A COMPENSATING NETWORK THEORY FOR
FEEDBACK-CONTROL SYSTEMS SUBJECT TO SATURATION

by

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Modern design theory for filters and feedback-control systems is based almost exclusively upon the mathematical techniques for treating linear systems. A linear system is an abstraction which exists only in the mathematical mind. All practical apparatus exhibits nonlinear behavior to some degree. Although a physical device can often be arranged so that the effects of nonlinearity are negligible over a certain range of operation, no way has been found for indefinitely extending this so-called linear operating range. Eventually any physical device will exhibit saturation if it is driven hard enough.

A rigorous design theory which includes saturation considerations would involve nonlinear system synthesis. Since nonlinear analysis is still in its infancy, it will be a long time before nonlinear synthesis can possibly be attempted by purely mathematical methods. The design theory presented in this paper is a modification of a current linear design theory and has been devised to at least take cognizance of the possible occurrence of saturation by attempting to avoid operation in the nonlinear ranges.

The work of this paper is based upon the theory of linear filter design originated by Dr. N. Wiener. As modified by Dr. Y. W. Lee, this theory determines the compensating network which must be cascaded with a fixed network to form an A filter such that the rms error for suitably specified inputs has the least possible value.
The scheme used in this paper for treating saturation is based on the assumption that limitation of the rms signal level at a point where saturation may occur in the fixed network is effective in controlling the probability that saturation actually occurs there. If it is assumed that the probability of saturation occurring is limited to low values at all points subject to saturation in the fixed network, then it is reasonable to assume that the rms signal levels occurring in the actual system are almost identical with those which would exist in a fictitious linear system corresponding to the actual one in the linear ranges but without limitations on the extent of the linear ranges. As long as discussion is limited to the fictitious linear system, it is shown that the filter and feedback-control problems are identical, and mathematical work is therefore limited to the filter arrangement of a compensating network cascaded with fixed network.

Because it is frequently desirable to inject transient performance considerations into the design of filters and feedback-control systems, a scheme for considering transient performance is incorporated into the design theory of this paper. This scheme comprises limitation of the integral-square errors for a set of transient test signals which are chosen to achieve the transient characteristics desired.

Use of these schemes for considering saturation and transient performance enables a mathematical formulation
of the compensating network design problem to be made. This formulation constitutes the major result of this paper. (See Subsection 3-1.)

After stating the mathematical problem, an integral equation form of solution (Eq. 3.18) is presented along with an explicit solution formula (Eq. 3.19). The derivation of these results is then outlined in considerable detail. The derivation uses mathematical techniques similar to those used to solve isoperimetric problems in the calculus of variations.

The body of the paper is closed by sections containing applications of the results to several simple examples and a concluding summary.

The first three sections of the Appendix of this paper comprise a summary of the specialized mathematics used, a review of Wiener's theory of filtering and Lee's extension thereof, and an analysis of the effects of disturbing signals in feedback-control systems in which it is shown that the results obtained with disturbing signals excluded can be modified to include their effects. The fourth section of the Appendix reports the results of an experimental check made on the saturation assumption mentioned above. It is found that controlling the rms value of a saturation signal in a fictitious linear system is indeed quite effective in controlling the probability of saturation occurring in an actual system. In a fifth section, some general comments on
equipment for statistical experiments in the servomechanism field may be of use to others who are planning experimental work in this area.
PREFACE

Why has the theory of compensating network design which is presented in this paper been devised? The author for the past eleven years has had an active interest in the practical development of feedback-control systems as well as in the theory behind them. During the early 1940's, a compensation theory for the design of such systems rapidly evolved as the result of the enormous concentration of effort in this field caused by the exigencies of the war. This theory, which is strictly a linear theory, presumes certain parts of a system to be fixed. It is the object of the theory to determine what characteristics the remainder of the system must have in order that the over-all system fulfill its performance specifications. As long as the performance specifications are amenable to linear analysis, the theory is quite successful, and with its aid many useful designs have been made. However, this compensation theory has at least one disturbing aspect. It is unable to detect any limit (except in the case of nonminimum-phase-fixed components) on the ultimate performance of a system incorporating fixed elements no matter how "bad" these fixed elements may be. According to it, sufficient compensation will always make possible unlimited performance. Practical designers know by bitter experience that definite limits are imposed on the performance of any system by the fixed component which it contains since any physical device will ultimately exhibit saturation if driven too hard. The lack of a clear theoretical understanding of the relationship between the
saturation limits of the fixed elements and the ultimate performance of the system was the principal reason for undertaking the work which lead to the design theory of this paper.

How should this paper be read? The main body of the paper is written for the specialist who is quite familiar with the current literature in this field. The specialist should read the main body (Sections 1 through 6), which is theoretical in nature, and Sections 10 and possibly 11 of the Appendix which are concerned with experimental work. On the other hand, the general technical reader, who is not a specialist in the fields of feedback-control or filter theory, should read this paper in the following order: Sections 1, 2, 7 (Subsection 7-1 only), 8, 3, 4 (interrupting at the proper point to read Subsection 7-2), 5, 6, 10, and 11. Both the specialist and general reader may read or omit Section 9, depending upon the depths of their individual interests. If Section 9 is of interest, it may be read any time after the first 4 sections have been completed.

The author has received assistance from many individuals at the Massachusetts Institute of Technology during the course work reported in this paper. Dr. A. C. Hall supervised the research and made available certain facilities in the Dynamic Analysis and Control Laboratory for conducting the experimental work reported in Section 10. Dr. G. S. Brown arranged for sponsorship of the research in the Servomechanisms Laboratory
and acted as a reader. Dr. Y. W. Lee was consulted in connection with the mathematical aspects of the research and also acted as a reader. Prof. J. B. Wiesner of the Research Laboratory of Electronics acted as an unofficial adviser during this research, particularly with respect to the experimental portion. Advice on mathematical matters was obtained from Dr. P. Franklin, Dr. G. P. Wadsworth, and Mr. J. G. Bryan. Mr. W. W. Siefert helped the author in making use of the facilities of the Dynamic Analysis and Control Laboratory. Miss D. Hamilton and Mr. J. Fry did the major part of the computation work involved in the analysis of the experimental data. Mr. P. Travers helped in the use of the correlator used in the analysis of the experimental data and proof read portions of this paper. Miss D. Young did an excellent job in typing the manuscript. To these people, and the many others who helped in the course of this research, the author extends his sincere thanks.

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1. BACKGROUND OF THE COMPENSATING-NETWORK DESIGN PROBLEM

1-1. WHERE PROBLEM IS ENCOUNTERED

The problem of compensating-network design is met rather frequently in communication and control-engineering practice. In wire communication the design of equilizers is a special case of the general task of compensating-network design. In servomechanism synthesis the designer often faces the problem of designing compensating networks for the purpose of "getting the most out" of a specified servomotor under given circumstances. In the fields of instrumentation and scientific measurement, as well as in the field of process control, engineers and scientists are found to be designing an ever increasing variety of control systems which, more probably than not, involve compensation design. As evidence of the great practical importance of the compensation problem, Table 1.1 shows an incomplete list of industries which use, in one or more vital ways, apparatus whose design requires knowledge of compensation theory.

1-2. CONVENTIONAL SUBCLASSIFICATION OF PROBLEM

The compensation design problem has been recognized as peculiarly important in two branches of electrical engineering: filter theory (or network theory) and feedback-control theory. This has resulted in two apparently fundamental forms of the compensation problem: the filter problem and the feedback-control problem.
### TABLE 1.1

**INDUSTRIES WHICH EMPLOY APPARATUS DESIGNED WITH THE AID OF COMPENSATION THEORY**

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<tr>
<td>Communication</td>
<td>Radio and Television, Railroad Signal, Telegraph and Telephone</td>
</tr>
<tr>
<td>Computing Devices and Machinery</td>
<td></td>
</tr>
<tr>
<td>Process Industries</td>
<td>Chemical, Rubber and Plastics, Paper, Metallurgical, Textile</td>
</tr>
<tr>
<td>Military</td>
<td>Aircraft Navigation and Bombing, Conventional Fire Control, Missile Guidance, Submarine and Torpedo Control</td>
</tr>
<tr>
<td></td>
<td>Power Generation and Distribution</td>
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1.3

Figure 1.1 is the fundamental block diagram of the filter system. The fixed network represents any apparatus connected to the output which is not subject to the control of the designer and is therefore a specification of the problem. The essential feature of the filter system is that the compensating network, which is under the control of the designer, is placed in simple cascade ahead of the fixed network.

Figure 1.2 is the fundamental block diagram of the feedback-control system. Again the fixed network represents any apparatus connected to the output which is not freely chosen by the designer, but which is essentially a specification of the problem. The essential feature of the feedback-control system is that the designer accomplishes his ends by actuating the fixed network by a signal formed as the resultant of the input minus the return signals. This actuating signal is modified by the compensating controller, which is freely chosen by the designer, to form the input to the fixed network. The return signal is formed from the output signal by the feedback network which is also freely chosen by the designer. The designer thus has, in general, two means for controlling the performance of the system: the feedback network and the compensating controller.

No cognizance of load effects or other disturbances has been taken in the fundamental block diagrams for the filter and feedback-control system, Figs. 1.1 and 1.2.
Fig. 1.1 Block Diagram of Filter System.

Fig. 1.2 Block Diagram of Feedback-Control System.
In the Appendix of this paper, Section 9, it is shown that disturbing signals occurring within the fixed network, under certain conditions, can be accounted for by modifying other signals, thus making it unnecessary to include them in the fundamental block diagrams.

If all the elements of a component network in either the filter or feedback-control systems are linear and have constant parameters, then this component may be completely described by a system (or transfer) function. System functions for each component of the filter and feedback-control systems are indicated by suitable letters in the blocks of Figs. 1.1 and 1.2. These system functions are to be understood as having existence only when the particular block in question is linear and has constant parameters.

It is recognized that problems involving no fixed components, such as often arise in filter design, are a special case of the more general problem of compensating network design. This special case is handled by making the system function of the fixed network equal to unity.

In Subsection 2-2 of this paper, it is shown that under certain conditions the filter and feedback-control problems are identical, and that either one may be used as the basis of a design for itself or the other. Thus it is possible to speak of compensating-network design rather generally without reference to either a filter or a feedback-control system. The word "system" without a modifier will be used to mean either a feedback-control system or a filter system.
1.3. SIMPLIFICATIONS MADE FOR SYSTEM ANALYSIS

It cannot be overemphasized that physical systems have been mathematically analyzed to the extent that perfect correlation exists between their predicted and their actual behavior. When attempting to design physical apparatus, it is essential that simplifications be made in the mathematical analysis. This is done by postulating mathematical models to replace the actual physical system under consideration. These mathematical models are idealizations of the actual system which may be treated by analytical methods and which represent the behavior of actual systems to greater or lesser degree, depending upon the skill with which they are chosen, the degree of approximation used, and the criterion used to judge agreement.

Various classes of mathematical models have been used to represent actual systems for purposes of compensating-network design. Examples are the:

1. Constant parameter, linear class
2. Time-varying parameter, linear class
3. Constant parameter, nonlinear class
4. Time-varying parameter, nonlinear class

The order of complexity of the mathematical analyses for these classes is presumably in the order of their listing with the constant parameter, linear class, the least complicated and the time-varying parameter, nonlinear class, the most complicated.

All discussion in this paper will be limited
to constant parameter models. Much of the discussion will be for constant-parameter, linear models even though one of the objectives is to include saturation considerations in compensating network design. From now on models of systems will be termed simply "linear" or "nonlinear" with "constant parameter" understood.

1-4. SURVEY OF LINEAR DESIGN PROCEDURES IN CURRENT USE

Before stating the objective of this paper, it is desirable to survey the linear design procedures that are now in use and make a rough appraisal of their applicability to practical problems.

Historically, the first procedure advanced for filter problems was what we now know as "conventional filter theory." This theory presumes the problem specifications to be a desired amplitude response for the over-all system. No fixed components are contemplated. The problem is solved by standardized sections with one or two adjustable parameters which are cascaded to form the over-all filter.

As an outgrowth of the conventional filter theory, a new, more general theory of network synthesis is gradually evolving. Here the problem specifications are presumed to be a desired amplitude response or a desired phase response or some combination. Various procedures are available for achieving a linear model with the desired characteristics to a degree of approximation limited only by the number of elements used.

More recently a re-examination of the whole
question of problem specification has been taking place. What is the system designer trying to accomplish? How do his ultimate objectives relate to amplitude and phase response? This is stimulating a rapid development of a new aspect of communication and control—information theory—and is making statistical approaches to the whole question seem more advantageous.

Dr. N. Wiener in Reference 23.6 has put forth a theory for system design which employs a statistical approach. This theory (as extended by Lee and others) postulates the rms error between the desired output and the actual output to be the criterion of the goodness of the system. It determines the system function which minimizes this error as a function of the problem specifications. A brief presentation of this theory is included in the Appendix, Subsection 8-1.

Dr. Y. W. Lee (see Reference 12.3) has extended Wiener's theory to include the design of a compensating network to work with a specified fixed network. In its extended form the theory determines the optimum linear compensating network under the following assumptions:

One, the input, output, and desired output signals are all stochastic variables or other functions possessing suitable correlation functions.

Two, the desired output signal is related to the input signal.

Three, the transfer characteristics of the fixed network are linear and known.
Four, the optimum compensating network is that one which minimizes the rms error between the desired output and actual output signals.

In applications of this theory, the input signal is frequently considered to be the sum of a message component and a noise component with the desired output related to the message in some way. In Wiener's original work, the desired output was usually taken to be the message part of the input with either a lead (prediction) or a lag (filtering). Actually any linear relationship between the input message and the desired output can be encompassed by the theory. For example, the desired output could be the derivative of the message or the result of passing the message through any arbitrary network.

So far the survey of current, linear-design procedures has omitted theories of feedback-control system synthesis. Modern authors on the subject, for example Ahrendt and Taplin (Reference 1.3), Brown and Campbell (Reference 2.6), Hall (Reference 8.2), Harris (Reference 8.4), James, Nichols and Phillips (Reference 10.5), and MacColl (Reference 13.1), all use methods similar to conventional filter theory, network synthesis theory, or statistical theory with rms error as a criterion. The basic problem is to provide compensation for fixed elements so that the over-all system is stable and has suitable performance characteristics. Frequently, degree of stability and static errors (such as load torque error, position, velocity or acceleration error) are considered to be sufficient specifications for these
systems. In the not too distant past, systems have been designed on a transient performance basis largely by using trial and error methods. But the recent trends have been toward the steady-state frequency response methods typical of conventional filter or network synthesis theories and toward the statistical approach introduced by Wiener.

1-5. PROS AND CONS OF CURRENT DESIGN PROCEDURES

With respect to conventional filter theory, it has the beautiful advantage of simplicity and standardization of approach. It has the disadvantage of not being a rigorous method in the sense of starting from the desired end and proceeding directly to the final design. If accurate results are desired, a certain amount of trial and error procedure must be used.

If conventional filter theory is not satisfactory, the more modern network synthesis theory may be used in which a realizable, desired frequency response is achieved as accurately as one pleases. The trouble with both conventional filter theory and network synthesis theory is that they assume half of the solution of the system design problem at the very outset. How to find the desired frequency response they do not say.

The statistical approach used by Wiener and Lee is somewhat more realistic than either the conventional filter or network synthesis theories in that it starts at the beginning of the problem. It only asks the designer to give a statistical specification of his input signal and his desired
output signal. It then tells him what frequency response he needs, and, if the various ramifications used by Lee are employed, how to make approximations so that only simple network structures are required to realize this response. Perhaps it should be said that the statistical methods of Wiener and Lee and conventional filter theory or network synthesis theory supplement each other in that the latter can be used after the former has suitably interpreted the specifications. In spite of its achievements, the statistical theory of Wiener and Lee is restricted by its assumption of the rms error criterion. Other criteria of at least equally limited scope may be used, but it appears that those criteria which permit mathematical progress may be disadvantageous in certain problems.

All the methods used so far for linear system design have in common at least two disadvantages. First, they fail to consider the fact that practical devices have only finite ranges of linearity. This means the designer has to first design on a linear basis and then attempt to find components with sufficiently wide ranges of linearity to permit practical construction of the system. In the case of feedback-control systems, this may be an especially poor way of proceeding because the power amplifying components of such systems often have ranges of linearity that are relatively independent of physical size or power capacity. The question then arises as to what is the best that can be done with such "limited" components. Into this question none of
the theories discussed so far gives a sufficient insight.

The second disadvantage of all of the modern design procedures discussed above is that they fail to consider transient performance. In feedback-control systems this can be objectionable, since transient specifications often may exist even when message information is lacking. Some procedure should be possible under these conditions, but current techniques do not work without message or frequency response specifications.

1-6. NONLINEAR SYSTEM DESIGN

Why not develop design procedures for nonlinear systems analogous to those for linear systems? Such procedures would certainly be able to cope with the saturation question raised above. The answer to this is that nonlinear system design is many orders of magnitude more difficult than linear system design. For every element, which in linear systems is specified by a single parameter, at least a curve is necessary in a nonlinear system. The most general curve would require an infinite number of parameters to specify it. Therefore, just on the basis of the number of parameters involved, it can be seen that nonlinear system design is certain to be very complicated indeed.

Because nonlinear systems are so complicated, their mathematical treatment has progressed only as far as analysis, and at that only the simplest systems have been analyzed. (See References 13.9 and 1.6.) It is therefore concluded that development of nonlinear design procedures
with any degree of generality is not feasible at the present time.

1-7. OBJECTIVE OF THIS INVESTIGATION

The purpose of the investigation reported in this paper is to extend the statistical methods of compensating network design used by Lee so that the two practical objections, lack of cognizance of saturation and failure to consider transient performance, are overcome to a sufficient degree to make possible wider application of these methods to filter and feedback-control problems. The next section will describe the scheme by which it is proposed to accomplish this objective.
2. PROPOSED SCHEME FOR CONSIDERING SATURATION AND TRANSIENT PERFORMANCE

2-1. SATURATION

In the preceding discussion, we concluded that saturation is an important factor in filters and feedback-control systems, that current linear design theory does not take cognizance of this factor, and that nonlinear methods at the present time generally are confined to analysis and are not powerful synthesis tools. The following paragraphs disclose the basic scheme by which it is hoped that saturation can be taken into account in system design to a greater degree than heretofore has been feasible.

Two very different approaches to the saturation question are conceivable. One approach is to regard a combination of linear and saturating elements as an essentially nonlinear system. This approach immediately requires nonlinear methods and thereby suffers from the lack of generality characteristic of these methods. The work of Hurewicz and Nichols in References 8.6 and 8.8 may be cited as an example of this approach in the design of servomechanisms. A second approach is to view a combination of linear and saturating elements as an essentially linear system. By appropriate design of the linear portions of the system, it is conceivable that operation of the saturating elements in their nonlinear ranges may be largely avoided. If this can be done, then linear methods may be used, thereby introducing great generality after once accepting the initial restriction of operating the saturating
elements substantially within their linear ranges only.

The second approach to systems containing saturating elements seems to be quite unexplored in the literature. Travers in Reference 20.5 is the only author discovered who utilizes this basic concept. This he does in a thesis research in which the objective is to design servomechanisms containing saturating servomotors so that the largest possible transients may be accepted without saturation occurring.

To understand better the meaning of phrases like "operation of a saturating element substantially within its linear range only" consider this simple example. Let the response of a saturating element to its excitation be represented by the solid curve in Fig. 2.1. This curve could be characteristic of the induced emf versus field current relation for a shunt d-c generator if hysteresis is neglected. It could also characterize the output versus input voltage behavior of a simple vacuum tube amplifier. In fact, this type of saturation curve is widely experienced in filter and feedback-control system components. Suppose now that the excitation is a stochastic signal. Then the probability density curve of Fig. 2.2 could conceivably correspond to it. 2.1 This probability density curve of the

2.1 It should not be implied that only stochastic signals could have such probability density curves. Obviously any signal (periodic, aperiodic, or stochastic) will have a single probability density curve which characterizes it over an infinite time interval.
FIG. 2.1 TYPICAL SATURATION CURVE

FIG. 2.2 TYPICAL PROBABILITY DENSITY CURVE FOR EXCITATION.
excitation in conjunction with the saturation curve will yield the probability density curve of the response; a typical response density curve is shown in Fig. 2.3. The sum of the shaded areas equals the probability of finding the response somewhere outside the linear range.

By the phrase "operation of a saturating element substantially within its linear range only" is meant "operation such that the probability of finding the response somewhere outside of its linear range is small enough to be negligible." By "small enough to be negligible" we mean so small that the departure from linear operation causes negligible deviation of actual system operation from that which would occur for a fictitious system identical to the actual system except for replacement of the saturating elements by linear models. The linear model of a saturating element is a fictitious element whose saturation curve is identical with that of the saturating element in the linear range but is extended with the initial slope in both directions without limit. Figure 2.1 shows in dashed lines the saturation curve of the linear model of the saturating element characterized by the solid curve.

Throughout this paper all analytical work is based upon the fictitious linear system obtained from the actual system by replacement of all of the actual saturating elements with their linear models. This fictitious system will be so designed that the root-mean-square response amplitude in this idealized system for each linear model representing
**Fig. 2.3** Typical probability density curve for response.

**Fig. 2.4** Probability of saturation in simple filter having one saturating element only when driven by signal which produces an excitation for saturating element which has normal probability density curve.
a saturating element in the actual system is maintained below specifiable limits. That this is useful derives from the basic assumption of this scheme for treatment of saturation, viz.:

Basic assumption in the treatment of saturation used in this paper

In the fictitious linear system obtained by replacing the saturating elements of the actual system by their linear models, limiting the rms responses of these linear models is effective in limiting the probabilities of finding the responses of the saturating elements of the actual system outside of the linear ranges characterizing these elements.

This assumption in no way establishes the exact relationships between rms responses in the fictitious system and the probabilities of saturation in the actual system. The exact determination of these relationships would require a nonlinear analysis involving the probability density curves for the input signals and the exact knowledge of the nonlinear characteristics of all the nonlinear elements. Such an analysis is obviously extremely complex and is just what we are trying to avoid by use of the premise. It is therefore fair to ask what good is this assumption if its "truth" cannot be rigorously evaluated. The answer to this may be in two different directions. The first is the hope that experimental experiences will tend to confirm it and will give insight into just how the rms responses in the fictitious system have to be limited to make the probabilities
of saturation in the actual system sufficiently small. A simple experiment has been performed with this in mind, and the results are recorded in the Appendix, Section 10. Without the adoption of such an assumption, no analysis and no experiments could be made and therefore no progress could result. The second direction is that the assumption may be viewed as induced by "engineering intuition" from the following simple situation.

