between the interacting modes. It was also possible to see energy transferred across the spectrum from the unstable wave modes to the modes that were predicted to be linearly stable. With this class of nonlinear mode coupling, bi-exponential growth of linearly stable modes was documented. Furthermore, the energy provided to the resonant modes by the linear instability allows the amplitudes to grow significantly larger than in the linearly stable resonant triad cases.

Within the context of this paper, we shall consider all possible near-resonant triads in which the wavenumbers and frequencies of the interacting waves satisfy the resonance condition

\[
\begin{aligned}
\begin{cases}
    k_2 - k_1 = k_3 \\
    \omega_2 - \omega_1 = \omega_3 + \sigma
\end{cases}
\end{aligned}
\]  

(7.25)

where \(|\sigma| \ll 1\) represents the frequency detuning parameter. Note that the case of a perfect resonance is obtained when \(\sigma = 0\). The special case of the second harmonic (or subharmonic) resonance examined by Campbell & Liu [12] are also captured by this condition in the limit that \(k_1 \to k_3\) and \(\omega_1 \to \omega_3\).

### 7.4 Nonlinear Transition Criterion

In this section, a nonlinear slug transition criteria which couples the effects of the linear instability derived in §7.2 and the nonlinear resonant interaction theory described in §7.3. To justify our new transition mechanism, several general observations are made. First, many slug transition criteria are formulated off of some form of the Kelvin-Helmholtz mechanism. Many of them incorporate a range of physics such as interfacial and wall stress models, surface tension, gravity along with other heuristic corrections which are believed to account for the role of nonlinearity. With these models, a critical velocity is sought that produces linearly unstable long wave modes. This class of linear model inherently assumes that the only way for a long wave mode
to become unstable is through the inherent linear instability. By failing to account for the contributions from resonant interaction theory, many powerful energy transfer processes are neglected.

Second, resonant interaction theory provides an efficient mechanism for the transfer of energy across the spectrum. When the modes are linearly stable, modal energy exchange occurs but the extent to which the modes can grow is bounded by the initial energy. When some of the resonant modes fall in the linearly unstable portion of the spectrum, the rate of energy transfer and the maximum modal amplitudes increase dramatically.

Therefore, we propose a novel slug transition criteria which couples the effects of a linear instability and resonant interaction theory. In general, it is proposed that in order to minimize the amount of energy available among the long wave modes, the number of modal components which are members of resonant triads which fall in the linearly unstable part of the spectrum must be minimized.

For a given set of flow conditions, the frequency and growth rate spectrum can be determined from eqns. (7.20 & 7.21). It is possible for this growth rate curve to contain several neutral points, defined by $\omega_i = 0$; therefore, in this work we defined the neutral point with the highest wavenumber as $k_N$ as shown in figure 7-2. Using the frequency spectrum, a grid search may be carried out to identify all of the wave modes that satisfy the general triad resonance condition given by eqn. (7.25) to within a threshold $|\sigma| < \sigma_{tol}$. Since according to the resonance condition, $k_2$ is the shortest wave component, all of the triads are sorted by $k_2$. Examination of the distribution of the $k_2$ modes shows that it is possible for there to be several different bands of bunched $k_2$-modes. One set, defined as $\mathcal{K}_s$, contains the band of small wavenumbers in the vicinity of $k = 0$. The next set, defined as $\mathcal{K}_l$, contains the large wavenumber modes. Given the definition of these two sets, we define $\kappa \equiv \min_{k_2 \geq k_N} \{k_2 \in \mathcal{K}_l\}$.

Using this nomenclature, a search for the critical gas velocity (above which slugging may occur) can be carried out. Initial work showed that the search for the critical gas velocity may exhibit a number of possible interactions between the linearly unstable modes and the resonant modes. The different classes of interactions are shown.
in figure 7-1. Since this transition criterion is focused on the transfer of energy from linearly unstable short waves to linearly stable long waves, only the band of short \( (k_2) \) waves are examined. The typical location of the \( k_2 \)-wave bands are denoted in black in figure 7-1 while the growth rate curve is plotted in orange.

Starting the search from a sufficiently small gas velocity, it is expected that the growth rate spectrum is linearly stable as shown in figure 7-1(a). As the gas velocity is increased an unstable mode may be identified. Experiments have suggested that this velocity is usually too small to initiate slugging, so the search is allowed to continue on to higher gas velocities. As the velocity is increased, a finite band of wavenumbers becomes linearly unstable as shown in figure 7-1(b). If the unstable modes are identified as \( (k_1, k_3) \)-modes or are part of the \( K_s \) set, then the gas velocity is allowed to increase further. Instability of the \( (k_1, k_3) \)-modes will result in their growth and resonant exchange of energy to the linearly stable higher wavenumber components in the \( K_l \) set. Propagation of energy in this direction toward higher wavenumber modes may result in short wave growth which becomes bounded by surface tension, nonlinearity\[61, 67\], or result in energy losses due to dissipation and/or wave breaking among short waves. While it is possible that some of the energy may be transferred to the short waves only to be backscattered back to the long wave components, it is expected that this process will experience energy loss due to the aforementioned damping/bounding mechanisms. Given this logic, the gas velocity is allowed to increase until the gas velocity satisfies the condition \( U_c = U(k_N = \kappa) \). Allowing the gas velocity to increase above this condition, shown in figure 7-1(c), permits members of the \( K_l \) set to become linearly unstable. As a consequence, the resonant coupling permits that linearly unstable energy to be passed directly to the long wave components resulting in rapid wave growth. If members of the \( K_s \) set are already unstable, this resonant energy cascading from the \( K_l \) set will only enhances their existing instability among the long wave modes. This is why the intent of this new transition criterion is to act as a bounding energy condition for the gas velocity and only limit the number of resonant wave modes which can receive energy from the linear instability. A description of how to implement this criterion is summarized in
Figure 7-1: Demonstration of the different resonance-instability combinations that may be observed in the search for the critical gas velocity. Case (a) shows a gas velocity which produces a linearly stable spectrum, (b) shows a narrow band of linearly unstable wave modes which do not intersect the $K_s$ set, and (c) shows a high gas velocity which has excited linearly unstable $k_2$-modes in the $K_s$ set which will result in the transfer of energy directly to linearly stable long waves.

the Nonlinear Slug Transition Criterion Algorithm.

This condition may easily be summarized pictorially. If the growth rates curve $\omega_l(k)$ is plotted while the $k_2$ modes are indicated along the horizontal axis, the critical velocity is identified when the neutral mode $(k_N)$ intersects $\kappa$. Figure 7-2 shows three gas velocities which correspond to subcritical (too slow), critical (correct), and supercritical (too fast) conditions. Occasionally, small bands of $k_2$ modes appear and judgement must be used to determine if they belong to the $K_s$ or $K_l$ set. Experience has suggested, when in doubt, include those small bands in the $K_s$ set and error towards a slightly larger gas velocity. If these small bands are assigned to the wrong set, then the calculated critical velocity usually stands out as an outlier on the final transition curve. Figure 7-2(b) shows such a $k_2$-band and it has been assigned to the $K_l$ set.

Algorithm Nonlinear Slug Transition Criterion

**Input:** $r, Fr, Re, \mathcal{W}, h_g, s, U_0, U_{g,\text{guess}}, k_{\text{min}}, k_{\text{max}},$ and $k_{\text{inc}}$

1. $U_g \leftarrow U_{g,\text{guess}}$
2. $K = k_{\text{min}} : k_{\text{inc}} : k_{\text{max}}$
3. **while** found = 0
4. **do** Calculate $\omega_1 = \omega(k_1), \omega_2 = \omega(k_2),$ and $\omega_3 = \omega(k_3)$ using eqn. (7.20)
5. Identify all triads for which $\sigma \equiv \omega_3 - (\omega_2 - \omega_1) < \sigma_{\text{tol}}$
6. Of the triads which satisfy $tol$-condition, $\kappa \equiv \min_{k_2 > k_N} \{ k_2 \in K_l \}$
Figure 7-2: Demonstration of the selection of the critical velocity. The (———) denotes the growth rate curve $\omega_i(k)$ and the $\times$’s denote the location of the discrete $k_2$-modes which are involved in a resonant triad where $|\sigma| < tol = 10^{-3}$. Case (a) shows a subcritical $U_s < U_{crit}$ where $\kappa > k_N$, (b) shows the critical velocity $U_s = U_{crit}$ where $\kappa \equiv k_N$, and (c) shows a supercritical $U_s > U_{crit}$ where $\kappa < N$

