Feasibility of Electrostatic Bearings

for Micro Turbo Machinery

by

José Oscar Mur Miranda

B.S.E.E., Massachusetts Institute of Technology (1995)

Submitted to the Department of Electrical Engineering and Computer Science in partial fulfillment of the requirements for the degree of

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at the

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Abstract

The goal of this thesis was to explore the feasibility of augmenting gas journal bearings in the rotor of a microcompressor with electrostatic bearings. To this end, models were developed for the fluid in the gas bearings at low rotor eccentricities and the dynamics of the rotor. An electrostatic bearing consisting of four electrodes surrounding the rotor was designed to optionally provide additional stiffness and damping through electrostatic forces, and it was also modeled. The electrostatic forces were severely limited due to electric breakdown and highly directional. Different equilibrium positions for the rotor were considered and a static point at low eccentricities was finally chosen for detailed study. The stability of the equilibrium in the absence of electrostatic assist was characterized and found to be unstable. Next, the nonlinear dynamics of the augmented bearing were linearized around the equilibrium position in order to design a linear controller for the electrostatic bearing. The linear controller was designed to minimize an error function proportional to both the distance of the rotor from equilibrium, and the magnitude of the actuating electrostatic force. The linear controller was simulated together with the nonlinear characteristic of the electrostatic force using Matlab to assess the performance and limitations of the closed-loop augmented bearing. The augmented bearing could only provide stability for small perturbations. Various suggestions were made on how to increase the electrostatic force of the bearing and whether alternate equilibrium positions for the rotor could work.

Thesis Supervisor: Jeffrey H. Lang
Title: Associate Director, Laboratory for Electromagnetic and Electronic Systems
To my parents, Rogelio Mur and María Teresa Miranda. Without you, I would not be; thanks for being the best parents I could wish for and for your unwavering support of my studies throughout my life.

To my teachers, Jorge de Jesús, for making me the mathematician I am today, and Lydia González, for showing me how to see the world. You are and will always be my teachers.

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Table of Contents

1 Introduction ................................................................................................................. 13
  1.1 Description of Applications ........................................................................... 15
  1.2 Description of Problem .................................................................................. 18
  1.3 Electrostatic Actuation ................................................................................... 19
  1.4 System Block Diagram ..................................................................................... 21
  1.5 Thesis Outline .................................................................................................. 23
2 Electrostatics .............................................................................................................. 25
  2.1 Introduction ....................................................................................................... 25
  2.2 Capacitance ...................................................................................................... 31
  2.3 Electrostatic Force ............................................................................................ 33
  2.4 Inverse Force Conversion ................................................................................ 36
  2.5 Voltage Limiter .................................................................................................. 40
  2.6 Summary ............................................................................................................ 46
3 Dynamics .................................................................................................................... 49
  3.1 Fluid Dynamics .................................................................................................. 49
  3.2 Equations of Motion ......................................................................................... 51
  3.3 Equilibrium Point .............................................................................................. 53
  3.4 Jacobian Matrix ................................................................................................ 56
  3.5 Controller ......................................................................................................... 58
  3.6 Non-Linear Model ............................................................................................ 65
  3.7 Controller with Voltage Conversion ................................................................. 70
  3.8 Summary ............................................................................................................ 70
4 Summary, Conclusions, and Suggestions For Future Work ................................... 73
  4.1 Summary ......................................................................................................... 73
  4.2 Conclusions ..................................................................................................... 76
  4.3 Suggestions For Future Work ........................................................................... 78
5 Simulation Code .......................................................................................................... 83
  5.1 Capacitance ...................................................................................................... 83
  5.2 Electrostatic Forces ......................................................................................... 84
  5.3 Force Inversion and Limiter ............................................................................. 86
  5.4 Natural Dynamics ............................................................................................. 88
  5.5 Jacobian Matrix ............................................................................................... 90
  5.6 Dynamics Simulation ....................................................................................... 91
  5.7 Nonlinear Model .............................................................................................. 93
  5.8 Controller with Voltage Conversion ............................................................... 95

References .................................................................................................................... 97
List of Figures

Figure 1.1: Schematic concept of micro-turbine generator .................................................16
Figure 1.2: Silicon-etched micro-turbine prototype..............................................................17
Figure 1.3: Simplified cross-sectional side view of the rotor inside the stator (compressor design) and relevant degrees of freedom .................................................................18
Figure 1.4: Cross-sectional top view of the rotor and electrostatic bearing elements ....19
Figure 1.5: Block diagram of the proposed control scheme for the electrostatic journal bearing ..........................................................................................................................22
Figure 2.1: Electrode setup for electrostatic bearings..............................................................26
Figure 2.2: Electrical breakdown voltage for various media (Paschen’s curves) ..............28
Figure 2.3: Offset rotor inside stator with θ dependent air gap ............................................32
Figure 2.4: Typical values for the capacitance as a function of rotor position ..................34
Figure 2.5: Electrostatic force on the rotor as it sweeps the line $\alpha=0$ from $r=-9$ mm to $r=9$ mm .................................................................................................................................37
Figure 2.6: Voltage-limited set of forces ...............................................................................42
Figure 2.7: Resulting forces for a set of limited voltages. The position of the rotor is fixed at $\varepsilon=0.01, \alpha=0$ ........................................................................................................43
Figure 2.8: Resulting forces for a set of limited voltages. The position of the rotor is fixed at $\varepsilon=0.5, \alpha=0$ ........................................................................................................44
Figure 2.9: Resulting forces for a set of limited voltages. The position of the rotor is fixed at $\varepsilon=0.5, \alpha=-\pi/4$ ........................................................................................................45
Figure 3.1: Offset rotor inside stator. The relevant state variables and the forces acting on the rotor are shown ........................................................................................................52
Figure 3.2: Stability calculations for the rotor position with different loading forces ....55
Figure 3.3: Eigenvalues of the Jacobian matrix for the incompressible gas bearing model as a function of rotor position ........................................................................59
Figure 3.4: Block diagram of the closed-loop system which combines gas and electrostatic journal bearings ..................................................................................................................60
Figure 3.5: Solution for the incompressible gas bearing model with unlimited permissible electrostatic force. The initial condition is $\varepsilon=0.7, \alpha=-7\pi/8, \nu=0, \omega=0$ ........................................................................66
Figure 3.6: Solution for the incompressible gas bearing model with unlimited permissible electrostatic force. The initial condition is $\varepsilon=0.2, \alpha=-7\pi/8, \nu=0, \omega=0$ ..................................................67
Figure 3.7: Solution for the incompressible gas bearing model using magnitude-limited electrostatic force. Feedback forces were limited to 2 mN. The initial condition is $\varepsilon=0.7, \alpha=-7\pi/8, \nu=0, \omega=0$ ..................................................68
Figure 3.8: Solution for the incompressible gas bearing model using magnitude-limited electrostatic force. Feedback forces were limited to 2 mN. The initial condition is $\varepsilon=0.2, \alpha=-7\pi/8, \nu=0, \omega=0$ ..................................................69
List of Tables

Table 1.1: μEngine and Battery Performance Comparison [1]. ...........................................14
Chapter 1

Introduction

The goal of this thesis is to examine the feasibility of an active electrostatic controller which will assist the stability of the gas bearing in a micro gas turbine engine-generator. The engine-generator is the focus of a multi-disciplinary project at the Massachusetts Institute of Technology. Until now, electric energy generators have been typically large, leading to centralized power generation and large networks to distribute this power. However, modern portable devices require more power than it is possible to carry compactly. Electrochemical cells have been typically used for these applications, but their power density is low, so that the power consumption of portable devices is highly constrained. When larger amounts of power are required, the size and weight of portable equipment are dominated by the power sources. Fuels cells are currently under research, as well as energy storage devices such as flywheels. These methods offer some advantages as well as drawbacks, but still no technology has proven to be a final solution. The micro gas turbine engine-generator project proposes another solution: use a jet engine of less than one cubic centimeter in volume to convert the chemical energy of high energy density materials, such as hydrocarbon fuels or hydrogen, into electricity. Simplified calculations show that such a device would outperform electrochemical cells, as shown in Table 1.1.

Modern fabrication techniques suggest that fabricating the micro jet engine-generator might be possible, but not without considerable engineering effort. The project, then, intends to create a jet engine turbine, the same type as used in airplanes, with a 1 cm diameter and a 3 mm thickness, made out of silicon carbide (SiC), which will generate “10-20 W of electric power while consuming 10 grams/hr of H₂” [2]. The primary sponsor
<table>
<thead>
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<th></th>
<th>μEngine + Fuel</th>
<th>LiSO₂ Battery (BA5590)</th>
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Table 1.1: μEngine and Battery Performance Comparison [1].

for this project, the Army Research Office, intends to use such a device to power the portable electronic systems an infantry soldier could carry into battle.

Many other applications have surfaced while designing the micro jet engine. The ARO has also funded the construction of an electric microcompressor. In the micro gas turbine engine-generator, the turbine blades use a pressure gradient to induce rotation, and this rotation is then converted into electricity through an electrostatic generator. With minor modifications, the generator can be turned into a motor and the turbine into a compressor, thus creating a micro-electric air compressor. This is the first expected product in the project agenda, so that this thesis will in fact use the dimensions relevant to the compressor, with the understanding that the micro-engine and its derived devices offer a similar challenge and the bearing designed here can be easily adapted to any of them.

NASA is funding a project in which the microengine compressor is used as pump, enabling the creation of a rocket motor for space that is a fraction of the weight and size of the ones currently in use. Such rockets have applications as directional thrusters in satellites and other spacecraft, where both space and weight are at a premium.

Finally, the electric microcompressor can be designed to actuate a thermodynamic refrigeration cycle, which can be used to cool down portable or small devices. Applica-
tions include cooling of missiles prior to launching them, and cooling of modern high-frequency microchips.

Although not under current research, other applications seem possible. In fact, the micro gas turbine engine without the generator is projected to produce, as a micro jet engine, 0.125 N of thrust with a mass of about 100 mg (the engine itself weighs less than 20 mg). These numbers yield an amazing 100:1 thrust-to-weight ratio. The best modern aircraft engines have a 10:1 thrust-to-weight ratio [3]. Simplified calculations show that 2,000 microengines operating in parallel could lift masses of around 60 pounds, such as small boxes or luggage. Similarly, 100,000 microengines can lift about 3,000 pounds, which is the mass of a modern automobile, and about 10 million could propel an object of similar mass into orbit. The added advantage is that such propulsion systems are small and lightweight, and, since they are a distributed system, offer a much greater reliability.

1.1 Description of Applications

The micro gas turbine engine-generator concept, shown in Figure 1.1, internally houses a rotor composed of five distinct micro-fabricated layers bonded together. The microscale gas turbine engine generator requires combustor exit temperatures of 1300-1700 K and rotor peripheral speeds of 300-600 m/s in order to maintain the same power density as a macroscale gas turbine [4]. The wafers will be made out of silicon carbide, to provide appropriate resistance to the high temperatures (around 1700 K) expected during operation. It is worthwhile to note that other microdevices, such as the electric microcompressor, will not need to withstand extreme temperature and can be constructed out of silicon. Each wafer has a thickness of 0.3 mm, giving the rotor a total axial length of 1.5 mm. The design diameter of both the microengine compressor and turbine blades is 4 mm. The shaft connecting the microengine compressor blades and the turbine blades is thinner, giv-
ing the rotor a dumbbell shape. This rotor is housed within a stator. The gap between the stator and the rotor will be in the order of 10 micrometers. The microengine follows the same basic thermodynamic cycle as any other jet engine: air flows through an intake, and then passes through a compressor which increases its pressure in a 4:1 ratio [2]. High pressure air is mixed with hydrogen and ignited in a annular combustion chamber surrounding the stator. Through the combustion, the gas expands and provides the energy to rotate the shaft at 2.4 million rpm by passing through turbine blades. A 2.4 million rpm rotational speed implies a tip speed of 500 m/s, which is transonic operation, a thermodynamic requirement.

The design of the electric microcompressor is derived directly from the microengine. However, since no combustion is involved, the structure does not have to withstand high temperature and can be constructed using silicon and more established microfabrication techniques than those required by the silicon carbide used in the microengine. The radius
of the electric microcompressor is kept at 2 mm, but its length is decreased to 0.3 mm, since it can be constructed out of a single silicon wafer.

The rocket pump and the refrigerator are also derived from the microengine design, thus retaining most of its virtues and challenges. In tackling the different engineering challenges necessary for the operation of these devices, similar techniques are used, and the design solutions can typically be shared between designs. For instance, the strength-to-density ratio of single crystal silicon is superior to that of macroscopic materials and can withstand higher stresses [3]. New techniques in micro-fabrication using silicon carbide and other materials are also being developed [2], which allow deeper etching and tighter tolerances. Figure 1.2 shows a micro-fabricated prototype of the micro gas turbine etched in silicon.
Figure 1.3: Simplified cross-sectional side view of the rotor inside the stator (compressor design) and relevant degrees of freedom. Dimensions are not to scale. The air gap has been greatly enlarged for clarity.

1.2 Description of Problem

One of the main concerns of the project is how to keep the rotating and stationary elements of the micro-devices apart. Rolling element bearings cannot be constructed easily using micro-fabrication techniques and would be unreliable and short-lived at 2.4 million rpm due to wear and friction. Air bearings have a clear advantage and have been successfully used in devices of similar size. Air bearings provide both high loading forces and ease of fabrication. However, the theoretical and experimental studies of air bearings are for speeds and geometries much different than those encountered in this project. Current research shows that air bearings may be unstable in these conditions. Another form of bearing may be necessary to assist and provide stability [5].

Any bearing system, in general, needs to control the position of the rotor in space to keep it from touching the wall of the stator. This includes six degrees of freedom: displacement in x, y and z, and rotation in $\theta_x$, $\theta_y$ and $\theta_z$, as shown in Figure 1.3. The rotational angle of the rotor, $\theta_z$, will be unconstrained. It is assumed that another mechanism, not related to bearings, will control the rotor speed, $\omega_z$. Gas journal and thrust bearings
Figure 1.4: Cross-sectional top view of the rotor and electrostatic bearing elements. The air gap and the spacing between the ends of the electrodes in the stator has been greatly exaggerated for clarity.

have been designed for the remaining five degrees of freedom. The gas thrust bearing appears stable and strong enough to provide adequate control of vertical displacement ($z$), and wobbling ($\theta_x$ and $\theta_y$). However, the gas journal bearing, designed to control transverse displacement ($x$ and $y$), may not be sufficiently stable. Therefore, this thesis explores the concept of an electrostatic journal bearing as a backup technology, which should operate in parallel with the gas journal bearing to provide sufficient overall stability in the $x$ and $y$ directions only.

