A Nondimensional Model for Axial Digging in Granular Materials

by

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Abstract

We investigate the mechanics of thin diggers in a packing of granular materials. Experiments were conducted by pushing diggers of varying thickness into the granular packing and recording the force-depth data. The digger is modeled as a column in buckling; its column effective length factor is calculated.

We solve two optimization problems. First, we limit the axial force the digger is allowed to experience and analytically solve for the maximum depth dug. This analysis is compared to experimental data. Second, we fix the digger’s final depth and estimate the thickness that minimizes digging energy. A stochastic algorithm is proposed to model the digger’s axial force as a function of depth, taking into account the force distribution at each depth. We then calculate the energy required for a digger of each thickness to reach a fixed depth and compare with experimental data.

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Chapter 1

Introduction

Digging is the process of purposefully displacing granular materials. One indicator of the economic importance of this common activity is the revenue of a single manufacturer of earthmoving equipment, Caterpillar, Inc., which reached over $60 billion in 2012.

Digging can take many forms; in most cases, grain-scale phenomena are irrelevant due to the large size of digging equipment. In this thesis, however, we investigate the process of thrusting a slender object (digger) along its axis into a loose packing of macroscopic grains. In this case, the tip of a thin digger interacts with a small number of grains, making grain-scale phenomena important. We show that for sufficiently thin and flexible diggers, the variations in contact forces between individual grains can significantly influence the outcome of the digging process, giving an example of how the details of the microscopic arrangement of a granular packing can affect its macroscopic properties.

The relation between macroscopic properties of granular materials and the grain scale phenomena has long been a subject of research. Of greatest relevance is the work of Liu and Mueth, who described the distribution of inter-grain forces when external pressure is applied to a grain packing [1, 2]. Later, Majmudar and Behringer, having developed a method to visualize forces acting on packed grains, found granular ‘force chains’ comprising an intricate, anisotropic web of grains that supports the majority of forces within a packing [3]. Geng et al. investigated the force chain response to
point forces, showing that the response is highly dependent on the grain packing [4].

We investigate the effects of the inhomogeneities in stress distribution within granular packings, uncovered by the studies above, on the process of axial digging. In particular, we examine the hypothesis that a flexible digger may avoid the localized areas of high stress, i.e. the load-bearing force chains (Figure 1-1), and thus lower the total energy required to reach the desired depth. Our experiments and a stochastic model of the force acting on the digger indicate that in both energy- and force-constrained scenarios, elastic buckling of the digger poses a limit to the depth that can be reached. However, due to the stochastic nature of the forces exerted on the digger, there is a finite probability of successfully digging to a depth greater than would be possible in a deterministic medium.

Figure 1-1: Left: dimensions of the digger. Its width \( w \) (not shown) is measured into the page. Right: Schematic of the idealized routes taken by diggers of various flexibility in a granular packing. Stiff diggers penetrate force chains, requiring the most energy. Flexible diggers can avoid force chains, saving energy. Excessively flexible diggers buckle and do not make substantial downward progress.
Chapter 2

Experimental setup

Our experiments were conducted in a flat acrylic box of dimensions 100mm × 120mm × 9mm, filled with a granular substrate of soda lime glass beads 0.4–0.6mm in diameter, resulting in a quasi 2-dimensional packing (Figure 1-1). A 100mm × 100mm extender piece was attached to the top of the box to house a guide track. This track prevented the digger from buckling above the grain surface. The box was filled with beads up to the junction between the box and extender piece. See Figure 2-1.

The cantilevered diggers were constructed from plastic shim from Artus Corporation, with length $L = 125$mm, width $w = 8$mm and thickness $t$ ranging from 0.00508 – 0.762mm, such that $L \gg w \gg t$. The modulus of elasticity of the 0.00508 – 0.254mm shims was $E = 4.48$GPa, and that of the 0.3175 – 0.762mm diggers was $E = 2$GPa.