7. Calculate $\omega_i(K)$ from eqn. (7.21)
8. Set $k_N \equiv \max_k \omega_i(k) > 0$
9. if $|\kappa - k_N| \leq k_{inc}$
10. then
11. found=1
12. else
13. Increase $U_g$
14. return $U_{crit} = U_g$

When numerically implementing this algorithm, the resonant detuning parameter needed to be specified by the user. The value of the detuning tolerance needed to be sufficiently small so as to determine the location of the strongest resonant triads within the system; however, permitting a value that was too large would allow nearly all wavenumbers to participate in a weak near-resonant interaction. Validation tests against experimental measurements showed that the value of $\sigma_{tol} = 10^{-3}$ was sufficiently robust. To confirm that this parameter would not drive the solution, tests were carried out which examined the sensitivity of the value $\kappa$ to the specified value of $\sigma_{tol}$. In one such test, the Nonlinear Slug Transition Criterion Algorithm was using the values $\sigma_{tol} = \{0.05, 0.01, 0.005, 0.001\}$ and the resulting values of $\kappa$ at the critical velocity were determined. For $\sigma_{tol} = 0.05$, $\kappa \equiv 0$ which showed that nearly every mode in the wavenumber spectra was found to be involved in a weak triad resonance.
For the remaining values $\sigma_{tol} \sim (0.01 - 0.001)$, the value of $\kappa$ changed by less than one percent showing that within that range, the prediction was invariant of the particular value of $\sigma_{tol}$.

The algorithm also requires a range of wavenumbers to be specified so that the growth rate curves and resonant triads can be identified. Typically, the wavenumber range was selected to be $k \in [0, 25]$ with $k_{inc} = 0.01$; however, for some of the pipe flow tests shown in §7.5.3 the wavenumbers needed to go as high as $k_{max} = 50$. The value of $k_{max}$ is not important as long as the location of $\kappa$ is well established. Additionally, specifying a value of $k_{inc}$ which is too large reduces the number resonant triads and may cause a shift in $\kappa$. Making the value of $k_{inc}$ too small may significantly slow down the search algorithm.

The gas pressure and shear stress models $(\dot{P}, \dot{T})$ also require closure models. Cohen & Hanratty suggested that because of the small viscosity of air, the shear stress variations are generally much smaller than the pressure variations, given by eqn. (7.1a). Therefore in this work, the shear stress variations are neglected, $\dot{T} = 0$, and reasonable agreement is observed in the comparisons against experiments. The derivation included this effect such that for liquid-liquid systems or higher viscosity gases, a future model can be added to account for these effects. In this work, the pressure term, given by eqn. (7.1a) included a real part corresponding to the Kelvin-Helmholtz model described in §7.2.3 and an imaginary part corresponding to the Jeffreys' sheltering model described in §7.2.3.

The last set of terms that require closure are the $U_0$ and $\dot{U}_0'$. Given that most experimental measurements report either the volumetric flow rate in the liquid or the superficial liquid velocity, closing these terms is not exact. Two methods were proposed and tested. In the first test, the experimental value was used to determine a measure of the average velocity. Then, assuming a two-phase laminar Poiseuille velocity profile, the average velocity profile can be used to determine an applied pressure gradient. Knowing an estimate of the pressure gradient allows a theoretical velocity profile to be constructed which then supplies the values of $U_0$ and $\dot{U}_0'$ analytically. The second approach again uses the experimental value to determine a measure of the
average velocity; however, this time $U_0 = U_{L,\text{mean}}$ and $\bar{U}_0' \approx 0$. For a turbulent gas flowing over a slower moving liquid layer, this may be a realistic approximation for the liquid layer solution. If a better model is available, they can easily be substituted to provide an improved estimate of $U_0$ and $\bar{U}_0'$. It should be noted in most experiments, the value of $Re$ is large. Additionally, the $\bar{U}_0'$ term appears in the dispersion relationship, eqn. (7.20), and the growth rate expression eqn. (7.21). Generally, in the liquid layer, the value of $\bar{U}_0$ is less than one. In eqn. (7.20), $\bar{U}_0'$ is normalized by the wavenumber which in the vicinity of $k$ is $O(10)$ while in eqn. (7.21) it is normalized by either the wavenumber or the $Re \gg 1$. Therefore, $\bar{U}_0'$ usually has a second order effect on the dispersion relationship allowing for these simple models to be sufficiently accurate for this problem. The differences in the Nonlinear Slug Transition Criterion Algorithm predictions are most extreme as the liquid layer becomes thin. A comparison of these two methods is shown in §7.5.

7.5 Evaluation of Nonlinear Transition Criteria Against Experimental Measurement

In order to demonstrate that our proposed Nonlinear Slug Transition Criterion Algorithm is effective, several comparisons are made against experimental measurements. In §7.5.1, experimental measurements carried out in a square channel are considered and comparisons of our prediction of the critical gas velocity as a function of gas void fraction are reported. Good agreement between the experiments and our algorithm is observed. Since circular pipe geometries are more common in industrial applications, in §7.5.2, a simple heuristic method is proposed which make our slug transition criterions applicable for horizontally oriented pipes. Section 7.5.3 shows our theoretical predictions compared with experimental measurements and once again, good agreement is demonstrated.
7.5.1 Comparison With Wallis & Dobson Square Channel Experiments

In this section, our Nonlinear Slug Transition Criterion Algorithm is used to make theoretical predictions on the critical gas velocity for flows through horizontal channels. We use the experimental measurement carried out by Wallis & Dobson [78]. Comparing against these square channel tests is desirable because this geometry is most consistent with our two-dimensional theoretical analysis.

The first validation tests corresponds to the measurement reported in their Figure 8 [78, (pp. 181)]. For this test, air and water ($\rho_a = 1.23 \ \frac{kg}{m^3}$, $\rho_t = 999 \ \frac{kg}{m^3}$, $\mu_a = 1.79 \times 10^{-5} \ \frac{N\cdot m^2}{m^3}$, $\mu_t = 1.12 \times 10^{-3} \ \frac{N\cdot m^2}{m^3}$, and a surface tension coefficient of $\sigma = 0.07$ $N/m$) flow through a 5 ft long one-inch square channel. The liquid volumetric flow rate was $Q_L = (1.51 - 1.81) \times 10^{-4} \ \frac{m^3}{s}$ which corresponds to the “+” data marker in their figure. Measurements of the gas velocity responsible for the onset of slugging was reported as a function of the gas void fraction $\alpha \equiv h_a/H$ (where $h_a$ is the equilibrium gas depth and $H$ is the channel depth). Therefore, for this test (and subsequent tests), the experimentally measured void fraction is supplied as input along with the fluid properties ($\rho_a, \rho_t, \mu_t, U_l, W, Fr, etc$) to the Nonlinear Slug Transition Criterion Algorithm to determine the critical gas velocity.

The theoretical predictions for this test case are shown in Figure 7-3. The plotted results show the dependence of the dimensionless volumetric flux $j_G^* = j_G \sqrt{\frac{\rho_g}{gH(\rho_l-\rho_g)}}$ on the void fraction ($\alpha$) where $j_G = Q_G/H^2$ and $Q_G$ is the volumetric flow rate. There is good agreement between our nonlinear theoretical prediction and the experimentally measured values. For comparison, the transition criteria developed by Taitel & Dukler[73], $j^* = \alpha^{5/4}$, and Lin & Hanratty[43, eqn. (49)] (described in section 1.2) are displayed in Figure 7-3. There is a fair bit of disagreement between the experimental values and the Taitel & Dukler model; however, the agreement improves as the liquid layer becomes thinner. The Lin & Hanratty models shows similar trends to that of the Taitel & Dukler model; however the values of $j^*$ remain significantly smaller than the experimentally reported values. For these tests, the Nonlinear Slug Transition
Figure 7-3: Theoretical prediction of the critical gas velocity corresponding to the $Q_L = (1.51 - 1.81) \times 10^{-4}$ experimental conditions from Figure 8 of Wallis & Dobson [78, (pp. 181)]. The (●) denotes the experimentally measured value while the (----) represents the predictions of Nonlinear Slug Transition Criterion Algorithm, (-----) denotes the Taitel & Dukler [73] condition $j^* = \alpha^{5/4}$, and (---) represents the viscous Kelvin-Helmholtz theory from Lin & Hanratty[43, eqn. (49)].