1.3 Electrostatic Actuation

The gas/electrostatic journal bearing is shown in Figure 1.4. The center of the rotor might
be offset from the center of the housing if an impulse or other disturbance is applied. It is assumed that the gas thrust bearing will keep the rotor centered in z, consequently, all displacements studied here will be in the x and y directions alone. The electrostatic bearing is formed in parallel with the gas bearing by placing three or more electrodes in the inner surface of the stator. The electrodes are driven by voltage sources to produce electrostatic forces between the electrodes and the rotor. The direction and magnitude of the electrostatic force can be controlled by adjusting the voltage sources such that the rotor will travel back to a stable equilibrium position or trajectory.

The voltage across the gap is limited to 300 V to avoid electric breakdown. Assuming an average gap distance of 3 μm, this yields an electric field of $10^6$ V/m, which is typical for a micron-scale gap [6,7]. The maximum electrostatic pressure which can be exerted on the surface of the rotor by this electric field is given by [8]

$$P = \frac{1}{2} \varepsilon_0 |E|^2 = 4.4 \cdot 10^4 \text{N/m}^2$$

It is important to note that this pressure is a theoretical maximum, and realizable pressures will often be smaller. In particular, electrostatic forces as used here are always attracting, and it is reasonable to expect that electrostatic forces will be desired were the gap is largest, in order to pull the rotor back toward the center. As a worst case scenario, consider the case in which the rotor is fully offset and its gap is as large as 20 μm. In this case, assuming 300 V limit, the restoring electric field across the enlarged gap is reduced to $1.5 \cdot 10^7$ V/m. This field yields an electrostatic pressure of 996 N/m². Since typical gas bearing pressures range from $2.5 \cdot 10^3$ N/m² to $2.4 \cdot 10^4$ N/m² at medium eccentricities, the electrostatic bearing may provide limited assistance to the stability of the rotor position assuming incompressible behavior for the fluid in the bearing.
1.4 System Block Diagram

To control the voltage on the electrodes, and thus, the electrostatic force on the rotor, a linear controller will be employed. This controller is the simplest and most common controller. Considerable amounts of research exists to support its analysis and design in a linear setting [9]. However, the electrostatic journal bearing application is nonlinear, and a linear controller is only guaranteed to work in a linear system. Furthermore, the linear controller assumes the availability of an infinite amount of controlling force, and such a force is not available in the electrostatic journal bearing. Given these limitations, there are more advanced control schemes which may work better than a linear controller. However, in general, nonlinear systems can be treated as linear system for sufficiently small perturbations, and similar control designs have worked effectively [9]. Furthermore, the linear controller should not be discarded immediately under the assumption that it will fail for any particular reason. Given the tractability of linear systems, the performance the linear controller can be exhaustively analyzed. From this analysis, a decision can be made whether the linear controller is appropriate or a more advanced control scheme is required.

A block diagram of the system to be explored is shown in Figure 1.5. The control loop will begin with a sensor which will give the position of the rotor. The design on a position sensor will not be explored in this thesis. However, it is assumed that such a sensor exists. The rotor position is then compared with a predetermined position, or setpoint. Unless the rotor is precisely at the setpoint, there will be a difference between the setpoint and the estimated position, or error. The error is fed to a function which produces an output proportional to the error. This function is the linear controller. The output will be a commanded force which will decrease the error for a local linear system. However, this force must be actuated through a set of voltages applied to the electrodes. These voltages will have limitations, and they will be accordingly scaled, if need be, to meet these limitations.
Figure 1.5: Block diagram of the proposed control scheme for the electrostatic journal bearing.

Thus, the commanded force must go through a lookup table in order to produce a set of voltages that best compromises the commanded force with the system limitations. Through the electrodes, these voltages will turn into an effective force applied on the rotor. Any other forces or disturbances may be added to the control force if needed. The dynamic response of the rotor is then calculated using these forces and its current position.
and velocity, or *state*, as input. With this information, the new rotor velocity and position can be computed, thus closing the loop. Artificial noise may also be added to the model to account for a noisy measurement. This model will serve to support the design of the system.

1.5 Thesis Outline

Chapter 2 of the thesis will develop the necessary mathematical tools to understand the voltage to force conversion shown in Figure 1.5 and the limitations imposed by electric breakdown (Paschen’ curves), the capacitance between the actuating electrodes and the rotor, and Gauss’ law. Mathematical transformations of voltage to force and force to voltage will be devised, together with the limitations imposed by the physics of the system. A simple characterization of the nonlinearities of the electrostatic actuators will be explored.

Chapter 3 will tackle the dynamics of the rotor motion inside the gas/electrostatic journal bearing. A simplified fluid model, appropriate for low eccentricities, will be justified and employed for the model of the dynamics of the rotor position. A linearization process for the nonlinear system will be explained and a linear controller will be designed and tested. Using the results of Chapter 2, a simplified nonlinear system will be tested to explore the effects of the nonlinearities present in the electrostatic journal bearing.

Chapter 4 will analyze the results of the previous two chapters in an attempt to obtain a global understanding of the gas/electrostatic journal bearing. A general methodology for designing and testing a linear controller will be outlined. Several suggestions for the improvements and design considerations of the electrostatic bearing will be made, as well as preliminary comments regarding equilibrium positions and fabrication process. To look forward, the electrostatic journal bearing will prove effective in providing stability to the gas journal bearing assuming a simplified nonlinear model. However, the electrostatic
forces available are highly directional and severely limited in magnitude. The reasons for these limitations are analyzed and suggestions are made regarding how to overcome or improve some of these limitations in future designs.

Chapter 5 contains all of the Matlab code employed in the analysis and modeling of the electrostatic bearing and the rotor dynamics. Its sections are referenced throughout the thesis as required.
Chapter 2

Electrostatics

2.1 Introduction

The basic idea behind electrostatic actuation is explained in Section 1.3. In this section, the explanation will be expanded to include greater physical detail and mathematical rigor. An accurate model for the electrostatic forces will be developed and the appropriate assumptions will be specified.

Recall that the electrostatic bearing consists of three or more electrodes placed in the stator around the rotor such that independent voltages may be applied to each electrode. For this thesis, the electrostatic bearing has four electrodes, providing a good balance between design simplicity and reliability, and limitations on the electrostatic force. Figure 2.1 shows the basic setup for the electrostatic journal bearing using four electrodes.

The electrostatic forces in Figure 2.1 are always attracting. This implies that the electrostatic forces cannot "push" the rotor; they can only "pull" it. Furthermore, the electrostatic forces increase as the inverse square distance. This means that for a fixed voltage, electrostatic forces are unstable. If the rotor moves closer to any electrode with a voltage differential across the gap, the electrostatic forces will increase dramatically so that the rotor will be pulled even harder toward the electrode. For these forces to be stabilizing, the time response of the controller and the electrodes must be much faster than the dynamics of the rotor. Ignoring the assistance of gas or any other type of journal bearing, if the rotor is moving toward a charged electrode, the voltage across the gap must decrease much faster than the force on the rotor due to the electrode. That is, the bandwidth of the controller must be much larger than the dynamics of the rotor. For this thesis, the controller band-
Figure 2.1: Electrode setup for electrostatic bearings. The air gap and the spacing between the ends of the electrodes in the stator has been greatly exaggerated for clarity.

width will be assumed to be as large as necessary, but it must be verified that the electronic implementation of the final design does fulfill this requirement.

Furthermore, if the electrodes are turned off, electric fields must not be present in the gap. This, in fact, may happen if the rotor is electrically charged. Such a charge on the rotor will always create electric fields across the gap. This is equivalent to having nonzero voltages across the gap, and for the reasons explained above, forces created by this field will make the position of the rotor unstable. Educated assumptions regarding both the bandwidth of the controller and the presence of free charges on the rotor will be made in this thesis.
Another major limitation of the system is electric field breakdown. If the field exceeds a certain critical value, the electrons will be ripped from the material and “jump” through the gap. If there are sufficient gas molecules in the gap, then this process will avalanche and generate a spark. The associated thermal energy can easily destroy the rotor. In order to avoid this problem, the gap voltages must be maintained below this critical voltage, or breakdown voltage. This voltage can be computed from Paschen’s curves in Figure 2.2 [10]. These curves show the maximum voltage that can be applied across a gap without breaking down the gas in the gap. The gas in the microengine bearing will be air, and since the gap is of 3-10 micrometers, the voltage must be kept below 375V to avoid breakdown. In order to provide a safety margin, the voltages considered here will be limited to 300V. In this way, it is certain that the voltage across the gap will not generate sparks.

Another constraint imposed on the air-gap voltages is Gauss’ law. It states that the net displacement flux through a closed surface is proportional to the charge inside the surface, or

\[ \oint_{\text{surface}} \epsilon_0 \mathbf{E} \cdot d\mathbf{A} = Q_{\text{inside surface}} \] \hspace{1cm} (2.1)

To apply Gauss’ law, end effects will be ignored and the field will be treated as radial since the distance between the electrodes and the rotor is on the order of 10 micrometers. Another implication of this assumption is that there will not be cross-capacitance between adjacent electrodes. The electric field in the gap is therefore

\[ \mathbf{E} \mathbf{=} \frac{-V}{g(\theta)} \hat{r} \hspace{1cm} (2.2) \]

where \( V \) is the voltage difference across the gap from the electrode to the rotor, and \( g(\theta) \) is the gap distance as a function of rotor position. Substituting this electric field into Gauss’ law yields
**Figure 2.2:** Electrical breakdown voltage for various media (Paschen’s curves). The units are pressure times distance. The air gap will be as small as 1 μm at a pressure of 4 atm. Converting units, 4 μm•atm equals 3.04 mm•mm Hg. Figure taken from [10].
\[ \oint_{g_{\text{gap}}} \varepsilon_0 \frac{-V}{g(\theta)} dA_{\text{gap}} = Q_{\text{inside rotor}} \] (2.3)

The gap voltage remains constant across each electrode. Therefore, the gap voltage can be extracted from the integrand for each electrode according to

\[ -V_1 \int_{\text{electrode 1}} \varepsilon_0 \frac{dA}{g(\theta)} - V_2 \int_{\text{electrode 2}} \varepsilon_0 \frac{dA}{g(\theta)} - V_3 \int_{\text{electrode 3}} \varepsilon_0 \frac{dA}{g(\theta)} - V_4 \int_{\text{electrode 4}} \varepsilon_0 \frac{dA}{g(\theta)} = Q_{\text{inside rotor}} \] (2.4)

It will be shown that

\[ \int_{\text{electrode}} \varepsilon_0 \frac{dA}{g(\theta)} = C_{\text{electrode}} \] (2.5)

so that Equation 2.4 turns into

\[ -C_1 V_1 - C_2 V_2 - C_3 V_3 - C_4 V_4 = Q_{\text{inside rotor}} \] (2.6)

or,

\[ -C^T V = Q_{\text{inside rotor}} \] (2.7)

in matrix notation.

It was discussed that charge present in the rotor leads to instability. To simplify the analysis, it will be assumed that the rotor will remain uncharged, that is, the charge inside the rotor will be zero at all times. However, it cannot be assured that the rotor will remain uncharged during its operation. In fact, it is likely that the rotor might become charged through triboelectrification with the air. That is, the spinning rotor might pull off electrons from the air as it spins by, in much the same way a balloon can become charged if it is rubbed against a sweater. Such analysis involving a charged rotor may have to be performed during later stages of design of the microdevices as it may have an important effect on the behavior of the bearing. This assumption leads to the final form of Gauss’ law for the electrostatic journal bearing:

\[ C^T V = 0 \] (2.8)
In the above analysis, the rotor is assumed to be a perfect conductor. Since the rotor is spinning at 2.4 million rpm, the surface which the electrode sees is moving at approximately 500 m/s. In order to assume that the rotor is a perfect conductor, the time it takes for the electrons to conduct over the surface of the rotor must be much faster than the surface speed of the rotor so that charges easily mirror those on the controlled electrodes. A rotational speed of 2.4 million rpm means the rotor will make 40,000 revolutions per second, or one revolution in 25 μs. If it is required that the electrons relax at least a hundred times faster than 25 μs, and it is assumed that the electrons settle within five time constants, a time relaxation constant of at most 0.05 μs is required, or,

\[ 5τ < 0.25μs \] (2.9)

The charge relaxation time constant can be approximated using the capacitance of the gap and the resistance of the surface of the rotor. The largest possible capacitance is computed by assuming a gap distance of 1 micron and treating the gap as a parallel plate capacitor according to

\[ C = \frac{ε_0 A}{g} = \frac{8.85 \cdot 10^{-12} F/m \cdot 0.3 \cdot 10^{-3} m \cdot (π/2 \cdot 2 \cdot 10^{-3} m)}{1 \cdot 10^{-5} m} = 8.3 \cdot 10^{-12} F \] (2.10)

Using Equations 2.9 and 2.10, and the fact that \( τ=R/C \),

\[ R < \frac{0.05 \cdot 10^{-6} g}{8.3 \cdot 10^{-12} F} = 6 KΩ \] (2.11)

where R is an effective rotor resistance. Since this is an order of magnitude calculation, the simple geometry of a rectangle of length l and cross-section A will be used to estimate R according to

\[ R = \frac{V}{I} = \frac{EL}{JA} = \frac{ρ}{l} \] (2.12)

where E is the electric field and J is the current density through the rotor. The last step is justified by noting that the path of the current (l) is proportional to the radius of the rotor,
while the area of the conductor (A) is proportional to the radius and the length (L) of the rotor. The radius proportionality cancels, leaving a resistance proportional to resistivity over length. Substituting this proportionality back into (2.11), and using a rotor length of 0.3 mm yields

\[
\frac{\rho}{L} < 6K\Omega \Rightarrow \rho < 6K\Omega \cdot 0.03\text{cm} = 180\Omega \cdot \text{cm}
\]

(2.13)

Wafers can easily be obtained with impurity concentrations on the order of \(10^{17} \text{ cm}^{-3}\). At this concentration, Silicon doped with a p-type material (Boron) has a resistivity of 0.2 \(\Omega\cdot\text{cm}\) at 300 K, which is 900 times less than required. Silicon doped with a n-type material (Phosphorous) has a resistivity of 0.09 \(\Omega\cdot\text{cm}\) at 300 K, which is 2,000 times less than required [11]. Thus, this analysis shows that the rotor may be assumed to be a perfect conductor.

### 2.2 Capacitance

If the assumptions of Section 2.1 hold true and, thus, the rotor may be treated as a perfect conductor, then the four electrodes will create four capacitors with the rotor being the common node to all. Recall that the capacitance of each electrode must be computed in order to fulfill Gauss' law (Equation 2.7).