The diggers were attached to a flat acrylic piece that slid freely within the box extender. This piece was then attached to the Micro-Texture Analyzer (Ta.XT.Plus). The Texture Analyzer was then used to push down the digger. Force data was sent from the Analyzer to a computer with Exponent software for analysis. For a few calibration trials, digging speeds from 0.25mm/s to 8mm/s were attempted. It was found that the digging speed does not significantly affect digging force. Subsequently, trials of all digger thickness were run at a constant speed of 1mm/s. To ensure consistent initial packing of beads, the beads were poured out, the box tilted at 20 degrees, and the beads slowly poured back into the box before each trial.
Each trial was conducted by pushing the digger into the granular packing until the digger buckled or it reached the maximum digging depth allowed by our setup, 75mm. Typical data is shown in Figure 2-2. When the digger buckles, the tip does not advance but the Texture Analyzer continues to push downward. The extra length of shim accumulated causes the first $\approx 10$mm of shim immediately below the surface to curve. Force data recorded after buckling is not analyzed. Diggers with thicknesses $0.0508 - 0.1905$mm buckled before reaching the maximum depth, while diggers with thicknesses $0.254 - 0.762$mm dug to 75mm without buckling. A typical buckled response is shown in Figure 2-3.

Data collected over multiple trials is shown in Figure 4-5. Note the fluctuations in the force of order 0.1N. This is much greater than the 0.001N sensitivity of the Micro-Texture Analyzer, indicating that the fluctuations result from the properties
of the granular packing. The axial force average and its standard deviation increase linearly with depth.
Albert et al. found that the drag force on an object in a granular packing is proportional with an object’s frontal area, and to first order, the object’s depth [5]. We can apply this to calculate the digging force on a digger of width $w$ and thickness $t$ (frontal area $wt$), with the tip at depth $x$ in a packing.

$$F_{dig} = C wt x$$

(3.1)

where $C$ is an constant with dimension force/volume. $C$ is assumed to be dependent upon the digger material and glass bead type, but is independent of $w$, $t$ and $x$. The standard deviation of $F_{dig}$ was estimated from experimental data to be $2C wt x$.

Let $E$, $K$, and $D$ be the digger’s Young’s Modulus, column effective length ratio, and target depth, respectively. The design of the box (Figure 2-1) ensured that the length of digger available for buckling ($L$) is equal to the depth of the digger tip ($D$), for both diggers that attain their target depth and diggers that buckle before reaching that depth. The moment of inertia of the beam is $\frac{wt^3}{12}$. The force required to buckle the digger is

$$F_{buck} = \frac{\pi E wt^3}{12(KL)^2},$$

(3.2)

from the Euler-Lagrange beam buckling formula. Using data from trials where the digger buckled and known values of $F_{dig}$, $w$, $t$, $E$, and $L = D$, we estimated $K = 0.110$. 

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3.1 Friction on digger sides is insignificant

Friction acts on the sides of the digger as it digs downwards. We would like to estimate the scaling of this friction force on depth $x$. The local normal force $F_N$ is taken as a hydrostatic force, which is proportional to the depth $x$. This force acts over an area of $2wx$, accounting for both sides of the digger. Thus the friction force is

$$F_f = \mu F_N \propto \mu x(2wx) \propto wx^2,$$  \hspace{1cm}(3.3)

where $\mu$ is the coefficient of friction. Combining Equations 3.3 with 3.1, the total force accounting for friction on digger sides is

$$F_{\text{dig}} = C_1 wt x + C_2 wx^2.$$  \hspace{1cm}(3.4)

However, the typical data shown in Figure 2-2 shows a linear dependence on depth. All our data were consistent with linear depth dependence. Therefore, we can take $C_2 = 0$ in Equation 3.4. This leaves us with Equation 3.1, which we will use from this point onward.

3.2 Calculating the axial drag force constant ($C$)

We took the digging force at the digger’s maximum depth, $F_{\text{dig}}(x = L)$, and plotted against $wtL$ for all experimental trials. $C$ depends on the interaction between the grains and the digger and is material dependent. There are two digger materials, one for diggers of thickness .0508-.3175mm and another for .381-.762mm, so two values for $C$ were found according to Equation 3.1. $F_{\text{dig}}$ was taken at the maximum digger tip depth, $L$. A best fit line with $(0, 0)$ intercept gave $C = 9.32 \times 10^6 N/m^3$ and $5.82 \times 10^6 N/m^3$ (see Figure 3-1).
Figure 3-1: \( C = 6.37 \times 10^6 \text{N/m}^3 \) and \( 5.82 \times 10^6 \text{N/m}^3 \) are the slopes of the best fit lines with intercept (0,0).

### 3.3 Calculating the digger’s column effective length factor (K)

We rearrange Equation 3.2 to

\[
\frac{\pi E w t^3}{12(L)^2} = K^2 F_{\text{buck}}. \tag{3.5}
\]

We took the LHS of Equation 3.5 and plotted against \( F_{\text{buck}} \) for experimental trials where the digger buckled at depths less than 75mm. \( F_{\text{buck}} \) was taken as the maximum force sustained before the digger buckled (see Figure 2-3). These trials were comprised of diggers of thicknesses .0508mm-.1905mm. \( K = .110 \) was obtained.