Imagine a cascaded group of elements forming a filter which contains only one saturating element. Suppose this filter is driven by a signal which results in an excitation amplitude for the saturating element which has a normal (Gaussian) probability density curve. Such signals are not infrequently met in practice, particularly in situations where the signal is a composite signal formed from many random sources; an example is a "noise" signal. The response for the linear model of the saturating element in the fictitious linear system is obviously normal. The probability of finding the actual response of the saturating element outside of the linear range is therefore rigorously defined in this case. Let

\[ \Theta_{SL} \triangleq \text{maximum linear response of the saturating element} \]
\[ \sigma_s \triangleq \text{rms response of the linear model of the saturating element in the fictitious linear system} \]
\[ P_s \triangleq \text{probability of finding the actual response outside the linear range} \]
Then for this case

\[ P_s \triangleq 1 - \frac{1}{\sqrt{2\pi}} \int_{-\theta_{SL}/\sigma_s}^{\theta_{SL}/\sigma_s} \frac{dy}{\sigma_s} e^{-\frac{y^2}{2\sigma_s^2}} \]  

(2.1)

A plot for this case of the probability of saturation \( P_s \) as a function of the ratio \( \theta_{SL}/\sigma_s \) is shown in Fig. 2.4.2.2. This plot shows that for the normal distribution the probability of saturation is a rapidly decreasing function of the maximum linear response for a fixed rms excitation. Similar plots for other nonnormal distributions which may be met in practice would generally be expected to show at least as rapid a decrease of the probability of saturation after a certain value of maximum linear response is past. We thus observe for the simple filter having only one saturating element that no question exists concerning the validity of the basic assumption.

2-2. EQUIVALENCE OF FILTER AND FEEDBACK-CONTROL PROBLEMS

At the beginning of this paper, mention was made that the filter and feedback-control problems would be shown to be equivalent to each other under certain conditions. It is the purpose of this section to do just that.

The conditions under which these two problems are equivalent are:

2.2 See Cramer, Reference 3.8, for more details concerning the normal distribution and related functions. Pages 557 and 558 show tables of the normal distribution functions.
(1) Systems must be linear (with constant parameters)

(2) Over-all systems must be stable

The basic assumption used in the treatment of saturation permits the first condition to be met in all the analytical work to follow. The second condition is obviously not a severe restriction— who wants to design unstable systems?

Figure 1.1 at the beginning of this paper shows the basic block diagram of the filter. The compensating-network system function is designated as \( H_c(\omega) \); the fixed network as \( H_f(\omega) \). The over-all system function \( H(\omega) \) is given by

\[
H(\omega) = H_c(\omega)H_f(\omega)
\]  

(2.2)

In designing a filter which is simply a compensating network cascaded with a fixed network, it is obvious that over-all system stability requires both networks to be individually stable.

Figure 1.2 at the beginning of this paper shows the basic block diagram of the feedback-control system. The fixed network is designated as \( H_f(\omega) \); the compensating controller as \( G_c(\omega) \); the feedback network as \( G_f(\omega) \). The over-all system function \( H(\omega) \) is given by

\[
H(\omega) = \frac{G_c(\omega)H_f(\omega)}{1 + G_c(\omega)H_f(\omega)G_f(\omega)}
\]  

(2.3)

In contrast to the simple filter, it is evident in designing feedback-control systems that any one, two, or all of the component networks need not be stable. It would therefore
appear that the feedback-control system may have more freedom of adjustment than a simple filter. We wish to show that this is not the case and that, therefore, a general solution of the filter design problem is also a general solution of the feedback-control problem.

Consider first the situation for stable fixed networks. If the feedback-control system is to have more freedom of design than the filter in this situation, it must arise through the possibility of achieving an equivalent-filter compensating-network function $H'_c(w)$ which is unstable. But the equivalent-filter compensating-network function is merely the over-all system function divided by the fixed network function. Thus from Eq. 2.3

$$H'_c(w) = \frac{G_c(w)}{1 + G_c(w)H_f(w)G_f(w)} \quad (2.4)$$

Suppose $H'_c(w)$ is unstable, then it must have at least one pole in the lower half plane. But if it has a pole in the lower half plane, so must $H(w)$, since cancellation of a lower-half-plane pole of $H'_c(w)$ by a lower-half-plane zero of $H_f(w)$ is forbidden if we assume stability to mean stability within small perturbations of parameter values. But $H(w)$ cannot have poles in the lower half plane because of the hypothesis of over-all system stability. Hence $H'_c(w)$ must have no poles in the lower half plane and therefore must be stable. Thus the feedback-control system has no more freedom of design than the filter for stable fixed networks.
The situation for unstable fixed networks is simply this. Such networks are obviously active networks, and the truly fixed components are the sources and passive elements—not the feedback connections. The placing of additional feedback paths around the unstable fixed network, or the alteration of internal feedback paths, or both can always convert such a network into a stable one. Thus the situation posed by an unstable fixed network is not a very material one as it can be circumvented in practice.

In view of the equivalence of the filter and feedback-control problems, all analytical work to follow will be based on the simple filter. If a feedback-control system is to be designed, the $H_c(\omega)$ of an equivalent filter will be designed first, and then a $G_c(\omega)$ and a $G_f(\omega)$ will be found which satisfy Eq. 2.4.

2.3. TRANSIENT PERFORMANCE

The desirability of having some control over the transient performance of a system was pointed out in Section 1. The purpose of these paragraphs is to explain how we propose to take transient performance into account. First, the meaning of transient performance will be clarified. Then, the measure of transient performance to be used here will be presented. Finally, the basic scheme for controlling the transient performance will be disclosed.

Transient performance refers to the ability of a system to respond in some desired fashion to an input signal which is neither periodic nor stochastic, but which
is a definite function of time which is aperiodic within an infinite range of time. Such input signals are called transient signals. The unit step function is a typical example. A transient input function will be denoted by $C_1(t)$. When the signal $C_1(t)$ is applied to the system, the desired response will be defined as the function $C_d(t)$.

How should the transient performance of the system be measured? If the system output $C_o(t)$ existing for the input $C_1(t)$ is exactly equal to the desired output $C_d(t)$, then there is no question but that the transient performance is "ideally" perfect. The difference, $C_d(t) - C_o(t)$, will be designated by the function $C_e(t)$. It is apparent that this difference function is a measure of the departure of the actual performance from the ideal. However, it is a complicated measure since it is a function of time. In order to simplify the measure, the integral-square value of $C_e(t)$ will be adopted. We define the integral-square transient error $I_e$ as

$$I_e = \int_{-\infty}^{\infty} dt \ C_e^2(t) \quad (2.5)$$

In this manner a single number is arrived at which characterizes the transient performance of the system for a given disturbance.

The use of the integral-square error as a measure of the transient performance of a system is justified on the basis of expediency. Any other criteria that can be devised appears to involve greater difficulties in its...
mathematical formulation. The integral-square error has been used before in a different connection as a measure of the transient performance of systems by Hall (see Chapter II of Reference 2.2).

In the formulation of the compensating-network design problem used in this paper, K pairs of transient-input and desired-output signals with K corresponding to integral-square errors will be considered. These will be designated as $C_{ik}(t)$, $C_{dk}(t)$, and $I_{ek}$, respectively, for the kth transient-input function or test signal, the kth transient desired-output function, and the kth integral square of the kth error $C_{ek}(t)$.

The basic scheme for bringing transient performance considerations to bear in the design of compensating networks will be to so design these networks that the over-all system produces, for K pairs of specified transient-input and desired-output functions, integral-square errors equal to or less than specified limits $I_{emk}$. The limitation of integral-square errors for a number of transient test signals is the fundamental idea used to inject transient-performance considerations into the compensating-network design problem.

The introduction of transient-performance considerations into the design problem is useful in at least
three ways. First, it allows a certain degree of control of the system settling time. This can be achieved by limiting the integral-square errors for transient test signals which have a duration of the same order as the desired settling time. Second, it permits control of the error coefficients characterizing the system. For example, requiring the integral-square error to be finite for a step-input, step-output pair as the transient-input, desired-output pair will ensure that the position error coefficient is zero. Third, it permits the substitution of certain transient performance requirements in lieu of message specifications, thereby making design solutions possible in the presence of incomplete input data. For an illustration of this see Example 3 of Section 5.

2.3 For a solution of a somewhat different problem than is considered in this paper under the constraint of a finite settling time see Zadeh, Reference 26.5.

2.4 The error coefficients $d_i$ are defined (only for systems where the desired output equals the input) as

$$d_1 = \frac{-(-1)^i}{1!} \int_{-\infty}^{\infty} dt \ t^n h(t) \quad i = 1, 2, 3, \ldots$$

$$d_0 = 1 - \int_{-\infty}^{\infty} dt \ h(t)$$

in which $h(t)$ is the impulse response of the system. These error coefficients may be thought of as the coefficients of a series expansion for $H(w)$

$$1 - H(w) = d_0 + d_1 jw + d_2 (jw)^2 + d_3 (jw)^3 + \ldots$$

$d_0$ is called the position error coefficient; $d_1$, the velocity error coefficient, etc.
3. RESULTS OBTAINED

3.1. THE MATHEMATICAL PROBLEM

In view of the scheme presented in the preceding section for treating saturation and for introducing transient-performance considerations, the problem of designing a compensating network can be formulated mathematically, if certain additional assumptions are made. These assumptions are four in number:

One, the normal input and normal desired output are both functions of time possessing suitable correlation functions.

Two, the desired output is related to the input (or a portion of the input such as a message component).

Three, the transfer characteristics of the fixed network are known, including the over-all transfer characteristic and the transfer characteristics through to the responses for the models of the saturable elements of the actual fixed network.

Four, the optimum compensating network is that one which minimizes the rms error between the normal desired output and the normal actual output. (The compensating network is to be built so that no saturation occurs within it.)

(The adjective "normal" is used to distinguish the several signals of the system when under the normal excitation of a stochastic input signal from these same signals when the system is actuated by a transient-input test function for
purposes of transient-performance determination.) The first three assumptions are generally not very restrictive. The fourth assumption, however, is sometimes a questionable one. Some problems do exist where minimizing the rms error is equivalent to maximizing success in terms of more meaningful criteria. (For a particular example of this see Ragazzini and Zadeh, Reference 18.5.) Other problems occur in which minimization of the rms error is not the best criterion. Often for these latter, the proper criterion is either not known or is very difficult to apply, and the use of the rms error criterion is then justified on the basis of expediency with the thought that a solution on the basis of rms error is better than no solution at all.

As an aid in stating the mathematical problem of this paper, the block diagram of Fig. 3.1 is useful. In view of the equivalence of the filter and feedback-control problems pointed out in the preceding section, and since the filter formulation is simpler than that for feedback control, this block diagram is based upon the block diagram for the filter problem, Fig. 1.1. The input, usually considered to be a stochastic signal, is denoted by \( \Theta_1(t) \). The compensating network, described completely by its impulse response \( h_c(t) \) or alternatively by its system function \( H_c(w) \), modifies the input to form the input signal to the fixed network \( \Theta_f(t) \). In the real system, the fixed network contains saturable elements, but in this block diagram the linear model of the actual fixed network is employed. From now on the term...
FIG. 3.1 BLOCK DIAGRAM USED IN FORMULATING THE MATHEMATICAL PROBLEM
"fixed network" will mean the linear model of the real fixed network, and the term "real fixed network" will be used to refer to the network containing saturable elements. The fixed network is thus describable in terms of an impulse response $h_f(t)$ or system function $H_f(w)$. By means of the fixed network $\Theta_f(t)$ is connected to the system output $\Theta_o(t)$. The output is schematically shown as subtracted from the desired output $\Theta_d(t)$ to form the error $\Theta_e(t)$, even though the last two quantities are not actually present in the physical system but rather are conceptual in nature.

Also shown in Fig. 3.1 is a group of auxiliary blocks used to relate the saturation signals in the fixed network directly to the input $\Theta_f(t)$ of the latter. The $N$ saturation signals $\Theta_{sn}(t)$ are the responses of those $N$ elements in the fixed network which correspond to the $N$ saturable elements of the real fixed network. It facilitates the analysis to have these signals explicitly represented as functions of the $N$ transfer relationships $h_{sn}(t)$ or $H_{sn}(w)$ between them and $\Theta_f(t)$. To indicate these transfer relationships on the block diagram, auxiliary blocks called "saturation signal computers" are used. These blocks, of course, do not actually exist outside of the fixed network but are really contained inside of it.

We now state the mathematical problem to which the compensating-network design problem is reduced by means of the scheme that was outlined in Section 2 for dealing with saturation and considering transient performance and by
means of the assumption stated above. This problem will be
first stated in words and then repeated in terms of symbols.
In words, the problem is as follows:

Given:

1. Information about the normal input signal.
2. Information about the relationship between
   the normal-input and desired-output signals.
3. One over-all and N saturation-signal
   transfer characteristics for the fixed network.
4. Limits upon the rms values of the N
   saturation signals.
5. A set of K pairs of transient-input test
   signals and transient-desired-output signals
   together with limits on the K corresponding
   integral-square errors.

Find:

The transfer characteristic of the com-
pensating network which must be cascaded with
the fixed network in order to produce an over-
all system such that
1. the N saturation signals have rms values
   equal to or less than the given limits.
2. the K integral-square errors resulting
   from the transient tests are equal to or less
   than the given limits.
3. the rms error between the normal desired
   output and the real output is the minimum
possible in the presence of the first two requirements.

In this statement of the problem, it must be emphasized that the transient tests are made with the normal input removed and that the rms saturation-signal requirements are not imposed during these tests. In theory, the transient tests can always be made without saturating the system simply by using small enough amplitudes.

A restatement of the mathematical problem is now made in terms of symbols. (To do this certain results are anticipated by assigning symbols to quantities before their pertinence to the problem has been demonstrated. For example, part of the fundamental problem is to find what information concerning the normal input signal is required to determine the compensating network; this knowledge would not be available until the solution is partly worked out.)

These symbols are:

\[ \phi_{ii}(\tau) \triangleq \text{autocorrelation function of the normal input signal } \theta_i(t) \]  \hspace{1cm} (3.1)

\[ \Delta \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \theta_i(t)\theta_i(t + \tau). \]  \hspace{1cm} (3.2)

\[ \phi_{id}(\tau) \triangleq \text{cross-correlation function of the normal input signal } \theta_i(t) \text{ and desired output signal } \theta_d(t) \]  \hspace{1cm} (3.3)

\[ \phi_{id}(\tau) \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \phi_i(t)\phi_d(t + \tau). \]  \hspace{1cm} (3.4)
\[ h_f(t) = \text{the over-all impulse response of the fixed network} \quad (3.5) \]
\[ h_{sn}(t) = \text{the impulse response of the nth saturation signal computer} \quad (3.6) \]
\[ \sigma_e = \text{the rms value of the error } \theta_e(t) \text{ between the desired output and the actual output when the system is driven by its normal input signal} \quad (3.7) \]
\[ \sigma_{sn} = \text{the rms value of the nth saturation signal } \theta_{sn}(t) \quad (3.8) \]
\[ \sigma_{snm} = \text{the limit upon the rms value } \sigma_{sn} \text{ of the nth saturation signal} \quad (3.9) \]
\[ I_{ilk}(\tau) = \text{autotranslation function of the kth transient input test signal } C_{ik}(t) \quad (3.10) \]
\[ = \int_{-\infty}^{\infty} dt \ C_{ik}(t) C_{ik}(t + \tau) \quad (3.11) \]
\[ I_{idk}(\tau) = \text{cross-translation function of the kth transient input signal } C_{ik}(t) \text{ and desired output signals } C_{dk}(t) \quad (3.12) \]
\[ = \int_{-\infty}^{\infty} dt \ C_{ik}(t) C_{dk}(t + \tau) \quad (3.13) \]
\[ I_{ek} = \text{the kth integral-square of the error } C_{ek}(t) \text{ between the transient desired output } C_{dk}(t) \text{ and the transient actual output } C_{ok}(t) \quad (3.14) \]
\[ C_{ek}(t) = C_{dk}(t) - C_{ok}(t) \quad (3.15) \]
\[ I_{ek} = \int_{-\infty}^{\infty} dt \ C_{ek}^2(t) \quad (3.16) \]
I_{emk} = \text{the limit upon the kth integral-square error}
I_{ek} \quad \quad \quad \quad \quad \quad \quad \quad (3.17)

(Note that \( C_{ek}(t) \) is \( \Theta_e(t) \) for the kth transient test; a separate symbol is used to clearly distinguish the transient error from the normal error. Also note that the translation-function definitions restrict the class of transient test functions to those for which the defining integrals exist. This is not a severe restriction since convergence factors can be used; as an illustration see Example 3 of Section 5.)

In symbols the problem is as follows:

**Given:**

1. \( \phi_{ii}(\tau) \)
2. \( \phi_{id}(\tau) \)
3. \( h_f(t) \) and the set of N responses \( h_{sn}(t) \)
4. The set of N limits \( \sigma_{smn} \)
5. The two sets of K-translation functions each, \( I_{iik}(\tau) \) and \( I_{idk}(\tau) \), and the set of K limits \( I_{emk} \)

**Find:**

The \( h_c(t) \) which yields a cascaded system such that

1. the set of N conditions \( \sigma_{sn} \leq \sigma_{smn} \) are met.
2. the set of K conditions \( I_{em} \leq I_{emk} \) are met.
3. \( \sigma_e \) is the minimum possible in the presence of the first two requirements.
The $h_c(t)$ which satisfies the requirements of this problem will be called the optimum compensating-network impulse response and will be denoted by $h_{co}(t)$.

The above statement of the mathematical problem is the principal result of this paper. The solution of the problem is, of course, important but not nearly so important as the idealization of the physical problem which made possible the formulation of the solvable mathematical problem. What follows from here on are the results which any reasonably competent engineer or mathematician could derive from the problem statement made above and the results already published in the literature.

3-2. A SOLUTION IN THE FORM OF AN INTEGRAL EQUATION

A solution for the sought for compensating-network impulse response $h_c(t)$ can be expressed in terms of an integral equation. This equation is obtained by first formulating the normal rms error $\sigma_e$ as a function of the two impulse responses $h_c(t)$ and $h_f(t)$ and the two correlation functions $\phi_{11}(\tau)$ and $\phi_{1d}(\tau)$. In a similar way the $N$ rms values $\sigma_n$ of the saturation signals are determined as functions of $h_c(t)$ and $h_f(t)$ and $\phi_{11}(\tau)$. Next, the $K$-integral square errors $I_{ek}$ are found as functions of $h_c(t)$ and $h_f(t)$ and the $K$ pairs of translation functions $I_{11k}(\tau)$ and $I_{1dk}(\tau)$. By assuming an optimum $h_c(t)$ to exist, a necessary condition upon it is found by handling the situation as an isoperimetric problem in the calculus of variations with $N + K$ constraints. This necessary condition is also sufficient and represents
an integral equation form of solution for the optimum compensating-network impulse response $h_{co}(t)$:

A statement of this solution is

$$\left\{ \int_{-\infty}^{\infty} dy_{f}(y) \int_{-\infty}^{\infty} dv_{f}(v) \left[ \phi_{11}(x + y - u - v) \right] \right\}$$

$$\int_{-\infty}^{\infty} du_{co}(u) + \sum_{k=1}^{K} \rho_{ck} I_{11k}(x + y - u - v)$$

$$+ \sum_{n=1}^{N} \rho_{sn} \int_{-\infty}^{\infty} dy_{sn}(y) \int_{-\infty}^{\infty} dv_{sn}(v) \phi_{11}(x + y - u - v)$$

$$- \int_{-\infty}^{\infty} dy_{f}(y) \left[ \phi_{1d}(x + y) + \sum_{k=1}^{K} \rho_{ck} I_{1dk}(x + y) \right] = 0 \text{ for } x > 0$$

(3.18)

That this can possibly form a solution for $h_{co}(t)$ can be seen if it is recognized that the left-hand side is simply a function of $x - u$, say $f_1(x - u)$, and the whole expression is merely a function of $x - u$, say $f_2(x)$, so that the left-hand side is $\int_{-\infty}^{\infty} du_{co}(u) f_1(x - u) = f_2(x)$, which is a function of $x$, say $f_3(x)$. Now $f_3(x)$ depends upon the shape of $h_{co}(t)$ and Eq. 3.18 says that this shape must be such that $f_3(x)$ is zero for $x$ positive. (The variables $x$, $y$, $u$, and $v$ are all integration variables with the dimensions of time.)

The only quantities appearing in Eq. 3.18 which have not been defined so far are the $\rho_{sn}$'s and $\rho_{ck}$'s. These are the Lagrangian multipliers which permit the $h_{co}(t)$ to be adjusted to meet the $N + K$ constraints on the saturation signals and integral-square transient errors. The
h_{co}(t) obtained from Eq. 3.18 is a function of these N + K Lagrangian multipliers. By substituting the h_{co}(t) obtained from Eq. 3.18 into the N + K equations for the rms values of the saturation signals and for the integral-square transient errors, it is in theory at least possible to find values for the Lagrangian multipliers which will allow all of the N + K limit specifications, \sigma_{smn} and I_{emk}, to be met. For illustrations of such practical details, see the examples of Section 5.

3-3. AN EXPLICIT SOLUTION FORMULA

The integral equation form of the solution (Eq. 3.18) is rather cumbersome to use in practice. If the normal input and desired output signals are characterized by correlation functions which are Fourier transformable, and if the translation functions of transient test signals are transformable, it is possible to solve the integral equation and to obtain an explicit formula for the compensating-network system function. The Fourier transforms of the correlation functions are referred to as power-density spectra; the transforms of the impulse responses are equal to the corresponding system functions divided by 2\pi. The method of solving the integral equation is one of spectrum factorization first used by Wiener (see Reference 23.6).