In order to find the capacitance of each electrode, the curvature is ignored since the gap is much smaller than the radius of the rotor, and a parallel plate capacitor is integrated across the arc length of each electrode according to

\[
dC = \frac{\varepsilon_0 dA}{g} \Rightarrow C = \int_{\text{Arc length}} \frac{\varepsilon_0 LR}{g(\theta)} d\theta
\]

(2.14)

where L is the axial length of the rotor.

An expression for the gap as function of rotor position is now derived. Figure 2.3 defines the relevant variables to be used in this derivation. Define
\[ R_O - R_i \equiv \bar{g} \]
\[ \varepsilon \equiv \frac{s}{\bar{g}} \]  
(2.15)

Then, the outer edge of the gap is described by
\[ r = R_O \]  
(2.16)

and the inner edge is
\[ (x - s \cos(\alpha))^2 + y - s \sin(\alpha))^2 = R_i^2 \]  
(2.17)

Substituting polar expressions for \( x \) and \( y \), and after some manipulation,
\[ r = s \cos(\theta - \alpha) + R_i \sqrt{1 - \left( \frac{s}{R_i} \right)^2 (\sin(\theta - \alpha))^2} \]  
(2.18)

over the rotor surface. Since
\[ \frac{s}{R_i} \ll 1 \Rightarrow r \equiv s \cos(\theta - \alpha) + R_i \]  
(2.19)
Then

\[ g(\theta) = R_0 - r = R_0 - R_1 - s \cos(\theta - \alpha) = \tilde{g} - \varepsilon \tilde{g} \cos(\theta - \alpha) \]  \tag{2.20}

Therefore,

\[ g(\theta) \equiv \tilde{g}(1 - \varepsilon \cos(\theta - \alpha)) \]  \tag{2.21}

Thus, the only \( \theta \)-dependent element is the gap, and the integral of Equation 2.14 becomes:

\[ C_n = \frac{\varepsilon_0 LR}{\tilde{g}} \int_{\frac{\pi}{2}}^{\frac{\pi}{2} \pi} \frac{1}{1 - \varepsilon \cos(\theta - \alpha)} \, d\theta \]  \tag{2.22}

This integral evaluates to

\[ C_n = \frac{\varepsilon_0 LR}{\tilde{g}} \left( \frac{2}{\sqrt{1 - \varepsilon^2}} \arctan \left( \frac{1 + \varepsilon \tan(\theta - \alpha)}{\frac{1}{\varepsilon} \tan \left( \frac{\theta - \alpha}{2} \right)} \right) \right)_{\frac{\pi}{2}}^{n \frac{\pi}{2}} \]  \tag{2.23}

The Matlab code used to evaluate the capacitance is given in Section 5.1.

Figure 2.4 shows the capacitance of one electrode as a function of rotor position. A negative distance implies that the rotor is moving away from the electrode. Notice that the capacitance increases dramatically as the eccentricity approaches one. The large capacitances will lead to large voltages across the gap in order to fulfill Gauss' law, and this will limit the maximum forces which may be applied on the rotor.

2.3 Electrostatic Force

The electrostatic force on the rotor as a function of rotor position and the electrode voltages must be found in order to obtain a dynamic model. Ignoring curvature and end effects as before, the field generated across the gap will be

\[ \tilde{E} = \frac{-V}{g(\theta)} \hat{r} \]  \tag{2.24}

where \( V \) is the differential voltage across the gap, \( g(\theta) \) is the gap distance as a function of
Figure 2.4: Typical values for the capacitance as a function of rotor position. The Matlab code used to generate this plot is in Section 5.1.
angle, and the electric field is purely radial. The electrostatic pressure on the rotor due to this field will be

\[ P = \frac{1}{2} \varepsilon_0 \left| \mathbf{E} \right|^2 \rho \]  

(2.25)

In order to find the force on the rotor, the rotor is assumed to be a rigid body, since its bending modes have frequencies on the order of megahertz [1] rendering the modes unobservable at the frequencies of interest. The forces are decomposed in the x and y directions and integrated across the capacitor to obtain the total force on the rotor. Thus, for each electrode,

\[ F_x = \int_{\theta_1}^{\theta_2} \frac{1}{2} \varepsilon_0 \left( \frac{V}{g(\theta)} \right)^2 \cos(\theta) LR d\theta \]

\[ F_y = \int_{\theta_1}^{\theta_2} \frac{1}{2} \varepsilon_0 \left( \frac{V}{g(\theta)} \right)^2 \sin(\theta) LR d\theta \]  

(2.26)

Since both the electrode and the rotor are perfect conductors, and the curvature can be ignored, the voltage across each capacitor is independent of \( \theta \). After substituting the previous expression for the gap separation from Equation 2.21,

\[ F_x = \frac{\varepsilon_0 LRV^2}{2g^2} \int_{\theta_1}^{\theta_2} \frac{\cos(\theta)}{(1 - \varepsilon \cos(\theta - \alpha))^2} d\theta \]

\[ F_y = \frac{\varepsilon_0 LRV^2}{2g^2} \int_{\theta_1}^{\theta_2} \frac{\sin(\theta)}{(1 - \varepsilon \cos(\theta - \alpha))^2} d\theta \]  

(2.27)

These integrals do have closed form solutions, but they are too complex to be manageable. In this thesis, these integrals will be evaluated numerically every time it is necessary.

The square of voltage may be extracted as an external factor from Equation 2.27. To do so, define

\[ h_{1n} = \frac{\varepsilon_0 LR}{2g^2} \int_{(n-1)\frac{\pi}{2}}^{n\frac{\pi}{2}} \frac{\cos(\theta)}{(1 - \varepsilon \cos(\theta - \alpha))^2} d\theta \]  

(2.28)
and similarly,

\[ h_{2n} = \frac{\varepsilon_0 LR}{2g^3} \int_{\frac{n\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(\theta)}{(1 - \varepsilon \cos(\theta - \alpha))^2} d\theta \]  

(2.29)

For a given rotor position, Equation 2.28 and 2.29 define linear transformations from the square of the voltage differences across the capacitor to the resultant forces on the rotor according to

\[
\begin{bmatrix}
F_x \\
F_y
\end{bmatrix} =
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & h_{14} \\
h_{21} & h_{22} & h_{23} & h_{24}
\end{bmatrix}
\begin{bmatrix}
V_1^2 \\
V_2^2 \\
V_3^2 \\
V_4^2
\end{bmatrix} \Leftrightarrow F = H
\begin{bmatrix}
V_1^2 \\
V_2^2 \\
V_3^2 \\
V_4^2
\end{bmatrix}
\]  

(2.30)

Figure 2.5 shows a simple test case for the force conversion in order to get an estimate of the available electrostatic forces. Notice that if the rotor moves along the line \(\alpha = 0\) that cuts the stator in two, the "mirror" electrodes are going to see equal capacitances. Thus, equal and opposite voltages may be applied to two of them to see how much force can be exerted on the rotor. For this example, \(V_1 = 300\) V and \(V_4 = -300\) V. Even though the force approaches 20 mN in this example, typical capacitances and Gauss' law limit the electrostatic force to approximately 2 mN, as will be seen in Section 2.5. The Matlab code used to generate the H matrix is given in Section 5.2.

2.4 Inverse Force Conversion

The transformation from the voltages applied across the gap to the forces acting on the rotor was derived in Section 2.3. However, the linear controller will actually specify that certain forces be applied to the rotor. Thus, the inverse transformation must be derived in order to find the voltages necessary to produce a given force. These necessary voltages are not unique. However, it is at least desired to minimize the maximum gap voltage among
Electrostatic Force on the Rotor with $V_1=300$, $V_4=-300$

![Graph showing electrostatic force on the rotor](image)

**Figure 2.5**: Electrostatic force on the rotor as it sweeps the line $\alpha=0$ from $r=-9 \mu m$ to $r=9 \mu m$. Even though the force approaches 20 mN in this example, typical capacitances and Gauss’ law limit the electrostatic force to approximately 2 mN, as will be seen in Section 2.5. The Matlab code used to generate this plot is in Section 5.2.

all solutions to avoid breakdown. This reduces the solution set, but still the solution is not unique.

The solution is not unique because there are only three constraints, $F_x$, $F_y$ (Equation 2.30) and Gauss’ law,

$$C^T V = 0$$

(2.31)

yet there are four gap voltages which must be determined from the constraints. Further-
more, since the sign of all voltages may be negated without altering the force, the solution is not unique even after imposing the desired constraint of minimizing the maximum gap voltage. To complicate matters more, the problem of inversion is significantly nonlinear. Consequently, an analytical solution to this problem has not been found at this point in time due to the complexity of the problem. Instead, the inversion is computed numerically whenever needed using Matlab's least-squares minimization function \texttt{fmins}.

For a given rotor position, the capacitances are fixed, and the set of voltages that fulfill Gauss' law can be found. Given that Gauss' law, Equation 2.31, is of rank one, and that the space of possible voltages is four-dimensional, the solution lies in a three-dimensional subspace. This solution is any linear combination of three independent vectors in the voltage space. For example, the vector $V$ of four air gap voltages can be expressed as

$$ V = \begin{bmatrix} V_{\text{null } 1} & V_{\text{null } 2} & V_{\text{null } 3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} $$ (2.32)

where $V_{\text{null } 1}$, $V_{\text{null } 2}$, and $V_{\text{null } 3}$ are linearly independent vectors of air gap voltages which span the nullspace of Gauss' law, Equation 2.31, such that

$$ C^T \begin{bmatrix} V_{\text{null } 1} & V_{\text{null } 2} & V_{\text{null } 3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} $$ (2.33)

Substituting these vectors in the voltage-to-force transformation of Equation 2.30,

$$ F = HG \begin{bmatrix} V_{\text{null } 1} & V_{\text{null } 2} & V_{\text{null } 3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} $$ (2.34)

where every null vector in the equation is a vector of four air gap voltages, $F$ is the vector of the two electrostatic forces in cartesian coordinates, and $G$ is a nonlinear function defined as
\[
G \begin{bmatrix} V_A \\ V_B \\ V_C \\ V_D \end{bmatrix} = \begin{bmatrix} V_A^2 \\ V_B^2 \\ V_C^2 \\ V_D^2 \end{bmatrix} \tag{2.35}
\]

The problem of inverting Equation 2.34 for \( a, b, \) and \( c \) is a nonlinear problem involving three unknowns and two constraints. Thus, the solution is not unique and has at least one degree of freedom. Ideally, the solution process should yield the "optimal" solution. The "optimal" solution is that which has the smallest possible maximum air gap voltage among the four voltages to minimize the effects of electrical breakdown. At this point in time, finding this optimal solution involves time-consuming algorithms inappropriate for a first design of the electrostatic journal bearing. Instead, this equation is solved by performing a three-dimensional minimization over the space of square differences between the required forces and those created by the electrode voltages. The Matlab code used to perform this minimization is in Section 5.3, and it seeks to find the voltage combination vector \([a \ b \ c]^T\) which minimizes

\[
\left( HG \begin{bmatrix} V_{\text{null}1} & V_{\text{null}2} & V_{\text{null}3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} - F \right)^2 \tag{2.36}
\]

over the space of all air gap voltage vectors \( V \) which satisfy Gauss' law, as defined in Equation 2.32. There are an infinite number of solutions to this problem, and the minimization will again be attracted to a nearby solution, though not necessarily the "optimal" solution. In an effort to find the smallest possible solution, the initial condition for the voltages are set to zero.

After Equation 2.34 is properly inverted, the resulting voltage vector may contain air gap voltages which exceed the breakdown voltage of the gap. That is, it is possible for the controller to request a force that cannot be implemented given the air gap voltage limit.
imposed by electrical breakdown. To treat this case, the resulting voltage vector is then scaled down so that its largest air gap voltage equals the permissible gap voltage, taken here to be 300 V. This scaling is done in such a way so as to reduce the force magnitude but preserve its direction, and is described further in Section 2.5.

The numerical implementation of minimizing Equation 2.36 is ill-conditioned in the sense that there is no guarantee of finding the "optimal" solution involving the minimization of the maximum air gap voltage. The solution found by this procedure fulfills Gauss' law and matches the forces $F_x$ and $F_y$, but does not necessarily minimize the maximum absolute gap voltage. Thus, undesirably large voltages result in the solution. When these voltages are scaled down to 300 V, the resulting magnitude of force is considerably smaller, and unnecessarily so. However, the set of numerical results which are attracted to an "optimal" solution are sufficient to clearly outline the achievable forces within a 300 V limit. This limit on achievable electrostatic forces is then used in the dynamic simulations in place of a real-time limited force inversion which may be overly pessimistic. As Figures 2.8 and 2.9 show, a reasonable maximum force for eccentricities near 0.5, for example, is 2 mN; and this limit will be used in the dynamic simulations in Chapter 3. The electrostatic force does have a peak force considerably larger than 2 mN, but the force is achievable in the worst possible direction, namely that direction in which the rotor is closest to the stator. It is in the other directions that the larger forces are needed.

2.5 Voltage Limiter

Since the voltage difference across the gap can never exceed 300V, this sets a hard limit on the maximum force the electrodes can apply to the rotor. This limit cannot be kept by restricting each individual gap voltage since the resulting set of voltages would likely violate Gauss' law. In reality, applying a set of voltages that appears to violate Gauss' law
will change the voltage of the rotor such that Gauss' law is preserved. This change of voltage in the rotor will not only decrease the magnitude of the applied force, but will also change the direction of the applied force. This might be catastrophic. A better approach is to compute the maximum magnitude of the force that may be applied within these limits, keeping the direction of the applied force fixed. This is done by recognizing that any set of voltages may be multiplied by a constant without changing its direction, and it will still fulfill Gauss' law. For example,

\[
\begin{align*}
\text{If } C^T V = 0 & \quad \Rightarrow \quad C^T aV = aC^T V = 0, \quad a \in \mathbb{R} \\
\begin{bmatrix}
V_1^2 \\
V_2^2 \\
V_3^2 \\
V_4^2
\end{bmatrix} & \Rightarrow \\
\begin{bmatrix}
(aV_1)^2 \\
(aV_2)^2 \\
(aV_3)^2 \\
(aV_4)^2
\end{bmatrix} & = H a^2 \begin{bmatrix}
V_1^2 \\
V_2^2 \\
V_3^2 \\
V_4^2
\end{bmatrix} = a^2 H a^2 \begin{bmatrix}
V_1^2 \\
V_2^2 \\
V_3^2 \\
V_4^2
\end{bmatrix} = a^2 F, \quad a \in \mathbb{R}
\end{align*}
\]

Thus, if there exists a set of voltages that fulfills a given required force magnitude and direction, but one or more of its voltages exceed 300V, the voltages are multiplied by the ratio between 300V and the maximum required voltage in order to arrive at another set of voltages that do not exceed 300V and whose resultant force is of lesser magnitude but identical direction. The Matlab code used to compute the resulting set of limited voltages is in Section 5.3.