Figure 3-3 shows common end conditions for column buckling. Theoretical values of \( K \) range from .5-2. The calculated \( K = .110 \) for this granular-constrained digger is significantly lower, implying that the constraint of immersion in grains is tighter than common end conditions.
Figure 3-2: $K^2$ is the slope of the best fit line with intercept (0,0). Using the buckling data, we get $K^2 = 0.0120$, so $K = 0.110$. The group of data points in the upper right was taken from .1905mm thick diggers.

### 3.4 Calculating the maximum depth attained

Biological organisms and robots can be force-limited. Consider the case where force is limited to $F_{\text{max}}$. What digger thickness maximizes the depth dug? Our goal is to solve for the maximum depth at thickness $t$, $D(t)$. Equations 3.1 and 3.2 give expressions for $F_{\text{dig}}$ and $F_{\text{buck}}$, respectively.

Above a certain thickness $t_1$, $F_{\text{dig}}$ is larger than $F_{\text{buck}}$, so the digger is limited by the drag force. We can substitute $x = D$ into Equation 3.1 and solve for

$$F_{\text{max}} = C wt D(t)$$

$$\Rightarrow D(t) = \frac{F_{\text{max}}}{C wt}.$$  \hspace{1cm} (3.6)

Below thickness $t_1$, $F_{\text{dig}}$ is smaller than $F_{\text{buck}}$, so the digger is limited by the buckling force. This means the digger buckles before $F_{\text{max}}$ can be attained. When
the digger buckles at depth $D(t)$,

$$F_{\text{dig}} = F_{\text{buck}} = C_w t D(t).$$ (3.7)

We can substitute $L = D(t)$ and Equation 3.7 into Equation 3.2 to obtain

$$C_w t D(t) = \frac{\pi E wt^3}{12(KD(t))^2}$$

$$\Rightarrow D(t) = \left( \frac{t}{K} \right)^{\frac{2}{3}} \left( \frac{\pi E}{12C} \right)^{\frac{1}{3}}.$$ (3.8)

Combining Equations 3.6 and 3.8, we obtain:

$$D(t) = \begin{cases} \left( \frac{t}{K} \right)^{\frac{2}{3}} \left( \frac{\pi E}{12C} \right)^{\frac{1}{3}} & t \in [0, t_1) \\ \frac{F_{\text{max}}}{C_w t} & t \in [t_1, \infty) \end{cases}$$ (3.9)

The graph of this function is shown as the “analytical model” in Figure 3-4.

$D(t)$ represents the maximum depth a digger of thickness $t$ can dig given a force...
constraint of $F_{\text{max}}$. Note that $D(t)$ is a piecewise function with the left piece increasing and the right piece decreasing, so $D(t)$ is maximum at $t_1$. Next, we solve for $t_1$ by setting Equation 3.6 to Equation 3.8 and $t = t_1$:

$$
\frac{F_{\text{max}}}{C_w t_1} = \left( \frac{t_1}{K} \right)^{\frac{3}{2}} \left( \frac{\pi E}{12C} \right)^{\frac{1}{2}}
$$

$$
\Rightarrow t_1 = \left( \frac{F_{\text{max}}}{C_w} \right)^{\frac{2}{3}} K^{\frac{2}{3}} \left( \frac{12C}{\pi E} \right)^{\frac{1}{3}}.
$$

(3.10)

Finally, we can solve for the maximum depth $D_{\text{max}}$ in terms of $F_{\text{max}}$. Set $D(t) = D_{\text{max}}$ and substitute Equation into Equation 3.6 to obtain

$$
D_{\text{max}} = \frac{F_{\text{max}}}{C_w \left( \frac{F_{\text{max}}}{C_w} \right)^{\frac{2}{3}} K^{\frac{2}{3}} \left( \frac{12C}{\pi E} \right)^{\frac{1}{3}}} = \frac{1}{C_w} \left( \frac{F_{\text{max}} \pi}{wK} \right)^{\frac{2}{3}} \left( \frac{E}{12} \right)^{\frac{1}{3}}.
$$

(3.11)

For Figure 3-4, we nondimensionalized by setting $t^* = \frac{t}{t_1}$ and $D^* = \frac{D}{D_{\text{max}}}$.