An explicit solution of the integral equation (3.18) is:

\[ H_{co}(w) = \frac{L + \left[ \frac{B(w)}{A^{-}(w)} \right]}{A^{+}(w)} \]  (3.19)
Here

\[ H_{oo}(w) \triangleq \text{system function of the optimum compensating network} \] (3.20)

\[ L \triangleq a \text{ constant which is adjusted to yield minimum rms error } \sigma_e \] (3.21)

\[ A(w) \triangleq H_f(w)H_f(w) \left[ \Phi_{11}(w) + \sum_{k=1}^{K} \rho_{ck}^{I_{11k}(w)} \right] \]

\[ + \sum_{n=1}^{N} \rho_{sn} H_{sn}(w) H_{sn}(w) \Phi_{11}(w) \] (3.22)

\[ B(w) \triangleq H_f(w) \left[ \Phi_{1d}(w) + \sum_{k=1}^{K} \rho_{ck}^{I_{1dk}(w)} \right] \] (3.23)

\[ A^+(w) \triangleq \text{factor of } A(w) \text{ which has all of the poles and zeroes of } A(w) \text{ which lie in the upper half plane} \] (3.24)

\[ A^-(w) \triangleq \frac{A(w)}{A^+(w)} \text{ and has all of the poles and zeroes of } A(w) \text{ which lie in the lower half plane} \] (3.25)

\[ \begin{bmatrix} B(w) \\ A^-(w) \end{bmatrix} \triangleq \text{component of } \frac{B(w)}{A^-(w)} \text{ which has all of its poles in the upper half plane, and such that} \]

\[ \frac{B(w)}{A^-(w)} - \left[ \frac{B(w)}{A^-(w)} \right]_+ \text{ has all of its poles in the lower half plane} \] (3.26)

\[ H_f(w) \triangleq \text{the system function for the fixed network} \] (3.27)

\[ H_{sn}(w) \triangleq \text{the system function for the nth saturation signal computer} \] (3.28)

\[ \Phi_{11}(w) \triangleq \text{the power density spectrum of the normal input signal } \Theta_1(t) \] (3.29)
\[ I_{ik}(w) \triangleq \text{the Fourier transform of } I_{ik}(\tau) \quad (3.30) \]
\[ \Phi_{id}(w) \triangleq \text{the cross-power density spectrum of the normal input and desired output signals} \quad (3.31) \]
\[ I_{idk}(w) \triangleq \text{the Fourier transform of } I_{idk}(\tau) \quad (3.32) \]

N.B. The Fourier transform of \( f(t) \), denoted by \( f(w) \), is taken to be \( \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-j\omega t} f(t) \) and the inverse transform as \( \int_{-\infty}^{\infty} dw e^{j\omega t} f(w) \) throughout this paper. The system function \( H(w) \) is related to the impulse response \( h(t) \) by \( H(w) = \int_{-\infty}^{\infty} dt e^{-j\omega t} h(t) \) and inversely by \( h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{j\omega t} H(w) \).

A bar over a quantity denotes the conjugate; thus \( \overline{H}_{f}(w) \) is the conjugate of \( H_{f}(w) \).

In using Eq. 3.19, \( \frac{B(w)}{A^{-}(w)} \) is frequently not obtainable by the partial-fraction expansion method. In this event an auxiliary equation for this quantity can be used, viz.:

\[ \left[ \frac{B(w)}{A^{-}(w)} \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-j\omega t} \int_{-\infty}^{\infty} dwe^{j\omega t} \frac{B(w)}{A^{-}(w)} \quad (3.33) \]

The constant \( L \) in Eq. 3.19 is determined by finding the value which produces a minimum rms error. The Lagrangian multipliers, \( \rho_{en} \) and \( \rho_{ck} \), in Eq. 3.19 are determined by the same procedure as described in connection with the integral equation of the preceding subsection.
4. DERIVATION OF SOLUTIONS

4-1. EXPRESSION FOR RMS ERROR $\sigma_e$

This section gives the detailed steps by which the solution for the compensating-network transfer relationships are derived. In the broad outline of the procedure discussed in Subsection 3-2, the first step was to obtain an expression for the rms error $\sigma_e$ in terms of the normal-input and desired-output correlation functions and the compensating- and fixed-network impulse responses. This is explained in detail in the following paragraphs.

First we need an expression for $Q_0(t)$. Referring to Fig. 3.1, we see that

$$\Theta_f(t - y) = \int_{-\infty}^{\infty} dx h_c(x) \Theta_1(t - x - y) \quad (4.1)$$

$$Q_0(t) = \int_{-\infty}^{\infty} dy h_f(y) \Theta_f(t - y) \quad (4.2)$$

Substituting the value of $\Theta_f(t - y)$ of Eq. 4.1 into Eq. 4.2 yields

$$Q_0(t) = \int_{-\infty}^{\infty} dy h_f(y) \int_{-\infty}^{\infty} dx h_c(x) \Theta_1(t - x - y) \quad (4.3)$$

From the definition of the error $Q_e(t)$ given by Fig. 3.1 (viz.: $Q_e(t) = \Theta_d(t) - Q_0(t)$), we write

$$Q_e^2(t) = \Theta_d^2(t) - 2\Theta_d(t)Q_0(t) + Q_0^2(t) \quad (4.4)$$

From Eq. 4.3 for $Q_0(t)$, we write for $Q_e^2(t)$
\[ \theta^2_0(t) = \int_{-\infty}^{\infty} dsf_r(y)\int_{-\infty}^{\infty} dxh_c(x)\Theta_1(t - x - y)\int_{-\infty}^{\infty} dvh_r(v)\int_{-\infty}^{\infty} duh_c(u)\Theta_1(t - u - v) \quad (4.5) \]

Eq. 4.4 we now write as

\[ \theta^2_e(t) = \theta^2_d(t) - 2\theta_d(t)\int_{-\infty}^{\infty} dsf_r(y)\int_{-\infty}^{\infty} dxh_c(x)\Theta_1(t - x - y) \]
\[ + \int_{-\infty}^{\infty} dsf_r(y)\int_{-\infty}^{\infty} dxh_c(x)\Theta_1(t - x - y)\int_{-\infty}^{\infty} dvh_r(v)\int_{-\infty}^{\infty} duh_c(u)\Theta_1(t - u - v) \quad (4.6) \]

The square of the rms value \( \sigma_e \) of the error \( \theta_e(t) \) is by definition

\[ \sigma^2_e \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt\theta^2_e(t) \quad (4.7) \]

Substituting the value \( \theta_e(t) \) of Eq. 4.6 into Eq. 4.7 yields upon interchanging orders of integration and limit processes

\[ \sigma^2_e = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt\theta^2_d(t) - 2 \int_{-\infty}^{\infty} dsf_r(y)\int_{-\infty}^{\infty} dxh_c(x)\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt\Theta_1(t - x - y)\Theta_d(t) \]
\[ + \int_{-\infty}^{\infty} dsf_r(y)\int_{-\infty}^{\infty} dxh_c(x)\int_{-\infty}^{\infty} dvh_r(v)\int_{-\infty}^{\infty} duh_c(u)\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt\Theta_1(t - x - y)\Theta_1(t - u - v) \quad (4.8) \]

Recall that the definition of the autocorrelation function of any function \( f_1(t) \) is

\[ \varphi_{f_1f_1}(\tau) \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dtf_1(t)f_1(t + \tau) \quad (4.9) \]
and that of the cross-correlation function of any pair of functions \( f_1(t) \) and \( f_2(t) \) is

\[
\phi_{f_1 f_2}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d t f_1(t) f_2(t + \tau) \quad (4.10)
\]

Then Eq. 4.8 can be written in terms of auto- and cross-correlation functions of \( \Theta_1(t) \) and \( \Theta_d(t) \) as

\[
\sigma_e^2 = \phi_{dd}(0) - 2 \int_{-\infty}^{\infty} dy h_f(y) \int_{-\infty}^{\infty} dx h_c(x) \phi_{1d}(x + y)
\]

\[
+ \int_{-\infty}^{\infty} dy h_f(y) \int_{-\infty}^{\infty} dx h_c(x) \int_{-\infty}^{\infty} dv h_f(v) \int_{-\infty}^{\infty} du h_c(u) \phi_{11}(x + y - u - v)
\]  

The "\( \Theta \)'s" in the subscripts for the correlation functions have been omitted for simplicity, since no confusion is caused thereby. Equation 4.8 is the desired expression for the rms error \( \sigma_e \) in terms of the normal-input and desired-output correlation functions and the compensating- and fixed-network impulse responses.

4.2. EXPRESSION FOR RMS SATURATION SIGNAL \( \sigma_{sn} \)

In order to obtain the necessary expression for the nth rms saturation signal \( \sigma_{sn} \) in terms of the input and desired-output correlation functions and the compensating-network and saturation-signal computer impulse responses, we first write an expression for the nth saturation signal \( \Theta_{sn}(t) \).

Figure 3.1 shows that this will be the same as the expression for \( \Theta_0(t) \) (Eq. 4.3) with \( h_{sn}(t) \) substituted for \( h_f(t) \). Thus

\[
\Theta_{sn}(t) = \int_{-\infty}^{\infty} dy h_{sn}(y) \int_{-\infty}^{\infty} dx h_c(x) \Theta_1(t - x - y) \quad (4.12)
\]
Hence

\[ e_{sn}^2(t) = \int_{-\infty}^{\infty} dyh_{sn}(y) \int_{-\infty}^{\infty} dxh_c(x) \Theta_1(t - x - y) \int_{-\infty}^{\infty} dyh_{sn}(v) \int_{-\infty}^{\infty} duh_c(u) \Theta_1(t - u - v) \]  \hspace{1cm} (4.13)

Now

\[ \sigma_{sn}^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \Theta_{sn}^2(t) \]  \hspace{1cm} (4.14)

Substituting the value of \( e_{sn}^2(t) \) from Eq. 4.13 into Eq. 4.14, interchanging orders of integration and limit processes, and using the definition of the autocorrelation function yields

\[ \sigma_{sn}^2 = \int_{-\infty}^{\infty} dyh_{sn}(y) \int_{-\infty}^{\infty} dxh_c(x) \int_{-\infty}^{\infty} dyh_{sn}(v) \int_{-\infty}^{\infty} duh_c(u) \Phi_{11}(x + y - u - v) \]  \hspace{1cm} (4.15)

This is the desired expression for \( \sigma_{sn} \).

4-3. EXPRESSION FOR INTEGRAL-SQUARE \( I_{ek} \) OF TRANSIENT ERROR \( C_{ek}(t) \)

Next we find the expression for the kth integral-square error. The transient error \( C_{ek}(t) \) is given by Eq. 4.6 with \( C_{ek}^2(t) \) replacing \( e^2(t) \), \( C_{dk}(t) \) replacing \( \Theta_d(t) \), and \( C_{ik}(t) \) replacing \( \Theta_1(t) \).

\[ C_{ek}^2(t) = C_{dk}^2(t) - 2C_{dk}(t) \int_{-\infty}^{\infty} dyh_f(y) \int_{-\infty}^{\infty} dxh_c(x) C_{ik}(t - x - y) \]

\[ + \int_{-\infty}^{\infty} dyh_f(y) \int_{-\infty}^{\infty} dxh_c(x) C_{ik}(t - x - y) \int_{-\infty}^{\infty} dyh_f(v) \int_{-\infty}^{\infty} duh_c(u) C_{ik}(t - u - v) \]  \hspace{1cm} (4.16)
Use of the definition of the integral-square transient error $I_{ek}$ (Eq. 3.16), and of the definition of the translation functions $I_{lik}$, $I_{idk}$ (Eqs. 3.11, 3.13, respectively) yields

\[
I_{ek} = I_{ddk}(0) - 2 \int_{-\infty}^{\infty} dyh_f(y) \int_{-\infty}^{\infty} dxh_c(x)I_{idk}(x + y)
+ \int_{-\infty}^{\infty} dyh_f(y) \int_{-\infty}^{\infty} dxh_c(x) \int_{-\infty}^{\infty} dvh_f(v) \int_{-\infty}^{\infty} duh_c(u)I_{lik}(x + y - u - v)
\tag{4.17}
\]

when integration orders are interchanged. $I_{ddk}(\tau)$ is defined as

\[
I_{ddk}(\tau) = \int_{-\infty}^{\infty} dtC_{dk}(t)C_{dk}(t + \tau)
\tag{4.18}
\]

Equation 4.17 gives the desired expression for the integral-square error $I_{ek}$ in terms of the transient-input and desired-output translation functions and the impulse responses of the compensating and fixed networks.

4.4. DERIVATION OF THE INTEGRAL EQUATION 3.18

Equations 4.11, 4.15, and 4.17 give the rms error $\sigma_e$, the N rms values $\sigma_{sn}$ of the saturation signals, and the K integral-square values $I_{ek}$ of the transient errors as functions of $h_c(t)$. The mathematical problem is to find the curve $h_c(t)$ which minimizes $\sigma_e$ and which at the same time satisfies the $N + K$ conditions $\sigma_{sn} \leq \sigma_{smn}$ and $I_{ek} \leq I_{emk}$. Also $h_c(t)$ must satisfy, of course, the condition that $h_c(t) = 0$ for $t < 0$ in order to ensure a realizable compensating network. Beyond these conditions $h_c(t)$ is unbounded.
The finding of $h_c(t)$ is effectively the solving of an isoperimetric problem in the calculus of variations. In References 3.6 and 13.3 and in the Appendix (Subsection 7-2), it is shown that this problem is equivalent to finding the $h_c(t)$ which minimizes the function

$$ F[h_c(t)] = \sigma_e^2 + \sum_{n=1}^{N} \rho_{sn} \sigma_{sn}^2 + \sum_{k=1}^{K} \rho_{ck} I_{ek} \quad (4.19) $$

where $\rho_{sn}$ and $\rho_{ck}$ are $N + K$ Lagrangian multipliers. (To be precise, the $h_c(t)$ which minimizes the function $F$ of Eq. 4.19 is that which makes $\sigma_e^2$ a minimum within the $N + K$ constraints $\sigma_{sn} = \sigma_{spn}$ and $I_{ek} = I_{epk}$, where $\sigma_{spn}$ and $I_{epk}$ are constants which are functions of the $N + K$ $\rho_{sn}$ and $\rho_{ck}$ multipliers. It is evident that this is the $h_c(t)$ which we desire, since suitable adjustment of the multipliers will ensure that the limit conditions on the rms saturation signals and integral-square transient errors are met.)

The procedure for finding the $h_c(t)$ which minimizes the $F$ of Eq. 4.19 is to assume that it exists. Denote this optimum value of $h_c(t)$ as $h_{co}(t)$. Then consider an $h_c(t)$ different from this optimum formed by

$$ h_c(t) = h_{co}(t) + \epsilon h_{cl}(t) \quad (4.20) $$

where $h_{cl}(t)$ is any fixed, arbitrary, realizable impulse response and $\epsilon$ is a variable introduced for convenience. Use of this $h_c(t)$ in Eq. 4.19 will yield an $F$ which is a function of $\epsilon$. Since $h_{co}(t)$ is the optimum, we know that $F(\epsilon)$ should be zero for $\epsilon = 0$. Thus a necessary condition
on \( h_{co} (\tau) \) is obtained by setting

\[
\frac{dF(\varepsilon)}{d\varepsilon} = 0 \quad \text{at } \varepsilon = 0 \quad (4.21)
\]

This condition in conjunction with the limit conditions proves to be sufficient to determine \( h_{co}(t) \).

Consider now the components of \( F \). The derivative of each one with respect to \( \varepsilon \) is to be determined and the results added in order to form Eq. 4.21. Differentiating Eq. 4.11 and setting \( \varepsilon \) equal to zero yields

\[
\frac{d\sigma^2}{d\varepsilon} \bigg|_{\varepsilon=0} = -2 \int_{-\infty}^{\infty} dy_f(y) \int_{-\infty}^{\infty} dx_{cl}(x) \phi_{id}(x + y) \\
+ \int_{-\infty}^{\infty} dy_f(y) \int_{-\infty}^{\infty} dx_{cl}(x) \int_{-\infty}^{\infty} dv_f(v) \int_{-\infty}^{\infty} du_{co}(u) \phi_{ii}(x + y - u - v) \\
+ \int_{-\infty}^{\infty} dy_f(y) \int_{-\infty}^{\infty} dx_{co}(x) \int_{-\infty}^{\infty} dv_f(v) \int_{-\infty}^{\infty} du_{co}(u) \phi_{ii}(x + y - u - v) \quad (4.22)
\]

Because \( \phi_{ii}(\tau) \) is an even function, the last two terms are equal. Making use of this fact and interchanging orders of integration gives

\[
\frac{d\sigma^2}{d\varepsilon} \bigg|_{\varepsilon=0} = 2 \int_{-\infty}^{\infty} dx_{cl}(x) \int_{-\infty}^{\infty} du_{co}(u) \int_{-\infty}^{\infty} dy_f(y) \int_{-\infty}^{\infty} dv_f(v) \phi_{ii}(x + y - u - v) \\
- 2 \int_{-\infty}^{\infty} dx_{cl}(x) \int_{-\infty}^{\infty} dy_f(y) \phi_{id}(x + y) \quad (4.23)
\]
Equation 4.15, when differentiated and when \( \varepsilon \) is set equal to zero, results in a sum of two terms similar to the last two terms of Eq. 4.22. Making use of the fact that these terms are equal and interchanging orders of integration yields

\[
\frac{d \sigma^2}{d \varepsilon} \bigg|_{\varepsilon=0} = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dh_{cl}(x) duh_{co}(u) dyh_{sn}(y) dvh_{sn}(v) \phi_{11}(x + y - u - v) \quad (4.24)
\]

Treating Eq. 4.17 in an identical fashion to Eq. 4.11 will give by analogy to Eq. 4.23

\[
\frac{d I_{ek}}{d \varepsilon} \bigg|_{\varepsilon=0} = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dh_{cl}(x) duh_{co}(u) dyh_{sn}(y) dvh_{sn}(v) I_{11k}(x + y - u - v)
\]

\[
- 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dh_{cl}(x) dyh_{sn}(y) I_{1dk}(x + y) \quad (4.25)
\]

Adding together the components of \( \frac{d F(\varepsilon)}{d \varepsilon} \bigg|_{\varepsilon=0} \) (in accordance with Eq. 4.19) produces the following necessary condition:

\[
2 \int_{-\infty}^{\infty} dh_{cl}(x) \left[ \text{Left side of Eq. 3.18} \right] = 0 \quad (4.26)
\]

But \( h_{cl}(x) \) is arbitrary except that it must be zero for \( x < 0 \). This means that the brackets must be zero for \( x > 0 \), since if they were not an \( h_{cl}(x) \) could be found which would invalidate Eq. 4.26. Thus Eq. 3.18 is shown to be a necessary condition on \( h_{co}(\tau) \). (From what has been said so far, this could be a condition for either a
a maximum or minimum value of $\sigma_0^2$. Consideration of $\frac{d^2 F(\epsilon)}{d\epsilon^2} \bigg|_{\epsilon=0}$ will show that we have a condition for a minimum since this quantity is guaranteed to be positive.) That Eq. 3.18 is a sufficient condition (in conjunction with the rms saturation signal and integral-square-error limit conditions) to determine $h_0(\tau)$ is evident from the fact that unique solutions are obtained.

4.5. DERIVATION OF THE SOLUTION FORMULA 3.19

The solution formula represented by Eq. 3.19 is obtained from the integral equation 3.18 by considering the Fourier transform of the left side of the latter. It is evident that the left side of Eq. 3.18 is a function of time which can only be non-zero for negative time; let this function of time be $f(x)$. Thus

$$f(x) = \int_{-\infty}^{\infty} du h(x+u)$$

On the assumption that transforms exist for the correlation and translation functions, we can transform both sides of Eq. 4.27. This gives, upon interchanging orders of integration and rearranging,
Identifying the integrals in Eq. 4.28 as system functions, transforms of translation functions, and power-density spectra permits writing

\[
\ell(w) = H_c^*(w) \left\{ \overline{H_f}(w) \overline{D_1}(w) + \sum_{k=1}^{K} \rho_{ck} \overline{D}_{11k}(w) \right\} - \overline{H_f}(w) \left[ \overline{D}_{1d}(w) + \sum_{k=1}^{K} \rho_{ck} \overline{D}_{1dk}(w) \right] \tag{4.29}
\]

(See Eqs. 3.29, 3.30, 3.31, and 3.32 for definitions of \( \overline{D}_{11}(w) \), \( \overline{D}_{11k}(w) \), \( \overline{D}_{1d}(w) \), and \( \overline{D}_{1dk}(w) \).) Using the definitions of A(w) and B(w) given by Eqs. 3.22 and 3.23,
respectively, Eq. 4.29 can be written as

$$\tau(w) = H_{co}(w)A(w) - B(w)$$  \hspace{1cm} (4.30)

We recognize that, since $\tau(t)$ is zero for positive time, $\tau(w)$ has no poles in the UHP. (UHP is an abbreviation for upper half plane; LHP, for lower half plane.)

Let $A(w)$ be split into two factors: $A^+(w)$ with poles and zeros in the UHP only and $A^-(w)$ with poles and zeros in the LHP only, such that $A^+(w) \times A^-(w)$ is equal to $A(w)$. Divide both sides of Eq. 4.30 by $A^-(w)$ to obtain

$$\frac{\tau(w)}{A^-(w)} = H_{co}(w)A^+(w) - \frac{B(w)}{A^-(w)}$$ \hspace{1cm} (4.31)

$\tau(w)$ still has no poles in the UHP. Next let $\frac{B(w)}{A^-(w)}$ be separated into two parts; $[\frac{B(w)}{A^-(w)}]_+$ with poles in the UHP and $[\frac{B(w)}{A^-(w)}]_-$ with poles in the LHP, such that $[\frac{B(w)}{A^-(w)}]_+ + [\frac{B(w)}{A^-(w)}]_-$ is equal to $\frac{B(w)}{A^-(w)}$. Add to both sides of Eq. 4.31 the quantity $[\frac{B(w)}{A^-(w)}]_-$ to get

$$\frac{\tau(w)}{A^-(w)} + [\frac{B(w)}{A^-(w)}]_- = H_{co}(w)A^+(w) - \frac{B(w)}{A^-(w)}$$ \hspace{1cm} (4.32)

The left side has no poles in the LHP. The right side has no poles in the UHP, since $H_{co}(w)$ is a realizable system function. Thus each side of the equation is a function with no poles and is therefore constant. Let this constant be $L$.

Then

$$L = H_{co}(w)A^+(w) - \frac{B(w)}{A^-(w)}$$ \hspace{1cm} (4.33)
and rearranging gives the solution formula for $H_{\infty}(\omega)$, Eq. 3.19.

It should be noted that the derivations in this section are purely formal and are not proofs in the strict mathematical sense. In applied mathematics, particularly as used in engineering, formal derivations are usually satisfactory if the results are used only for solution of "natural" problems so that only the better-behaved classes of functions are met. The final "proof" for the engineer is usually whether "it works or not" and is not a logical proof based upon some minimum number of axioms. What is said here in no way should be interpreted as "belittling" the role of the mathematician; rather it is an engineer's apology for not rigorously proving his results. Undoubtedly the above results could be mathematically proven by following the paths of Kolmogoroff and Wiener (References 11.6 and 23.6), since these results are extensions of their work.
5. APPLICATION OF RESULTS TO THREE SIMPLE EXAMPLES

5-1. BACKGROUND FOR FIRST TWO EXAMPLES

The major objective of this research is to develop a design procedure for compensating networks which takes cognizance of the possibility that certain components of a physical network may impose limitations because of their finite ranges of linearity. Up to the present time all network synthesis theory has presupposed components which are linear over infinite ranges. It was left to the application engineer to select physical parts with sufficiently wide ranges of linearity so that the designed system would not operate with any part in the saturation region. For much engineering work this is a satisfactory procedure. However, in certain applications, particularly in automatic-control applications, it would seem that using this procedure could result in requiring excessively large components, since component operating range plays no part in establishing the design. In certain situations it would seem possible to reduce linear range requirements by permitting the system error to increase slightly from the minimum value possible with linear range considerations excluded from the system determination. One can conceive of a "trading curve" for system design which governs the exchange of reduction of linear range requirements for increase in system error. If specifications are such that the "trading curve" permits a large reduction of linear range for a small increase in error, one is inclined to make use of this fact in making his design.
This is particularly true in servomechanisms where the size of the servomotor is a measure of system size and cost in weight or money.