Figure 2.6 shows the result of the minimization procedure for a force of 5 mN which rotates from -\(\pi\) to \(\pi\), with the rotor fixed at \(\varepsilon=0.01\) and \(\alpha=0\). The resulting set of voltages is shown, marked with x's. This set of voltages exceeds 300V. It is then limited as described above, and the resultant voltages, marked with o's, are shown superimposed. Figure 2.7 shows the forces produced by the set of limited voltages shown in Figure 2.6. The original required force of 5 mN is shown as a reference.
Force to Voltage Conversion with Limiter, $\varepsilon=0.01$, $\alpha=0$, $|F|=5\, \text{mN}$

**Figure 2.6**: Voltage-limited set of forces. Values marked with x's are not limited. Values marked with o's are voltage-limited to 300V as explained above. The Matlab code used to generate this plot is in Section 5.3.

In Figures 2.8 and 2.9, the rotor is held at $\varepsilon=0.5$. The force is shown for $\alpha=0$ and $\alpha=-\pi/4$. Then, the inverse transformation is carried out for a force of 15 mN which rotates from $-\pi$ to $\pi$. The voltages are limited as described above, and the resultant magnitude-limited force is computed and plotted using o's. The magnitude of the original required force is shown as a reference.

If the rotor is centered, a force of 3-4 mN can be applied in any direction. However, if the rotor is offset, the capacitances for the electrodes grow dissimilar. Recall that the
Figure 2.7: Resulting forces for a set of limited voltages. The position of the rotor is fixed at $\epsilon=0.01$, $\alpha=0$. The resulting force direction rotates from $-\pi$ to $\pi$. The requested force magnitude is 5 mN. The code used to generate this plot is in Section 5.3.
Figure 2.8: Resulting forces for a set of limited voltages. The position of the rotor is fixed at $\varepsilon=0.5$, $\alpha=0$. The resulting force direction rotates from $-\pi$ to $\pi$. The requested force magnitude is 15 mN. The code used to generate this plot is in Section 5.3.
Figure 2.9: Resulting forces for a set of limited voltages. The position of the rotor is fixed at $\varepsilon=0.5$, $\alpha=-\pi/4$. The resulting force direction rotates from $-\pi$ to $\pi$. The requested force magnitude is 15 mN. The code used to generate this plot is in Section 5.3.
capacitance is not a linear function of distance. Each approaches infinity as its distance between the rotor and the electrode approaches zero. In this situation, Gauss’ law becomes quite problematic, since a small voltage in the closest electrode implies a large voltage in the farthest electrodes in order for the net field to be zero. But by having to increase the voltages, the direction or magnitude of the applied force is also altered. The solution easily grows to voltages much larger than 300 V, thus limiting the maximum force which can be applied.

Another experiment was done using only three electrodes. The advantage was that the solution is unique, so that the minimization technique may be eliminated. However, the equation left to solve is also nonlinear. The solution had to be solved using a nonlinear equation solver which took comparable time to the least squares solver used in the minimization technique. More importantly, the limitations imposed by Gauss’ law grew more drastic, and the force that could be exerted on the rotor was even smaller.

The simulation will test if these limitations on the actuating force available to the controller, along with the nonlinear nature of the plant, still yield acceptable control of the position of the rotor.

2.6 Summary

Section 2.1 began the technical development of the electrostatic bearing by explaining the physics of electrostatic actuation, and the assumptions adopted in this thesis. In particular, the bandwidth of the controller was assumed to be as large as necessary so that the response of the controller was much faster the dynamics of the physical system. Next, the rotor was assumed to be uncharged. This insured that all electric fields were created as a consequence of voltages applied to the stator electrodes, and the resulting gap voltages were limited to 300 V in order to avoid electric field breakdown. The constraint imposed
by Gauss' law, together with the assumption of an uncharged rotor, was expressed in Equation 2.8. Finally, a first order analysis was performed to insure that the rotor could be treated as a perfect conductor.

The capacitance between the electrodes and the rotor was derived in Section 2.2 and expressed in Equation 2.23. The conversion of gap voltage to force on the rotor was derived in Section 2.3 and expressed in Equation 2.27, and again in Equation 2.30 in matrix form. The assumption of the rotor acting as a rigid body was used in the derivation of these equations. An algorithm for the inverse transformation from force to voltage was derived in Section 2.4, using the minimization procedure of Equation 2.34. A voltage limiter, expressed in Equation 2.35, was derived in Section 2.5 to insure that the applied voltages never exceed 300 V. However, due to this constraint, the permissible force applied to the rotor was limited.

The available electrostatic force acting on the rotor was next computed for different eccentricities. Figure 2.7 showed that, for low eccentricities (ε=0.01), between 3-4 mN may be applied in any direction. Figures 2.8 and 2.9 showed that, if the rotor is moved to higher eccentricities (ε=0.5), the available force that may be applied is 2 mN, with a peak of 15 mN in the direction of the offset rotor. In later sections, these forces will prove inadequate to provide stability for initial conditions or disturbances that move the rotor far away from its equilibrium position.
Chapter 3

Dynamics

3.1 Fluid Dynamics

The gas bearings in the microdevices considered here operate well off the known scale of operation for gas bearings. Thus, bearing models found in the literature may not be applicable, and the performance of the gas bearing itself is uncertain. To characterize gas bearings, fluid dynamicists define characteristic numbers which describe the behavior of the fluid in the bearing. These numbers are: L/D, the ratio of axial length of the bearing to the diameter of the rotor; the clearance ratio ψ, defined as <g>/R, the ratio of the nominal (average) gap clearance to the radius of the rotor; the Mach number M, the ratio of the speed of the fluid to the speed of sound (γRT)^1/2; the inertial parameter χ, the product of the clearance ratio ψ and the Reynolds number Re, defined as ρωγR<g>/η; and the compressibility number Λ, defined as 6γM^2/ψRe.

The gas bearing in the microdevices studied here has characteristic numbers much different than those of gas bearings found elsewhere in the literature. For example, Piekos et al. specifies these numbers for the microcompressor as “L/D=0.075, M=1.09, ψ=0.005 and χ=2.9 at Λ=3.5,” and comments further that “This L/D is one third the smallest value commonly found in the literature, the Mach number is at least an order of magnitude larger than for typical bearings, the clearance ratio is five times greater than the usual minimum, and the inertial parameter is approximately three orders of magnitude greater than typically treated.” [5]

Further analysis shows that the microcompressor bearing behaves like an incompressible fluid for eccentricities ranging from 0 to about 0.6. Piekos et al. explains: “it may be concluded that side leakage effects dominate the short bearing, causing it to behave like an
incompressible bearing over much of its operating range. This is unfortunate because truly incompressible gas bearings would be unconditionally unstable." [5] These problems emphasize the need for electrostatic bearing assistance.

To overcome the stability problem when using only a gas bearing, the bearing is forced to operate at high eccentricities, where compressibility effects are more pronounced. However, with the use of electrostatic forces, the bearing will be stabilized at a low eccentricity (ε=0.1). To determine the fluid forces present at high bearing eccentricities, where the gas behaves like a compressible fluid, requires time consuming numerical simulations. However, the fluid forces present at low eccentricities, where the gas behaves like an incompressible fluid, can be obtained in closed form. For low eccentricities, gas bearing performance is obtained from Pinkus and Sternlicht [12], where all necessary assumptions have been made to solve the Navier-Stokes Equations for the case of incompressible fluid bearings. The expressions below yield the radial and tangential components of the fluid force exerted on the rotor as a function of the position and velocity of the rotor. The position of the rotor is captured by the eccentricity ε, defined as the distance between the center of mass of the rotor and the center of the bearing divided the average gap distance <g>; in the equations below, <g> is represented as a g with a bar over it. The velocity of the rotor is expressed by ω, the rotational frequency of the center of mass of the rotor around the center of the bearing, and v, the radial speed of the center of mass of the rotor directly away from the center of the bearing. With these definitions,

\[
\begin{align*}
f_{\text{radial}} & = -\frac{\pi \eta_0 RL^3}{8 g^2} \frac{1 + 2\varepsilon^2}{(1 - \varepsilon^2)^{5/2}} v \\
f_{\text{tangential}} & = \frac{\pi \eta_0 RL^3}{8 g^2} \frac{\varepsilon}{2(1 - \varepsilon^2)^{3/2}} (\omega_r - 2\omega)
\end{align*}
\]  

(3.1)

where \(\omega_r\) is the rotational speed of the rotor \((\omega_r=251,000 \text{ rad/s})\), R is the rotor radius.
(R=0.002 m), L is the length of the rotor (L= 0.0003 m), \( <g> \) is the average gap clearance (\( <g> = 10 \, \mu m \)), and \( \eta_0 \) is the absolute viscosity of the fluid, assumed to be constant, and is given by Sutherland's law as

\[
\eta_0 = 1.458 \cdot 10^{-6} \frac{(T/(1K))^{1.5}}{(T/(1K)) + 110.4}
\] (3.2)

where \( T \) is temperature in Kelvin and \( \eta_0 \) has units of Kg/(m*s). For \( T=524 \, K \), \( \eta_0=2.76\cdot10^{-5} \, Kg/(m*s) \).

Equation 3.1 shows that there will be no radial restoring force unless the rotor has a non-zero velocity. On the other hand, there will be a tangential force for any nonzero eccentricity and rotor whirl speed (\( \omega \)) not equal to the rotational half speed (\( \omega_r/2 \)). This force will drive a rotation of the center of the rotor about the center of the bearing, or whirl, which will increase the eccentricity of the rotor until it hits the wall.

### 3.2 Equations of Motion

In this section, the equations of motion that govern the dynamics of the rotor will be derived, so that the effect of the different forces applied to the rotor can be seen. Figure 3.1 shows the state variables for the position and velocity of the rotor and the direction of the forces to be considered. The rotor will be approximated as a point mass by assuming the rotor to be a rigid body with uniform density distribution. The forces acting on the rotor include fluid forces, electrostatic forces, and constant loading forces. Furthermore, each of these forces will be decomposed into a radial and a tangential component. To simplify the notation throughout the thesis, the following definitions will be followed:

\[
\begin{align*}
F_r & = \text{radial electrostatic force} & F_\alpha & = \text{tangential electrostatic force} \\
f_r & = \text{radial fluid force} & f_\alpha & = \text{tangential fluid force} \\
f_{lr} & = \text{radial loading force} & f_{la} & = \text{tangential loading force}
\end{align*}
\] (3.3)
Figure 3.1: Offset rotor inside stator. The relevant state variables and the forces acting on the rotor are shown. Both radial and tangential forces include fluid, loading and electrostatic forces. The air gap and the spacing between the ends of the electrodes in the stator has been greatly exaggerated for clarity.

Equation 3.1 expresses the radial and tangential components of the fluid force on the rotor. A constant loading force may be applied to create an equilibrium position at any eccentricity. The necessity of this force will be argued in the next section. This loading force may be decomposed as constant forces in the tangential and rotational direction. Finally, time-varying electrostatic forces in both the tangential and radial directions may be applied.

Refer to Figure 3.1. Using Newton’s second law,

\[ m\dot{r} = f_r + f_{ir} + F_r + r\dot{\alpha}^2 \]

\[ m r^2 \ddot{\alpha} = (f_a + f_{ia} + F_a) r - 2 \dot{r} \dot{\alpha} \]

(3.4)
Equation 3.4 can be shown in state-space form as

\[
\begin{align*}
\dot{r} &= v \\
\dot{\alpha} &= \omega \\
\dot{v} &= r\omega^2 + \frac{f_r}{m} + \frac{f_{1r}}{m} + \frac{F_r}{m} \\
\dot{\omega} &= \frac{2v\omega}{r} + \frac{f_a}{mr} + \frac{f_{1a}}{mr} + \frac{F_a}{mr}
\end{align*}
\]

\[
\left[
\begin{array}{c}
\dot{r} \\
\dot{\alpha} \\
\dot{v} \\
\dot{\omega}
\end{array}
\right] = \left[
\begin{array}{c}
f_1 \\
f_2 \\
f_3 \\
f_4
\end{array}
\right]
\]

(3.5)

where \( x \) is the state vector which includes \( r, \alpha, v \) and \( \omega \), and \( u \) is the input vector which includes the electrostatic forces \( F_r \) and \( F_\alpha \). This system will be simulated using Matlab's \textit{ode45} differential equation solver, which uses fourth and fifth order Runge-Kutta numerical integration with variable step size to solve an ordinary differential equation. The Matlab code used to compute \( x^* \) is given in Section 5.4.

3.3 Equilibrium Point

A linear controller for the bearing will be created by linearizing the dynamic system (Equation 3.5) around a static equilibrium point. The equilibrium solution does not need to be a static point; however, dynamic solutions will not be considered in this thesis for simplicity. In order to reduce the electrostatic forces required, the equilibrium point will be defined by a constant fluid loading force. The electrostatic forces will only be used to make the equilibrium point stable. Thus, for this calculation, the electrostatic forces, \( F_r \) and \( F_\alpha \), will be set to zero. A static equilibrium point is defined as one having all state derivatives equal to zero (\( x^* = 0 \)). Thus, the left side of Equation 3.5 is set equal to zero in order to find all possible equilibrium states.

\[
\begin{align*}
0 &= v \\
0 &= \omega \\
0 &= r\omega^2 + \frac{f_r}{m} + \frac{f_{1r}}{m} \\
0 &= \frac{2v\omega}{r} + \frac{f_a}{mr} + \frac{f_{1a}}{mr}
\end{align*}
\]

(3.6)
The second and third equations of the system together imply that

\[ f_{tr} = -f_r \]  

(3.7)

but the first equation constraints \( v \) to zero. Recalling Equation 3.1, Equation 3.7 implies that the fluid radial force is zero. Thus, the radial loading force is zero. The second and fourth equations together imply that

\[ f_{ta} = -f_a \]  

(3.8)

Recalling Equation 3.1, Equation 3.8 implies that

\[ f_{ta} = -\frac{\pi \eta_0 R L^2}{g^2} \frac{\varepsilon}{2(1 - \varepsilon^2)^{3/2}} \omega_r \]  

(3.9)

If the loading force is zero, then the only equilibrium point in the system is at the center (\( \varepsilon = 0 \)). The stability of the gas bearing may be enhanced by positioning the rotor at higher eccentricities. Since the gas bearing will be designed with loading ports, the equilibrium point can be moved to any desired position by applying the appropriate loading force. The constant force may be created via a pressure differential across the rotor [5]. The equations above allow us to model these loading forces and therefore move the equilibrium point of the system.