Figure 3-4: $D^*$ reached when force is limited to 1N. When $t^* < 1$, depth is limited by buckling. When $t^* > 1$, depth is limited by drag force. There is an optimum thickness that maximizes the digging depth.
Chapter 4

Digging to a constant depth

We set the depth $D = L$ constant and vary the thickness $t$ to minimize the digging energy $U$. Let the characteristic thickness $t_{\text{char}}$ occur when the digger buckles at depth $L$. That is, $F_{\text{buck}} = F_{\text{dig}}(x = L)$. Equating 3.1 and 3.2 setting $t = t_{\text{char}}$, and solving for the thickness $t_{\text{char}}$, we obtain

$$Cwt_{\text{char}}L = \frac{\pi E wt^3_{\text{char}}}{12(KL)^2}$$

$$\Rightarrow t_{\text{char}} = \frac{K}{\pi} L^\frac{3}{2} \left( \frac{12C}{E} \right)^\frac{1}{2}.$$  \hspace{1cm} (4.1)

Substituting 4.1 into 3.1 gives us the characteristic force:

$$F_{\text{char}} = C wt_{\text{char}} L = \frac{wK}{\pi} L^{\frac{3}{2}} C^{\frac{3}{2}} \left( \frac{12}{E} \right)^\frac{1}{2}.$$ \hspace{1cm} (4.2)

Integrating 3.1 from $x = 0$ to $x = L$ gives us the characteristic digging energy:

$$U_{\text{char}} = \int_0^L F_{\text{dig}} dx = \frac{F_{\text{char}}L}{2} = \frac{wK}{2\pi} L^{\frac{3}{2}} C^{\frac{3}{2}} \left( \frac{12}{E} \right)^\frac{1}{2}.$$ \hspace{1cm} (4.3)

For Figure 4-4, we used the nondimensional variables $t^* = \frac{t}{t_{\text{char}}}$, $U^* = \frac{U}{U_{\text{char}}}$. 

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4.1 Force distribution in granular media

In order to simulate the digger, we must first find the probability distribution function of $F_{\text{dig}}$. Mueth et. al. [2] found the force distribution in packings of soda lime glass beads (same substrate as our experiments). The probability $P$ of a grain experiencing a normalized force $f$ is

$$P(f) = 3(1 - 0.75e^{-f^2})e^{-1.5f}. \quad (4.4)$$

This result is not immediately useful; while Mueth studied the forces acting on single grains, hundreds of grains act on our digger. We therefore modify Mueth’s model. We are interested in large forces in the distribution—forces that could buckle the digger. Therefore, we neglected small forces and focused on the exponential tail of the distribution as a first approximation. The quantity $0.75e^{-f^2}$ was dropped in Equation 4.4 and the resulting probability density function was normalized. This we [7] simplified the distribution to an exponential:

$$P_{\text{exp}}(f) = \frac{1}{\theta} e^{-\frac{f}{\theta}}. \quad (4.5)$$

Equations 4.4 and 4.5 are compared in Figure 4-1.

Next, we generalize the distribution to best describe the forces on the digger. The sum of $k$ independent random variables, each with an exponential distribution, is distributed according to the Erlang distribution:

$$P_{\text{Erlang}}(f; k, \theta) = \frac{f^{k-1}}{\theta^k (k-1)!} e^{-\frac{f}{\theta}} \quad (4.6)$$

The Erlang distribution is a special case of the gamma distribution with integer $k$. We generalize for real $k$ and use the gamma distribution to model the forces on the digger:

$$P_{\text{gamma}}(f; k, \theta) = \frac{f^{k-1}}{\theta^k \Gamma(k)} e^{-\frac{f}{\theta}} \quad (4.7)$$

Distribution (4.7) was verified against experimental data in [7] and reproduced in
Figure 4-1: A comparison of the probability density functions Equation 4.4 (red) and 4.5 (blue). In Equation 4.5, $\theta = \frac{3}{4}$ was chosen to match values at $f = 0$.

Figure 4-2. We now proceed to calculate the relationship between $t^*$ and $U^*$ for a variety of digger thicknesses and lengths.