The first two examples to follow seek to establish the "trading curves" relating linear range requirement to error for two sets of specifications differing only in the variable considered to be critical as to linear range. The specifications for these examples have been purposely selected to make the analytical work as simple as possible and still illustrate the fact that "trading-curve" shape favorable to reduction of linear range may or may not exist, depending upon the problem specifications. In all examples, the rms values of the saturation signals are considered to be a measure of the linear range requirement.

5.2. EXAMPLE 1

Given a servomechanism problem in which the input is pure message and the desired output is just the input message. Given further that the fixed elements of the problem impose no limitations other than the possibility of output position saturation. Consider the input to be a stationary, stochastic function characterized by the autocorrelation function:

\[ \phi_{11}(\tau) = \sigma_{ip}^2 e^{-\frac{1}{\tau}} \]  

5.1 Such an autocorrelation function could result from an input function which is constant except at change points having a Poisson distribution in time. The constant amplitudes between changes could be governed by a Gaussian distribution of standard deviation \( \sigma_{ip} \).
Find the optimum system function as a function of the limit on the rms value of the saturation signal. Determine the function relating the limit on the rms saturation signal to the rms error for the optimum system.

For this problem, since the desired output is the same as the input,

$$\Phi_{ii}(w) = \Phi_{id}(w) \quad \text{(5.2)}$$

Now from the definition of the input power-density spectrum

$$\Phi_{ii}(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-j\omega\tau} \sigma_{ip}^2 e^{-|f\tau|} \quad \text{(5.3)}$$
or

$$\Phi_{ii}(w) = \sigma_{ip}^2 \frac{1}{\pi} \frac{f}{w^2 + f^2} \quad \text{(5.4)}$$

Also for this problem $H_f(w)$ and $H_{sl}(w)$ can be chosen to be

$$H_f(w) = 1 \quad \text{(5.5)}$$
$$H_{sl}(w) = 1 \quad \text{(5.6)}$$

since the fixed elements are premised to impose no limitations other than possible output-position saturation.

In the general solution formula (Eq. 3.19), the values of $A(w)$ and $B(w)$ from Eqs. 3.22 and 3.23, respectively, are

$$A(w) = (1 + \rho_{sl}) \sigma_{ip}^2 f \frac{1}{\pi} \frac{1}{w^2 + f^2} \quad \text{(5.7)}$$

$$B(w) = \frac{\sigma_{ip}^2 f}{\pi} \frac{1}{w^2 + f^2} \quad \text{(5.8)}$$

since only one point of saturation is involved and no
transient specification is imposed.

We may take

$$A^{-}(w) = \frac{1}{-j\omega + f} \quad (5.9)$$

and

$$A^{+}(w) = \frac{(1 + \rho_{sl})\sigma_{ip}f}{\pi} \frac{1}{j\omega + f} \quad (5.10)$$

Accordingly,

$$\frac{B(w)}{A^{-}(w)} = \frac{\sigma_{ip}f}{\pi} \frac{1}{j\omega + f} \quad (5.11)$$

and

$$\left(\frac{B(w)}{A^{-}(w)}\right)^{+} = \frac{\sigma_{ip}f}{\pi} \frac{1}{j\omega + f} \quad (5.12)$$

Thus

$$H_{co}(w) = \frac{\pi L(j\omega + f)}{(1 + \rho_{sl})\sigma_{ip}f} + \frac{1}{1 + \rho_{sl}} \quad (5.13)$$

Because the rms level of the output (and also of the error) would approach infinity if \(L\) were different than zero, it is evident that \(L\) must be zero. Since \(H_{f}(w)\) is unity, the overall optimum system function \(H_{o}(w)\) is equal to \(H_{co}(w)\). Thus

$$H_{o}(w) = \frac{1}{1 + \rho_{sl}} \quad (5.14)$$

which is a simple attenuator.

Having found the proper form of \(H_{o}(w)\), it is now necessary to eliminate \(\rho_{sl}\) and find \(H_{o}(w)\) in terms of the rms saturation signal \(\sigma_{sl}\). In this case, since the output is the saturation signal,
\[ \sigma_{sl}^2 = \int_{-\infty}^{\infty} \Phi_{11} \] 

This basis for this equation is given in the Appendix, Subsection 7-1. Integration yields

\[ \sigma_{sl}^2 = \frac{\sigma_{ip}^2}{(1 + \rho_{sl})^2} \] 

\( H_o(w) \) in terms of \( \sigma_{sl} \) is then simply

\[ H_o(w) = \frac{\sigma_{sl}}{\sigma_{ip}} \] 

for \( 0 \leq \sigma_{sl} \leq \sigma_{ip} \) (5.17)

Denoting the limit upon the rms saturation signal as \( \sigma_{sml} \), it is evident that \( H_o(w) \) should be

\[ H_o(w) = \frac{\sigma_{sml}}{\sigma_{ip}} \] 

for \( \sigma_{sml} \leq \sigma_{ip} \)

\[ = 1 \] 

for \( \sigma_{sml} \geq \sigma_{ip} \) (5.18)

for minimum rms error. This is the first result asked for.

In order to obtain the second required result, the rms error \( \sigma_e \) must be found in terms of \( \sigma_{sml} \) using the system function (Eq. 5.18). Because the input and desired output are identical, the minimum mean-square error is, in this case,

\[ \sigma_e^2 = \int_{-\infty}^{\infty} dw \left[ 1 - H_o(w) \right] \left[ 1 - H_o(w) \right] \Phi_{11} \] 

(5.19)

or

\[ \sigma_e^2 = \left( 1 - \frac{\sigma_{sml}}{\sigma_{ip}} \right)^2 \sigma_{ip}^2 \] 

for \( \sigma_{sml} \leq \sigma_{ip} \)

\[ = 0 \] 

for \( \sigma_{sml} > \sigma_{ip} \) (5.20)
Hence
\[ \sigma_e = \left( 1 - \frac{\sigma_{sml}}{\sigma_{ip}} \right) \sigma_{ip} \]
for \( \sigma_{sml} \leq \sigma_{ip} \)
\[ = 0 \]
for \( \sigma_{sml} \geq \sigma_{ip} \)  \hspace{1cm} (5.21)

is the second required result.

The foregoing results may be expressed in another form as
\[ H_0(w) = S_{ml} \]
for \( S_{ml} \leq 1 \)
\[ = 1 \]
for \( S_{ml} > 1 \)  \hspace{1cm} (5.22)
\[ E = 1 - S_{ml} \]
for \( S_{ml} \leq 1 \)
\[ = 0 \]
for \( S_{ml} > 1 \)  \hspace{1cm} (5.23)

where
\[ S_{ml} \]
is the limit on the rms saturation signal expressed in nondimensional terms as
\[ S_{ml} = \frac{\sigma_{sml}}{\sigma_{ip}} \]  \hspace{1cm} (5.24)

\( E \)
is the rms error expressed in nondimensional terms as
\[ E = \frac{\sigma_e}{\sigma_{ip}} \]  \hspace{1cm} (5.25)

Figure 5.1 shows how the limit on the saturation signal level varies as a function of the allowable error for this simple example. It is evident from this "trading curve" that the cost of reducing the saturation signal level, as measured in terms of error, is so great that once the "natural" saturation limit of unity is passed, it is not feasible to trade error for reduction of linear
range. This is what one would intuitively surmise in this example. All that has been done is simply to quantitatively confirm this guess.

5.3. **EXAMPLE 2**

This example is identical to the first example, except that the fixed elements impose no limitations other than the possibility of output velocity saturation (as distinguished from output position saturation in Example 1).

Here again $\Phi_1 (w) = \Phi_{id} (w)$, $\Phi_{11} (w) = \left[ \frac{\sigma_{ip}^2}{\pi} \right] \left[ \frac{1}{w^2 + f^2} \right]$, and $H_f (w) = 1$. To account for output velocity saturation, take

$$H_{sl} (w) = jw \quad (5.26)$$

Then by Eqs. 3.22 and 3.23

$$A(w) = \frac{\sigma_{ip}^2 f}{\pi} \left[ \frac{1}{w^2 + f^2} + \frac{\rho_{sl} w^2}{w^2 + f^2} \right] \quad (5.27)$$

$$B(w) = \frac{\sigma_{ip}^2 f}{\pi} \left[ \frac{1}{w^2 + f^2} \right] \quad (5.28)$$

Write

$$A(w) = \frac{\sigma_{ip}^2 f \rho_{sl}}{\pi} \frac{w^2 + \frac{1}{\rho_{sl}}}{w^2 + f^2} \quad (5.29)$$

and factor into

$$A^-(w) = \frac{-jw + \frac{1}{\sqrt{\rho_{sl}}}}{-jw + f} \quad (5.30)$$

$$A^+(w) = \frac{\sigma_{ip}^2 f \rho_{sl}}{\pi} \frac{jw + \frac{1}{\sqrt{\rho_{sl}}}}{jw + f} \quad (5.31)$$
Then

\[
\frac{B(w)}{A^-(w)} = \frac{\sigma_{ip}^2 f}{\pi} \frac{1}{(jw + f)(-jw + \frac{1}{\sqrt{\rho_{sl}}})}
\]  

(5.32)

and

\[
\left(\frac{B(w)}{A^-(w)}\right)_+ = \frac{\sigma_{ip}^2 f}{\pi} \frac{1}{f + \frac{1}{\sqrt{\rho_{sl}}}} jw + f
\]  

(5.33)

Thus, by Equation 3.19,

\[
H_{co}(w) = \frac{\pi L}{\sigma_{ip}^2 f \rho_{sl}} \frac{jw + f}{jw + \frac{1}{\sqrt{\rho_{sl}}}} + \frac{1}{f\sqrt{\rho_{sl}} + 1} \frac{1}{\sqrt{\rho_{sl}}} jw + 1
\]  

(5.34)

L must be zero for finite rms output velocity. Also \( H_f \) is unity. Therefore, the optimum over-all system function is

\[
H_o(w) = \frac{1}{f\sqrt{\rho_{sl}} + 1} \frac{1}{\sqrt{\rho_{sl}}} jw + 1
\]  

(5.35)

In this example, the saturation signal is the output velocity, and its mean square value is given by

\[
\sigma_{s1}^2 = \int_{-\infty}^{\infty} dw (-jw) H_o(w)(jw)H_o(w)D_{11}
\]  

(5.36)

or

\[
\sigma_{s1}^2 = \frac{\sigma_{ip}^2 f}{\pi} \left(\frac{1}{f\sqrt{\rho_{sl}} + 1}\right)^2 \int_{-\infty}^{\infty} dw \frac{w^2}{D(w)D(w)}
\]

(5.37)

where \( D(w) = \sqrt{\rho_{sl}} (jw)^2 + (f\sqrt{\rho_{sl}} + 1)jw + f \). Evaluation of the integral (James, Nichols, and Phillips, Reference 10.5, has an appendix which is useful in this regard) gives
\[ \sigma_{s1}^2 = \frac{\sigma_{1p}^2 f}{\sqrt{\rho_{s1}}(f\sqrt{\rho_{s1}} + 1)^3} \]  

(5.38)

It is impossible to solve this last equation for \( \sqrt{\rho_{s1}} \) and thereby obtain an explicit expression for \( H_o(\omega) \) in terms of \( \sigma_{s1} \). However, it is apparent that the rms error will be a monotonic increasing function and the rms saturation signal a monotonic decreasing function of \( \sqrt{\rho_{s1}} \) in the interval zero to infinity. It is further evident that rms saturation signal will range from infinity to zero as \( \sqrt{\rho_{s1}} \) varies from zero to infinity. Thus it will be possible for the rms saturation signal to reach any specified limit. This means that the limit on the saturation signal level \( \sigma_{sml} \) is continuously related to \( \sqrt{\rho_{s1}} \) by

\[ \sigma_{sml}^2 = \frac{\sigma_{1p}^2 f}{\sqrt{\rho_{s1}}(f\sqrt{\rho_{s1}} + 1)^3} \]  

(5.39)

The first result asked for is determined by Eqs. 5.35 and 5.39 through the elimination of \( \sqrt{\rho_{s1}} \). This can be done for any set of numerical values, but it cannot be done algebraically because of the fourth degree equation for \( \sqrt{\rho_{s1}} \).

Next, let the rms error be evaluated. The minimum mean-square error is given by Eq. 5.19, and for this example is

\[ \sigma_e^2 = \frac{\sigma_{1p}^2 f}{\pi} \int_{-\infty}^{\infty} dw \left[ 1 - \frac{1}{\sqrt{\rho_{s1}(\omega) + 1}} \right] \left[ 1 - \frac{1}{\sqrt{\rho_{s1}(\omega) + 1}} \right] \left[ \frac{1}{w^2 + r^2} \right] \]  

(5.40)
or

\[
\sigma_e^2 = \frac{\sigma_{ip} f \rho_{sl}}{\pi} \int_{0}^{\infty} dw \frac{w^2 \left( f \sqrt{\rho_{sl} + 1} \right)^2}{D(w) D(w)}
\]

(5.41)

where \( D(w) = \sqrt{\rho_{sl}}(jw)^2 + (f \sqrt{\rho_{sl}} + 1)jw + f \). Integration yields

\[
\sigma_e^2 = \sigma_{ip} f \sqrt{\rho_{sl}} \frac{1 + \left( f \sqrt{\rho_{sl}} + 1 \right)^2}{f \sqrt{\rho_{sl}} + 1}
\]

or

\[
\sigma_e^2 = \sigma_{ip} \frac{f \sqrt{\rho_{sl}}(f \sqrt{\rho_{sl}} + 1)^2 + (f \sqrt{\rho_{sl}})^2}{(f \sqrt{\rho_{sl}} + 1)^3}
\]

(5.42)

Again there is no direct way of eliminating \( \sqrt{\rho_{sl}} \) between Eqs. 5.42 and 5.39, so these two equations together must be considered as constituting the second required result.

The results of Example 2 may be summarized in slightly different form as

\[
H_0(w) = \frac{1}{\alpha + 1} \frac{1}{\frac{f}{\sqrt{\alpha(\alpha + 1)^{3/2}}}}
\]

(5.43)

\[
S_{ml} = \frac{1}{\sqrt{\alpha(\alpha + 1)^{3/2}}}
\]

(5.44)

\[
E = \frac{\gamma \alpha(\alpha + 1)^{2} + \alpha^2}{(\alpha + 1)^{3/2}}
\]

(5.45)

where

\[
\alpha = f \sqrt{\rho_{sl}}
\]

(5.46)
\[ S_{ml} = \frac{\sigma_{sml}}{\sigma_{ip}} \]  
\[ E = \frac{\sigma_{e}}{\sigma_{ip}} \]  

Curve A in Fig. 5.2 is for this example what Fig. 5.1 is for the first example. Note that here the "trading curve" is so shaped as to favor reduction of the limit on the saturation signal level through increasing the error, since the cost in error is not so great as in Example 1. Indeed, in view of these results, a designer for the specifications of this problem would be foolish not to reduce the burden on his servomotor by allowing a little error to occur. Although it is very simple and the results may seem extreme, this example appears to justify the "hunch" that advantages may be gained by injecting saturation considerations into a system design problem at the outset rather than at the end of its solution.

Example 2, Supplement 1

At this point the reader may ask, "How sensitive are these results to departures of the system function from its optimum form?" Some insight into this question may be gained by comparing results for a particular nonoptimum system with those above for the optimum system. For example, let the nonoptimum over-all system function be

\[ H(w) = \frac{1}{\left(\frac{jw}{\omega_n}\right)^2 + 2\xi \frac{jw}{\omega_n} + 1} \]  

instead of the optimum
FIG. 5.2  SATURATION LIMIT VERSUS ERROR FOR EXAMPLE 2.
By expressing the limit on the saturation signal level \( \sigma_{sml} \) and the rms error \( \sigma_e \) in terms of the parameters of \( H(\omega) \), and by expressing the results nondimensionally, one obtains

\[
S_{ml} = \frac{1}{\sqrt{2\zeta\alpha(1 + 2\zeta\alpha + \alpha^2)}}
\]

(5.50)

\[
E = \frac{\alpha[(2\zeta + \frac{1}{2\zeta}) + \alpha]}{1 + 2\zeta\alpha + \alpha^2}
\]

(5.51)

(both equations for \( \alpha \) such that \( E \leq 1.0 \)),

where

\[
\alpha = \frac{f}{\omega_n}
\]

(5.52)

\[
S_{ml} = \frac{\sigma_{sml}}{\sigma_{ip}}
\]

(5.47)

\[
E = \frac{\sigma_e}{\sigma_{ip}}
\]

(5.48)

Curve B in Fig. 5.2 is a plot \( S_{ml} \) versus \( E \) for \( \xi \) equal to one half. Notice that the same general relation is established between saturation limit and error for the nonoptimum system as for the optimum system; the only practical difference is that in the low error range about 40 per cent more error is necessary to obtain the same saturation limit.

**Example 2, Supplement 2**

As an aside, it is interesting to note that the results of this example may be used in a limiting sense.
to handle the problem resulting when the input power-density spectrum is changed from \((\sigma_{1p}^2/\pi)\left[f/(w^2 + f^2)\right]\) to

\[
\Phi_{11} = \frac{\nu}{\pi \omega^2}
\]  

(5.53)

with the other specifications remaining unchanged. This is done by considering the limit of the results (Eqs. 5.35, 5.39, and 5.42) as \(f \to 0\) with \(\sigma_{1p}^2 f = \nu\). These equations become, respectively,

\[
H_0(\omega) = \frac{1}{\sqrt{\rho_{s1} j\omega + 1}}
\]  

(5.54)

\[
\sigma_{sml}^2 = \frac{\nu}{\sqrt{\rho_{s1}}}
\]  

(5.55)

\[
\sigma_e^2 = \nu \sqrt{\rho_{s1}}
\]  

(5.56)

These results may be expressed as

\[
H_0(\omega) = \frac{1}{\frac{1}{S_{ml}^2} j\omega + 1}
\]  

(5.57)

\[
S_{ml} = \frac{1}{E}
\]  

(5.58)

where

\[
S_{ml} = \frac{\sigma_{sml}}{\nu\nu}
\]  

(5.59)

\[
E = \frac{\sigma_e}{\nu\nu}
\]  

(5.60)

(Note that \(S_{ml}\) and \(E\) are no longer nondimensional.)

Curve C in Fig. 5.2 is plot \(S_{ml}\) versus \(E\) for this limiting case. This result is explicitly stated here because the power-density spectrum of Eq. 5.53 may be obtained,
for laboratory purposes, by integration of "white noise."

5.4. BACKGROUND FOR THIRD EXAMPLE

Sometimes the designer is given rather meager information about the message component of the input to a system. This may be for lack of knowledge of the statistical properties of this component, or it may be because the actual message is not describable in terms which fit into the known schemes of analysis. Under such conditions it is not uncommon to specify the transient response of the system to some degree as a substitute for input message specification. In addition, certain of the system error coefficients $^5.2$ may be specified.

With respect to the noise component of the input to a system, not infrequently it is found that noise predominates over message in producing saturation effects. For example, in a radar tracking device, noise can become the predominant cause of saturation. Thus, even though the message component of the input need not be directly accounted for, the noise may very well have to be considered in a system design.

The following example considers a problem where the input noise is assumed to be the major cause of saturation, and where the input message is not specifiable and is therefore assumed to be zero. In lieu of the input message specification, requirements on the position-error coefficient and the transient response are laid down.

5.2 The error coefficients are defined in Footnote $^4$. page 2.14.
5-5. **EXAMPLE 3**

Given a servomechanism problem in which the normal input is pure noise (and the desired output is therefore zero). Given that the transient response of the servomechanism must be such that the integral-square error \( I_{e1} \) to a step-function input is equal to or less than a finite error \( I_{em1} \) to be determined on the basis that the input step is the desired output. (This automatically makes the position-error coefficient zero.) Consider the input noise to be "white noise" characterized by the autocorrelation function

\[
\phi_{nn}(\tau) = \frac{2\pi \delta(\tau)}{(1 \text{ unit of time})} \quad (5.61)
\]

where \( \delta(\tau) \) is the unit impulse function. Given further that the fixed elements of the problem impose no limitations other than the possibility of output velocity saturation.

First, find the optimum system function as a function of the limit on the rms saturation signal and the limit on the transient error. Second, find the relationship among rms error, saturation limit, and transient-error limit when the optimum system function is used.