Criterion Algorithm utilized a value of $s = 0.03$ for the Jeffreys’ sheltering coefficient. It is an order of magnitude smaller than the value that Jeffrey proposed for deepwater ocean waves; however, these predictions have shown that it is a robust value for air-water flows through horizontal channels. Therefore, this forcing coefficient value ($s = 0.03$) will be used for the remaining tests in this section.

In §7.4, the closure models used for $\bar{U}_0$ and $\bar{U}'_0$ were discussed. Two methods were proposed. In the first, the experimentally measured values of the volumetric flow rate was used to calculate and average liquid velocity. This average liquid velocity was then used to determine an average pressure gradient and in turn, a two-phase laminar Poiseille velocity from which the solution would provide $\bar{U}_0$ and $\bar{U}'_0$ analytically. The second was to use the high Reynolds number and large value $\kappa$ to neglect $\bar{U}'_0 \cong 0$ and assume $\bar{U}_0 \cong U_{l,mean}$. Scaling estimates were made which suggested that the solution would be relatively insensitive to small variations in these parameters. Figure 7-4 shows the transition curves calculated from each of these two models. For $\alpha \in [0, 0.4]$, the models predictions are virtually identical. As the void fraction
Figure 7-4: Comparison of the sensitivity of the theoretical predictions using the Nonlinear Slug Transition Criterion Algorithm to the different models for the $\bar{U}_0$ and $\bar{U}_0'$. The (●) denotes the experimentally measured values from Figure 8 of Wallis & Dobson [78, (pp. 181)]. The the $\bar{U}_0$ and $\bar{U}_0'$ is defined by (—) laminar theory and (— — —) $\bar{U}_0 = Q_L/H^2$ and $\bar{U}_0' = 0$.

continues to increase, the two models begin to diverge from each other, but good agreement between the experimental measurements and the theoretical predictions are still observed. For the remaining results in this paper, the first method was used.

7.5.2 A Heuristic Condition For Accounting For Pipe Geometries

While good results are observed between our Nonlinear Slug Transition Criterion Algorithm and experiments in horizontal channels, industrial applications require that the model be applicable to two-phase flows through pipes. In this section, we propose a heuristic approach takes the pipe diameter, liquid hold-up, and superficial velocities and then transforms them to an “equivalent” channel flow case. Using this method, accurate slug transition predictions may be made.

For a given experiment, the typical measured quantities at the superficial gas and liquid velocities in the pipe which will be denoted as $U_{SG}^{(p)}$ and $U_{SL}^{(p)}$ with equilibrium gas and liquid depths of $h_{SG}^{(p)}$ and $h_{SL}^{(p)}$ respectively. An equivalent horizontal square
channel of height \( H \) with void fraction \( \alpha \equiv \frac{h_G}{h} \) is sought with mean gas and liquid velocities denoted as \( U_G^{(c)} \) and \( U_L^{(c)} \). In order to find the equivalent flow conditions for a pipe and square channel, two conditions are proposed:

1. The flux of the liquid and gas through a pipe must be equal to the flux of liquid and gas through a channel.

2. The cross sectional area of the liquid and gas in a pipe must be equal to the cross sectional area of the liquid and gas in a channel.

These conditions are not unique and lack support from a first principals based derivation; however, these conditions seem reasonable and will be shown to yield good qualitative comparisons against experimental measurements. Condition (1) can be expressed as

\[
Q_G \equiv U_G^{(c)} A_G^{(c)} = U_{SG}^{(c)} A^{(c)} = U_{SG}^{(p)} A^{(p)} \tag{7.26a}
\]

\[
Q_L \equiv U_L^{(c)} A_L^{(c)} = U_{SL}^{(c)} A^{(c)} = U_{SL}^{(p)} A^{(p)} \tag{7.26b}
\]

Additionally, condition (2) is expressed as

\[
A_G^{(p)} = A_G^{(c)} = H^2 \alpha \tag{7.27a}
\]

\[
A_L^{(p)} = A_L^{(c)} = H^2 (1 - \alpha) \tag{7.27b}
\]

Solving (7.27) produces the geometric conditions on the channel depth and void fraction

\[
\alpha = \frac{A_G^{(p)}}{A_G^{(p)} + A_L^{(p)}} \tag{7.28a}
\]

\[
H = \sqrt{\frac{A_G^{(p)}}{\alpha}} \tag{7.28b}
\]

while (7.26), along with (7.28), produces the equivalent mean velocities in a square
7.5.3 Comparison With Hurlburt & Hanratty Pipe Experiments

In this section, the heuristic condition derived in §7.5.2 is implemented into our Nonlinear Slug Transition Criterion Algorithm so that comparisons can be made against experimental measurements carried out in horizontal pipes. Within this section, we make comparisons against the measurements reported by Hurlburt & Hanratty [34] which were generated from experiments which utilized pipes of several different diameters along with a range of fluid combinations.

For the first set of validations tests, an air-water flow is transported through a 2.52 cm pipe. The superficial gas and liquid velocities at transition are reported along with the corresponding liquid layer heights. Using the superficial liquid velocity and liquid hold-up as input parameters, our Nonlinear Slug Transition Criterion Algorithm was used to determine the critical superficial gas velocity. The resulting transition curves are shown in Figure 7-5. An initial test found that the value of $s = 0.03$ from §7.5.1 is too small for air-water flows through pipes and resulted in critical gas velocities which were larger than the experimental values. Using one of the experimental measurement points of the liquid hold-up and superficial liquid velocity, the forcing coefficient was varied until it was close to the experimental value of the critical gas velocity. This test found that the value of $s = 0.1$ was appropriate for air-water flows through horizontal pipes. Using this value, figure 7-5 shows that there is good agreement between the remaining experimentally measured values and the theoretical predictions.

A second solution is also depicted with $s = 0.3$ so that sensitivity of the solution to the sheltering coefficient can be seen. To confirm that $s = 0.1$ is a robust value, a second test was carried out for an air-water flow through a 9.53 cm pipe. Figure 7-6
Figure 7-5: Theoretical prediction of the critical superficial gas velocity for an air-water flow through a 2.52 cm pipe. The (●) denotes the experimentally measured values from Figure 3 of Hurlburt & Hanratty [34, (pp. 719)] while the solutions using Nonlinear Slug Transition Criterion Algorithm with (—) $s = 0.1$ and (— —) $s = 0.3$ shows the resulting transition curves and the good agreement between the numerical and experimental transition conditions confirms that the sheltering constant appears to be consistent for air-water flows through pipes.

Despite the pipe to channel conversion scheme being based on a heuristic approximation, our transition scheme appears to give consistently accurate solutions. The predictions shown in Figure 7-6(a) capture the correct slopes and also resolve the point where the slope transition occurs. Similar changes in behavior are also captured in Figure 7-6(b) demonstrating that the dominant physics of the system are incorporated into our Nonlinear Slug Transition Criterion Algorithm.

For the next set of tests cases, the effect of liquid viscosity on the transition curves is examined. For these tests, air flows over a glycerine-water solution of density $\rho_l = 1220 \, \text{kg/m}^3$ with a surface tension coefficient of $\sigma = 0.066 \, \text{N/m}$. For the first case, shown in Figure 7-7, the pipe has a diameter of 2.52 cm and the glycerine-water solution has a viscosity of $\mu_l = 0.07 \, \text{N} \cdot \text{s/m}^2$. While in the second case, shown in Figure 7-8, the pipe has a diameter of 9.53 cm and the glycerine-water solution has a viscosity of $\mu_l = 0.1 \, \text{N} \cdot \text{s/m}^2$. As one might expect, the large increase in liquid viscosity required that the sheltering coefficient be increased slightly from the previous cases. One pair of liquid hold-up and superficial liquid velocity values were used to identify the correct
Figure 7-6: Theoretical prediction of the critical superficial gas velocity for an air-water flow through a 9.53 cm pipe. The (●) denotes the experimentally measured values from Figure 3 of Hurlburt & Hanratty [34, (pp. 719)] while the solutions using Nonlinear Slug Transition Criterion Algorithm with (——) \( s = 0.1 \).

coefficient magnitude. The remaining value and trends were then captured without any additional parameter tuning demonstrating the fidelity of the solution scheme.