Matlab simulations were used to verify the stability of the system. In Figure 3.2, the system was simulated using only static loading, that is, using only a constant force. In the first run, no loading force is provided. Thus, the equilibrium position is at \( r = 0, \alpha = 0 \). The initial condition is set at \( r = 2 \mu m, \alpha = 0 \text{ rad}, v = 0 \text{ m/s}, \omega = 0 \text{ rad/s} \). The solution starts to rotate counterclockwise until the rotor hits the wall. On the second run, a static load for an equilibrium position at 0.4 eccentricity is provided. The initial condition is again set at \( r = 2 \mu m, \alpha = 0 \text{ rad}, v = 0 \text{ m/s}, \omega = 0 \text{ rad/s} \). The solution starts to rotate counterclockwise around the equilibrium point until the rotor hits the wall. On the third run, a static load for an equilibrium position at 0.1 eccentricity is provided. The initial condition is set at \( r = 0.5 \mu m, \alpha = 0 \text{ rad}, v = 0 \text{ m/s}, \omega = 0 \text{ rad/s} \).
Natural Dynamics, Zero Load
Initial Condition=[2 μm 0 0 0]

Natural Dynamics, ε=0.4 Load
Initial Condition=[2 μm 0 0 0]

Figure 3.2: Stability calculations for the rotor position with different loading forces. On the first run, no loading force is provided. Thus, the equilibrium position is marked by a small circle at r=0, α=0. The initial condition is set at r=2 μm, α=0 rad, v=0 m/s, ω=0 rad/s. On the second run, a static load for an equilibrium position at 0.4 eccentricity, marked by small circle, is provided. The initial condition is again set at r=2 μm, α=0 rad, v=0 m/s, ω=0 rad/s. The solution starts to rotate counterclockwise around the equilibrium point until the rotor hits the wall. On the third run, a static load for an equilibrium position at 0.1 eccentricity, marked by a small circle, is provided. The initial condition is set at r=0.5 μm, α=0 rad, v=0 m/s, ω=0 rad/s. In all cases, the solution starts to wander around the equilibrium position, marked by an o, but it is eventually driven against the stator. The Matlab code used to generate this plot is in Section 5.4.
$\alpha=0$ rad, $v=0$ m/s, $\omega=0$ rad/s. The solution again starts to rotate counterclockwise around the equilibrium point until the rotor hits the wall. Other simulations not shown using even smaller departures from equilibrium were run. In every case, any small departure from the equilibrium position sends the rotor to the inside wall of the stator. Thus, the system is unconditionally unstable. Notice that the solution will not be valid for high eccentricities, since compressibility effects appear there, and the fluid is assumed to be incompressible. However, the effects of compressibility do not change stability, but rather may prevent the final crash by providing an increased radial force. The code used to generate this simulation is given in Section 5.4.

3.4 Jacobian Matrix

Once an equilibrium position has been established, linearized dynamics around that equilibrium may be derived using the nonlinear equations of motion. Recalling the system as defined in Equation 3.5, the linearized dynamics are described by the Jacobian matrix which is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial \alpha} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial \omega} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial \alpha} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial \omega} \\
\frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial \alpha} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial \omega} \\
\frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial \alpha} & \frac{\partial f_4}{\partial v} & \frac{\partial f_4}{\partial \omega}
\end{bmatrix}_{\alpha, \omega}$$

(3.10)

where $x^*$ and $u^*$ are the equilibrium state and input vector. Substituting Equation 3.5 into Equation 3.10 yields
\[
\frac{\partial f}{\partial x_{x',\omega}} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\omega^2 + \frac{1}{m \partial r} & 0 & \frac{1}{m \partial v} & 2 \omega r \\
- \frac{2 \nu \omega}{r^2} - \frac{f_{\alpha}}{m r^2} \frac{f_{\alpha}}{m r^2} + \frac{1}{m r \partial r} & \frac{2 \omega}{r} & \frac{2 \nu}{r} + \frac{1}{m r \partial \omega} \\
\end{bmatrix}
\]

(3.11)

Evaluating the matrix in Equation 3.11 for the equilibrium conditions yields

\[
x^* = \begin{bmatrix}
\begin{bmatrix} r \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\end{bmatrix}, \quad f^*_{\alpha} = -f^*_{\omega}
\]

(3.12)

Further, notice that if

\[
\nu = 0 \Rightarrow \frac{\partial f_r}{\partial r} = 0
\]

(3.13)

so that

\[
\frac{\partial f}{\partial x_{x',\omega}} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & \frac{1}{m \partial v} & 0 \\
\frac{1}{m r \partial r} & 0 & 0 & \frac{1}{m r \partial \omega} \\
\end{bmatrix}
\]

(3.14)

Finally, after substituting Equation 3.1,

\[
\frac{\partial f}{\partial x_{x',\omega}} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & - \frac{\pi \eta_0 R L^3}{m g^3} \frac{1 + 2 \epsilon^2}{\left(1 - \epsilon^2\right)^{3/2}} & 0 \\
0 & 0 & \frac{\pi \eta_0 R L^3}{m g^3} \frac{\epsilon^2 \omega_r}{\left(2 \left(1 - \epsilon^2\right)^{3/2} + 3 \left(1 - \epsilon^2\right)^{5/2}\right)} & 0 \\
\end{bmatrix}
\]

(3.15)

Throughout the thesis, the values for the different quantities in Equation 3.15 are
\[ L = 0.0003 \text{ m} \]
\[ R = 0.002 \text{ m} \]
\[ m = 8.78 \cdot 10^{-6} \text{ Kg} \]
\[ \bar{g} = 10^{-5} \text{ m} \]
\[ \omega_r = 2.5133 \cdot 10^5 \text{ rad/s} \]
\[ T = 524 \text{ K} \]
\[ \eta_0 = 2.7567 \cdot 10^{-5} \text{ Kg/(ms)} \] (3.16)

The eigenvalues of the system can be seen by substituting these numerical values of (3.16) into Equation 3.15. Figure 3.3 shows the eigenvalues as a function of equilibrium eccentricity. The Matlab code used to evaluate this matrix is in Section 5.5. All eigenvalues are purely real; two of them are always negative, the other two are always zero. If all four eigenvalues had negative real parts for all eccentricities, then the system would be stable. However, this system has zero eigenvalues, and in the presence of zero eigenvalues, stability cannot be determined by looking only at the eigenvalues. In fact, the Matlab simulation shows that the system is unstable.

### 3.5 Controller

Figure 3.4 shows the controller which is proposed to stabilize the bearings. For simplicity in this thesis, the sensor noise and system disturbances will be ignored. Also, this thesis will assume full-state feedback. Thus, the controller will only be required to provide stability to the rotor position given any arbitrary initial condition. As shown in Figure 3.4, the controller will have the difference between the actual state and the desired equilibrium, or error, as an input. Its output will be the necessary force to be applied on the rotor to achieve stability of the rotor position. The controller will be linear and created for the linearized model of the rotor behavior developed in Section 3.4. Recall that the canonical state-space formulation for a linear dynamic system is given by
Figure 3.3: Eigenvalues of the Jacobian matrix for the incompressible gas bearing model as a function of rotor position. All eigenvalues are real; two of them are always zero. The Matlab code used to generate this plot is in Section 5.5.

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]  
(3.17)

The state vector, \(x\), contains all the necessary information to uniquely characterize the system at any point in time. The state derivatives vector, \(x^*\), expresses how the state evolves through time. The evolution of \(x\) can depend on the current state of the system, \(x\), or on external inputs, \(u\). Matrices \(A\) and \(B\) express the dependence of \(x^*\) on \(x\) and \(u\), respectively. The output vector, \(y\), can also depend on \(x\) and \(u\). Matrices \(C\) and \(D\) express the depen-
Figure 3.4: Block diagram of the closed-loop system which combines gas and electrostatic journal bearings.

The dependence of $y$ on $x$ and $u$. In our system, $x$ is composed of our two degrees of freedom, $r$ and $\alpha$, and their derivatives, $r^*$ and $\alpha^*$. Our inputs will be $F_\alpha$ and $F_r$, the electrostatic forces in rotational coordinates. Note that $u$ is the vector of electrostatic forces in rotational coordinates, while $F$, as derived in Sections 2.3-5, is the vector of electrostatic forces in cartesian coordinates.
The current system is nonlinear. That is, the dependence of \( x^* \) on \( x \) and \( u \) cannot be expressed as a linear combination of \( x \) and \( u \), as shown in Equation 3.5. Instead, the system can be written as \( x^* = f(x,u) \) allowing for a general function to operate on \( x \) and \( u \). The localized linear system that describes the evolution of \( x^* \) as a function of \( x \) was calculated using the Jacobian in Section 3.4. This description is only useful for small deviations of \( x \) from the linearized equilibrium \( x^* \). By definition, the Jacobian of \( f \) at \( x^* \) is the system matrix \( A \) such that

\[
\dot{x} = Ax \tag{3.18}
\]

where \( x \) is a small perturbation from \( x^* \). A similar linearizing procedure for \( B \) yields

\[
B = \frac{\partial f}{\partial u}\bigg|_{x^*} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\frac{1}{m} & 0 \\
0 & \frac{1}{mr}
\end{bmatrix} \tag{3.19}
\]

As mentioned above, this thesis assumes full-state feedback for simplicity. Whether the state is directly measured or reconstructed via a state observer is not relevant. This assumption implies that \( C=I \) and \( D=0 \).

From Figure 3.4 and Equation 3.17, the linear controller produces a commanded force vector in rotational coordinates \( (u) \), by taking linear combinations of the error difference between the state and the equilibrium position \( (x) \). Thus, the controller can be expressed as

\[
u = -Kx \tag{3.20}
\]

One way to assign the gains in the matrix \( K \) is to create a cost function and minimize this function over \( K \). A linear quadratic regulator is designed by one of these procedures. It seeks to produce an optimal feedback matrix \( K \) which minimizes the cost function

\[
J = \int_0^T (x^T Q x + u^T R u) dt \tag{3.21}
\]
This cost function increases as the square of error (x) or commanded force (u). The relative importance of excursions in each state or commanded force can be altered by changing the entries in the Q and R matrices. Once the cost function has been defined, the linear quadratic regulator procedure will produce the optimal feedback matrix K subject to the constraint equation of the dynamic system

$$\dot{x} = Ax + Bu$$  \hspace{1cm} (3.22)

Note that the minimization of the cost function \( J \) over all feedback matrices \( K \) is independent of the initial condition.

Matrices \( A \) and \( B \) are given in Equations 3.15 and 3.19. As explained above, matrices \( Q \) and \( R \) express the cost of the excursion of each state and input variable from zero. In the case of the state variables, a high cost \( Q \) implies that the control gains will be designed to prevent the state from departing from equilibrium. In the case of the inputs, a high cost \( R \) implies that the control gains will be designed to prevent the control from using excessive force. Thus,

$$Q = \begin{bmatrix}
\frac{1}{r_{\text{weight}}} & 0 & 0 & 0 \\
0 & \frac{1}{\alpha_{\text{weight}}} & 0 & 0 \\
0 & 0 & \frac{1}{\nu_{\text{weight}}} & 0 \\
0 & 0 & 0 & \frac{1}{\omega_{\text{weight}}} \\
\end{bmatrix}$$  \hspace{1cm} (3.23)

$$R = \begin{bmatrix}
\frac{1}{f_{\text{weight}}} & 0 \\
0 & \frac{1}{f_{\text{weight}}} \\
\end{bmatrix}$$  \hspace{1cm} (3.24)

In order to compute matrices \( Q \) and \( R \), costs must be associated with deviations of each state and control variable. In general, the behavior of the state variables other than the
radial position is not important. However, a high rotational speed requires a large centripetal force. The centripetal force required by the rotor whirling at frequencies in the order of the rotational frequency (ωr) is much larger than the maximum electrostatic force available. Therefore, the whirl frequency (ω) must be controlled. However, controlling the whirl frequency competes with the control of the angular position (α). Furthermore, proportional control of the angular position is integral control of the whirling frequency, since ω=α. Therefore, for the case of an equilibrium point, r and α must both be controlled. By loosening the requirements on ω and v, the force commanded by the controller is minimized. Also, there is a limit on how much force may be applied on the rotor, so that another high cost is associated with the control forces. The lqr command in Matlab creates the optimal feedback matrix K as described above. The Matlab code used to compute the feedback matrix is given in Section 5.6.