For this problem, \( \phi_{11}(w) = \phi_{nn}(w) \), and therefore

\[
\phi_{11}(w) = \frac{\eta}{\pi} \quad (5.62)
\]

Because the desired output is zero,

\[
\phi_{1d}(w) = 0 \quad (5.63)
\]

In order to handle the transient constraint, the translation functions, \( I_{11k}(\tau) \) and \( I_{1dk}(\tau) \), have to be transformable (see
Eqs. 3.11 and 3.13 for definitions), and this fact will prevent direct use of step functions. However, let the input transient be

\[ C_{il}(t) = Vu(t)e^{-at} \]  \hspace{1cm} (5.64)

where \( u(t) \) is the unit step function, and "a" is a positive number which can approach zero in the limit (\( e^{-at} \) may be thought of as a convergence factor). Then from the definition (Eq. 3.11)

\[ I_{iil}(\tau) = v^2 \int_{-\infty}^{\infty} dt u(t)e^{-at}u(t+\tau)e^{-a(t+\tau)} \]  \hspace{1cm} (5.65)

or

\[ I_{iil}(\tau) = \frac{v^2 e^{-a|\tau|}}{2a} \]  \hspace{1cm} (5.66)

and

\[ I_{iil}(w) = \frac{v^2}{2\pi} \frac{1}{w^2 + a^2} \]  \hspace{1cm} (5.67)

Because the input and desired output are the same for the transient-response test

\[ I_{id1}(w) = I_{iil}(w) \]  \hspace{1cm} (5.68)

\( H_f(w) \) and \( H_{sl}(w) \) for this problem may be taken as

\[ H_f(w) = 1 \]  \hspace{1cm} (5.69)

\[ H_{sl}(w) = jw \]  \hspace{1cm} (5.70)

Referring to the solution form (Eq. 3.19), the values of \( A(w) \) and \( B(w) \) from Eqs. 3.22 and 3.23, respectively, are
\[ A(w) = \frac{n}{\pi} + \frac{\rho_{cl} \nu^2}{2\pi} \frac{1}{w^2 + a^2} + \frac{\rho_{sl} \eta}{\pi} w^2 \]  
(5.71)

\[ B(w) = \frac{\rho_{cl} \nu^2}{2\pi} \frac{1}{w^2 + a^2} \]  
(5.72)

Rewrite \( A(w) \) as

\[ A(w) = \frac{\rho_{sl} \eta}{\pi} \frac{w^4 + (a^2 + \frac{1}{\rho_{sl}})w^2 + \frac{a^2}{\rho_{sl}} + \frac{\rho_{cl} \nu^2}{2\rho_{sl} \eta}}{w^2 + a^2} \]  
(5.73)

Factor \( A(w) \) into

\[ A^{-}(w) = \frac{(-jw + c + jd)(-jw + c - jd)}{-jw + a} \]  
(5.74)

\[ A^{+}(w) = \frac{\rho_{sl} \eta}{\pi} \frac{(jw + c + jd)(jw + c - jd)}{jw + a} \]  
(5.75)

where

\[ c = \sqrt{\frac{1}{\pi} \left( a^2 + \frac{1}{\rho_{sl}} \right) + \frac{1}{2} \left( a^2 + \frac{\rho_{cl} \nu^2}{2\rho_{sl} \eta} \right)} \]  
(5.76)

\[ d = \sqrt{\frac{1}{2} \left( \frac{a^2}{\rho_{sl}} + \frac{\rho_{cl} \nu^2}{2\rho_{sl} \eta} - \frac{1}{4} \left( a^2 + \frac{1}{\rho_{sl}} \right) \right)} \]  
(5.77)

Hence

\[ \frac{B(w)}{A^{-}(w)} = \frac{\rho_{cl} \nu^2}{2\pi} \frac{1}{(jw + a)(-jw + c + jd)(-jw + c - jd)} \]  
(5.78)

and

\[ \left( \frac{B(w)}{A^{-}(w)} \right)^{+} = \frac{\rho_{cl} \nu^2}{2\pi(a^2 + 2ca + c^2 + d^2)} \frac{1}{jw + a} \]  
(5.79)
Thus

\[ H_{co}(\omega) = \frac{\pi L}{\rho_{s1} \eta} \left( \frac{j\omega + a}{(j\omega + c + j\omega)(j\omega + c - j\omega)} \right) + \frac{\rho_{cl} V^2}{2\rho_{s1} \eta (a^2 + 2ca + c^2 + d^2)} \frac{1}{(j\omega + c + j\omega)(j\omega + c - j\omega)} \]  

(5.80)

L must be zero since, if it were not, the saturation signal level would be infinite. Therefore, \( H_0(\omega) \), which is the same as \( H_{co}(\omega) \) in this problem where \( H_f(\omega) = 1 \), is

\[ H_0(\omega) = \frac{\rho_{cl} V^2}{2\rho_{s1} \eta (a^2 + 2ca + c^2 + d^2)(c^2 + d^2)} \frac{1}{(j\omega)^2 + \frac{2c}{c^2 + d^2} j\omega + 1} \]  

(5.81)

Consider now the limit of \( H_{co}(\omega) \) as \( \alpha \) approaches zero.

\[ \lim_{\alpha \to 0} (c^2 + d^2) = \sqrt{\frac{\rho_{cl} V^2}{2\rho_{s1} \eta}} \]  

(5.82)

\[ \lim_{\alpha \to 0} \frac{2c}{c^2 + d^2} = \sqrt{\frac{2\eta}{\rho_{cl} V^2} + 2\sqrt{\frac{2\rho_{s1} \eta}{\rho_{cl} V^2}}} \]  

(5.83)

Let the following abbreviations be used

\[ b_2 = \sqrt{\frac{\rho_{cl} V^2}{2\rho_{s1} \eta}} \]  

(5.84)

\[ b_1 = \sqrt{\frac{2\eta}{\rho_{cl} V^2} + 2\sqrt{\frac{2\rho_{s1} \eta}{\rho_{cl} V^2}}} \]  

(5.85)

Then the limit of \( H_0(\omega) \) becomes
\[ H_0(\omega) = \frac{1}{b_2(j\omega)^2 + b_1 j\omega + 1} \quad (5.86) \]

This is the optimum form of the over-all system function for limited integral-square error to step-function disturbances and limited velocity saturation signal when the normal input is white noise only. This form will be used from here on to the end of this problem.

The integral-square error \( I_{e1} \) for the step input \( V_u(t) \) is now evaluated in terms of the parameters of \( H_0(\omega) \). A corollary of the real convolution theorem states that

\[ \int_{-\infty}^{\infty} dt f^2(t) = 2\pi \int_{-\infty}^{\infty} dw F(w)F(w) \quad (5.87) \]

Letting \([1 - H_0(\omega)]V/2\pi j\omega = F(\omega)\) in Eq. 5.87 gives

\[ I_{e1} = \frac{V^2}{2\pi} \int_{-\infty}^{\infty} dw \frac{b_2^2 \omega^2 + b_1^2}{D(\omega)D(\omega)} \quad (5.88) \]

where \( D(\omega) = b_2(j\omega)^2 + b_1 j\omega + 1 \). Integrating yields

\[ I_{e1} = \frac{V^2}{2} \left( \frac{b_2}{b_1} + b_1 \right) \quad (5.89) \]

The rms saturation-signal level \( \sigma_{s1} \) is next evaluated.

\[ \sigma_{s1}^2 = \frac{\eta}{\pi} \int_{-\infty}^{\infty} dw \frac{\omega^2}{D(\omega)D(\omega)} \quad (5.90) \]

where \( D(\omega) = b_2(j\omega)^2 + b_1(j\omega) + 1 \). Hence

\[ \sigma_{s1} = \sqrt{\frac{\eta}{b_1 b_2}} \quad (5.91) \]
Finally, the rms error \( \sigma_e \) is determined.

\[
\sigma_e^2 = \frac{\eta}{\pi} \int dw \frac{1}{D(w)D(w)}
\]  \hspace{1cm} (5.92)

where \( D(w) = b_2(jw)^2 + b_1(jw) + 1 \). Therefore

\[
\sigma_e = \sqrt{\frac{\eta}{b_1}}
\]  \hspace{1cm} (5.93)

Inspection of Eqs. 5.89, 5.91, and 5.93 indicates that the error can always be minimized by pushing the saturation level \( \sigma_{sl} \) up to its prescribed limit \( \sigma_{sml} \) and by pushing the integral-square error \( I_{el} \) for a step input up to its prescribed limit \( I_{eml} \). Therefore, the first result asked for may be summarized as

\[
H_0(w) = \frac{1}{b_2(jw)^2 + b_1(jw) + 1}
\]  \hspace{1cm} (5.86)

\[
I_{eml} = \frac{\nu^2}{2} \left( \frac{b_2}{b_1} + 1 \right)
\]  \hspace{1cm} (5.94)

\[
\sigma_{sml} = \sqrt{\frac{\eta}{b_1b_2}}
\]  \hspace{1cm} (5.95)

These equations relate the optimum system function to the limits set upon the integral-square transient error and the rms saturation signal. It is not convenient to algebraically solve for the \( b \)'s because of the cubic equation involved; numerical solutions for particular cases can always be handled, however.

The second required result, the relationship among error, saturation limit, and transient error limit is
contained in Eqs. 5.93, 5.94, and 5.95. Eliminating $b_2$ and $b_1$ yields

\[ \left( \frac{\sigma_e^2}{\eta} \right)^3 + \frac{\sigma_{sml}^2}{\eta} \left( 1 - \frac{2I_{eml} \sigma_e^2}{\eta} \frac{\sigma_e^2}{v^2} \right) = 0 \] (5.96)

or

\[ \frac{\sigma_{sml}^2}{\eta} = \left( \frac{\sigma_e^3}{\eta} \right) \frac{2I_{eml} \sigma_e^2}{\eta} \frac{\sigma_e^2}{v^2} - 1 \] (5.97)

This result may be expressed in another form as

\[ S_{ml} = \frac{E^3}{\sqrt{2Q_{ml}E^2 - 1}} \] (5.98)

where

\[ S_{ml} = \frac{\sigma_{sml}}{\sqrt{\eta}} \] (5.99)

\[ E = \frac{\sigma_e}{\sqrt{\eta}} \] (5.100)

\[ Q_{ml} = \frac{I_{eml}}{v^2} \] (5.101)

Since $Q_{ml}$ is normally the quantity which is most likely to be specified in advance and held constant throughout a problem, it is convenient for plotting purposes to write Eq. 5.98 as

\[ \frac{Q_{ml}^3/2S_{ml}}{\sqrt{2(Q_{ml}^{1/2}E)^2 - 1}} = \frac{(Q_{ml}^{1/2}E)^3}{(Q_{ml}^{1/2}E)^2 - 1} \] (5.102)
Figure 5.3 is a plot showing how the saturation-signal limit varies with error for this problem. Note that only a small increase in the rms error of about 23 per cent will buy a reduction of the saturation-signal limit from infinity to less than one of the nondimensional units. Note further that it is impossible to reduce the saturation-signal limit much below unity on the scale used. This is because the integral-square error requirement sets up a lower bound on the saturation limit.

Example 3, Supplement 1

A numerical example will illustrate the use of the above results. A simplified radar tracking servo-mechanism has an input which is substantially pure noise, possessing the power spectrum of Fig. 5.4. For a 1-mil step-input function (without noise), the integral-square error shall not exceed 1 mil² second. Ignoring saturation considerations, find the minimum rms error possible and the system function which yields this minimum error.

The solution is as follows. In terms of the symbols of the preceding paragraphs, we have as given data

\[ V = 1 \text{ mil} \]  
\[ \eta = 200 \text{ mil}^2 \text{ sec} \]  
\[ I_{eml} = 1 \text{ mil}^2 \text{ sec} \]  

\[ \text{(5.103)} \]  
\[ \text{(5.104)} \]  
\[ \text{(5.105)} \]
FIG. 5.3 SATURATION LIMIT VERSUS ERROR FOR EXAMPLE 3.
From the given data, we compute by Eq. 5.101

\[ Q_{ml} = 1 \text{ sec} \]

From Fig. 5.3, the minimum rms error will occur for

\[ Q_{ml}^{1/2} = 0.707 \]  \hspace{1cm} (5.106)

Thus by Eq. 5.100 the minimum rms error \( \sigma_e \) is

\[ \sigma_e = \frac{0.707}{\sqrt{1.000}} \]

\[ \sigma_e = 10 \text{ mil} \]  \hspace{1cm} (5.107)

The saturation-signal limit at which this error occurs is infinite. By Eqs. 5.93 and 5.95 it is then found that

\[ b_1 = 2 \text{ sec} \]  \hspace{1cm} (5.108)

\[ b_2 = 0 \]  \hspace{1cm} (5.109)

Thus

\[ H_0(w) = \frac{1}{2jw + 1} \]  \hspace{1cm} (5.110)

and the problem is solved.

Suppose now that an additional question is asked for the above servomechanism. What is the lowest limit that can be imposed upon the rms output velocity; what is the minimum rms error when this limit is imposed; and what is the optimum system function under these conditions?

Solving, we have by Fig. 5.3 that the lowest saturation-signal limit occurs for

\[ Q_{ml}^{3/2} = 0.918 \]  \hspace{1cm} (5.111)

By Eq. 5.99, the lowest limit on the rms output velocity is
Also from Fig. 5.3 we note that for the lowest saturation-signal limit

\[ q_{ml}^{1/2} = 0.866 \]  

so that the rms error is

\[ \sigma_e = \frac{0.866 \sqrt{200}}{\sqrt{1.000}} = 12.25 \text{ mil} \]  

Using Eqs. 5.93 and 5.95

\[ b_1 = 1.333 \text{ sec} \] 
\[ b_2 = 0.888 \text{ sec}^2 \]  

Hence

\[ H_0(\omega) = \frac{1}{0.888(j\omega)^2 + 1.333j\omega + 1} \]  

This completes the numerical supplement to Example 3.

5-6. SUMMARY FOR EXAMPLES

In this section we have illustrated the use of the compensating-network design procedure outlined in the preceding sections by three simple examples. These examples are indeed so simple that an algebraic study (as contrasted from a numerical study) of the relationship between the rms saturation-signal limit and the rms error was possible. The examples and results are summarized in Table 5.1.
<table>
<thead>
<tr>
<th>REF.</th>
<th>INPUT</th>
<th>PROBLEM SPECIFICATIONS</th>
<th>SATURATION</th>
<th>TRANSIENT</th>
<th>SYSTEM FUNCTION</th>
<th>SYMBOLS</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXAMPLE 1</td>
<td>( \Phi_{ll} = \sigma_p e^{-</td>
<td>ft</td>
<td>} )</td>
<td>MESSAGE ONLY</td>
<td>( \text{INPUT RMS OUTPUT POSITION, } \sigma_p )</td>
<td>NONE</td>
<td>OPTIMUM</td>
</tr>
<tr>
<td>EXAMPLE 2</td>
<td>( \Phi_{ll} = \sigma_p e^{-</td>
<td>ft</td>
<td>} )</td>
<td>MESSAGE ONLY</td>
<td>( \text{INPUT RMS OUTPUT VELOCITY, } \sigma_p )</td>
<td>NONE</td>
<td>OPTIMUM</td>
</tr>
<tr>
<td>EXAMPLE 2 SUPPLEMENT 1</td>
<td>( \Phi_{ll} = \sigma_p e^{-</td>
<td>ft</td>
<td>} )</td>
<td>MESSAGE ONLY</td>
<td>( \text{INPUT RMS OUTPUT VELOCITY, } \sigma_p )</td>
<td>NONE</td>
<td>NON-OPTIMUM</td>
</tr>
<tr>
<td>EXAMPLE 2 SUPPLEMENT 2</td>
<td>( \Phi_{ll} = \lim_{\tau \to \infty} \frac{1}{2\pi} \Phi_{ll}(\tau) e^{-</td>
<td>ft</td>
<td>} )</td>
<td>MESSAGE ONLY</td>
<td>( \text{INPUT RMS OUTPUT VELOCITY, } \sigma_p )</td>
<td>NONE</td>
<td>OPTIMUM</td>
</tr>
<tr>
<td>EXAMPLE 3</td>
<td>( \Phi_{ll} = \eta \delta(t) ) (1 UNIT OF TIME)</td>
<td>NOISE ONLY</td>
<td>( \text{ZERO RMS OUTPUT VELOCITY, } \sigma_p )</td>
<td>( \Phi_{\text{em}} )</td>
<td>OPTIMUM</td>
<td>( H_0 = \frac{1}{b_2(j\omega)^2 + b_1 j\omega + 1} )</td>
<td>( \frac{\sigma_{\text{mi}}}{\sqrt{\eta}} ) ( \frac{\sigma_{e}}{\sqrt{\eta}} ) ( \frac{\Phi_{\text{em}}}{\sqrt{\gamma}} )</td>
</tr>
</tbody>
</table>

* \( \Phi_{ll} = \text{INPUT AUTOCORRELATION FUNCTION} \)
** \( \sigma_{e} = \text{RMS ERROR} \)
Although the results presented here fit into the pattern intuitively expected by the system designer, it is felt that they are original in the sense that, within the writer's knowledge, this is the first time that they have been mathematically formulated.

In conclusion it should be remarked that the algebraic neatness with which these examples were solved is not typical of all problems met in practice. Complications arise because of the factoring higher-order polynomials, if for no other reason. When the normal desired output is the normal input with pure lead or lag, or when the correlation functions cannot be expressed as sums of exponentials, \( H_{oo}(\omega) \) will be transcendental, and approximations must be used to permit \( H_{oo}(\omega) \) to be expressed as a rational function. Even when the factoring or approximation problems are not encountered, the structure of intermediate expressions becomes quite complicated. Therefore, to work problems only a little more complicated than these examples, we should not be discouraged if numerical analysis has to be resorted to throughout.
6. CONCLUSION

This paper has presented a theory for the design of compensating networks in filters and feedback-control systems. In formulating this theory, which is based upon the work of Wiener and Lee, an attempt is made to recognize the possibility of saturation in the fixed components of the system and, in addition, to consider the transient performance of the overall system. Both the saturation and the transient performance considerations have been omitted in the previous generalized theories of filter and feedback-control system design in spite of the fact that these considerations are of great practical import.

The treatment for compensating-network design used in this paper developed from two fundamental ideas. The first is that, pending the development of a generalized approach to nonlinear system design, the practical designer is interested in building systems in which the saturable components are operated substantially in their linear ranges so that the overall performance can be predicted on the basis of linear theory. This leads to the principle of designing for a limited probability of saturation. If it is assumed that the probability of saturation of a real system can be controlled by controlling the rms values of the saturation signals that would exist in a linear model of the real system, then it is shown that a fairly simple mathematical formulation of the problem is possible.
The second fundamental idea is that possibilities for the control of transient performance exist if the integral-square errors to a set of transient test signals can be controlled. It is shown to be entirely feasible to incorporate such control into the mathematical formulation of the compensating-network design problem.

The assumptions made on the basis of the two fundamental ideas concerning saturation and transient performance, together with those used by Wiener and Lee, have been shown to lead to a rather general solution of the compensating-network design problem. This solution has been demonstrated for several specific examples showing that concrete results can be obtained by means of it.

In the appendix of this paper, experimental results are given which tend to substantiate the assumption used for the mathematical treatment of saturation (viz., that limitation of the rms saturation signals in a linear model is helpful in controlling the probability of saturation in a real system which contains saturable elements among its fixed parts).

The results of this paper should be helpful to servomechanism designers, especially in evaluating the ultimate limitations placed upon a system's performance by its fixed components such as its servomotor. Although it has been long recognized that limits exist upon what can be accomplished by means of compensation devices for systems
such as servomechanisms, little quantitatively exact knowledge of these limits has been available. It is hoped that the methods of this paper will help to improve this situation.

Designers of active networks for filters and other applications may find the methods outlined above useful in their problems when large latitude with respect to saturation or transient behavior is not allowed by the problem specifications.

It is regretted that all the results of this paper have to be premised upon the rms error as a criterion of system goodness. Certain extensions to other criteria may be possible, but these all would appear to have at least as limited usefulness as the rms criterion.

Opportunities for interesting investigations should develop from the work recorded here. For example, much additional work should be done to increase our knowledge of the relationships of the rms saturation signals in linear models to the probability of saturation in real systems. It is expected that much of this work will necessarily be of an experimental nature. Nevertheless, for sufficiently simple systems, analytical investigations should eventually prove to be possible. Another example is the further exploitation of transient performance control through limitation of integral-square errors (or other devices such as limitation of errors to periodic test signals of various wave shapes). A third example is the search for criteria other than rms error.
7. SUMMARY OF SPECIALIZED MATHEMATICS

7-1. STOCHASTIC VARIABLES

It has been assumed that the reader of this paper is familiar with Fourier integrals as well as with certain topics in the mathematics of stochastic processes and the calculus of variations. The purpose of this section is to collect together for reference the parts of the mathematics of stochastic processes and the calculus of variations which are used in this paper and which may not be familiar to the general reader. Some attempt will be made to explain the relations set forth here, but any formal derivations used in the discussion are not to be construed as "proofs" in the mathematical sense of the word. For a more comprehensive mathematical treatment of stochastic processes, the works of Wiener (References 23.2 and 23.4) and Chandrasekhar (Reference 3.2) are cited.

What is a stochastic variable? For purposes of this paper, a stochastic variable is a time function generated by a stochastic process in time. Broadly, any process whose outcome is not a definitely determined function of the independent variable is a stochastic process. For example, the sequence of sums of the numbers resulting from throws of a pair of dice is commonly assumed to be a stochastic process; the numbers representing the sums comprise a stochastic variable which is a function of the independent variable, the number of the throw. If the throws are made at instants of time determined according to
some prearranged schedule, then we have a stochastic variable which is a function of time. Such variables are frequently called time series. Another example of stochastic variables is shot noise in vacuum tubes or the effects of such noise in associated circuits. A comprehensive discussion of such phenomena has been published by Rice (Reference 18.8). Goldman has published an engineering approach to noise phenomena (Reference 7.6).

Stochastic processes may be either stationary or nonstationary. A stationary stochastic process is one whose statistical properties are invariant with time. Many physical phenomena approximate this characteristic. The probability of a die showing a certain number in any throw is approximately constant. Under given conditions, the shot noise of a vacuum tube is constant and characterized by constant probability distributions. A nonstationary process, on the other hand, may be thought of as one whose statistical properties vary in such a manner that a distinction between past and future exists. One example of a nonstationary process is that of throwing loaded dice in which the loading weights suddenly shift at some instant of time to new locations. A second example is the emission of alpha particles by a radioactive source which is not continuously replenished to maintain constant mass. A third example is the output of a circuit connected to a stationary noise generator when some parameter of the circuit is varied exponentially or as a step function. Evidently all physical stochastic phenomena are,
in the last analysis, nonstationary. It is only when process variations are small during the necessary observation intervals that the abstraction of stationarity can be introduced.

A stationary stochastic variable is one generated by a stationary process. The mathematics of stationary stochastic variables are simpler than those of the more general nonstationary variables. This is especially so when probability distribution functions are involved in the calculations. In the brief presentation given here, the probability aspects of stochastic variables will not be discussed, and consequently the assumption of stationarity will not be needed. Hence no distinction will be made between stationary and nonstationary variables. In this regard, however, it must be emphasized that the rms error criterion for systems driven by nonstationary variables may be quite unsatisfactory, even though it is satisfactory for stationary variables. For nonstationary variables one might find large percentage errors for regions of low signal level and small percentage errors for those of large signal level when the rms error over the infinite interval is minimized. Thus the very use of the rms error criterion may effectively limit practical applications to stationary inputs, even though the basic methods could be used to minimize rms error for nonstationary inputs.

In addition to the possible difficulty with the rms error criterion, there is another
aspect of nonstationary variables that can cause trouble. This is the matter of measuring or otherwise determining bases for the correlation functions defined below. In practice relatively short samples must necessarily be used in determining these functions. This makes it nearly impossible to collect enough information to determine the correlation functions for nonstationary variables. We conclude that, even though large classes of both stationary and nonstationary variables in theory may possess the correlation functions discussed below, experimental ascertainment of these functions in most instances will be confined to variables assumed to be stationary.