For the 70 cP case, shown in Figure 7-7(a) and 7-7(b), excellent agreement between the two solutions is observed. This transition criterion captures the linear dependence between the liquid hold-up and the superficial gas velocity shown in figure 7-7(a). The computed results also capture the complicated curvature and behavior changes in the relationship between the superficial gas and liquid velocity. Both plots exhibit a small lag in the superficial gas velocity, but this is likely due to the sheltering coefficient \( s = 0.3 \) not being perfectly tuned. As the liquid viscosity was increased further, the sheltering coefficient was once again increased to \( s = 1 \). Figure 7-8(a) & 7-8(b) shows that for small superficial liquid velocities and large liquid hold ups, there is high correlation between the experimental measurements and the numerical predictions. As the superficial liquid velocity increased, the \( k_2 \) spectrum became broadbanded, \( \kappa_l = \kappa_s \). This made the resulting transition condition be the minimum velocity for which the any part of the growth rate becomes positive. This left the computed critical velocities being well below the experiential results. However, it should be noted that the experimental findings of Fan [24] suggested that nonlinear resonant interactions were the mechanism responsible for slug formation for \( U_{SG} < O(3) \) m/s.
Figure 7-7: Theoretical prediction of the critical superficial gas velocity for high viscosity liquid flows through a horizontal pipe. The (●) denotes experimental values for the 2.52 cm case with the $\mu_l = 0.07 \frac{N\cdot m}{s}$ liquid values from Figure 3 of Hurlburt & Hanratty [34, (pp. 719)]. The theoretical solutions using Nonlinear Slug Transition Criterion Algorithm, denoted by (—--) curves, have a sheltering coefficient value of $s = 0.3$.

Figure 7-8: Theoretical prediction of the critical superficial gas velocity for high viscosity liquid flows through a horizontal pipe. The (●) denotes experimental values for the 2.52 cm case with the $\mu_l = 0.1 \frac{N\cdot m}{s}$ liquid experimental values from Figure 3 of Hurlburt & Hanratty [34, (pp. 719)]. The theoretical solutions using Nonlinear Slug Transition Criterion Algorithm, denoted by (—--) curves, have a sheltering coefficient value of $s = 1.0$. 

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The results shown in figure 7-8(a) & 7-8(b) are accurate within this range. It is only when $U_{SG} > 5 \text{ m/s}$ that the our nonlinear transition criterion begins to fail which may signify the transition from a wave-resonance mechanism to a wave-coalescence slug transition mechanism.

7.6 Conclusions

The work described in this chapter proposed a novel method for predicting the formation of slugs in horizontal channels and pipes. The previous methods which have been developed utilize linear stability analysis, often with a long wavelength assumption, to predict the formation of slugs. However, experiments have shown that one of the common methods of slug formation is through nonlinear wave-wave interactions resulting in an energy cascade from unstable short waves to more stable long waves. Therefore, in this chapter a novel nonlinear slug transition criterion was proposed that couples the effects of a linear instability with that of nonlinear resonant interactions. An energy bounding condition was proposed for which the number of resonant modes that are linearly unstable is minimized. Comparisons were made against experiments that were carried out in horizontal square channels and good agreement was observed.

In order to make this method applicable to pipe flow configuration, a heuristic method was proposed that allows for channel flow conditions to be obtained which are representative of the original pipe flow conditions. Using this approximation, comparisons against experimental measurements conducted over a range of pipe diameters, flow conditions, and fluid combinations have shown consistently good agreement.

This nonlinear slug transition criterion has been shown to be an effective method for predicting the onset of slugging. The theoretical analysis supporting this technique contain several general model terms for the pressure and shear stress fluctuations in the gas. The Jeffreys sheltering model was implemented to account for the normal interfacial pressure fluctuations while the shear stress fluctuations were neglected in the results shown in this work. This work showed that for a given fluid combination and/or geometry, once the sheltering coefficient was selected, the results were accur-
racte regardless of the pipe diameter or flow conditions; however, a sample data set was necessary to aid in identifying an appropriate value for the sheltering coefficient. As more accurate models become available or more data is made available for closure modeling, the simple empirical sheltering constant ($s$) can be replaced with a more robust, physics based parametrization resulting in improved theoretical prediction and less user involvement.

While the results show that there is a weak dependence of the solution on the sheltering coefficient, it is interesting to note that the overall solution trends are consistent with the experimental measurements. The correct transition curve slopes and regime transitions were correctly identified. This is in contrast to some of the other slug transition criteria proposed in the literature which may have the transition curve near the measured values, but the inflection of the curve may be incorrect. Additionally, our transition criteria has been shown to be accurate for both high and low viscosity fluids. Several of the commonly utilizable transition schemes described in the literature struggle with predicting the behavior of high viscosity fluids.

The current procedure has been based on supplying the liquid hold-up and gas velocity (or superficial gas velocity) as inputs to our slug transition algorithm. Comparisons against experimental measurements have shown that this procedure is accurate. However, for more general problems, it is desirable to supply a range of superficial gas velocities and seek the resulting superficial liquid velocities (or vice versa). Such a test removes the liquid hold-up as an input variable. The theoretical solution derived in §7.2 does not introduce any functional dependence between the gas and liquid velocities and the liquid hold-up. Therefore, this test requires a closure model. Following the procedures utilized in other methods, such as in Taitel & Dukler [73], Lin & Hanratty [43], etc., a momentum balance of the unperturbed system can provide the necessary closure, but tests are necessary to validate that this closure modeling is accurate and doesn’t corrupt the inputs to our slug transition criterion.
Chapter 8

Conclusions and future work

The work described in this thesis carried out both theoretical and computational investigations of some nonlinear mechanisms governing the interfacial stability and evolution of stratified two-phase flows through horizontal channels and pipes. This thesis consisted of three key focus areas.

8.1 Thesis Contributions

The first section developed a nonlinear potential flow analysis to identify a mechanism composed of a triad of resonantly interacting interfacial waves which are influenced by the Kelvin-Helmholtz interfacial instability. The mechanism that was identified permitted the rapid energy transfer from linearly unstable short waves to stable long waves through nonlinear resonant wave interactions. Based on a multiple-scale analysis, nonlinear interaction equations were derived which govern the time evolution of the amplitude for the interacting waves in the resonant triad. An effective numerical method allowed for the discrete or broadbanded wave spectra and accounted for the high order simulation of nonlinear wave interactions. It was found that, depending on the flow conditions, it is possible for linearly stable waves to achieve bi-exponential growth due to the resonant coupling. Extensions of this mechanism to broadbanded wave interactions were found to be in close agreement with experimental measurements. The theoretical analysis was extended to examine the special case of
sub-harmonic resonant interactions which have been observed in many experimental measurements and it was shown that this special case could still effectively create rapid long wave growth with up to bi-exponential growth rates.

The second focus area examined the robustness of the aforementioned potential flow mechanism. Investigations were carried out to identify if a linear instability could be effectively coupled with resonant interactions in the presence of viscosity and flow turbulence. A theoretical analysis was conducted to examine the interfacial stability of two-phase laminar/turbulent flows. Direct comparisons were made against experimental measurements and the analysis was able to accurately identify the linearly unstable interfacial modes indicated by experiments. Additionally, the theoretical linear dispersion relationship was also able to be used to predict the existence of the strong sub-harmonic resonances among modes which were reported in the experimental observations. Direct numerical simulations were carried out which confirmed the results of the linear stability analysis and the resulting nonlinear interfacial evolution. The numerical simulation results provided high-resolution data sets for which the interfacial stress distributions could quantified and described providing insights into the necessary behavior of future interfacial stress modeling.