For this next example, the equilibrium is set at ε=0.1, α=0. This equilibrium point insures that the assumption of incompressibility for the fluid is valid. Substituting these equilibrium values in Equations 3.15 and 3.19 yields

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -413.5830 & 0 \\
5.1972 \cdot 10^{13} & 0 & 0 & -401.4188
\end{bmatrix}
\quad B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1.2626 \cdot 10^5 & 0 \\
0 & 1.2626 \cdot 10^{11}
\end{bmatrix}
\]  

(3.25)

As discussed above, relatively high costs for r and α, and low costs for v and ω are used to compute the matrix Q. Zero costs are avoided so that both Q and R are full rank. Substitution of the costs in Equation 3.23 yields the Q matrix,

\[
r_{weight} = 1 \mu m
\]
\[
\alpha_{weight} = \pi \text{ radians}
\]
\[
v_{weight} = 100 \text{ m/s}
\]
\[
\omega_{weight} = 2 \cdot 10^5 \text{ radians/s}
\]

\[
\Rightarrow Q = \begin{bmatrix}
10^{12} & 0 & 0 & 0 \\
0 & 0.1013 & 0 & 0 \\
0 & 0 & 10^{-4} & 0 \\
0 & 0 & 0 & 2.5 \cdot 10^{-11}
\end{bmatrix}
\]

(3.26)
Likewise, a high cost for the electrostatic force is used in matrix R. Substitution of the cost in Equation 3.24 yields the R matrix,

\[ f_{weight} = 5mN \Rightarrow R = \begin{bmatrix} 40000 & 0 \\ 0 & 40000 \end{bmatrix} \]  

(3.27)

These weights imply that a deviation of 1 μm from equilibrium and the use of 5 mN to control are penalized equally. As explained above, the Matlab command \textit{lqr} performs all the necessary calculations and yields the feedback matrix K which minimizes the cost J defined in Equation 3.21. This yields

\[ K = \begin{bmatrix} 5.0161 \cdot 10^3 & 2.8847 \cdot 10^{-5} & 0.2786 & 6.2567 \cdot 10^{-9} \\ 320.6878 & 0.0016 & 0.0063 & 1.5745 \cdot 10^{-7} \end{bmatrix} \]  

(3.28)

which is then used to calculate the electrostatic force to be applied on the rotor. Before including a nonlinear electrostatic force limiter, a few solutions not shown were computed using linear feedback. Without an electrostatic force limiter, the electrostatic force applied on the rotor can be arbitrarily high. However, even with unlimited electrostatic force available, the controller fails to stabilize the position of the rotor. If the initial conditions are too far away from the equilibrium point, the linear approximation of the system is not accurate enough, rendering the linear controller useless. The Matlab code created for this simulation is given in Section 5.6. Large whirl frequencies required centripetal forces so large that the trajectory is uncontrollable. Thus, the initial conditions used have zero initial radial and angular velocities. Recall that in this model compressibility effects that will be present at high eccentricities are neglected. These effects will help since they increase the gas bearing force on the rotor and produce a net inward force [5]. These forces are expected to redirect the instability of the rotor into modes which the electrostatic bearing may be able to control. Specifically, the gas bearing is expected to redirect the whirl frequency and radial velocity into high eccentricities.
Figures 3.5 and 3.6 show the linear controller given by Equation 3.28 acting on the nonlinear system given by Equation 3.5. The idealizations in this model include that the electrostatic force can be arbitrarily large and incompressible behavior of the fluid everywhere. Given that arbitrarily large electrostatic forces can be applied, it is not surprising to see in Figure 3.5 that the rotor is quickly returned to its equilibrium position from a large initial eccentricity. The applied force on the rotor is as large as 15-20 mN. These force magnitudes cannot be achieved using electrostatic forces. Figure 3.6 starts from a smaller eccentricity. Thus, the required force on the rotor does not exceed 5mN, except at the beginning of the trajectory. Section 3.6 will incorporate a non-linear limit on the electrostatic force which may be applied. This limit will affect how fast the trajectories return to equilibrium, and whether the controller can drive the trajectory to equilibrium given its limited force.

3.6 Non-Linear Model

In Figures 3.7 and 3.8, the feedback force is bounded using a force magnitude limiter. As an approximation to the exact limit, Figures 2.8 and 2.9 show that there is a maximum electrostatic force of 2 mN at medium eccentricities given air gap voltages limited to 300 V. Thus, the magnitude of the feedback force is limited to 2 mN. The Matlab code used for these simulations is given in Section 5.7. The initial conditions for these simulations are the same initial conditions studied in Section 3.5. Not surprisingly, the controller is now unable to hold the rotor, and the whirl frequency reaches uncontrollable values in Figure 3.7. The jump in force magnitude and direction at time 0.18 ms is due to the crossing of the rotor over $\alpha=-\pi$ line, where the angular position controller reacts immediately to an apparent "jump" in position from $\pi$ to -$\pi$. In Figure 3.8, the controller still manages to stabilize the trajectory, although the time required is longer than in the linear case.
Figure 3.5: Solution for the incompressible gas bearing model with unlimited permissible electrostatic force. The initial condition is $\varepsilon=0.7$, $\alpha=-7\pi/8$, $v=0$, $\omega=0$. The Matlab code used to generate this plot is in Section 5.6.
Figure 3.6: Solution for the incompressible gas bearing model with unlimited permissible electrostatic force. The initial condition is $\varepsilon=0.2$, $\alpha=-7\pi/8$, $v=0$, $\omega=0$. The Matlab code used to generate this plot is in Section 5.6.
Figure 3.7: Solution for the incompressible gas bearing model using magnitude-limited electrostatic force. Feedback forces were limited to 2 mN. The initial condition is $\varepsilon=0.7, \alpha=-7\pi/8, \upsilon=0, \omega=0$. The Matlab code used to generate this plot is in Section 5.7.
Figure 3.8: Solution for the incompressible gas bearing model using magnitude-limited electrostatic force. Feedback forces were limited to 2 mN. The initial condition is $\varepsilon=0.2$, $\alpha=-7\pi/8$, $\nu=0$, $\omega=0$. The Matlab code used to generate this plot is in Section 5.7.
3.7 Controller with Voltage Conversion

Section 5.8 has a fully nonlinear model for the dynamics which includes the voltage inverter and limiter as described in Chapter 2. However, this model was not used to run simulation in this thesis. The computation of the H matrix and the minimization procedure were too time consuming, rendering it useless for an iterative design procedure were several runs would have to be made in the process. Also, the minimization procedure is unreliable; there are many instances were it fails to find a suitable solution as shown in Figures 2.7-9. However, better algorithms which minimize computing time and optimize the solution can easily be placed in the code when they are developed. Also, fast computers may be used to run simulations using this code. In any case, note that improving computing performance will not change the conclusions of this thesis.

3.8 Summary

The forces exerted by the gas on the rotor were derived in Section 3.1 and expressed in Equation 3.1 for the case where the gas behaved as an incompressible fluid. The incompressible fluid assumption was valid for eccentricities smaller than 0.5. The effects of the applied forces on the dynamics of the rotor was expressed as a system of equations in Equation 3.5 of Section 3.2. The dynamic system had a natural equilibrium point at the center of the stator were the eccentricity is zero, but the equilibrium point could be moved by applying a loading force. The force required to move the equilibrium point of the system was derived in Section 3.3 and expressed in Equation 3.9. Also in Section 3.3, Figure 3.2 showed a simulation where only loading and fluid forces affect the dynamics of the system to verify that the system was unstable without a controller, thereby necessitating electrostatic bearings.
In order to design a linear controller for the system, a linearized model of the system was computed in Section 3.4 and expressed in Equation 3.15. The linearized system was then cast in the canonical linear state-space form of Equation 3.17. The system state matrix A and control matrix B are given in Equations 3.15 and 3.19. A controller was designed which minimized the cost function of Equation 3.21. This function depends on position, velocity and forces on the rotor. The necessary state cost matrix Q and input cost matrix R were given in Equations 3.23 and 3.24, and the optimal feedback matrix K (Equation 3.20) was obtained for this system. Figures 3.5 and 3.6 show the performance of the controller on the system assuming unlimited electrostatic forces. With unlimited electrostatic forces, the controller is able to stabilize the rotor position with large initial errors, but forces used by the controller are larger than can be applied by the electrostatic forces with the breakdown limitation.

Figures 3.7 and 3.8 in Section 3.6 repeat the simulations of Figures 3.5 and 3.6 but include a nonlinear limiter which does not permit control forces larger than the 2 mN limit derived in Section 2.5. The controller was then unable to stabilize large initial errors. It was able to stabilize medium errors, but the time required to drive the rotor position to equilibrium in Figure 3.8 was one millisecond, as compared to the time in Figure 3.6 of half a millisecond.

Finally, Section 3.7 noted that a full non-linear model which incorporated the voltage limiter as described in Section 2.5 was written and tested, but its performance was poor in terms of computation time, which made it an inappropriate tool for use here in designing the electrostatic bearings. The model was developed in Section 5.8.
Chapter 4

Summary, Conclusions, and Suggestions For Future Work

4.1 Summary

The goal of this thesis was to explore the feasibility of augmenting gas bearings in the microcompressor with electrostatic bearings. To this end, models were developed for the fluid in the gas bearings and the dynamics of the rotor. An electrostatic bearing was designed to optionally provide additional stiffness and damping through electrostatic forces, and it was also modeled. Different equilibrium positions for the rotor were considered and a static point at low eccentricities was finally chosen for detailed study. Next, the nonlinear dynamics of the augmented bearing were linearized around the equilibrium position in order to design a linear controller for the electrostatic bearing. The linear controller was designed to minimize an error function proportional to both the distance of the rotor from equilibrium, and the magnitude of the actuating electrostatic force. All models were simulated using Matlab to assess the performance and limitations of the closed-loop augmented bearing.

Section 2.1 began the technical development by explaining the physics of electrostatic actuation, and the assumptions adopted in this thesis. In particular, the bandwidth of the controller was assumed to be as large as necessary so that the response of the controller was much faster than the dynamics of the physical system. Next, the rotor was assumed to be uncharged. This insured that all electric fields were created as a consequence of voltages applied to the stator electrodes, and the resulting gap voltages were limited to 300 V in order to avoid electric field breakdown. The constraint imposed by Gauss' law, together with the assumption of an uncharged rotor, was expressed in Equation 2.8. Finally, a first
order analysis was performed to insure that the rotor could be treated as a perfect conductor.

The capacitance between the stator electrodes and the rotor was derived in Section 2.2 and expressed in Equation 2.23. The conversion of gap voltage to force on the rotor was derived in Section 2.3 and expressed in Equation 2.27, and again in Equation 2.30 in matrix form. The assumption of the rotor acting as a rigid body was used in the derivation of these equations. An algorithm for the inverse transformation from force to voltage was derived in Section 2.4, using the minimization procedure of Equation 2.34. A voltage limiter, expressed in Equation 2.35, was derived in Section 2.5 to insure that the applied voltages never exceed 300 V. However, due to this constraint, the permissible force applied to the rotor was limited.

The available electrostatic force acting on the rotor was next computed for different eccentricities. Figure 2.7 showed that, for low eccentricities ($\varepsilon=0.01$), between 3-4 mN may be applied in any direction. Figures 2.8 and 2.9 showed that, if the rotor is moved to higher eccentricities ($\varepsilon=0.5$), the available force that may be applied is 2 mN, with a peak of 15 mN in the direction of the offset rotor. In later sections, these forces proved inadequate to provide stability for initial conditions or disturbances that moved the rotor far away from its equilibrium position.

The forces exerted by the gas on the rotor were derived in Section 3.1 and expressed in Equation 3.1 for the case where the gas behaved as an incompressible fluid. The incompressible fluid assumption was valid for eccentricities smaller than 0.5. The effects of the applied forces on the dynamics of the rotor was expressed as a system of equations in Equation 3.5 of Section 3.2. The dynamic system had a natural equilibrium point at the center of the stator where the eccentricity is zero, but the equilibrium point could be moved by applying a loading force. The force required to move the equilibrium point of the sys-
tem was derived in Section 3.3 and expressed in Equation 3.9. Also in Section 3.3, Figure 3.2 showed a simulation where only loading and fluid forces affect the dynamics of the system to verify that the system was unstable for low eccentricities without a controller, thereby necessitating electrostatic bearings.

In order to design a linear controller for the system, a linearized model of the system was computed in Section 3.4 and expressed in Equation 3.15. The linearized system was then cast in the canonical linear state-space form of Equation 3.17. The system state matrix A and control matrix B are given in Equations 3.15 and 3.19. A controller was designed which minimized the cost function of Equation 3.21. This function depends on position, velocity and forces on the rotor. The necessary state cost matrix Q and input cost matrix R were given in Equations 3.23 and 3.24, and the optimal feedback matrix K (Equation 3.20) was obtained for this system. Figures 3.5 and 3.6 show the performance of the controller on the system assuming unlimited electrostatic forces. With unlimited electrostatic forces, the controller is able to stabilize the rotor position with large initial errors, but the forces used by the controller are larger than can be applied by the electrostatic forces with the breakdown limitation.

Figures 3.7 and 3.8 in Section 3.6 repeat the simulations of Figures 3.5 and 3.6 but include a nonlinear limiter which does not permit control forces larger than the 2 mN limit derived in Section 2.5. The controller was then unable to stabilize large initial errors. It was able to stabilize medium errors, but the time required to drive the rotor position to equilibrium in Figure 3.8 was one millisecond, as compared to the time in Figure 3.6 of half millisecond.

Finally, Section 3.7 noted that a full non-linear model which incorporated the voltage limiter as described in Section 2.5 was written and tested, but its performance was poor in
terms of computation time, which made it an inappropriate tool for use here in designing the electrostatic bearings. The model was developed in Section 5.8.

The general process for designing the controller for the electrostatic bearing is divided in three basic steps. First, the nonlinear characteristics of the gap voltage to force conversion must be identified. Specifically, the model should express the highest magnitude of the force which may be applied, as constrained by Paschen's curves, and how directional the available force is, as constrained by Gauss law' and the electrode geometries. Next, a linear controller can be designed using a linearized model (Jacobian matrix) of the system of forces on the rotor and the limitations imposed by the nonlinear characteristic of the electrostatic force. Finally, the design should be tested with models which increase their complexity by adding nonlinear elements. If at any point the design does not perform adequately, a better controller should be designed, if it exists, taking into consideration the new information. This new design should then be tested in the same manner.

4.2 Conclusions

The simulations of Section 3.6 have shown that the electrostatic forces available to the controller limit the stability that the controller can provide. The controller can stabilize medium disturbances if the static equilibrium is close to the center, for example, for eccentricities less than approximately 0.2, as shown in Figure 3.8. The controller cannot stabilize small disturbances if the static equilibrium is far from the center, for example, for eccentricities greater than approximately 0.5, as shown in Figure 3.7. The controller cannot stabilize disturbances with rotational speeds (ω) in the order of magnitude of the operation of the rotor. It may be possible for a controller to provide stability to a limit cycle equilibrium around 0.5 eccentricity. The fundamental limitation in the controller performance is the limit imposed on the actuating force by electric breakdown.
Figure 2.7 shows that the electrostatic bearing can provide between 3-4 mN of force on a centered rotor in almost any direction. However, Figures 2.8 and 2.9 also show that the force is highly directional at eccentricities between 0.5-0.8. The force increases to around 15 mN in the direction where the rotor is closest to the wall of the stator, but decreases to 2 mN in the opposite direction. Although no simulations were done at higher eccentricities, the same pattern is expected: higher peak forces, but these forces will be even more directional.

Gauss’ law, together with the nonlinear dependence of the air gap capacitances on position and the electrode geometry are responsible for creating a highly directional force. At zero eccentricity, the capacitances are equal. Thus, the voltages across the gap must simply add to zero in order to fulfill Gauss’ law. However, if the rotor is moved to an eccentricity of 0.8 towards electrode 1, the capacitance of electrode 1 will be seven times larger than the capacitance of electrode 3, which is directly opposite to electrode 1. This implies that any voltage applied across the gap of electrode 1 will produce seven times more electric field than the same voltage applied in electrode 3. In general, in order to fulfill Gauss’ law, electrodes that are far from the rotor must provide large voltages across their respective gaps, while trying to maintain the correct magnitude and direction for the required force on the rotor. The solution set of voltages easily exceeds 300 V and therefore must be limited. Thus, the resulting force is of smaller magnitude than the requested force. Figure 3.5 shows that the force necessary to stabilize reasonable disturbances is ten times larger than that which can be reasonably applied, on the order of 20 to 50 mN.