Whenever rms errors are discussed, the correlation functions for the variables between which the error is computed are found to be significant rather than the variables themselves. The correlation function $\phi_{ab}(\tau)$ is defined for two time functions $f_a(t)$ and $f_b(t)$ as follows:

$$
\phi_{ab}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt f_a(t)f_b(t + \tau) \quad (7.1)
$$

If $f_b(t) = f_a(t)$, the name "autocorrelation function" is given to this quantity which is then denoted as $\phi_{aa}(\tau)$. Otherwise the function is referred to as the "cross-correlation function." The definition in Eq. 7.1 is quite general and applies to stationary and nonstationary stochastic variables, and aperiodic and periodic variables of a nonstatistical nature. The only requirement for a finite $\phi_{ab}(\tau)$ is that the product of the functions $f_a(t)$ and $f_b(t + \tau)$ be
of finite average value. For stationary stochastic variables $\phi_{ab}(\tau)$ is approximated by

$$\phi_{ab}(\tau)\Big|_{\text{approx.}} = \frac{1}{2T} \int_{-T}^{+T} dt f_a(t)f_b(t + \tau) \quad (7.2)$$

in which $T$ is finite; that is, the integration is performed only for a finite sample. The length of sample required is difficult to predict but is easily found by trial for specific data by observing how $\phi_{ab}(\tau)\Big|_{\text{approx.}}$ converges for increasing lengths of sample. Often the integration is replaced by a summation when treating laboratory data. The continuous functions $f_a(t)$ and $f_b(t)$ are replaced by their sequences $f_a(nAT)$ and $f_b(nAT)$ where $\Delta T$ is a small fraction of the total duration of the sample, $n$ is an integer $1, 2 \ldots N$, and $N\Delta T = 2T$. Then

$$\phi_{ab}(s\Delta T)\Big|_{\text{approx.}} = \frac{1}{N} \sum_{n=1}^{N} f_a(n\Delta T)f_b[(n + s)\Delta T] \quad (7.3)$$

where $s = 1, 2, \ldots$ and $s\Delta T = \tau$. The shift index $s$, of course, is usually limited in maximum value to a small fraction of $N$.

Certain properties of the correlation functions should be noted. First,

$$\phi_{ab}(\tau) = \phi_{ba}(-\tau) \quad (7.4)$$

This is demonstrated by replacing $t + \tau$ in Eq. 7.1 by $t_1$ and interchanging $f_a$ and $f_b$. Equation 7.4 follows because of the infinite range when the limit is taken. Second, as a special case of Eq. 7.4, we note

$$\phi_{aa}(\tau) = \phi_{aa}(-\tau) \quad (7.5)$$
that is, the autocorrelation is always an even function.

Third, we have

\[ \phi_{aa}(0) \geq \phi_{aa}(\tau) \quad (7.6) \]

This follows from the inequalities

\[
\left[ f_a(t) + f_a(t + \tau) \right]^2 \geq 0
\]
\[
\left[ f_a^2(t) + 2f_a(t)f_a(t + \tau) + f_a^2(t + \tau) \right] \geq 0
\]
\[
f_a^2(t) + f_a^2(t + \tau) \geq 2f_a(t)f_a(t + \tau)
\]

Integrating and averaging of both sides yields Inequality 7.6.

Frequently it is necessary to find the correlation functions of output signals of networks in terms of the correlation functions of the input signals. Figure 7.1 is a block diagram of this situation. Here two networks "A" and "B" have input signals \( f_{al}(t) \) and \( f_{bl}(t) \) and output signals \( f_{a2}(t) \) and \( f_{b2}(t) \), respectively. The networks are characterized by impulse responses \( h_a(t) \) and \( h_b(t) \) or alternatively by system functions \( H_a(\omega) \) and \( H_b(\omega) \), respectively. We wish to find the correlation function of \( f_{a2}(t) \) and \( f_{b2}(t) \) in terms of the network characteristics and the input functions. We have

\[ \phi_{ab2}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} dt f_{a2}(t)f_{b2}(t + \tau) \quad (7.7) \]

From

\[ f_{a2}(t) = \int_{-\infty}^{\infty} dx h_a(x)f_{al}(t - x) \quad (7.8) \]

and

\[ f_{b2}(t + \tau) = \int_{-\infty}^{\infty} dy h_b(y)f_{bl}(t + \tau - y) \quad (7.9) \]
FIG. 7.1 SIGNALS AND NETWORKS USED IN COMPUTING CORRELATION FUNCTIONS OF SIGNALS AFTER PASSING THROUGH NETWORKS.
we have

$$\phi_{ab2}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dxh_a(x) f_{a1}(t-x) \int_{-\infty}^{\infty} dyh_b(y) f_{b1}(t+\tau-y)$$

(7.10)

Interchanging orders of integration and limit processes yields

$$\phi_{ab2}(\tau) = \int_{-\infty}^{\infty} dxh_a(x) \int_{-\infty}^{\infty} dyh_b(y) \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} dt f_{a1}(t-x) f_{b1}(t+\tau-y)$$

(7.11)

But

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} dt f_{a1}(t-x) f_{b1}(t+\tau-y) = \phi_{abl}(\tau + x - y)$$

(7.12)

This follows from

$$\phi_{abl} \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} dt f_{a1}(t) f_{b1}(t+\tau)$$

(7.13)

Therefore we may write Eq. 7.11 as

$$\phi_{ab2}(\tau) = \int_{-\infty}^{\infty} dxh_a(x) \int_{-\infty}^{\infty} dyh_b(y) \phi_{abl}(\tau + x - y)$$

(7.14)

This is a rather general formula for relating the correlation function at one pair of points in a complex system to that at another.

When correlation functions are Fourier transformable, it frequently simplifies manipulations to use, in lieu of the correlation functions themselves, their transforms. The transform of a correlation function is

$$\phi_{ab}(w) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{jwt} \phi_{ab}(\tau)$$

(7.15)
The inverse transform of $\Phi_{ab}(\omega)$ is therefore

$$\phi_{ab}(\tau) = \int_{-\infty}^{\infty} d\omega e^{j\omega \tau} \Phi_{ab}(\omega) \quad (7.16)$$

If $\tau$ is set equal to zero in Eq. 7.16, we have

$$\phi_{ab}(0) = \int_{-\infty}^{\infty} d\omega \Phi_{ab}(\omega) \quad (7.17)$$

In the case of the autocorrelation function, Eq. 7.1 shows that

$$\phi_{aa}(0) = \text{mean-square value of } f_a(t) \quad (7.18)$$

Equation 7.17 permits us also to say

$$\int_{-\infty}^{\infty} d\omega \Phi_{aa}(\omega) = \text{mean-square value of } f_a(t) \quad (7.19)$$

that is, the integral with respect to the frequency $\omega$ of the transform of the autocorrelation function of a signal over the complete frequency range is equal to the mean-square value of that signal.

Transformation of Eq. 7.14 yields the general formula relating the transformed correlation function at one pair of points in a complex system to that at another.

Define

$$\Phi_{ab2}(\omega) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-j\omega \tau} \phi_{ab2}(\tau) \quad (7.20)$$

and

$$\Phi_{abl}(\omega) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-j\omega \tau} \phi_{abl}(\tau) \quad (7.21)$$

Transforming Eq. 7.14 gives
\[ \phi_{ab2}(w) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dx h_a(x) \int_{-\infty}^{\infty} dy h_b(y) \phi_{abl}(\tau + x - y) \]

(7.22)

Interchanging orders of integration and rearranging yields

\[ \phi_{ab2}(w) = \int_{-\infty}^{\infty} dx e^{jwx} h_a(x) \int_{-\infty}^{\infty} dy e^{-jwy} h_b(y) \int_{-\infty}^{\infty} d\tau e^{-jw(\tau + x - y)} \phi_{abl}(\tau + x - y) \]

(7.23)

In view of the infinite limits, the last integral in Eq. 7.23 can be replaced by \( \phi_{abl}(w) \). Furthermore,

\[ H_a(w) \triangleq \int_{-\infty}^{\infty} d\tau e^{-j\omega t} h_a(t) \]

(7.24)

\[ H_b(w) \triangleq \int_{-\infty}^{\infty} d\tau e^{-j\omega t} h_b(t) \]

(7.25)

Since \( h_a(t) \) is real,

\[ H_a(w) = \int_{-\infty}^{\infty} d\tau e^{+j\omega t} h_a(t) \]

(7.26)

where \( H_a(w) \) is the conjugate of \( H_a(w) \). Thus we can write for Eq. 7.23

\[ \phi_{ab2}(w) = H_a(w) H_b(w) \phi_{abl}(w) \]

(7.27)

as the transform equivalent of Eq. 7.14. Note the simplicity of this latter expression as compared with the former.

As a special case of Eq. 7.27, we have the relation between the transform of the autocorrelation function of the output signal from a network and the transformed autocorrelation function of the input signal. This relation is

\[ \phi_{aa2}(w) = H_a(w) H_a(w) \phi_{aa1}(w) \]

(7.28)
Further light on the nature of the transform of an autocorrelation function is shed by considering Eq. 7.28 in conjunction with Eq. 7.19. The latter applies to both the input and output. Any $\hat{\Phi}_{\omega\omega}(\omega)$ satisfying Eq. 7.19 can be regarded as a mean-square-value density function. A little reflection will reveal that if Eq. 7.28 is assumed valid for all density functions, then there is only one input-density function possible which will satisfy both Eqs. 7.19 and 7.28 and that is $\hat{\Phi}_{\omega\omega}(\omega)$. The assumption that Eq. 7.28 is valid for all density functions seems reasonable in view of two well-known facts. One, for a single, discrete, frequency input signal, a linear system with constant coefficients produces an output signal whose amplitude is the input amplitude multiplied by $|H(\omega)|$ and whose frequency is exactly the same as the input frequency. Two, the mean-square value of a signal composed of several frequency components is the sum of the mean-square values of these components. Since only one density function is possible, it is logical to refer to the transform of an autocorrelation function as the mean-square-value density function.

If the output signal of a network were a voltage or current applied to a one-ohm resistance, its mean-square value would be numerically equal to the power absorbed in the resistance. The transform of the autocorrelation function of the output signal would then be equivalent to the power-density function of the power dissipated in this resistance. This is the basis for the common practice of
calling the transform of the autocorrelation function the power-density spectrum (or sometimes more briefly, the power spectrum). By analogy, the transform of the cross-correlation function is termed the cross-power-density spectrum.

It should be recognized that the class of function for which power-density spectrums exist is far narrower than the class for which autocorrelation functions exist. Any stochastic functions with periodic or constant components will have correlation functions which are not transformable without the use of special convergence factors. Such convergence factors are usually not allowed when the spectrum factorization method is used to solve integral equations such as Eq. 3.18. This means that solution formulas of the type represented by Eq. 3.19 are usually limited to aperiodic functions with zero mean value in so far as the normal input and desired output signals are concerned.

7-2. MINIMIZATION UNDER CONSTRAINTS

In Subsection 4-4 dealing with the derivation of Eq. 3.18, it was stated that the problem of minimizing the rms error $\sigma_e$ under the N conditions on the rms saturation signals, $\sigma_{sn} \leq \sigma_{smn}$, and the K conditions on the integral-square errors, $I_{ek} \leq I_{emk}$, was equivalent to minimizing the function

$$ F = \sigma_e^2 + \sum_{n=1}^{N} \rho_{sn} \sigma_{sn}^2 + \sum_{k=1}^{K} \rho_{ck} I_{ek} $$

where $\rho_{sn}$ and $\rho_{ck}$ are $N + K$ constants (called Lagrangian multipliers). This subsection has the purpose of explaining the stated equivalence
in more detail than is found in the references on this subject quoted in Subsection 4-4.

We first recognize that if a solution can be obtained for the auxiliary problem of minimizing $\sigma_e^2$ with $\sigma_{sn}^2 = \sigma_{spn}^2$ and $I_{ek} = I_{epk}$ where $\sigma_{spn}^2$ and $I_{epk}$ are arbitrary constants, we then effectively have solved the main problem, since the arbitrary constants can be adjusted to meet the limit conditions in such a way that an absolute minimum on $\sigma_e$ is obtained. It may happen that many of the arbitrary constants will be set at values well below the specified limits when the absolute minimum is reached. Our problem is thus equivalent to minimizing $\sigma_e^2$ under $N + K$ constraints which may be briefly summarized as $F_j = 0$, $j = 1, 2, 3, \ldots$ $N + K$ where the first $N F_j$ are $\sigma_{sn}^2 - \sigma_{spn}^2$ and the last $K F_j$ are $I_{ek} - I_{epk} = 0$. $\sigma_e^2$ and $F_j$ are, of course, functions of $h_c(t)$.

Let the optimum function for $h_c(t)$, i.e., the one which makes $\sigma_e^2$ a minimum, be denoted by $h_{co}(t)$ and form the special $h_c(t)$

$$h_c(t) = h_{co}(t) + \sum_{1}^{N+K+1} \epsilon_i h_{ci}(t)$$  \hfill (7.29)

where the $h_{ci}$ are any other arbitrary realizable impulse responses and the $\epsilon_i$ are variables introduced for convenience. Further let the $\epsilon_i$ be functions of another variable $\epsilon$, such that all the $\epsilon_i = 0$ when $\epsilon = 0$. It is evident that substitution of this special $h_c(t)$ into the expressions for $\sigma_e^2$ and the $F_j$ make all of these functions of $\epsilon$. 
We observe that a necessary condition for $\sigma_\varepsilon^2$ to be a minimum is
\[ \frac{d\sigma_\varepsilon^2}{d\varepsilon} = 0 \quad \text{at } \varepsilon = 0 \quad (7.30) \]

But because the $F_j$ are constant, we also have $N + K$ relations
\[ \frac{dF_j}{d\varepsilon} = 0 \quad \text{for all values of } \varepsilon \text{ and } \varepsilon = 0 \text{ in particular} \quad (7.31) \]

In general, it is true that
\[ \frac{d\sigma_\varepsilon^2}{d\varepsilon} = \sum_{i=1}^{N+K+1} \frac{\sigma_\varepsilon^2}{\varepsilon_i} \frac{d\varepsilon_i}{d\varepsilon} \quad (7.32) \]

and
\[ \frac{dF_j}{d\varepsilon} = \sum_{i=1}^{N+K+1} \frac{F_j}{\varepsilon_i} \frac{d\varepsilon_i}{d\varepsilon} \quad (7.33) \]

Therefore, in particular, we have at $\varepsilon = 0$ (and hence $\varepsilon_1 = 0$) that
\[ \sum_{i=1}^{N+K+1} \frac{\sigma_\varepsilon^2}{\varepsilon_i} \frac{d\varepsilon_i}{d\varepsilon} = 0 \quad (7.34) \]

and $N + K$ equations
\[ \sum_{i=1}^{N+K+1} \frac{F_j}{\varepsilon_i} \frac{d\varepsilon_i}{d\varepsilon} = 0 \quad (7.35) \]

But at $\varepsilon = 0$ all the $\frac{\partial F_j}{\partial \varepsilon_1}$ will be functions of $h_{co}(t)$ (a definite function) and the arbitrary $h_{ci}(t)$. For a given selection of the $h_{ci}(t)$, the $\frac{\partial F_j}{\partial \varepsilon_1}$ at $\varepsilon = 0$ will have certain values. Therefore, by Eq. 7.35, $N + K$ of the $N+K+1(\varepsilon_1/d\varepsilon)$ at $\varepsilon = 0$ are determined in terms of the remaining one and the $N + K$ values of $\frac{\partial F_j}{\partial \varepsilon_1}$ at $\varepsilon = 0$. We select
\[ \frac{d\varepsilon_1}{d\varepsilon} = \left[ \sum_{j=1}^{N+K} \frac{\Delta_{ji}}{\Delta} \frac{\partial F_i}{\partial \varepsilon} \right] \frac{d\varepsilon_1}{d\varepsilon} = 0 \quad i = 2, 3 \ldots N+K+1 \quad (7.36) \]

where \( \Delta_{ji} \) is cofactor complimenting the \( j \)th row and \((i-1)\)th column and \( \Delta \) is the determinant of the simultaneous equation set \( (\varepsilon = 0) \).

\[
\frac{\partial F_1}{\partial \varepsilon} \frac{d\varepsilon_1}{d\varepsilon} + \frac{\partial F_2}{\partial \varepsilon} \frac{d\varepsilon_2}{d\varepsilon} + \ldots + \frac{\partial F_{N+K}}{\partial \varepsilon} \frac{d\varepsilon_{N+K}}{d\varepsilon} = \frac{dF_1}{d\varepsilon_1} \frac{d\varepsilon_1}{d\varepsilon} \]

\[
\frac{dF_2}{d\varepsilon_2} \frac{d\varepsilon_2}{d\varepsilon} + \ldots + \frac{dF_{N+K}}{d\varepsilon_{N+K}} \frac{d\varepsilon_{N+K}}{d\varepsilon} = \frac{dF_2}{d\varepsilon_1} \frac{d\varepsilon_1}{d\varepsilon} \]

\[
\frac{dF_{N+K}}{d\varepsilon_{N+K}} \frac{d\varepsilon_{N+K}}{d\varepsilon} = \frac{dF_{N+K}}{d\varepsilon_1} \frac{d\varepsilon_1}{d\varepsilon} \quad (7.37) \]

Substituting the \( \frac{d\varepsilon_i}{d\varepsilon} \) of Eq. 7.36 into Eq. 7.34 yields

\[
\left[ \frac{\partial \sigma^2_e}{\partial \varepsilon_1} + \frac{\sum_{j=1}^{N+K} \frac{\Delta_{ji}}{\Delta} \frac{\partial F_i}{\partial \varepsilon_1}}{N+K} \right] \frac{d\varepsilon_1}{d\varepsilon} = 0 \quad \text{at} \quad \varepsilon = 0 \quad (7.38) \]

Since \( \frac{d\varepsilon_1}{d\varepsilon} \) in general will not be zero, the brackets must be zero. Interchanging order of summation permits writing

\[
\frac{\partial}{\partial \varepsilon_1} \left[ \sigma^2_e + \frac{\sum_{j=1}^{N+K} \frac{\Delta_{ji}}{\Delta} F_j \frac{\partial \varepsilon^2}{\partial \varepsilon_1} \right] = 0 \quad \text{at} \quad \varepsilon = 0 \quad (7.39) \]

But

\[
\sum_{j=2}^{N+K+1} \frac{\partial \sigma^2_e}{\partial \varepsilon_1} \frac{\Delta_{ji}}{\Delta} \quad \text{at} \quad \varepsilon = 0 \quad \text{for the given selection of} \quad h_{c1}(t) \]

will be constant; let this constant be denoted by \( \varepsilon_j \). Thus we have as a necessary condition on \( h_{c0}(t) \) that

\[
\frac{\partial}{\partial \varepsilon_1} \left[ \sigma^2_e + \sum_{j=1}^{N+K} \frac{\partial \varepsilon_1 F_j}{\partial \varepsilon_1} \right] = 0 \quad \text{at} \quad \varepsilon = 0 \quad (7.40) \]

But this is identical to the condition that would be established
if we set about to minimize the quantity $\sigma_e^2 + \sum_{j=1}^{N+K} e_j F_j$.

In view of the definitions of $F_j$ and the statements made in the second paragraph of this subsection, we have hereby established the equivalence mentioned in the first paragraph.
W. WIENER'S THEORY OF FILTERING AND LEE'S EXTENSIONS THEREOF

8-1. WIENER'S THEORY OF FILTERING AS PRESENTED BY LEE

The primary purpose of this section is to furnish additional background to the procedure for compensating-network design which is reported in this paper. This is done by reviewing briefly the works of others, notably those of Wiener and Lee, which form the foundation of this procedure. A secondary objective of this section is to acquaint the general reader with the fundamental techniques employed in developing the above design procedure by presenting the solutions of the somewhat simpler problems considered by Wiener and Lee.

The results of this paper are to be viewed as extensions of the approach originally developed by Wiener and generalized by Lee in his course, Optimum Linear Systems, given in the Department of Electrical Engineering at the Massachusetts Institute of Technology (see Reference 12.3). Wiener's original filter theory was first published in 1942 as a classified document and then republished during 1949 in unrestricted form as a book, Reference 23.6. Wiener also presents the essentials of his theory in his book Cybernetics, Reference 23.8. Other people have published simpler explanations of Wiener's theory. Levinson in Reference 12.6 and Grant in Reference 7.8 may be cited as two. Still others have published various extensions and modifications of Wiener's original theory. In particular, Phillips and Weiss in Reference 16.5 have considered the problem of optimum system synthesis.
with an input consisting of noise and a polynomial in time when the constraint of finite settling time is imposed. Zadeh in Reference 26.5 has generalized on the approach of Phillips and Weiss.

To be presented here is a slightly extended form of Wiener's original theory which employs the concept of a general desired output rather than the specialized desired output, which is used in the original work and which consists of a certain component of the input (the message) shifted backward or forward in time. The original conception of a general desired output appears to be due to Lee. The following presentation of this extended version of Wiener's theory is almost exactly the one used by Lee in his course, Reference 12.3.

In the extended form of Wiener's theory, the problem considered is the design of a linear filter without fixed components. It may be stated as follows:

\[
\begin{align*}
\text{Statement of problem solved by Wiener's filter theory} & \quad \text{Given:} \\
1. \text{Information about the input signal.} & \\
2. \text{Information about the relationship between the input and desired output signals.} & \\
\text{Find:} & \\
& \text{The linear transfer characteristics of the filter which makes the rms error between the desired output and the actual output signals a minimum.}
\end{align*}
\]
The filter which has the sought-for-transfer characteristic is called the optimum filter. In solving this problem, it is found that the only necessary information concerning the input and desired output signals is the autocorrelation function of the input and cross-correlation function of the input and desired output.

Figure 8.1 is a block diagram of the system treated by the Wiener filter theory. The transfer characteristic of the filter is described by its response to a unit impulse $h(t)$ or alternatively by its system function $H(w)$. An integral equation, known as the Wiener-Hopf equation, which determines the optimum $h(t)$ is derived as follows. The error $e(t)$ is

$$e(t) = d(t) - o(t)$$  \hspace{1cm} (8.1)

where $d(t)$ and $o(t)$ are, respectively, the desired and actual outputs. The output is given by the convolution integral.

$$o(t) = \int_{-\infty}^{\infty} dx h(x) e(t-x)$$  \hspace{1cm} (8.2)

Let $e$ stand for the rms error. By definition, the mean-square error $\sigma_e^2$ is

$$\sigma_e^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt e^2(t)$$  \hspace{1cm} (8.7)

Use of Eqs. 8.1 and 8.2 yields
FIG. 8.1 SYSTEM TREATED BY WIENER'S FILTER THEORY.