### 4.2 Stochastic algorithm for fixed digging depth

An algorithm was proposed [7] to simulate the stochastic effects of digging. This algorithm increments depth $x$ from 0 to the target depth $D$ by $\Delta x$ each iteration. There are $n$ depth steps, so $n\Delta x = D$. At iteration $i$, the digger is at depth $i\Delta x$, and the axial force $F_i$ is chosen from a Gamma distribution with mean $C_{wti}\Delta x$ and standard deviation $0.2C_{wti}\Delta x$. If $F$ is smaller than a fixed buckling force $F_{\text{buck}} = \frac{E t^3_w}{12(KL)^2}$, the depth $x$ increases by $\Delta x$, and the digging energy $U$ increases by $F\Delta x$. Otherwise, the depth stays constant, but $U$ still increases by $\alpha F_{\text{buck}}\Delta x$, where $\alpha$ is an energy penalty ratio. The algorithm is shown graphically in Figure 4-3. $\alpha$ values $1 \times 10^{-2}$ to 1 were used in the numerical simulations, shown in Figure 4-4. $\Delta x$ is
choosen to be the grain diameter, $d$.

To minimize the digging energy, the digger should not be made too thin—when $t^* \ll 1$, the energy penalty $\alpha F_{buck} \Delta x$ can applied an arbitrarily large number of times. Similarly, the digger should not be made too thick. When $t^* \gg 1$, $F_{dig}$ can be made arbitrarily large. We can therefore expect a thickness $t^*_{\text{min}}$ that minimizes energy $U^*$.

### 4.3 Analysis of algorithm

The total digging energy is the random variable $U = U_{\text{dig}} + U_{\text{buck}}$, where $U_{\text{dig}}$ is the contribution from digging, and $U_{\text{buck}}$ is the contribution due to buckling. We are interested in the expectation of the digging energy $E(U) = E(U_{\text{dig}}) + E(U_{\text{buck}})$.

Let there be $n$ depth steps so that $n \Delta x = D$. Then the mean axial digging force at step $i$ is $Cwti \Delta x$, so
Choose digger dimensions; compute $F_{\text{buck}}$, set depth $x = 0$.

Choose $F_{\text{dig}}$ from a normal distribution, mean $C_{\text{wtx}}$, stdev $0.405C_{\text{wtx}}$.

If $F_{\text{dig}} < F_{\text{buck}}$, increase $x$ by $\Delta x$ and energy $U$ by $F_{\text{dig}} \Delta x$.

If $F_{\text{dig}} \geq F_{\text{buck}}$, increase $U$ by $\alpha F_{\text{buck}} \Delta x$

Figure 4-3: Above left: at each iteration of the algorithm, the digging force $F$ is chosen from a probability distribution function. Below left: based on the size of $F$, the digger may buckle (red) or dig (blue). Buckling increases energy but not depth; digging increases both. Right: flowchart of stochastic algorithm.

\[
E(U_{\text{dig}}) = \sum_{i=1}^{n} C_{\text{wti}} \Delta x \cdot \Delta x = \frac{C_{\text{wti}} (n \Delta x)^2}{2} = \frac{F_{\text{char}} D}{2}.
\] (4.8)

Next, we find an expression for $E(U_{\text{buck}})$. At the $i$th depth step, let $p_i$ be the probability that $F_i < F_{\text{buck}}$. $F_i$ is from a Gamma distribution with mean $\mu = C_{\text{wti}} \Delta x$ and standard deviation $\sigma = 0.2 C_{\text{wti}} \Delta x$. The shape and scale parameters of the Gamma distribution are $k = \frac{\nu}{\sigma^2}$ and $\theta = \frac{\sigma^2}{\mu}$. Then $p_i = Pr(F < F_{\text{buck}}) = \frac{1}{\Gamma(k)} \gamma(k, \frac{F}{\theta})$.

This is the cumulative distribution function of the gamma distribution.

Let $N_i$ be the number of times the digger is stuck at depth step $i$. A digger can be stuck 0 times, 1 time, or any integer number of times. At each timestep, the digger has probability $p_i$ getting stuck again. Then $N_i$ is geometrically distributed, so $E(N_i) = \frac{p}{1-p}$. The force $F$ from consecutive “roll of the dice” are independent, so the buckling energy contribution at depth $i$ is $N_i \cdot \alpha F_i$.

The total energy due to buckling is $U_{\text{buck}} = \sum_{i=1}^{n} \alpha F_i N_i$. For $i \neq j$, $F_i$ and $F_j$ are independent, and $N_i$ and $N_j$ are independent. Similarly, $F_i$ and $N_j$ are independent. Therefore, we have
\[ E(U_{\text{buck}}) = \sum_{i=1}^{n} \alpha E(F_i) E(N_i) \Delta x = C wt \Delta x^2 \sum_{i=1}^{n} \frac{ip_i}{1 - p_i} \quad (4.9) \]

where

\[ p_i = \frac{1}{\Gamma(k)} \gamma(k, \frac{x}{\theta}) \quad (4.10) \]

This method was used to calculate the analytic \( t^* - U^* \) curve in Figure 4-4.