The final focus area was dedicated to developing a novel nonlinear slug transition criterion which couples the effects of a linear instability with that of nonlinear resonant interaction theory. In this work, an energy bounding condition was proposed in which the number of resonant modes which are linearly unstable is minimized allowing for a critical gas velocity to be identified. Comparisons were made against experiments carried out in horizontal channels and good agreement was observed. A heuristic method was proposed which allows for "equivalent" channel flow conditions to be obtained which are representative of the original pipe flow conditions. Comparisons against experimental measurements conducted over a range of pipe diameters, flow conditions, and fluid combinations have shown consistently good agreement.
8.2 Future Work

1. *Coupled viscous instability resonant interaction theory*

The work described in this thesis was able to identify a new nonlinear mechanisms capable of generating large amplitude nonlinear waves and slugs in horizontal channels. The nonlinear potential flow analysis produced a set of interaction equations which described the modal behavior and portion of the parameter space which would result in bi-exponential growth of the linearly stable long waves. The direct numerical simulations, carried out in §6 for a turbulent gas laminar liquid demonstrated that this mechanism would persist in more complex flows; however, it would be useful to carry out an analytic analysis to uncover the form of the nonlinear interaction coefficients such that the details of the modal evolution could be predicted without the use of expensive numerical simulations. Such an analysis, for laminar and/or turbulent flows, could yield analytic expressions for the distribution and time evolution of the interfacial stresses on weakly nonlinear waves.

2. *DNS of flow with strong instabilities and nonlinearity*

While in §6, it was useful to use the flow conditions reported in experiments such that direct comparisons could be made, the resulting wave field contained waves of small to moderate steepness. It would be beneficial to examine the evolution of flows with stronger instabilities which may produce waves with higher wave steepness and stronger nonlinearity. This would be a necessary step towards the ultimate goal of simulating the complete process of slug formation.

3. *Wind-wave generation*

The coupled linear instability resonant interaction mechanism may also prove to be an important process in generating waves in other systems beyond pipe/channel flows. For instance, a significant amount of research has been dedicated to the problem of ocean wave generation by wind. Detailed Orr-Sommerfeld analysis has been used to predict the formation of wind-waves; however, the theoretical predictions show poor agreement with field observations. Additionally, resonant
interaction theory has shown to be an efficient mechanism for the redistribution of energy from capillary scale to gravity waves; however, the linear growth rate provided by the resonance has been shown to be slower than the bi-exponential rates reported from field measurements. The coupled instability-resonant interaction model described in this thesis appears to address both of the dominant features observed in the formation and nonlinear evolution of wind waves. Following the analysis described in §2, it may be found that the wind-wave process is governed by this relatively simple nonlinear process.

4. Two-phase modulational instability analysis
While this novel nonlinear mechanism did prove to be an effective method of generating large amplitude long waves, it is not the only possible mechanism capable of resulting in slug formation. Experiments have shown that wave coalescence/superposition of long waves are an equally effective method of slug formation. Therefore, it would be useful to carry out a stability analysis which examines the modulational stability of interfacial waves. This could lead to a result which is similar to the classic Benjamin-Feir condition for the nonlinear development of water waves.

5. Interfacial stress modeling
The turbulent numerical simulations carried out in §6 characterized the evolution and distribution of normal and shear distributions across the wave field. This provided insight into the general features which would need to be captured in future model development. However, a larger volume of simulations are needed before robust stress models can be developed. Wave stress models which account for the modal wavelength and steepness could have a significant impact on characterizing the development of unstable nonlinear wave fields and could be directly applied to industrial slug simulators and slug transition criterion.

6. DNS of turbulent multiphase flows through pipes and rectangular channels
While the simulations carried out in this thesis did incorporate three-dimensional
effects, they invoked the assumption of spatial periodicity. With a moderate amount of work, these numerical capabilities could be modified to account for a rectangular channel cross section or cylindrical pipe geometry. Accounting for pipe/channel wall geometry is necessary for future studies of the wave-wall contact problem or wave roll up and transition to annular flow conditions. Additionally, classical slug stability methods and commercial solvers use simple geometric shape factors to account for wall and turbulent profile effects. Direct numerical simulations which account for pipe/channel geometries can be utilized to validate the use of those simple shape factors and reduced order models.

7. **Turbulent two-phase boundary conditions**

Carrying out simulations under the assumption of spatial periodicity was a necessary requirement in order to carry out simulations over a reasonably sized domain. This allowed for the initial evolution to be simulated and compared to theoretical predictions. A more sophisticated treatment of the problem would be to increase the domain size and switch to laminar/turbulent in-flow/out-flow boundary conditions. Assuming a properly sized domain, this analysis would capture the identification of dominant slug wavelength and more realistic slug evolution. However, before this class of simulations can be carried out, a realistic set of multiphase turbulent outflow boundary conditions must be developed. Using simple zero-derivative outflow conditions can often lead to backscatter which can contaminate the upstream turbulence and corrupt the simulation behavior. A robust set of boundary conditions could lead to large improvement in simulation capabilities.

8. **High-order multiphase flow algorithm**

Finally, one limitation of the turbulent simulations carried out in this work was the second order numerical algorithm. The current state of the art interface tracking/volume-of-fluid schemes have not achieved higher than second order accuracy in three-dimensional domains. This is in contrast to traditional single-
phase turbulence simulations which utilize very high-order numerical stencils which result in the rapid convergence of the statistical quantities. Even for the simpler case of two-phase laminar Orr-Sommerfeld instabilities, very fine resolution was necessary in order to accurately recover the behavior of the theoretical solutions. Similar problems were even more noticeable for two-phase turbulent flows. The development of higher-order interface tracking algorithms can lead to a significant improvement in quality of the simulations and level of complexity of the underlying physics which can be addressed in two-phase regimes.
Appendix A

Perturbation Equations For Triad Resonances

Based on the definitions of the two time scales, the differential time operator in (3.5 - 3.7), $\partial/\partial t$, is replaced by $\partial/\partial t + \Delta \frac{1}{\tau} \partial/\partial \tau$, which leads to:

$$\nabla^2 \phi_u = 0 \quad (A.1a)$$
$$\nabla^2 \phi_l = 0 \quad (A.1b)$$
$$\phi_{u,y} = 0 \quad (A.1c)$$
$$\phi_{l,y} = 0 \quad (A.1d)$$
$$\eta_t + \Delta \frac{1}{2} \eta_r + [U_l + U_c (1 + \Delta) + \phi_{u,x}] \eta_x - \phi_{u,y} = 0 \quad (A.1e)$$
$$\eta_t + \Delta \frac{1}{2} \eta_r + (U_l + \phi_{l,x}) \eta_x - \phi_{l,y} = 0 \quad (A.1f)$$
$$\mathcal{R}\{\phi_{u,t} + \Delta \frac{1}{2} \phi_{u,r} + \frac{1}{2} \phi_{u,x}^2 + [U_l + U_c (1 + \Delta)] \phi_{u,x} + \frac{1}{2} \phi_{u,y}^2 + \eta\}$$
$$- \left(\phi_{l,t} + \Delta \frac{1}{2} \phi_{l,r} + \frac{1}{2} \phi_{l,x}^2 + U_l \phi_{l,x} + \frac{1}{2} \phi_{l,y}^2 + \eta\right) + \eta_{xx} \frac{(1 + \eta_x^2)^{-3/2}}{\mathcal{W}} = 0 \quad (A.1g)$$

Expanding (A.1e), (A.1f) and (A.1g) in Taylor series about the undisturbed interface position and applying (2.14) gives rise to a sequence of governing equations for $\phi_u^{(m)}$, etc.
\( \phi^{(m)}_i \), and \( \eta^{(m)} \), \( m=1,\ldots,5 \):

\[
\nabla^2 \phi^{(m)}_i = 0 \quad (A.2a)
\]

\[
\nabla^2 \phi^{(m)}_i = 0 \quad (A.2b)
\]

\[
\phi^{(m)}_{u,y} = 0 \quad (A.2c)
\]

\[
\phi^{(m)}_{l,y} = 0 \quad (A.2d)
\]

\[
\eta^{(m)}_x + (U_l + U_c) \eta^{(m)}_x - \phi^{(m)}_{u,y} = f^{(m)}_1 \quad (A.2e)
\]

\[
\eta^{(m)}_y + U_l \eta^{(m)}_y - \phi^{(m)}_{l,y} = f^{(m)}_2 \quad (A.2f)
\]

\[
\mathcal{R} [\phi^{(m)}_{u,t} + (U_l + U_c) \phi^{(m)}_{u,x} + \eta^{(m)}] - [\phi^{(m)}_{i,t} + U_l \phi^{(m)}_{i,x} + \eta^{(m)}] + \frac{\eta^{(m)}_{xx}}{\mathcal{W}} = f^{(m)}_3 \quad (A.2g)
\]

where \( f^{(1)}_j = 0 \) and \( f^{(2)}_j = 0 \), \( j = 1, 2, 3 \), and \( f^{(m)}_j \) for \( m=3, 4 \) and 5 are functions of the lower order solutions.