FIG. 8.2 SYSTEM TREATED BY LEE'S EXTENSION OF WIENER'S FILTER THEORY.
\[ \sigma_e^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \theta_d^2(t) \]
\[ - 2 \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \theta_d(t) \int_{-\infty}^{\infty} dx h(x) \theta_1(t - x) \]
\[ + \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \int_{-\infty}^{\infty} dx h(x) \theta_1(t - x) \int_{-\infty}^{\infty} dy h(y) \theta_1(t - y) \]  

(8.3)

Correlation functions are defined as follows
\[ \phi_{11}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \theta_1(t) \theta_1(t + \tau) \]  

(8.4)

\[ \phi_{1d}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \theta_1(t) \theta_d(t + \tau) \]  

(8.5)

\[ \phi_{dd}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \theta_d(t) \theta_d(t + \tau) \]  

(8.6)

The first is the autocorrelation function of the input; the second, the cross correlation of the input and desired output; the third, the autocorrelation function of the desired output. In terms of these correlation functions, Eq. 8.3 may be rewritten upon interchanging orders of integration and limit processes as
\[ \sigma_e^2 = \phi_{dd}(0) - 2 \int_{-\infty}^{\infty} dx h(x) \phi_{1d}(x) \]
\[ + \int_{-\infty}^{\infty} dx h(x) \int_{-\infty}^{\infty} dy h(y) \phi_{11}(x - y) \]  

(8.7)

In order to find a necessary condition upon the optimum impulse response, we consider \( h(x) \) to be
\[ h(t) = h_o(t) + \epsilon h_1(t) \]  

(8.8)
where \( h_o(t) \) is the optimum-system impulse response, \( h_1(t) \)
is any arbitrary impulse response, \( \epsilon \) is a variable introduced
for convenience. (All impulse responses are assumed equal to
zero for \( t < 0 \) in accord with the idea of realizability used
here.) Substituting Eq. 8.8 into Eq. 8.7 will make \( \sigma^2_e \) a
function of \( \epsilon \). A necessary condition for the existence of a
minimum or maximum value of \( \sigma^2_e \) at a particular value of \( \epsilon \) is
that \( d(\sigma^2_e)/d\epsilon \) be zero at this value. According to Eq. 8.8,
the particular value of \( \epsilon \) involved is zero by definition.
Differentiating Eq. 8.7 with respect to \( \epsilon \), and substituting
Eq. 8.8 into the result yields, upon equating to zero and
setting \( \epsilon = 0 \),

\[
-2 \int_{-\infty}^{\infty} dh_1(x) \phi_{1d}(x) + \int_{-\infty}^{\infty} dh_1(x) \int_{-\infty}^{\infty} dyh_0(y) \phi_{11}(x - y)
\]

\[
+ \int_{-\infty}^{\infty} dh_0(x) \int_{-\infty}^{\infty} dyh_1(y) \phi_{11}(x - y) = 0 \quad (8.9)
\]
as a necessary condition. The second and third terms are
equal because \( \phi_{11}(x - y) \) is an even function. Thus a
necessary condition for \( \sigma^2_e \) to be a minimum or maximum is

\[
2 \int_{-\infty}^{\infty} dh_1(x) \left[ \int_{-\infty}^{\infty} dyh_0(y) \phi_{11}(x - y) - \phi_{1d}(x) \right] = 0 \quad (8.10)
\]
which must be true for all choices of \( h_1(x) \). This means the
brackets must be zero for all \( x \gg 0 \). (\( h_1(x) \) is zero for \( x < 0 \).)
Hence

\[
\int_{-\infty}^{\infty} dyh_0(y) \phi_{11}(x - y) - \phi_{1d}(x) = 0 \text{ for } x \gg 0 \quad (8.11)
\]
is a necessary condition for $\sigma_e^2$ to be a minimum or a maximum. This is a modified form of the original Wiener-Hopf equation.

Taking the second derivative of Eq. 8.7 with respect to $\epsilon$, substituting Eq. 8.8 into the result, and manipulating will show $d^2(\sigma_e^2)/d\epsilon^2$ to be positive at $\epsilon = 0$. Hence Eq. 8.11 is a necessary condition for a minimum. A unique solution of 8.11 for $h_o(y)$ generally exists for physical problems. This means Eq. 8.11 is a necessary and sufficient condition for a minimum $\sigma_e^2$. We therefore conclude that the Wiener-Hopf equation constitutes one form of the solution for the optimum filter.

When the correlation functions appearing in the Wiener-Hopf equation are Fourier transformable, an explicit solution for the optimum filter can be obtained by considering the Fourier transform of the left side of Eq. 8.11. This transform is a function with no poles in the upper half of the $\omega$ plane, since the left side is zero for $x > 0$. (Upper-half and lower-half planes will be denoted as UHP and LHP, respectively.) Let $\Phi_{11}(\omega)$ be the transform of $\phi_{11}(\tau)$, and $\Phi_{1d}(\omega)$ of $\phi_{1d}(\tau)$. $\Phi_{11}(\omega)$ and $\Phi_{1d}(\omega)$ are, respectively, the power-density spectrum of the input and the cross-power-density spectrum of the input and desired output. Let $\mathcal{L}(\omega)$ be the transform of the left side of Eq. 8.11. Then it follows after changing orders of integration that

$$\mathcal{L}(\omega) = H_o(\omega)\Phi_{11}(\omega) - \Phi_{1d}(\omega)$$  \hfill (8.12)

Let $\Phi_{1\pm}(\omega) = \Phi_{11}^+(\omega) \times \Phi_{11}^-(\omega)$, where "+" and "-" indicate
factors with poles and zeros in UHP and LHP, respectively. Dividing Eq. 1.12 by $\Phi_{11}^-(w)$ yields

$$\frac{\ell(w)}{\Phi_{11}^-(w)} = H_0(w)\Phi_{11}^+(w) - \frac{\phi_{11}^d(w)}{\Phi_{11}^-(w)} \quad (8.13)$$

where $\ell(w)/\Phi_{11}^-(w)$ has no poles in UHP. $\phi_{11}^d(w)/\Phi_{11}^-(w)$ may be decomposed into the sum of two parts, $\left[\phi_{11}^d(w)/\Phi_{11}^-(w)\right]_+$ and $\left[\phi_{11}^d(w)/\Phi_{11}^-(w)\right]_-$, which have poles in the UHP and LHP, respectively. Thus Eq. 8.13 becomes

$$\frac{\ell(w)}{\Phi_{11}^-(w)} + \left[\phi_{11}^d(w)/\Phi_{11}^-(w)\right]_- = H_0(w)\Phi_{11}^+(w) - \left[\phi_{11}^d(w)/\Phi_{11}^-(w)\right]_+ \quad (8.14)$$

The right side of this equation has no poles in the LHP because $h(t)$ is zero for $t < 0$. It also has no poles in the UHP because the left side has no poles in the UHP. Hence the right side has no poles and therefore must be equal to a constant. Let this constant be $L$. Then the solution of the Wiener-Hopf equation (Eq. 8.11) is

$$L + \left[\phi_{11}^d(w)/\Phi_{11}^-(w)\right]_+ \quad (8.15)$$

$$H_0(w) = \frac{\left[\phi_{11}^d(w)/\Phi_{11}^-(w)\right]_+}{\Phi_{11}^+(w)}$$

$\left[\phi_{11}^d(w)/\Phi_{11}^-(w)\right]_+$ may be determined from $\phi_{11}^d(w)/\Phi_{11}^-(w)$ by taking the transform of a function equal to the inverse transform of $\phi_{11}^d(w)/\Phi_{11}^-(w)$ for $t > 0$ and equal to zero for $t < 0$. This yields the following auxiliary formula

$$\left[\phi_{11}^d(w)/\Phi_{11}^-(w)\right]_+ = \frac{1}{2\pi} \int_0^\infty dt e^{-jwt} \int_{-\infty}^\infty dw e^{jwt} \frac{\phi_{11}^d(w)}{\Phi_{11}^-(w)} \quad (8.16)$$
We thus have found an explicit solution of the optimum system function for our filter.

In practical applications of Eq. 8.15, $\hat{\Phi}_{id}(w)$ and $\hat{\Phi}_{ii}(w)$ are derivable from specifications in the problem statement, and $L$ is adjusted to give minimum $\sigma^2_e$. In many problems $L$ is zero; in these cases $L$ is easily shown to be zero by assuming it to be not zero and showing that this leads to larger $\sigma^2_e$ than if it were zero. In cases where $L$ is not zero for minimum $\sigma^2_e$, its proper value is found by setting $d(\sigma^2_e)/dL$ equal to zero.

The optimum filter formula (Eq. 8.16) does not always give $H_o(w)$ in forms which are realizable in terms of lumped parameter circuits. In such cases approximations to $H(w)$ may be used. Two approaches to the approximation problem are:

1. Determine $H_o(w)$ by Eq. 8.15 as it stands and approximate in the frequency domain.

2. Determine an approximate $H_o(w)$ by Eq. 8.15 through the use of an approximate $\left[\Phi_{id}(w)/\Phi_{11}(w)\right]_+$ obtained by replacing $\int_{-\infty}^{\infty} d\omega e^{j\omega t}\left[\Phi_{id}(w)/\Phi_{11}(w)\right]$ in Eq. 8.16 by an approximation to it designed to make the resultant $H(w)$ realizable in terms of lumped parameter circuits. This is one type of approximation in the time domain. The first approach is in disfavor because of lack of control of the approximation error. The second approach is the one used by Lee in his course where considerable emphasis is placed upon it.
In summary, the Wiener filter theory postulates:

1. A linear system.
2. An input which possesses an autocorrelation function.
3. A desired output such that a cross-correlation function between the input and itself exists.
4. The rms error criterion.

From these premises is derived the Wiener-Hopf integral equation 8.11 for $h_o(t)$, the optimum-system impulse response. This integral equation may be solved by use of Fourier transforms to yield formula 8.15 for $H_o(\omega)$, the optimum-system function. Approximations are often necessary to obtain an $H(\omega)$ which is realizable in terms of lumped parameter circuits.

8.2. LEE'S EXTENSION OF WIENER'S FILTER THEORY

In this subsection, Lee's extension of Wiener's original theory to solve the problem of designing the optimum linear compensating network is discussed briefly. The problem which Lee has considered assumes certain components, forming a linear fixed network, are beyond the control of the designer. The problem statement is:

\[
\text{Given:} \quad \begin{align*}
\text{Statement of problem} & \quad \begin{align*}
1. \text{Information about the input signal.} \\
2. \text{Information about the relationship between the input and desired output signals.}
\end{align*}
\end{align*}
\]
Lee's extension of Wiener's filter theory

Find:

The linear transfer characteristic of the compensating network which must be cascaded with the fixed network in order to make the rms error between the desired output and actual output signals a minimum.

The compensating network which has the sought-for transfer characteristic is termed the optimum compensating network.

Figure 8.2 is a block diagram of the system treated by Lee's extension. By a procedure quite analogous to that used in the preceding subsection, Lee finds the equivalent of the Wiener-Hopf equation for this problem to be

\[
\int_{-\infty}^{\infty} dh_{co}(u) \int_{-\infty}^{\infty} dy_{f}(y) \int_{-\infty}^{\infty} dv_{f}(v) \phi_{11}(x + y - u - v)
\]

\[
- \int_{-\infty}^{\infty} dy_{f}(y) \phi_{1d}(x + y) = 0 \text{ for } x > 0 \quad (8.17)
\]

where \( h_{co}(t) \) is the impulse response of optimum compensating network, and \( h_{f}(t) \) is the impulse response of the fixed network. All other symbols are as defined in the preceding subsection.

In the case of transformable correlation functions, the explicit solution formula is found by the same method as before to be
where \( H_{co}(\omega) \) is the system function of the optimum compensating network, and \( H_f(\omega) \) is the system function of the fixed network. All other symbols are as defined in the preceding subsection.

Equation 8.17 and 8.18 are the two alternative solutions of the compensating-network design problem. For a minimum-phase fixed network, the system function for the compensating network obtained by Eq. 8.18 is just the same as that which could be obtained by dividing the system function of the fixed network into an overall system function that would be obtained by the simpler filter formula of Eq. 8.15 on the basis that no fixed network is present. When the fixed network is of the nonminimum phase variety, however, the more complicated formula of Eq. 8.18 yields a less trivial solution—one that cannot be obtained by dividing system functions.

The brief review of the Wiener filter theory and Lee's extension thereof, which has been given in this section, hardly does justice to these beautiful mathematical formulations. It is hoped that the interested reader will take the trouble to refer to the rapidly growing literature on this subject in order to get a more extensive view than it has been possible to give here.
9. DISTURBING SIGNALS IN FEEDBACK-CONTROL SYSTEMS

9-1. BASIC BLOCK DIAGRAM

Figure 9.1 is a block diagram for a feedback-control system with \( J + 1 \) disturbing signals applied at various points within the fixed network. In a practical system these disturbances could be load torques which are independent of output signal, frame of reference motion, etc. We wish to show that, even though disturbing signals are present in the fixed network, compensating networks can still be designed by the method given above for the simple cascade filter, providing the feedback network is specified beforehand, and providing certain trivial alterations are made which correspond to mere changes of variables. In the analysis to follow, it is assumed that each of the \( J \) disturbing signals \( D_j(t) \) is variable with suitable correlation functions. The general scheme of notation of Fig. 9.1 is the same as before and is self-explanatory.

9-2. ERROR COMPUTATION

As a first step toward computation of the error \( \theta_e \), it is convenient to use the block diagram of Fig. 9.2, which is equivalent to Fig. 9.1, if \( D(t) \) is suitably defined. The proper value of \( D(t) \) for equivalence is

\[
D(t) = D_0(t) + \sum_{j=1}^{J} \left[ \frac{1}{T} \int_{q=1}^{\infty} dt_q h_{qj}(t_q) \right] D_j(t - \sum_{i=1}^{J} t_i) \quad (9.1)
\]

where

\[
\left[ \frac{1}{T} \int_{q=1}^{\infty} dt_q h_{qj}(t_q) \right] = \int_{-\infty}^{\infty} dt_1 h_1(t_1) \int_{-\infty}^{\infty} dt_2 h_2(t_2) \ldots \int_{-\infty}^{\infty} dt_J h_J(t_J)
\]
FIG. 9.1  BASIC BLOCK DIAGRAM FOR FEEDBACK-CONTROL SYSTEM WITH DISTURBING SIGNALS.
FIG. 9.2 FIRST EQUIVALENT FEEDBACK CONTROL SYSTEM FOR ERROR COMPUTATION PURPOSES.

FIG. 9.3 SECOND EQUIVALENT FEEDBACK CONTROL SYSTEM FOR ERROR COMPUTATION PURPOSES.
Figure 9.3 is a second equivalent to Fig. 9.1 in which the effective disturbing signal has been taken outside the loop. For equivalence

\[ D'(t) = \int_{-\infty}^{\infty} dt_1 g_f(t_1) D(t - t_1) \]  

(9.2)

With the disturbing signals completely removed from the closed loop, the feedback-control system can be replaced by its equivalent filter. In carrying out the analysis for \( h_{co}(t) \), the error will be computed as usual as \( e(t) = \phi_d - \phi_o \) where \( \phi_d(t) \) is the desired output. Thus it is evident that Eq. 4.6 may be used for the error, providing \( \phi_1(t) \) is replaced by \( \phi_1 + D'(t) \) and \( \phi_d(t) \) by \( \phi_d + D(t) \). Since neither \( D(t) \) nor \( D'(t) \) is a function of \( g_\alpha(t) \), which is the only characteristic allowed to vary to obtain \( h_{co}(t) \), we have shown that merely changing the \( \phi_1(t) \) and \( \phi_d(t) \) variables permits all of the filter analysis to be used as before insofar as error considerations are concerned.

9-3. SATURATION SIGNAL COMPUTATION

The preceding equivalent block diagrams can be made suitable for saturation signal computation providing the effects of the disturbing signals are fully accounted for in computing the saturation signals. Using the same principle as before of moving the disturbing signals to the periphery of the block diagram enables the equivalent diagrams of Fig. 9.4 to be drawn. For equivalence the nth effective disturbing signal \( D_{an}(t) \) to be introduced at the output of the nth saturation signal computer is
FIG. 9.4 EQUIVALENT FEEDBACK-CONTROL SYSTEM FOR SATURATION SIGNAL COMPUTATION.
\[ D_{sn}(t) \triangleq \sum_{j=1}^{n-1} \int_{-\infty}^{\infty} dt_1 \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \left( \frac{H_{sn}(w)}{j} \right) \right] D_j(t-t_1) \]
\[ + \int_{-\infty}^{\infty} dt_1 h_{sn}(t_1)D_j(t-t_1) \quad (9.3) \]

in which "\(a_n\)" is lowest index of the group of disturbing signals anterior to the \(n\)th saturating element. \(D_{sn}(t)\) physically represents the \(n\)th saturation signal \(\Phi_{sn}(t)\) that would exist under open-loop conditions (\(g(t) = 0\)) if only the \(J+1\) disturbing signals \(D_j(t)\) were acting upon the system with the normal input zero.

Replacement of the closed loop of Fig. 9.4 by its equivalent filter makes it possible to write the analogy of Eq. 4.12 for the \(n\)th saturation signal as

\[ \Phi_{sn}(t) = \int_{-\infty}^{\infty} dy h_{sn}(y) \int_{-\infty}^{\infty} dx h_c(x) \Phi'_1(t-x-y) - D_{sn}(t) \quad (9.4) \]

where \(\Phi'_1(t) = \Phi_1(t) + D'_1(t)\). Squaring yields

\[ \Phi'^2_{sn}(t) = D'^2_{sn}(t) - 2D_{sn}(t) \int_{-\infty}^{\infty} dy h_{sn}(y) \int_{-\infty}^{\infty} dx h_c(x) \Phi'_1(t-x-y) \]
\[ + \int_{-\infty}^{\infty} dy h_{sn}(y) \int_{-\infty}^{\infty} dx h_c(x) \Phi'_1(t-x-y) \int_{-\infty}^{\infty} dv h_{sn}(v) \int_{-\infty}^{\infty} du h_c(u) \Phi'_1(t-u-v) \quad (9.5) \]

This leads to the following equation (analogous to Eq. 4.15) for the rms value of the \(n\)th saturation signal:

\[ \Phi'^2_{sn} = \Phi_{sn}^2(0) - 2 \int_{-\infty}^{\infty} dy h_{sn}(y) \int_{-\infty}^{\infty} dx h_c(x) \Phi_{1D_{sn}}(x+y) \]
\[ + \int dy h_{sn}(y) \int_{-\infty}^{\infty} dx h_c(x) \int_{-\infty}^{\infty} dv h_{sn}(v) \int_{-\infty}^{\infty} du h_c(u) \Phi'_{11}(x+y-u-v) \quad (9.6) \]
where
\[ \phi_D^{(\text{sn})}(\tau) \triangleq \text{autocorrelation function for } D_{\text{sn}}(t) \]  
(9.7)
\[ \phi_{1D}^{(\text{sn})}(\tau) \triangleq \text{cross-correlation function for } \theta_1^{(t)}(t) \text{ and } D_{\text{sn}}(t) \]  
(9.8)
\[ \phi_{11}^{(\text{sn})}(\tau) \triangleq \text{autocorrelation function for } \theta_1^{(t)}(t) \]  
(9.9)

Carrying through a procedure exactly the same as that in Section 4 gives as the integral equation form of the solution for \( h_{\text{co}}(t) \) in the presence of \( J \) disturbing signals

\[
\int_{-\infty}^{\infty} du h_{\text{co}}(u) \left\{ \int_{-\infty}^{\infty} dy h_{\text{f}}(y) \left[ \phi_{11}^{(\text{SN})}(x + y - u - v) + \sum_{k=1}^{K} \rho_{ck} I_{11k}(x + y - u - v) \right] \\
+ \sum_{n=1}^{N} \rho_{\text{sn}} \int_{-\infty}^{\infty} dy h_{\text{sn}}(y) \int_{-\infty}^{\infty} dv h_{\text{sn}}(v) \phi_{11}^{(\text{SN})}(x + y - u - v) \\
- \int_{-\infty}^{\infty} dy h_{\text{f}}(y) \left[ \phi_{1d}^{(\text{SN})}(x + y) + \sum_{k=1}^{K} \rho_{ck} I_{1dk}(x + y) \right] \\
- \int_{-\infty}^{\infty} dy h_{\text{sn}}(y) \sum_{n=1}^{N} \rho_{\text{sn}} \phi_{1D}^{(\text{SN})}(x + y) = 0 \right\} 
\]  
for \( x \gg 0 \)  
(9.10)

where the only previously undefined quantity is
\[ \phi_{1d}^{(\text{SN})}(\tau) \triangleq \text{cross-correlation function between } \theta_1^{(t)}(t) \text{ and } \theta_{d}^{(t)}(t) \]  
(9.11)
in which
\[ \theta_{d}^{(t)}(t) \triangleq \theta_{d}(t) + D(t) \]  
(9.12)
We note that Eq. 9.10 for $h_{co}(t)$ contains only one additional group of terms (the last group) beyond the previous Eq. 3.15. This new equation can be given an explicit solution in the case of transformable correlation functions just as easily as the old one. We therefore conclude that disturbing signals in the fixed network present no new problems in procedure for finding solutions; they merely increase slightly the complexity of the solutions which have been found when they are absent.
10. EXPERIMENTAL CHECK ON SATURATION ASSUMPTION

10-1. PURPOSE OF EXPERIMENTAL CHECK

The usefulness of the analysis presented in the main body of this paper depends upon the validity of the basic assumption concerning saturation made in Section 2. There this assumption is stated as follows:

In the fictitious linear system obtained by replacing the saturating elements of the actual system by their linear models, limiting the rms responses of these linear models is effective in limiting the probabilities of finding the responses of the saturating elements of the actual system outside of the linear ranges characterizing these elements.

As mentioned in Section 2, the only practical means currently available for checking the validity of the saturation assumption are experimental. For any specific system, experimental tests will show how the several ratios of the maximum linear responses of the saturating elements of the actual system to the rms values of the saturation signals in the fictitious linear system obtained by replacing the saturating elements by their linear models are correlated with actual performance figures. The probabilities of saturation and the ratio of rms error in the actual system to that in the fictitious linear system represent two typical performance figures which may be of interest.

Performing a large number of such experimental tests on a variety of systems will show under what circum-
stances, if any, suitable performance figures may be obtained. These experiments, in those cases where suitable performance figures are obtained, will also yield the design figures (ratios of maximum linear responses to rms values of saturation signals) needed to establish future designs when conditions are similar to those of the original experiments.

The object of this section is to describe a single, simple experiment which has been performed for the purpose of illustrating by example the above general remarks concerning experimental evaluation of the saturation assumption.

10-2. THE EXPERIMENT ATTEMPTED

The results reported in this section were obtained from an attempt to perform an experiment comprising three aspects:

One - Construction of a feedback-control system having the system function of Example 2, Subsection 5-3, in its linear range and exhibiting output velocity saturation of a very abrupt type.