### 4.4 Estimating the energy required to reach each depth in a discrete distribution

Diggers with thicknesses .0508-.1905mm all buckled at depths less than the 75mm limit of our experimental setup. For each of these diggers, five to seven trials were conducted, resulting in a discrete distribution of maximum depths dug. We propose a method to estimate the energy needed to reach each depth in the distribution.

Let \( D_1, D_2, \ldots, D_S \) be the maximum depths dug by a digger over \( S \) trials such that \( D_1 \leq D_2 \leq \cdots \leq D_S \). We can also assume that the difference in depths, \( D_S - D_1 \), is small compared to each of the \( D_i \).

We dig repeatedly until the depth \( D_i \) is reached. Let \( M \) be a random variable representing the number of trials required to reach \( D_i \), and \( E(M) \) the expected number of trials to reach \( D_i \). Then we can estimate the energy to reach \( D_i \) as

\[ U = E(M) U_i, \quad (4.11) \]

where \( U_i \) is the energy of the \( i \)th trial as calculated by integrating the force-depth plot (Figure 2-2).

The probability that \( M = j \) is the probability of hitting a depth of less than \( D_i \) on the first \( j - 1 \) dig attempts, \( (\frac{i-1}{S})^{j-1} \), multiplied by the probability of hitting a depth of \( D_i \) or greater on the last dig attempt, \( \frac{S-i+1}{S} \). Therefore,
Using the definition of expectation, we compute

\[ E(M) = \sum_{j=1}^{\infty} j \cdot \Pr(M = j) \]

\[ = \sum_{j=1}^{\infty} j \left( \frac{i - 1}{S} \right)^{j-1} \left( \frac{S - i + 1}{S} \right) \]

\[ = \left( \frac{S - i + 1}{S} \right) \sum_{j=1}^{\infty} j \left( \frac{i - 1}{S} \right)^{j-1} \]

\[ = \left( \frac{S - i + 1}{S} \right) \left( \frac{S}{S - i + 1} \right)^2 \]

\[ = \frac{S}{S - i + 1}. \quad (4.13) \]

Equations 4.11, and 4.13 were used to compute the data points for diggers with thicknesses .0508-.1905mm in Figure 4-4.
Figure 4.4: $t^* - U^*$ plot for fixed digging depth. Numerical simulations are shown with two curves, with $\alpha = 0.01, 1$. Data from 11 digger thicknesses are shown: diggers with thicknesses .0508-.1905mm buckled at depths less than 75mm, and those with thicknesses .254-.762mm reached 75mm depth without buckling.

Figure 4.5: Left: example force vs. depth graph with superimposed force probability distribution functions (pdfs). Right: force pdfs plotted separately. Note that as depth increases, more force variation occurs is observed, in addition to a larger mean force.
Chapter 5

Discussion and outlook

This thesis investigates axial digging in granular materials from mathematical and physical perspectives. First, an experiment was conducted with thin, plastic diggers to characterize the force response. The experiment was similar to that conducted by Wendell [7]. However, key differences (controlled depth and a guide track) allowed a more detailed investigation of buckling. Next, two analytical avenues of investigation were taken: fix the force and maximize digging depth, and fix the digging depth and minimize digging energy. Both methods gave formulas for calculating the optimal thickness. The method of constant digging depth allowed for a stochastic approach. Finally, these analyses were verified with experimental data. Modeling the digger stochastically was an interesting approach that led to an insightful, yet tractable set of equations that can be used to generate a simulation and process data (see Sections 4.3 and 4.4).

This work may spawn several interesting physical problems. Future work can focus on the following: first, a detailed frequency analysis can be done on the force-distance plot (Figure 2-2). To start, we can estimate the spacing of the force fluctuations to be on the order of grain size. This analysis can be done with different size grains.

Second, the energy required to separate a granular packing along a plane can be investigated. This is analogous to digging with a digger of zero thickness. The first order model used in this thesis gives an energy of zero, but higher order models may be more accurate.
Third, the experiment was run for at most eight trials per buckled digger. More trials, say, on the order of 10-100, can be run to characterize the shape of the depth distribution.

Finally, a model can be developed for a digger buckling with granular constraints. This model would predict the forces on the digger sides at each depth. It may involve modeling the soil as a series of spring and damper systems.
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