Even in the best case scenario, electrostatic forces do not have the necessary magnitude to provide adequate shock absorption by themselves. They can effectively provide stability to the rotor position at eccentricities smaller than 0.2, but the forces necessary to absorb shocks must be provided by the gas bearing. In a few experiments, it was found
here that the electrostatic bearings could only absorb shocks equivalent to dropping the 
microengine from less than one millimeter, assuming all the energy was transferred as a 
force impulse to the rotor.

Recall that the model used here for the gas bearing assumed incompressible behavior 
for the fluid in the bearing. Further analyses should be done incorporating more accurate 
models for the fluid mechanics. Section 3.1 explained that more accurate models of the 
fluid behavior will show that the fluid instability is not as large as given by the incompress-
ible model used here, and that the electrostatic forces necessary to provide stability may 
not be as large.

The two principal modes of instability for the gas bearing are half-speed whirl, 
induced by the fluid, and synchronous whirl, induced by an intrinsic moment of inertia of 
the rotor. The centripetal forces required to center the rotor at these whirling frequencies 
are much larger than those which the electrostatic bearing can provide. The electrostatic 
bearing must rely on compressibility effects and viscous dissipation from the gas bearing 
to redirect or diminish the whirling frequency. If the gas bearing can effectively redirect 
the rotational energy into another state, such as rotor position or radial velocity, the elec-
trostatic bearing may be able to provide the necessary force to pull the rotor toward its 
equilibrium.

If a better design of the electrostatic bearing could enable the electrostatic forces to be 
applied in any direction, and if the gas bearing can keep the rotor from whirling too fast, 
the controller may provide stability to the rotor position up to large eccentricities.

4.3 Suggestions For Future Work

There are several ways in which this thesis may be extended. First, the mathematical 
force-to-gap-voltage inversion must be better defined mathematically. The current defini-
tion does not insure a minimal solution, nor guarantee finding a solution, which it is known to exist. Better algorithms for the inversion should be used in order to minimize the computing time, guarantee finding a solution, and find the optimal solution. Ideally, a closed form solution may be developed, either empirically or from theory, in order to avoid the inversion computation every time a simulation is run.

The number of electrodes in the system should be revised. There must be at least three in order to produce arbitrary forces in the x-y plane. However, with more electrodes, the limitations imposed by Gauss' law soften, thus effectively increasing the magnitude of the available force, and providing these forces in any direction. However, increasing the number of electrodes leads to a decrease in reliability, a more difficult fabrication, and more pronounced cross-capacitance effects which were ignored for this thesis. In any case, increasing the number of electrodes should be explored.

The equilibrium solution to the position of the rotor inside the stator must be revised in accordance with the design of the gas bearing. A centered rotor is more attractive from an electrostatic bearing point of view, but the gas bearing will be inherently unstable in this location. An off-centered rotor leads to a more stable gas bearing, although at this point in time, it is not known whether the gas bearing can achieve full stability by itself. Thus, an electrostatic bearing may still be necessary for a non-centered rotor, with perhaps the added advantage that the required electrostatic forces may be smaller because the gas bearing may be only lightly unstable. However, fluid forces in the gas bearing do increase with eccentricity, so that the ability of the electrostatic bearing to provide stability decreases as the rotor moves to higher eccentricities due to the limited electrostatic force.

One equilibrium which was not covered by this thesis is a closed orbit. This may be the only stable solution for the gas bearing. An electrostatic bearing, with an intrinsic high frequency response may provide optimal augmentation in this case, although the complex-
ity of the corresponding controller would greatly increase. The electrostatic bearing would likely require a nonlinear controller.

The fluid model is currently being developed for the gas bearing. As better models become available, they should be included in the design of the electrostatic bearing, as they will have a profound effect on it.

If the equilibrium position is that of a static non-centered rotor, then the geometry of the electrodes should be optimized in order to minimize the nonlinear effects of the electrostatic forces. Optimizing for both Gauss' law (capacitance) and voltage-to-force conversion (H matrix) would likely imply an irregular array of electrodes, with shorter electrodes where the rotor is close to the stator wall, and longer electrodes where they are farther away. This would tend to equalize the capacitance seen by each electrode and the electrostatic force per electrode with the rotor in its equilibrium position. However, note that it is unlikely that the electrode geometry may be optimized for these two conditions simultaneously, since they differ in their dependence on space. For example, optimizing for a non-directional force (equal capacitances) at equilibrium may yield a directional force per voltage dependence in the H matrix. A compromise should be found in the design.

Preliminary options for the fabrication of the electrodes include covering the inside wall of the stator with conductor. The disadvantages to this include that any wall strike from the rotor or extreme fluid temperatures may damage the electrode and that it may be hard to provide adequate isolation between electrodes. Also, the fabrication steps required by this type of electrode, although possible, may add significant complexity to the fabrication process. On the other hand, the theoretical analysis of the electrostatic bearing using this type of electrode is quite simple since the physical electrode strongly resembles the idealized model. A second design involves the construction of a trench around the stator,
where conductor material may be deposited. The disadvantages of the type of electrode include concerns regarding the structural integrity of the inside wall of the stator and associated heat transfer issues, and the added complexity in the theoretical analysis since the effect of the wall on the electric field must be included. The advantages include that the electrodes are physically protected from the fluid and rotor, there is good isolation between electrodes, and that the fabrication steps for this type of electrode may be more amenable to the fabrication process. In any case, this thesis has not considered fabrication, and this must be addressed.
Chapter 5

Simulation Code

5.1 Capacitance

cap.m is a Matlab function which returns a vector containing the capacitances of the four electrodes as a function of rotor position. The first capacitance \( C_1 \) is defined to be for the electrode that extends from \( \alpha=0 \) to \( \alpha=\pi/2 \); the rest are defined following a rotation in the \( +\alpha \) direction. The capacitance of the electrodes is defined in Section 2.2.

```
%Capacitance=cap(offset<ecc*1e-5>,angle)
function c=c(r,a)
e=r/1e-5;
aux5=2/(1-e^2)^.5;
aux6=((1+e)/(1-e))^^.5;

aux=[atan(aux6*tan((-a)/2))
    atan(aux6*tan((pi/2-a)/2))
    atan(aux6*tan((pi-a)/2))
    atan(aux6*tan((-pi/2-a)/2))
    atan(aux6*tan((-a)/2))];

aux1=pi/2*(1-sign(aux(2)-aux(1)));
aux2=pi/2*(1-sign(aux(3)-aux(2)));
aux3=pi/2*(1-sign(aux(4)-aux(3)));
aux4=pi/2*(1-sign(aux(5)-aux(4)));
aux=aux+[0
    aux1
    aux1+aux2
    aux1+aux2+aux3
    aux1+aux2+aux3+aux4];
aux=aux5*aux;

c=[5.31e-13*(aux(2)-aux(1))
    5.31e-13*(aux(3)-aux(2))
    5.31e-13*(aux(4)-aux(3))
    5.31e-13*(aux(5)-aux(4))];

%Values used:
%e0=8.85e-12
%l(depth)=.0003
%r(radius)=.002
%g(gap)=10e-6=1e-5
%base cap=e0*1*r/g=5.31e-13
```
capproc.m is a Matlab program which generates a set of capacitance curves for electrode 1 as a function of rotor position. The resulting plot is used in Figure 2.4.

```matlab
clear
e=[0 0.5e-5 0.7e-5 0.9e-5 0.95e-5];
subplot 211
for j=1:5,
    k=0;
    for i=-pi:.01*pi:pi
        k=k+1;
        cd(:,k)=cap(e(j),i);
        a(k)=i;
    end
    plot(a/pi,cd(1,:)*1e12)
    hold on
end
hold off
grid on
xlabel('Angle of rotor (alpha, radians/pi)')
ylabel('Capacitance of C1 (picoFarads)')
title('Gap capacitance (C1) vs. angular position of rotor,
ecc=[0 0.5 0.7 0.9 0.95]')

subplot 212
for j=pi/4:pi/4:3*pi/4,
    k=0;
    for i=-.90e-5:.01e-5:.90e-5
        k=k+1;
        cd2(:,k)=cap(i,j);
        a2(k)=i;
    end
    plot(a2*1e6,cd2(1,:)*1e12)
    hold on
end
hold off
grid on
xlabel('Eccentricity of rotor (r, micrometers)')
ylabel('Capacitance of C1 (picoFarads)')
title('Gap capacitance (C1) vs. radial position of rotor,
alpha=[pi/4 pi/2 3*pi/4]')
```

### 5.2 Electrostatic Forces

hmat.m is a Matlab function which computes the matrix H, as defined in Section 2.3. This matrix transforms a vector of squared gap voltages into electrostatic forces in the x and y directions. The Matlab function `quad` uses an adaptive recursive Simpson's rule to numerically compute the integral of the function provided. The two integrands, $h_{1n}$ and $h_{2n}$ in
Section 2.3, are defined as auxiliary functions \texttt{hintx.m} and \texttt{hinty.m} below.

function h=hmat(r,a)
    precis=1e-2;
    h=2.655e-18*[quad(‘hintx’,0,pi/2,precis,[]),r,a]...
        quad(‘hintx’,pi/2,pi,precis,[]),r,a)...
        quad(‘hintx’,-pi,-pi/2,precis,[]),r,a)...
        quad(‘hintx’,-pi/2,0,precis,[]),r,a)
    quad(‘hinty’,0,pi/2,precis,[]),r,a)...
        quad(‘hinty’,pi/2,pi,precis,[]),r,a)...
        quad(‘hinty’,-pi,-pi/2,precis,[]),r,a)...
        quad(‘hinty’,-pi/2,0,precis,[]),r,a)];

%Values used:
%e0=8.85e-12
%l=.0003
%r=.002
%f=e0*1*r/2*v^2*int
% =8.85e-12*.0003*.002/2*v^2*int
% =2.6550e-18*v^2*int

\texttt{hintx.m} is an auxiliary Matlab function used in \texttt{hmat.m}.

function intx=hintx(z,r,a)
    intx=cos(z)./(1e-5-r*cos(z-a)).^2;

%Values used:
%g=1e-5

\texttt{hinty.m} is another auxiliary Matlab function used in \texttt{hmat.m}.

function inty=hinty(z,r,a)
    inty=sin(z)./(1e-5-r*cos(z-a)).^2;

%Values used:
%g=1e-5

\texttt{hmatproc.m} is a Matlab program used to generate electrostatic forces for a simple set of voltages as the rotor changes radial position. The resulting plot is used in Figure 2.5.

clear
for r=-.9e-5:.1e-5:.9e-5,
    rmat=[rmat r]
    force=[force hmat(r,0)*([300^2 0 0 (-300)^2]’)];
    gauss=[gauss cap(r,0)’*([300 0 0 -300]’)];
end

max(abs(gauss))

plot(rmat*1e6,force*1e3)
gtext on
5.3 Force Inversion and Limiter

`forceinvandlimiter.m` is a Matlab program which uses the Matlab command `fmins` to find the set of gap voltages that produce a required force. The Matlab `fmins` command performs a least squares iterative minimization on the function provided, as explained in Section 2.4. After obtaining the voltages, it limits them to 300 V as explained in Section 2.5. The position of the rotor is fixed in the program. The magnitude of the required force is also fixed, but the angle of the force sweeps from $-\pi$ to $\pi$.

clear
global force h nuc
fmax=15e-3;
r=.5e-5;
a=-pi/4;
c=cap(r,a);
nuc=1e14*[-c(4) 0 0 c(1);-c(3) 0 c(1) 0;-c(2) c(1) 0 0
0 0 -c(4) c(3);0 -c(4) 0 c(2);0 -c(3) c(2) 0]’;
h=hm(r,a);

for a=-pi:pi/64:pi,
amat=[amat a];
force=[fmax*cos(a) fmax*sin(a)]’;
options(1)=0;
options(2)=1e1;
options(3)=1e-19;
coeff=fmins(’forceerr’, [0 0 0 0 0], options);
v=nuc*coeff’;
vmat=[vmat v];
gauss=[gauss c’.v];
fmat=[fmat h.*v.^2];

red=norm(v,inf)/300;
if red>1,
    v=v/red;
end
vlim=[vlim v];
flim=[flim h.*v.^2];
end

save forceinvandlimiter4
forceerr.m is Matlab function which computes the square difference between the required force and the force created by a set of voltages which fulfill Gauss' law. It is called by the Matlab command fmins in forceinvandlimiter.m.

```matlab
function err=f(coeff)
global force h nuc
aux=h*(nuc*coeff').^2;
err=(aux-force)'*(aux-force);
```

forceinvandlimiterp.m is a Matlab program which takes the output of forceinvandlimiter.m and creates two plots. In the first plot, it shows each gap voltage, both before and after limiting the voltages. In the second plot, it shows the magnitude of both the original set of voltages and of the voltage-limited set using both cartesian and polar coordinates. The plots are used in Figures 2.6-9.