The \( O(\Delta) \) problem:

\[
\begin{align*}
\mathcal{A}^{(3)}_1 &= \eta^{(1)}_r - \phi^{(1)}_{u,x} \eta^{(1)}_{x} + \eta^{(1)} \phi^{(1)}_{u,y} \\
\mathcal{A}^{(3)}_2 &= -\eta^{(1)}_r - \phi^{(1)}_{l,x} \eta^{(1)}_{x} + \eta^{(1)} \phi^{(1)}_{l,y} \\
\mathcal{A}^{(3)}_3 &= \left[ \eta^{(1)} \phi^{(1)}_{u,x} + \phi^{(1)}_{u,r} + \frac{1}{2} \left( \phi^{(1)}_{u,x} \right)^2 + U_1 \eta^{(1)} \phi^{(1)}_{l,xy} + \frac{1}{2} \left( \phi^{(1)}_{l,y} \right)^2 \right] \\
&\quad - \mathcal{R} \left[ \eta^{(1)} \phi^{(1)}_{u,ty} + \phi^{(1)}_{u,r} + \frac{1}{2} \left( \phi^{(1)}_{u,x} \right)^2 + (U_l + U_c) \eta^{(1)} \phi^{(1)}_{u,xy} + \frac{1}{2} \left( \phi^{(1)}_{u,y} \right)^2 \right] \\
\end{align*}
\]

The \( O(\Delta^{\frac{5}{2}}) \) problem:

\[
\begin{align*}
\mathcal{A}^{(4)}_1 &= -\eta^{(2)}_r - \eta^{(2)}_{x} \phi^{(2)}_{u,x} - \eta^{(2)} \phi^{(2)}_{u,y} + \eta^{(2)} \phi^{(2)}_{u,yy} \\
\mathcal{A}^{(4)}_2 &= -\eta^{(2)}_r - \eta^{(2)}_{x} \phi^{(2)}_{l,x} - \eta^{(2)} \phi^{(2)}_{l,y} + \eta^{(2)} \phi^{(2)}_{l,yy} \\
\mathcal{A}^{(4)}_3 &= \left[ \eta^{(2)} \phi^{(2)}_{u,ty} + \eta^{(2)} \phi^{(2)}_{u,r} + \frac{1}{2} \left( \phi^{(2)}_{u,x} \right)^2 + U_1 \eta^{(2)} \phi^{(2)}_{l,xy} + \frac{1}{2} \left( \phi^{(2)}_{l,y} \right)^2 \right] \\
&\quad + \phi^{(2)}_{l,y} \phi^{(2)}_{l,y} - \mathcal{R} \left[ \eta^{(2)} \phi^{(2)}_{u,ty} + \eta^{(2)} \phi^{(2)}_{u,r} + \eta^{(2)} \phi^{(2)}_{u,xy} + \phi^{(2)}_{u,r} \phi^{(2)}_{u,x} \right] \\
&\quad + (U_l + U_c) \left( \eta^{(2)} \phi^{(2)}_{u,xy} + \phi^{(2)} \phi^{(2)}_{u,y} + \phi^{(2)} \phi^{(2)}_{u,x} \right) \\
\end{align*}
\]

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The $O\left(\Delta^3\right)$ problem:

\[ f_1^{(5)} = -\eta_x^{(3)} - U_x^{(1)} - \eta_x^{(1)}\phi^{(3)} - \eta_x^{(3)}\phi^{(1)} - \eta_x^{(1)}\phi^{(1)} - \phi^{(2)} - \phi^{(2)} \]
\[ + \eta^{(1)}\phi^{(3)} + \eta^{(3)}\phi^{(1)} + \phi^{(2)} + \frac{1}{2} (\eta^{(1)})^2 \phi^{(1)} \]  
(A.5a)

\[ f_2^{(5)} = -\eta_x^{(3)} - \eta_x^{(1)}\phi^{(3)} - \eta_x^{(3)}\phi^{(1)} - \eta_x^{(1)}\phi^{(1)} - \phi^{(2)} + \eta^{(1)}\phi^{(3)} \]
\[ + \eta^{(3)}\phi^{(1)} + \eta^{(2)}\phi^{(2)} + \frac{1}{2} (\eta^{(1)})^2 \phi^{(1)} \]  
(A.5b)

\[ f_3^{(5)} = \left[ \eta^{(1)}\phi^{(3)} + \eta^{(3)}\phi^{(1)} + \eta^{(2)}\phi^{(2)} + \frac{1}{2} (\eta^{(1)})^2 \phi^{(1)} + \phi^{(1)} + \phi^{(1)} \phi^{(1)} + \frac{1}{2} (\eta^{(1)})^2 \phi^{(1)} \right] \]
\[ + \phi^{(1)}\phi^{(3)} + \eta^{(1)}\phi^{(1)} + \frac{1}{2} (\phi^{(2)})^2 + U_t (\eta^{(1)}\phi^{(3)} + \eta^{(2)}\phi^{(2)} + \eta^{(3)}\phi^{(1)} + \frac{1}{2} (\eta^{(1)})^2 \phi^{(1)} \right] - \mathcal{R} \left[ \eta^{(1)}\phi^{(3)} + \eta^{(3)}\phi^{(1)} + \eta^{(2)}\phi^{(2)} + \frac{1}{2} (\eta^{(1)})^2 \phi^{(1)} \right] \]
\[ + \frac{1}{2} (\eta^{(1)})^2 \phi^{(1)} + \phi^{(3)} + \phi^{(1)}\phi^{(3)} + \eta^{(1)}\phi^{(1)} + \frac{1}{2} (\eta^{(1)})^2 \phi^{(1)} + U_t \phi^{(3)} \]
\[ + (U_t + U_c) (\eta^{(1)}\phi^{(3)} + \eta^{(2)}\phi^{(2)} + \eta^{(3)}\phi^{(1)} + \frac{1}{2} (\eta^{(1)})^2 \phi^{(1)} + \phi^{(1)}\phi^{(3)} \right] \]
\[ + \frac{1}{2} (\phi^{(2)})^2 + \frac{3}{2} \frac{\eta^{(1)}}{\eta^{(1)}} \]  
(A.5c)
Appendix B

Perturbation Equations For Sub-harmonic Resonances

In this paper several different formulations of the perturbation expansions were used. In each case, the two time scales \((t, \tau = \zeta t)\) were defined, with \(\zeta\) being specific to each problem. With these two time scales, the differential time operator in (3.5)-(3.7), \(\partial / \partial t\), is replaced by \(\partial / \partial t + \zeta \partial / \partial \tau\), which leads to

\[
\begin{align*}
\nabla^2 \phi_{u}^{(m)} &= 0 \quad \text{(B.1a)} \\
\nabla^2 \phi_{l}^{(m)} &= 0 \quad \text{(B.1b)} \\
\phi_{u,y}^{(m)} &= 0 \quad \text{(B.1c)} \\
\phi_{l,y}^{(m)} &= 0 \quad \text{(B.1d)} \\
\eta_{l, t}^{(m)} + U \eta_{l, x}^{(m)} - \phi_{u,y}^{(m)} &= f_1^{(m)} \quad \text{(B.1e)} \\
\eta_{l, t}^{(m)} + U \eta_{l, x}^{(m)} - \phi_{l,y}^{(m)} &= f_2^{(m)} \quad \text{(B.1f)} \\
\mathcal{R} \left[ \phi_{u,t}^{(m)} + U \phi_{u,x}^{(m)} + \eta^{(m)} \right] - \left[ \phi_{l,t}^{(m)} + U \phi_{l,x}^{(m)} + \eta^{(m)} \right] + \frac{\eta_{l,x}^{(m)}}{\mathcal{W}} &= f_3^{(m)} \quad \text{(B.1g)}
\end{align*}
\]

with \(\mathcal{U}\) being a measure of the mean velocity in the upper phase which is uniquely defined in each problem.
B.1 Case 1: Both The $k_1$ And The $k_2$ Modes Are Linearly Stable

In this problem, the perturbation expansions were carried out in terms of a measure of the wave steepness $\epsilon$ which generated a second time scale with $\zeta = \epsilon$. The expansions were carried out with $U \equiv U_u$ which yields forcing functions $f_j^{(m)}$ of the form:

Order $\epsilon^2$

\[
\begin{align*}
    f_1^{(2)} &= -\eta_{x}^{(1)} - \phi_{u,xx}^{(1)} \eta_{x}^{(1)} + \eta_{x}^{(1)} \phi_{u,yy}^{(1)} \\
    f_2^{(2)} &= -\eta_{y}^{(1)} - \phi_{u,xy}^{(1)} \eta_{y}^{(1)} + \eta_{y}^{(1)} \phi_{u,yy}^{(1)} \\
    f_3^{(2)} &= \eta_{x}^{(1)} \phi_{u,yy}^{(1)} + \phi_{u,xx}^{(1)} + \frac{1}{2} \left( \phi_{u,xx}^{(1)} \right)^2 + U_l \eta_{x}^{(1)} \phi_{u,xy}^{(1)} + \frac{1}{2} \left( \phi_{u,yy}^{(1)} \right)^2 \\
        & \quad - \mathcal{R} \left[ \eta_{x}^{(1)} \phi_{u,yy}^{(1)} + \phi_{u,xx}^{(1)} + \frac{1}{2} \left( \phi_{u,xx}^{(1)} \right)^2 + U_l \eta_{x}^{(1)} \phi_{u,xy}^{(1)} + \frac{1}{2} \left( \phi_{u,yy}^{(1)} \right)^2 \right] 
\end{align*}
\]

(B.2a)

(B.2b)

(B.2c)

B.2 Case 2: The $k_1$ Is Marginally Unstable And The $k_2$ Mode Is Linearly Stable

In this problem, the perturbation variable was chosen to be the deviation of the upper velocity away from the critical velocity which was measured by the variable $\Delta$, which generated a second time scale with $\zeta = \Delta^{1/2}$. The expansions were carried out with $U \equiv U_u + U_l$ which yields forcing functions $f_j^{(m)}$ of the form:

Order $\Delta$

\[
\begin{align*}
    f_1^{(3)} &= -\eta_{x}^{(1)} - \phi_{u,xx}^{(1)} \eta_{x}^{(1)} + \eta_{x}^{(1)} \phi_{u,yy}^{(1)} \\
    f_2^{(3)} &= -\eta_{y}^{(1)} - \phi_{u,xy}^{(1)} \eta_{y}^{(1)} + \eta_{y}^{(1)} \phi_{u,yy}^{(1)} \\
    f_3^{(3)} &= \left[ \eta_{x}^{(1)} \phi_{u,yy}^{(1)} + \phi_{u,xx}^{(1)} + \frac{1}{2} \left( \phi_{u,xx}^{(1)} \right)^2 + U_l \eta_{x}^{(1)} \phi_{u,xy}^{(1)} + \frac{1}{2} \left( \phi_{u,yy}^{(1)} \right)^2 \right] \\
        & \quad - \mathcal{R} \left[ \eta_{x}^{(1)} \phi_{u,yy}^{(1)} + \phi_{u,xx}^{(1)} + \frac{1}{2} \left( \phi_{u,xx}^{(1)} \right)^2 + \left( U_l + U_c \right) \eta_{x}^{(1)} \phi_{u,xy}^{(1)} + \frac{1}{2} \left( \phi_{u,yy}^{(1)} \right)^2 \right] 
\end{align*}
\]

(B.3a)

(B.3b)

(B.3c)

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\[ f_1^{(4)} = -\eta_{x,x} - \eta_{x,x} \phi_{u,x} - \eta_{x,z} \phi_{u,z} + \eta_{x,z} \phi_{u,z} + \eta(1) \phi_{u,y} + \eta(2) \phi_{u,y} \]  
(B.4a)

\[ f_2^{(4)} = -\eta_{x,x} - \eta_{x,z} \phi_{u,z} - \eta_{x,x} \phi_{u,x} + \eta_{x,z} \phi_{u,z} + \eta(2) \phi_{u,y} \]  
(B.4b)

\[ f_3^{(4)} = \left[ \eta(1) \phi_{u,y} + \eta(2) \phi_{u,y} + \phi_{u,x} + \phi_{u,z} + \phi_{u,z} \right] - \mathcal{R} \left[ \eta_{x,x} \phi_{u,y} + \eta_{x,z} \phi_{u,x} + \phi_{u,y} + \phi_{u,z} \phi_{u,z} \right] + (U_l + U_c) \left( \eta(1) \phi_{u,y} + \eta(2) \phi_{u,y} + \phi_{u,z} \phi_{u,z} \right) \]  
(B.4c)

Order \( \Delta^\frac{1}{2} \)

\[ f_1^{(5)} = -\eta_{x,x} - \eta_{x,x} \phi_{u,x} - \eta_{x,z} \phi_{u,z} + \eta_{x,z} \phi_{u,z} + \eta(1) \phi_{u,y} + \eta(2) \phi_{u,y} \]  
(B.5a)

\[ f_2^{(5)} = -\eta_{x,x} - \eta_{x,z} \phi_{u,z} - \eta_{x,x} \phi_{u,x} + \eta_{x,z} \phi_{u,z} + \eta(2) \phi_{u,y} \]  
(B.5b)

\[ f_3^{(5)} = \left[ \eta(1) \phi_{u,y} + \eta(2) \phi_{u,y} + \phi_{u,x} + \phi_{u,z} + \phi_{u,z} \right] - \mathcal{R} \left[ \eta_{x,x} \phi_{u,y} + \eta_{x,z} \phi_{u,x} + \phi_{u,y} + \phi_{u,z} \phi_{u,z} \right] + (U_l + U_c) \left( \eta(1) \phi_{u,y} + \eta(2) \phi_{u,y} + \phi_{u,z} \phi_{u,z} \right) \]  
(B.5c)

\section*{B.3 Case 3: Both The \( k_1 \) And The \( k_2 \) Modes Are Linearly Unstable}

This problem contains the same two time scales as was used in §B.2 which produced \( \zeta = \Delta^{1/2} \) and \( \mathcal{U} = U_c + U_l \). With this scaling, the forcing functions \( f_j^{(m)} \) were found...
to be of the form:

**Order $\Delta^2$**

\[ f_1^{(2)} = -\eta_T^{(1)} \quad \text{(B.6a)} \]
\[ f_2^{(2)} = -\eta_T^{(1)} \quad \text{(B.6b)} \]
\[ f_3^{(2)} = \phi_{l,T}^{(1)} - \mathcal{R}\phi_{u,T}^{(1)} \quad \text{(B.6c)} \]

**Order $\Delta^2$**

\[ f_1^{(3)} = -\eta_T^{(2)} - U_c\eta_{x,T}^{(1)} - \eta_{x,T}^{(1)} \phi_{u,x}^{(1)} + \eta_{x,T}^{(1)} \phi_{u,yy}^{(1)} \quad \text{(B.7a)} \]
\[ f_2^{(3)} = -\eta_T^{(2)} - \eta_{x,T}^{(1)} \phi_{l,x}^{(1)} + \eta_{x,T}^{(1)} \phi_{l,yy}^{(1)} \quad \text{(B.7b)} \]
\[ f_3^{(3)} = \left[ \eta_{x,T}^{(1)} \phi_{l,yT}^{(1)} + \phi_{l,r}^{(2)} + \frac{1}{2} \left( \phi_{l,x}^{(1)} \right)^2 + U_l\eta_{x,T}^{(1)} \phi_{l,xy}^{(1)} + \frac{1}{2} \left( \phi_{l,x}^{(1)} \right)^2 \right] \quad \text{(B.7c)} \]

\[-\mathcal{R} \left[ \eta_{x,T}^{(1)} \phi_{u,yT}^{(1)} + \phi_{u,r}^{(2)} + \frac{1}{2} \left( \phi_{u,x}^{(1)} \right)^2 + \left( U_l + U_{c} \right) \eta_{x,T}^{(1)} \phi_{u,xy}^{(1)} + \frac{1}{2} \left( \phi_{u,x}^{(1)} \right)^2 + U_{c}\phi_{u,xx}^{(1)} \right] \]
Appendix C