clear
load forceinvandlimiter01

for k=0:length amat/4
    amatsub=[amat amat (4*k+1)];
    vmatsub=[vmatsub vmat(:,4*k+1)];
    vlimsub=[vlimsub vlim(:,4*k+1)];
end

clf
subplot 221
plot(amatsub/pi,vmatsub(1,:))
hold on
plot(amatsub,300*ones(size amatsub))
plot(amatsub,-300*ones(size amatsub))
plot(amatsub/pi,vmatsub(1,:),’x’)
plot(amatsub/pi,vlimsub(1,:))
plot(amatsub/pi,vlimsub(1,:),’o’)
hold off
axis([-1 1 -600 600])
title(’Force to Voltage, ecc=.01, a=0, f=4mN’)
ylabel(’V1 (V)’)
grid on
subplot 222
plot(amatsub/pi,vmatsub(2,:))
hold on
plot(amatsub,300*ones(size amatsub))
plot(amatsub,-300*ones(size amatsub))
plot(amatsub/pi,vmatsub(2,:),’x’)
plot(amatsub/pi,vlimsub(2,:))
```

87
plot(amatsub/pi,vlimsub(2,:),'o')
hold off
axis([-1 1 -600 600])
ylabel('V2 (V)')
grid on
subplot 223
plot(amatsub/pi,vmatsub(3,:))
hold on
plot(amatsub,300*ones(size(amatsub)))
plot(amatsub,-300*ones(size(amatsub)))
plot(amatsub/pi,vmatsub(3,:),')x')
plot(amatsub/pi,vlimsub(3,:))
plot(amatsub/pi,vlimsub(3,:),')o')
hold off
axis([-1 1 -600 600])
ylabel('V3 (V)')
grid on
subplot 224
plot(amatsub/pi,vmatsub(4,:))
hold on
plot(amatsub,300*ones(size(amatsub)))
plot(amatsub,-300*ones(size(amatsub)))
plot(amatsub/pi,vmatsub(4,:),')x')
plot(amatsub/pi,vlimsub(4,:))
plot(amatsub/pi,vlimsub(4,:),')o')
hold off
axis([-1 1 -600 600])
ylabel('V4 (V)')
grid on
pause

subplot 211
plot amat/pi,abs(flim(1,:)+flim(2,:)*i)*1e3,'o')
hold on
plot amat/pi,abs(fmat(1,:)+fmat(2,:)*i)*1e3
hold off
ylabel('Force magnitudes (mN)')
xlabel('Angle (rad/pi)')
grid on
subplot 212
polar amat,abs(flim(1,:)+flim(2,:)*i)*1e3,'o')

5.4 Natural Dynamics

dynlin.m is a Matlab function which returns the vector of state derivatives $x'$, as defined in Section 3.2, for the linear case. Since the electrode voltages do not have to be computed or limited, it is much faster than the version which does compute the inverse transformation.

function xdot=f(t,x)
global m g wr ff xeq k flr fla %share with proc
global fprintf %share with proc (records)

aux=rem(x(2)/2+pi/2,pi);
x(2)=2*aux-pi*sign(aux);

e=x(1)/g;
fr=-ff*(1+2*e^2)/(1-e^2)^2.5*x(3)/g;
fa=ff/2*e/(1-e^2)^1.5*(wr-2*x(4));

force=-k*(x-xeq)'
fmt=[fmt [force; t]];

xdot(1)=x(3);
xdot(2)=x(4);
xdot(3)=x(1)*x(4)^2+fr/m+fr/m+force(1)/m;
xdot(4)=-2*x(4)*x(3)/x(1)+fa/(m*x(1))+fa/(m*x(1))+force(2)/
(m*x(1));

dynload.m is Matlab program which runs a simulation of the system in Section 3.2 using
the Matlab function ode45. This command uses fourth and fifth order Runge-Kutta
method with variable step size to solve an ordinary differential equation. From the desired
eccentricity, the necessary loading forces are computed. A sample trajectory is then com-
puted and plotted using an arbitrary initial condition and an ending time. The plots are
used in Figure 3.1.

clear global

clear

global m g wr ff xeq k flr fla %share with dynlin

global fprintf %share with dynlin

m=8.78e-6;
g=1e-5;
wr=2.4e6*2*pi/60;

%wr = 2.5133e+05

t=524;
nu=1.458e-6*t^1.5/(t+110.4);

%nu = 2.7567e-05

radius=.002;
length=.0003;

ff=pi*nu*radius*length^3/g^2;

%ff = 4.6767e-08

Equilibrium without feedback
weq=0;eeq=.4;

xeq=[eeq*1e-5 0 0 weq];

k=zeros(2,4);
flr=0;
fla=-ff/2*eeq/(1-eeq^2)^1.5*(wr-2*weq);
fprintf('Calculating natural dynamics...')
start=fixed(clock);
start=start(4:6)
tic
[te, xe]=ode45('dynlin', 0, 4.7e-4, [.2e-5 0 0 0]);
fprintf('done')
toc
subplot(211)
plot(te*1e6, xe(:,1)*1e6)
%xlabel('Time (us)')
%ylabel('Radial eccentricity (um)')
%title('Natural dynamics, ecc=.4 load, IC=[.2e-5 0 0 0]')
grid on
subplot(212)
polar(xe(:,2), xe(:,1)*1e6)
hold on
polar(0, 4, 'o')
hold off
%title('Polar view')
fprintf('Press any key to continue...')
pause
fprintf('ok')

5.5 Jacobian Matrix

jacobian.m is a Matlab function which returns the Jacobian Matrix evaluated at some state x of the system. However, it has been simplified for the case where \( x = [r^* 0 0 0] \), as shown in Section 3.4; see Equation 3.15.

```matlab
function j=f(x)
global m f f w r g
r=x(1); a=x(2); v=x(3); w=x(4); e=r/g;

j=[0 0 1 0
 0 0 0 1
 0 0 -f f/(m*g)*(1+2*e^2)/(1-e^2)^(5/2) 0
(1/2*f f/(1-e^2)^2)*(w r)+3/2*f f*e^2/(1-e^2)^(5/2)*(w r))/
(m*r*g)...
 0 0 -f f/(m*g)/(1-e^2)^1.5];
```

jacobianproc.m is a Matlab program which creates a plot of eigenvalues for the Jacobian Matrix of the system as a function of the rotor position. The plot is shown in Figure 3.2.

clear
global m f f w r g
m=8.78e-6;
g=1e-5;
wr=2.4e6*2*pi/60; %wr = 2.5133e+05
t=524;
u=1.458e-6*t^1.5/(t+110.4);  \%\nu = 2.7567e-05
radius=.002;
length=.0003;
ff=\pi*nu*radius*length^3/g^2;  \%\ff = 4.6767e-08

for i=0.001e-5:.01e-5:.9e-5
r=[r i];
eigre=[eigre real(eig(jacobian([i 0 0])))];
eigim=[eigim imag(eig(jacobian([i 0 0])))];
end
max(max(abs(eigim))
plot(r*1e6,eigre)
axis([0 9 -1e4 0])
\%xlabel('Offset distance (um)')
\%title('Real part of Jacobian eigenvalues')
\%ylabel('Re(eigenvalues')
grid on

5.6 Dynamics Simulation

dynfeedlinproc.m is a Matlab program which uses the linear simulation dynlin.m described in Section 5.4, but computes a feedback matrix K using Matlab’s lqr command, as explained in Section 3.5. The resulting trajectories are plotted in Figures 3.3 and 3.4.

clear global
clear
global m g wr ff xeq k flr fla  \%share with dynlin
global fmat  \%share with dynlin

m=7.92e-6;
g=1e-5;
wr=2.4e6*2*pi/60;  \%\wr = 2.5133e+05

t=524;
u=1.458e-6*t^1.5/(t+110.4);  \%\nu = 2.7567e-05
radius=.002;
length=.0003;
ff=\pi*nu*radius*length^3/g^2;  \%\ff = 4.6767e-08

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Equilibrium without feedback
weq=0;eeq=.1;
xeq=[eeq*1e-5 0 0 weq];
flr=0;
fla=-ff/2*eeq/(1-eeq^2)^1.5*(wr-2*weq);

a=jacobian(xeq)
beq=[0 0 0;1/m 0 0 1/(m*eeq*1e-5)];
rmax=.1e-5;amax=\pi;vmax=100;wmax=2e5;

91
q=eye(4); q(1,1)=1/rmax^2; q(2,2)=1/amax^2;
q(3,3)=1/vmax^2; q(4,4)=1/wmax^2
fmax=.005;
rs=eye(2); r(1,1)=1/fmax^2; r(2,2)=r(1,1)
[k,s,e]=lqr(a,b,e,q,r)
tend=.8e-3;
%height=.0006;
vshock=sqrt(2*9.81*height);
fprintf('Calculating feedback dynamics...'

start=fix(clock);
start=start(4:6)
tic
[te,x]=ode45('dynlin',0,tend,[.2e-5 -7*pi/8 0 0]);
fprintf('done')
toc
subplot 421
plot(te*1e3,x(:,1)*1e6)
xlabel('Time (ms)')
ylabel('r (um)')
title('Feedback dynamics, ecc=.1 load, IC=[.2e-5 -7*pi/8 0 0]')
grid on
subplot 423
plot(te*1e3,x(:,2)/pi)
xlabel('Time (ms)')
ylabel('alpha (rad/pi)')
grid on
subplot 425
plot(te*1e3,x(:,3))
xlabel('Time (ms)')
ylabel('v (m/s)')
grid on
subplot 427
plot(te*1e3,x(:,4))
xlabel('Time (ms)')
ylabel('w (rad/s)')
grid on
subplot 222
polar(x(:,2),x(:,1)*1e6)
title('Polar view')
subplot 426
plot(fmat(3,:)*1e3,abs(fmat(1,:)+fmat(2,:)*i)*1e3)
xlabel('Time (ms)')
ylabel('Force (mN)')
axis([0 tend*1e3 0 10])
grid on
subplot 428
plot(fmat(3,:)*1e3,unwrap(angle(fmat(1,:)+fmat(2,:)*i))/pi)
xlabel('Time (ms)')
ylabel('Angle (rad/pi)')
grid on
fprintf('Press any key to continue...')
pause
5.7 Nonlinear Model

dynnonlin.m is a Matlab function which returns the same dynamic model as dynlin.m of Section 5.4, but it adds a magnitude limiter on the force which the controller may apply, as explained in Section 3.6.

```matlab
function xdot = f(t,x)
    global m g wr ff xeq k flr fla     %share with proc
    global fmta                         %share with proc (records)

    aux = rem(x(2)/2*pi/2, pi);
    x(2) = 2*aux - pi*sign(aux);

    e = x(1)/g;
    fr = -ff*(1+2*e^2)/(1-e^2)^2.5*x(3)/g;
    fa = ff/2*e/(1-e^2)^1.5*(wr-2*x(4));

    force = -k*(x-xeq)';
    magforce = sqrt(force(1)^2 + force(2)^2);
    if magforce > 0.002,
        force = force .* 0.002 / magforce;
    end
    fmta = [fmta force; t];

    xdot(1) = x(3);
    xdot(2) = x(4);
    xdot(3) = x(1)*x(4) + 2 + flr/m + m*force(1)/m;
    xdot(4) = -2*x(4)*x(3)/x(1) + fla/(m*x(1)) + fa/(m*x(1)) + force(2)/
               (m*x(1));
```

dynfeednonlinproc.m is a Matlab program equivalent to dynfeedlinproc.m, but instead of calling dynlin.m, which has linear feedback, it calls dynnonlin.m. The resulting trajectories are shown in Figures 3.5 and 3.6.

```matlab
clear global
    clear
    global m g wr ff xeq k flr fla     %share with dynnonlin
    global fmta                         %share with dynnonlin

    m = 7.92e-6;
    g = 1e-5;
    wr = 2.4e6*2*pi/60;                 %wr = 2.5133e+05

    t = 524;
    nu = 1.458e-6*t^1.5/(t+110.4);      %nu = 2.7567e-05
```
radius = .002;
length = .0003;
ff = pi*nu*radius*length^3/g^2; \quad \% \text{ff} = 4.6767e-08
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Equilibrium without feedback
weq = 0; eeq = .1;
xeq = [eeq*1e-5 0 0 weq];

flr = 0;
fla = -ff/2*eeq/(1-eeq^2)^1.5*(wr-2*weq);

a = jacobiarn(xeq)
beq = [0 0; 0; 1/m 0; 0 1/(m*eeq*1e-5)];
rmax = .1e-5; amax = pi; vmax = 100; wmax = 2e5;
q = eye(4); q(1,1) = 1/rmax^2; q(2,2) = 1/amax^2;
q(3,3) = 1/vmax^2; q(4,4) = 1/wmax^2

fmax = .005;
r = eye(2); r(1,1) = 1/fmax^2; r(2,2) = r(1,1)
[k, s, e] = lqr(a, beq, q, r)

tend = 1.5e-3;
\% height = .0006;
\% vshock = sqrt(2*9.81*height);
fprintf('Calculating feedback dynamics...\n')
start = fix(clock);
start = start(4:6)
tic
[te, xe] = ode45('dymnonlin', 0, tend, [.2e-5 -7*pi/8 0 0]);
fprintf('done')
toc

subplot 421
plot(te*1e3, xe(:,1)*1e6)
\% xlabel('Time (ms)')
\% ylabel('r (um)')
\% title('Feedback dynamics, ecc=.1 load, IC=[.2e-5 -7*pi/8 0 0]')
grid on

subplot 423
plot(te*1e3, xe(:,2)/pi)
\% xlabel('Time (ms)')
\% ylabel('alpha (rad/pi)')
grid on

subplot 425
plot(te*1e3, xe(:,3))
\% xlabel('Time (ms)')
\% ylabel('v (m/s)')
grid on

subplot 427
plot(te*1e3, xe(:,4))
\% xlabel('Time (ms)')
\% ylabel('w (rad/s)')
grid on

subplot 222
polar(xe(:,2), xe(:,1)*1e6)
\% title('Polar view')
5.8 Controller with Voltage Conversion

* dyn.m is a Matlab function which returns the state derivatives vector (x'). However, it performs the full voltage inversion and limitation as described in Sections 2.4-5 and in Section 3.7. It is, however, too slow due to the force inversion using *fmins* to be of practical use in the design of electrostatic bearings. As better code for the force inversion is created, it should be replaced in the body of this program. The program does show the structure the full electrostatic bearing program should have, and what steps are required.

```matlab
function xdot=f(t,x)
global m g wr ff xeq k flr fla %share with proc
global force h nuc %share with fmins
global vmat fmat vlimmat flimmat %share with proc (records)
e=x(1)/g;
fr=-ff*(1+2*e^2)/(1-e^2)^2.*x(3)/g;
fa=ff/2*e/(1-e^2)^1.5*(wr-2.*x(4));
forcerot=-k*(x-xeq)';
fmt=[fmat forcerot];
force=[cos(x(2)) -sin(x(2));sin(x(2)) cos(x(2))]*forcerot;
h=hmat(x(1),x(2));

c=cap(x(1),x(2));
nuc=1e4*[-c(4) 0 0 c(1);-c(3) 0 c(1) 0;-c(2) c(1) 0 0
0 0 -c(4) c(3);0 -c(4) 0 c(2);0 -c(3) c(2) 0]';
options(1)=0;
options(2)=1e-1;
options(3)=1e-19;
coeff=fmins('forceerr',[0 0 0 0 0],options);
\texttt{v=nucoeff';}
\texttt{vmat=[vmat v];}
\texttt{red=norm(v,inf)/300;}
\texttt{if red>1,}
\texttt{v=v/red;}
\texttt{end}
\texttt{vlimmat=[vlimmat v];}
\texttt{flimrot=[\cos(x(2)) \sin(x(2)); -\sin(x(2)) \cos(x(2))]\*h\*v.^2;}
\texttt{flimmat=[flimmat flimrot];}
\texttt{xdot(1)=x(3);}
\texttt{xdot(2)=x(4);}
\texttt{xdot(3)=x(1)*x(4)^2+flr/m+fr/m+flimrot(1)/m;}
\texttt{xdot(4)=-2\*x(4)*x(3)/x(1)+fla/(m\*x(1))+fa/(m\*x(1))+flimrot(2)/(m\*x(1));}
References


New York, 1981.