Asymptotic Theory

The Orr-Sommerfeld equation, given by eqn. (7.9), can be written in an alternative form

\[
\frac{1}{ik^2 \mathcal{R}} \left[ D^4 - 2k^2 D^2 + k^4 \right] \dot{\nu} = \left( \frac{U}{c} - 1 \right) \left( \frac{D^2}{k^2} - 1 \right) \dot{\nu} - \mathcal{L} \dot{\nu} \tag{C.1}
\]

where \( \mathcal{L} = \frac{U'}{ck} \) and \( \mathcal{R} = kcRe \). In the experiments by Cohen & Hanratty [15], it was found that in the limit of large \( \mathcal{R} \), the wave velocities were always larger than the liquid velocity. Therefore, there are no critical points within the liquid film and there are only two viscous boundary layers (near the wall and near the interface) which need to be considered. In the limit of large \( \mathcal{R} \), the dominant behavior of (C.1) consists of two inviscid solutions which are governed by the Rayleigh equation

\[
\left( \frac{D^2}{k^2} - 1 \right) \dot{\nu} - \frac{1}{\left( \frac{U}{c} - 1 \right)} \mathcal{L} \dot{\nu} = 0 \tag{C.2}
\]

which can be solved through a perturbation expansion in terms of the small variable \( \frac{1}{c-U(0)} \); however, for \( |\mathcal{L}| << 1 \), (C.2) reduces to the much simpler form of

\[
\left( \frac{D^2}{k^2} - 1 \right) \dot{\nu} = 0 \tag{C.3}
\]
which has general solutions of the form

\[ \hat{v}_1 \sim e^{k_y} \]  
\[ \hat{v}_2 \sim e^{-k_y}. \]  

(C.4a)  
(C.4b)

In the limit of large \( R \), the viscous effects are negligible over the majority of the domain with the exception of in the vicinity of the wall and near the interface. Asymptotic boundary layer solutions were obtained following the strategy developed by Lock[46] for the problem of interfacial boundary layers. In this work, the wall boundary layer solution shall be examined first followed by the derivation of the interfacial boundary layer using a similar technique.

The wall layer is examined by applying the change of variable \( \zeta \rightarrow (1 + y) R^{1/2} \), which from chain rule causes the derivatives to be transformed as

\[ \frac{d^n}{dy^n} \rightarrow \frac{\mathcal{R}^{n/2} d^n}{d\zeta^n} \]

yielding a transformed Orr-Sommerfeld equation for \( \hat{\nu}(y) = \chi(\zeta) \) of the form

\[ \frac{1}{ik^2 R} \left[ \mathcal{R} \hat{\nu}^4 - 2k^2 \mathcal{R} \hat{\nu}^2 + k^4 \right] \chi(\zeta) = \left( \frac{\hat{U}(\zeta)}{c} - 1 \right) \left( \frac{\mathcal{R} \hat{\nu}^2}{k^2} - 1 \right) \hat{\chi}(\zeta) - \mathcal{R} \tilde{\mathcal{L}} \{ \chi(\zeta) \} \]

(C.5)

The mean flow \( \bar{U}(y) \) near the wall can be approximated through a Taylor series expansion about \( y = -1 \) yielding

\[ \bar{U}(y) \approx U_w + U'_w (y + 1) + \frac{1}{2} U''_w (y + 1)^2 + \ldots \]
\[ \bar{U}''(y) \approx U'''_w + \ldots \]

which becomes through the change of variable

\[ \bar{U}(\zeta) \approx U_w + \mathcal{R}^{-1/2} U'_w \zeta + \frac{1}{2} \mathcal{R}^{-1} U''_w \zeta^2 + \ldots \]

(C.7a)
\[ \bar{U}''(\zeta) \approx R^{-1} U'''_w + \ldots \]

(C.7b)
where \( U_w = \bar{U} (y = -1) = 0, U'_w = \bar{U}' (y = -1), \) and \( U''_w = \bar{U}'' (y = -1). \) This viscous solution is approximated as a perturbation expansion of the form of

\[
\dot{v} (y) = \chi (\zeta) = \chi^{(0)} (\zeta) + \mathcal{R}^{-rac{1}{2}} \chi^{(1)} (\zeta) + \mathcal{R}^{-1} \chi^{(2)} (\zeta) + \ldots \quad (C.8)
\]

which upon substitution into (C.5), along with the mean flow approximations, yields a series of boundary value problems for \( \chi^{(0)}, \chi^{(1)}, \ldots, \chi^{(r)} \) of the form

\[
\frac{d^4 \chi^{(r)}}{d\zeta^4} + i \frac{d^2 \chi^{(r)}}{d\zeta^2} = \mathcal{F}^{(r)} \quad (C.9)
\]

with \( \mathcal{F}^{(0)} = 0, \mathcal{F}^{(1)} = \frac{i}{c} \zeta U'_w \frac{d^2 \chi^{(0)}}{d\zeta^2}, \) and more generally \( \mathcal{F}^{(r)} = f (\chi^{(r-1)}, \ldots, \chi^{(0)}). \) The no-flux boundary condition requires that \( \dot{v} (y = 0) = \chi (\zeta = 0) = 0 \) which requires \( \chi^{(r)} (0) = 0. \) This allows the leading order general solution to be of the form

\[
\chi^{(0)} (\zeta) \sim e^{-\theta \zeta} \quad (C.10)
\]

where \( \theta = \exp \left( \frac{-\pi i}{4} \right). \) The remaining orders of the expansion, can be obtained through the method of variation of parameters yielding the approximate solution

\[
\dot{v}_3 (y) = \chi (\zeta) \sim e^{-\theta \zeta} \left[ 1 + \frac{U'_w}{c \mathcal{R}^{1/2}} \left( \frac{5 \zeta}{4} + \frac{\sqrt{2} (1 - i) \zeta^2}{8} \right) \right] + \ldots \quad (C.11)
\]

An analogous expression can be found for the viscous solution in the vicinity of the interface. Near the interface, the solution should be scaled through the transformation \( \xi = -y \mathcal{R}^{1/2} \) such that the viscous solution is given by

\[
\dot{v}_4 (y) = \Phi (\xi) = \Phi^{(0)} (\xi) + \mathcal{R}^{-1/2} \Phi^{(1)} (\xi) + \mathcal{R}^{-1} \Phi^{(2)} (\xi) + \ldots \quad (C.12)
\]

and satisfies

\[
\frac{d^4 \Phi^{(r)}}{d\xi^4} + i \gamma \frac{d^2 \Phi^{(r)}}{d\xi^2} = \mathcal{G}^{(r)} \quad (C.13)
\]

with \( \mathcal{G}^{(0)} = 0, \mathcal{G}^{(1)} = \frac{i}{\epsilon} \zeta U'_0 \frac{d^2 \Phi^{(0)}}{d\zeta^2}, \) and more generally \( \mathcal{G}^{(r)} = f (\Phi^{(r-1)}, \ldots, \Phi^{(0)}). \) By applying the far field boundary conditions (\( y = -1 \)), the leading order solution is
found to be

\[ \Phi^{(0)}(\xi) \sim e^{-\gamma^{1/2}\xi} \quad (C.14) \]

where \( \gamma = \frac{c - U_0}{c} \). As in the previous case, the remaining orders can be determined through the method of variation of parameters producing the viscous solution in the vicinity of the interface

\[ \hat{v}_4(y) = \Phi(\xi) \sim e^{-\gamma^{1/2}\xi} \left[ 1 + \frac{U_0}{cR^{1/2}} \left( \frac{5\xi}{4\gamma} + \sqrt{\frac{2(1-i)\xi^2}{\gamma}} \right) + \ldots \right] \quad (C.15) \]

Given these four components of the solution, (C.4, C.11, & C.15), the complete asymptotic solution to the Orr-Sommerfeld equation is

\[ \hat{v} = d_1 \hat{v}_1 + d_2 \hat{v}_2 + d_3 \hat{v}_3 + d_4 \hat{v}_4 \quad (C.16) \]

with the \( d_j \) being the constants which enforce the interfacial and wall boundary conditions given by (7.10), (7.16), and (7.19). Upon substituting (C.16) into the boundary conditions, a set of homogenous algebraic equations is formed with a unique solution existing only if the determinant of the coefficients is equal to zero. By retaining the three highest orders in \( R \), analytic approximations were obtained. By requiring that the imaginary part of the phase velocity be zero, the imaginary part of the determinant produces the phase velocity given by eqn. (7.20). The growth rate, denoted by \( \omega_1 \), was also approximated by evaluating the determinant under the assumption that \( c_i \ll c_r \) and \( c_i^2 \approx 0 \) which yields eqn. (7.21)


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