Two - Excitation of this system by a driving function possessing an autocorrelation function of the type used in Example 2 and possessing a normal probability density distribution of amplitudes.

Three - Measurement of actual system performance as a function of the ratio of the maximum linear range of the actual system to the rms value of the saturation signal of the fictitious linear system corresponding to the actual system.
It will be recalled that the statement of the problem considered in Example 2 reads as follows:

Given — Input is pure message; the desired output is the input. The input autocorrelation function is

\[ \phi_{11}(\tau) = \sigma_{1p}^2 e^{-|\tau|} \]  

(5.1)

The fixed elements impose no limitations other than possible output velocity saturation, so that in the fictitious linear system used as a basis of the system design the saturation signal is the output velocity. No transient performance specification is imposed.

Find — The optimum (least rms error) system function as a function of the limit \( \sigma_{gm} \) imposed on the rms value of the saturation signal.

The function relating the limit on the rms value of the saturation signal to the rms error for the optimum system.

Figure 10.1 is a block diagram of a feedback-control system having the properties required for the first aspect of the attempted experiment, provided the amplifier has the characteristic indicated. When operated in its linear range, this system has the system function

\[ H(\omega) = \frac{1}{\alpha + \frac{1}{\beta} \frac{1}{\omega^2 + 1}} \]  

(10.1)

This is the same as the optimum system function for Example 2
FIG. 10.1 FEEDBACK-CONTROL SYSTEM HAVING
SYSTEM FUNCTION OF EXAMPLE 2 IN
ITS LINEAR RANGE.
as given by Eq. 5.43, provided the parameter \( f \) is identical to that appearing in the autocorrelation function of the input signal (see Eq. 5.1). When this is true, the parameter \( \alpha \) is related to the limit \( \sigma_{\text{sml}} \) on the rms value of the saturation signal for the fictitious linear system by

\[
\sigma_{\text{sml}} = \frac{\sigma_{1p} f}{\sqrt{\alpha (\alpha + 1)^{3/2}}} \tag{10.2}
\]

and to the rms error \( \sigma_e \) by

\[
\sigma_e = \sqrt{\frac{\alpha(\alpha + 1)^2 + \alpha^2}{(\alpha + 1)^{3/2}}} \sigma_{1p} \tag{10.3}
\]

These last two relations are obtained from Eqs. 5.39 and 5.42 by use of the definition of \( \alpha \) given by Eq. 5.46.

With respect to the second aspect of the attempted experiment, a driving signal with the required exponential autocorrelation function may be generated by passing "white noise" through a single-stage low-pass R-C filter. The parameter \( f \) of the autocorrelation function is then identical to the reciprocal of the R-C product.

Among the measures of actual performance to be considered under the third aspect of the attempted experiment are:

One - Typical samples of oscillograms for the several quantities \( \Theta_1, \Theta_e, \Theta_s, \Theta_0 \) (see Fig. 10.1).

Two - Probability of saturation.

Three - Ratio of the actual rms error to that for the fictitious linear system.
Four - Ratio of the actual rms saturation signal to that for the fictitious linear system.

Probability density plots for the amplitude distributions of \( \Theta_1, \Theta_0, \mathcal{E}, \Theta_0, \) and \( \Theta_0 \) will also be of auxiliary interest.

10-3. IMPLEMENTATION

Because all practical mechanical servomotors have more complicated transfer characteristics than pure gain or pure integration, it was decided to use an all-electronic representation of the feedback-control system of Fig. 10.1. For the servomotor, an electronic integrator was used. This integrator was driven by an amplifier having an abrupt saturation characteristic just as depicted in Fig. 10.1. In this way a feedback-control system having output velocity saturation was achieved.

Although electronic apparatus was used, it was decided to keep the time scale in the same range as that of mechanical positional servomechanisms. This policy was adopted in order that the measurement problems encountered here would be representative of those to be encountered in the future when, and if, positional servomechanisms are designed and tested by the methods of this paper. The power-density spectrum (Eq. 5.4), corresponding to the autocorrelation function of the input, has the property that the power between the frequency limits \(-\omega_0\) and \(+\omega_0\) is equal to \(2\pi^2/\pi \tan^{-1} \omega_0/f\). Thus 84% of the input power lies between \(-4f\) and \(+4f\). Most positional servomechanisms are designed for inputs having the bulk of their power below limits of.
the order of zero to 10 radians per second. As it was desirable to have $f$ as large as possible, in order to avoid excessive d-c amplifier gain after the R-C low-pass filter, $f$ was chosen to be $10/4$ or $2.5$ seconds$^{-1}$.

The selection of the parameter $\alpha$ in a practical design is based upon the allowable limit $\sigma_{\text{sml}}$ on the rms value of the saturation signal for the fictitious linear system or upon the allowable rms error. In this particular experiment, however, these specifications are arbitrary, and therefore the selection of $\alpha$ is based on other considerations. A little reflection will show that the smaller $\alpha$ is, the larger will be the system loop gain. Also, the rms saturation signal is increased relative to the rms input as $\alpha$ is decreased, whereas the rms error is decreased. It was desired to have the loop gain under $300/j\omega$ in order to avoid appreciable errors in the performance of the electronic system caused by imperfect behavior of the components at high frequencies. This consideration plus the desire for reasonably large errors to facilitate measurement lead to the choice of $\alpha = 0.01$.

This value in conjunction with the value of $f$ selected above gives $\sigma_{\text{sml}} = (2.63 \text{ sec}^{-1})\sigma_{\text{ip}}$ and $\sigma_{\text{e}} = 0.1000 \sigma_{\text{ip}}$ (by Eqs. 10.2 and 10.3, respectively). The corresponding numerical values of the gains in Fig. 10.1 are for the forward amplifier $[f/(\alpha + 1)$ in the linear range] $247.5$ and for the feedback network $(\alpha + 1) 1.01$. This gives a loop gain of $250/j\omega$.

Figure 10.2 is a consolidated block diagram of the system actually tested. This system differs from the
FIG. 10.2 FEEDBACK-CONTROL SYSTEM ACTUALLY TESTED.
"optimum design" of Fig. 10.1 in that the feedback network has been arbitrarily assigned the gain of unity and the forward amplifier the gain of \( f/\alpha \) or 250. This leaves the loop gain unchanged but does make the system actually tested slightly "nonoptimum." These departures from the "optimum design," necessitated by lack of sufficient equipment, were justified on the basis that the input signal would undoubtedly have an autocorrelation function not precisely of the exponential form (Eq. 5.1) upon which the "optimum design" was based. It is to be emphasized that the experimental verification of the saturation assumption in no wise requires the use of "optimum" systems. The attempt to set up the situation of Example 2 was made for reasons of simplicity and niceness, and we should not be too disappointed if it proves to be impossible to carry out this attempt in all its details.

The system function for the feedback-control system of Fig. 10.2, when operated in its linear range, is

\[
H(w) = \frac{1}{\alpha \frac{f}{jw} + 1} \quad (10.4)
\]

Using the methods of Section 5, it is easily shown for an input signal having the autocorrelation function of Eq. 5.1 that the rms value \( \sigma_s \) of the saturation signal is

\[
\sigma_s = \frac{f}{\sqrt{\alpha(\alpha + 1)}} \sigma_{ip} \quad (10.5)
\]

and that the rms error \( \sigma_e \) is

\[
\sigma_e = \sqrt{\frac{\alpha}{\alpha + 1}} \sigma_{ip} \quad (10.6)
\]
since the desired output is taken to be the input as in Example 2. Note that the actuating signal $\mathcal{E}$ is identical to the error $\theta_e$ for this system, since the desired output is equal to the input. For the values of $f$ and $\alpha$ given above, we have $\sigma_s = (24.88 \text{ sec}^{-1})\sigma_{ip}$ and $\sigma_e = 0.0995\sigma_{ip}$. These values are not significantly different from those for the "optimum" system.

Figure 10.3 is a detailed block diagram of the experimental setup and includes the equipment used to generate the input signal $\Theta_i$ as well as that used to form the feedback-control system under test. With the exception of the Noise Generator, each component block in this block diagram is a standard Servo Simulator unit manufactured by the Dynamic Analysis and Control Laboratory of the Massachusetts Institute of Technology and described in a thesis by Johnson, Reference 10.8. The Noise Generator, developed by Mr. W. W. Siefert of the Dynamic Analysis and Control Laboratory, has the drawing number B-10663-SM and is a standard component of the Fight Simulator developed there. The noise generator uses a 6Q5-G thyratron, operating as a diode with the grid and cathode tied together, as the noise source. The noise signal passes successively through a cathode follower, a wide-band filter (having a pass-band from approximately 25 to 8900 cycles per second), an a-c amplifier of gain 100, and a narrow-band filter which has a pass-band from approximately 380 to 420 cycles per second with the filter adjustment used.
Fig 10.3 Detailed Block Diagram of Experimental Setup
The operation of the input signal generator portion of Fig. 10.3 is as follows. The noise signal from the Noise Generator is demodulated and passed through a Ripple Filter designed to suppress frequency components which are 800 cycles per second and higher. The Ripple Filter has no appreciable effect on the amplitudes and only slight effect on the phase of signals below 30 cycles per second. The Ripple Filter is needed to suppress high-frequency spurious signals developed by the Demodulator. At the output of the Ripple Filter, we have substantially flat noise over a frequency range of zero to 20 cycles per second. The R-C Low-pass Filter operates upon the filtered noise signal to produce a signal having very nearly the autocorrelation function of Eq. 5.1. This signal is amplified to form the input signal $Q_1$ for the feedback-control system. The adjustable gains, $K_1$ in the Demodulator and $K_2$ in the Amplifier, permit the rms value $\sigma_{ip}$ of the input signal $Q_1$ to be adjusted.

The operation of the feedback-control system is self-evident. The Limiting Amplifier was adjusted to saturate at $\pm 25$ volts, corresponding to a maximum linear range $Q_{SL}$ for the saturation signal $Q_s$ of 125 volts per second. This value was chosen to be as high as possible without requiring excessive d-c amplification after the R-C Low-pass Filter or excessive signal level at the output of the Demodulator in the input signal generator. With the Limiting Amplifier
adjusted this way, no other component in the system ever exhibited appreciable saturation during any of the tests conducted. The saturation of the limiting amplifier appeared to be fairly abrupt when observed on a cathode-ray oscillograph; the corners were not rounded by more than 1/2 volt.

The principal quantitative measuring instrument used in making observations on the performance of the feedback-control system was a two-channel direct-writing oscillograph manufactured by the Brush Development Company. This equipment comprised two d-c amplifiers, model BL913 (one for each channel), and one two-channel recorder equipped with Penmotors, model 902A. Although this equipment was operated from a regulated voltage power source and was calibrated for each run against a Ballantine Model 300 vacuum tube alternating-potential voltmeter, its stability, frequency response, and linearity were such that all observations made with it must be considered accurate to within only +10 per cent or so. It is regretted that such an inaccurate instrument had to be used. No other suitable instruments exist for this type of work.

Other instruments used for auxiliary purposes included a rectifier-type voltmeter, a vacuum tube direct-potential voltmeter, and a special thermocouple voltmeter. None of these were depended upon for quantitative results. Each was used in a monitoring or checking capacity and therefore will not be discussed.
10.4. PROCEDURE

Before taking any runs, the feedback-control system of Fig. 10.3 was checked to be sure that it performed the way the nominal characteristics of the several blocks indicate that it should. This check comprised driving the input $Q_1$ by an audio oscillator and measuring the magnitude and phase of the error $Q_e$ and the output $Q_o$ when the saturation signal $Q_s$ was below the maximum linear range $Q_{5L}$. This check indicated that the actual behavior was identical to the nominal behavior at 20 cycles per second within the error of measurement (0.02 on magnitude ratio and 2.5 degrees on phase angle). Next, the Limiting Amplifier was adjusted to have the characteristic shown in Fig. 10.3. This was done by placing the $Q_s/(5 \text{ sec}^{-1})$ and $50Q_e$ signals on the vertical and horizontal axes of a cathode-ray oscillograph with the peak value of $50Q_e$ equal to 25 volts and adjusting the Limiting Amplifier to just barely limit. The error of this adjustment should not have been greater than 0.75 volts.

As part of the standard operating procedure, it should be mentioned that all d-c amplifiers of Fig. 10.3 were checked for balance (zero) by means of a vacuum tube direct-potential voltmeter before and after each set of runs. The maximum unbalance tolerated at the end of a set of runs was 0.05 volts. This precaution was taken in order to avoid unbalanced operation of the Limiting Amplifier. The effects of the larger unbalances occurring in the d-c amplifiers of
the oscillograph equipment were eliminated to a large extent in the analysis of the data by subtracting the square of the average value from the mean square of the observed value of a particular quantity in the formation of its "measured" rms value. Because the balance tended to drift only slightly during the 30-second period of a run, and because the stochastic component of any signal would be expected to have zero cross-correlation with an average value resulting from unbalance, this procedure of compensating the mean-square value is justified.

The results of the next subsection are derived from 21 runs of approximately 30 seconds duration each. The 21 runs comprise 5 groups of 4 runs each plus an additional run for determining the autocorrelation function of the input $\Theta_1$. In the four runs of each of the 5 groups, the following variables were recorded: $\Theta_1$ and $\Theta_1'$; $\Theta_1$ and $-\Theta_o$; $\Theta_1$ and $\Theta_o$; $\Theta_e$ and $\Theta_o$. For the input autocorrelation run, $\Theta_1$ and $\Theta_1$ were recorded, since the apparatus used for determining the autocorrelation function requires a pair of identical oscillograms in its operation. (The actual variables recorded were $\Theta_1$, $5\Theta_o$, $-\Theta_o$, and $\Theta_o/(5 \text{ sec}^{-1})$. See Fig. 10.3. To get $\Theta_e$ and $\Theta_o$, appropriate scale factors were used.) The groups of runs differed from one another only by the input signal levels used. The input level was controlled by adjusting the gains $K_1$ and $K_2$ of Fig. 10.3, and their products were in the ratios 0.1, 0.2, 0.35, 0.5, and 1.0. The gains were selected so that the lowest produced negligible probability of
saturating the Limiting Amplifier and the highest, a large probability. Throughout the period during which the runs were taken, a rectifier-type voltmeter monitoring the output of the Noise Generator indicated no detectable variation of signal level at this point. Thus the rms values of the input $\Theta_1$ were constrained in the ratios of 0.1, 0.2, 0.35, 0.5, and 1.0.

Among the results of the next subsection is a plot of the autocorrelation function $\phi_{11}$ of the input $\Theta_1$. The values of this function used to make the plot were obtained with the aid of a special correlating machine designed by Mr. P. Travers of the Servomechanisms Laboratory at the Massachusetts Institute of Technology. Figure 10.4 is a schematic diagram of this machine. To operate it two persons track the two oscillograms $f_1(t)$ and $f_2(t)$, respectively, by actuating indices while the paper is mechanically moved at constant speed. A shift $\tau$ may be introduced by displacing the $f_1$ index backwards before starting a run. The value of the correlation function is approximated by dividing the reading in the $\int dt f_1(t)f_2(t + \tau)$ register by the length tracked and multiplying by appropriate scale factors. The length required to get a satisfactory approximation is determined by trying increasing lengths and observing how the results converge. When functions with zero mean value are to be correlated, it is standard practice to set the machine up so that zero value is accumulated in the $\int dt f_2(t + \tau)$ register in order to avoid errors resulting from poor zeroing of the machine or the oscillograms. The accuracy of the correlator is of the order
FIG. 10.4 SCHEMATIC DIAGRAM OF CORRELATOR.
per cent of $2^{\lambda}$ of the mean-square value when the rms value is of the order of $1/4$ full scale. A report on the correlator by Travers will be published by the Servomechanisms Laboratory in the near future.

In the results given below, rms values of a number of quantities are used. These have been obtained from the oscillograms of these quantities with the aid of the correlator. When used for determining rms values, the two indices of the correlator are tied together and only a single function is tracked, thereby saving on operator time. When functions of zero mean value are involved, the mean-square value obtained from the $\int dt f_1(t) f_2(t + \tau)$ register is decreased by the square of the average value obtained from the $\int dt f_2(t + \tau)$ register in order to obtain a truer estimate of the actual mean-square value.

The probability-of-saturation figures recorded below are obtained by measuring the fraction of the time that the saturation signal $\Theta_s$ appeared in either the plus or minus limits $\Theta_{SL}$. This was done by direct observation of the oscillograms for this quantity.

Probability-distribution functions for several quantities are given in the results to follow. Points for these distribution functions were obtained by direct observation of the corresponding oscillograms. The procedure consisted of simply finding the fractions of time the oscillogram remained below a number of arbitrarily chosen levels.
The resultant distribution functions theoretically could be differentiated to obtain the density (frequency) functions. This is not done here because accuracy considerations hardly permit differentiation of distribution functions based on only 9 levels. Also, distribution functions are more fundamental than density functions in the last analysis (see Reference 3.8).

10-5. RESULTS

The results described in this subsection comprise a series of eleven figures numbered 10.5 through 10.15. The first of these compares the actual autocorrelation function of the input signal with the one which is theoretically expected. The next five figures show typical samples from the oscillograms which form the basis for all of the subsequent results. Figure 10.11 is a compact summary of the principal results of the experiment. The last four figures give the probability-distribution functions of the several variables.

Figure 10.5 shows how the actual autocorrelation function of the input signal $\Theta_1$ corresponds with the autocorrelation function theoretically expected on the basis of "white noise" shaped by an R-C filter with a time constant of 0.4 seconds. In this figure the autocorrelation functions have been divided by the mean-square values of the input signals. The solid curve is the theoretical function. The actual autocorrelation function, as nearly as it can be calculated from a thirty-second oscillogram of the input signal, is represented
Observed RMS value of input signal is $V_{rms} = 16.5$ volts at highest gain used.

Points calculated from oscillograms - O

Theoretical curve based upon $\phi_{le} = e^{-2.5 t}$

Figure 10.5 Normalized autocorrelation function of input signal $\theta(t)$
by the encircled points. For values of $\tau$ below 0.8 second, the agreement is about as close as can be expected in view of the known fact that the R-C filter was not supplied with "white noise" but rather with a noise signal which was flat only to about 15 or 20 cycles per second. The deviations of the measured points for the actual signal in the region of $\tau$ greater than 0.8 second is attributed to lack of convergence caused by the limited length of the oscillogram. These deviations cannot be charged to lack of precision in the correlator used in making the correlations since points can be repeated to within better than 0.02 on the ordinate scale.

The object in presenting the autocorrelation functions of Fig. 10.5 is to help describe the input signal upon which the results of Fig. 10.11 are founded. Figure 10.5 is one of a number of possible partial descriptions of the input. As such it is statistical in nature. Another statistical, partial description of the input is its probability distribution function given by Fig. 10.12. Additional input description which is qualitative in nature is furnished by inspection of the oscillograms of the input signal shown in Figs. 10.6 through 10.10.

Each of the five figures, Figs. 10.6 through 10.10, is a reproduction in reduced size of sections from the four oscillograms taken for each of the five different input signal levels used to explore the effects of saturation. In order of increasing figure numbers, the measured rms values $\sigma_{ip}$
FIG. 10.6 TYPICAL OSCILLOGRAMS FOR $\frac{\sigma_s}{\sigma_{sl}} = 0.165$
FIG. 10.8  TYPICAL OSCILLOGRAMS FOR $\frac{L_s}{L_{SL}} = 0.578$
FIG. 10.9  TYPICAL OSCILLOGRAMS FOR $\frac{\theta_s}{\theta_{sl}} = 0.825$
of the input signal are 1.68 volts, 3.36 volts, 5.88 volts, 8.4 volts, and 16.8 volts, respectively. Corresponding to each of these input signal levels, there is an rms value of the saturation signal $\sigma_s$ that would exist in the fictitious linear system corresponding to the actual system tested. The ratio $\sigma_s/\Theta_{SL}$ of the rms saturation signal $\sigma_s$ of the fictitious linear system to the maximum saturation signal $\Theta_{SL}$ for linear operation in the actual system is used as a measure of how hard the actual system is driven, and this ratio is referred to as the drive intensity. This ratio is a better measure of the severity of drive than the input signal level itself since it has a more direct bearing on the probability of saturation. The values of this ratio corresponding to the several levels of input signal used are 0.165, 0.330, 0.578, 0.825, and 1.65. These drive-intensity values appear in the titles of the figures.

The four sample oscillograms of each of the five figures from top to bottom are $\Theta_I$ and $\Theta_e$, $\Theta_I$ and $-\Theta_o$, $\Theta_I$ and $\Theta_s$, and $\Theta_e$ and $\Theta_s$. Although it would have been preferable to have recorded $\Theta_I$, $\Theta_e$, $\Theta_s$, and $\Theta_o$ simultaneously, this was impossible to do since only a two-channel oscillograph was available. Inspection of these figures reveals several facts. First, the error is more or less proportional to the velocity of the input for low values of drive intensity and becomes increasingly sensitive to the input velocity as the drive intensity increases. Second, the output is almost
identical to the input for low values of drive intensity. At high values of drive intensity the output experiences rather lengthy periods of running at maximum speed. During these periods the error may become very large. Third, the saturation signal exhibits negligible limiting at low values of drive intensity and increasing periods of limiting with increasing drive intensity. At the largest drive intensity, Fig. 10.10, the saturation-signal limits about 70 per cent of the time. Note that the oscillograph was unable to follow the abrupt changes of slope of the saturation signal when it limits.

Figure 10.11 summarizes the principal results of the experiment. Here the error ratio \( \sigma^1_e / \sigma_e \) of the rms error \( \sigma^1_e \) in the actual system to the rms error \( \sigma_e \) in the fictitious linear system, the corresponding saturation signal ratio \( \sigma^1_s / \sigma_s \), and the probability of saturation \( P_s \) are plotted as functions of the drive intensity \( \sigma_s / \sigma_{SL} \). The values of the rms error \( \sigma_e \) and the rms saturation signal \( \sigma_s \) for the fictitious linear system are based on observations of the actual rms error \( \sigma^1_e \) and actual rms saturation signal \( \sigma^1_s \) for the two lowest input signal levels used, \( \sigma^1_{ip} = 1.68 \) and \( \sigma^1_{ip} = 3.36 \) volts, respectively. This is proper because the probability of saturation is negligible for these two levels of input signal. Under the condition of negligible probability of saturation, \( \sigma^1_e = \sigma_e, \sigma^1_s = \sigma_s \), and \( \sigma^1_e / \sigma^1_{ip} \) should be independent of signal level. Furthermore, \( \sigma^1_s \) should be 250 second\(^{-1}\) times as great as \( \sigma^1_e \) (see Fig. 10.3). Use of these facts made
FIG. 10.11  PROBABILITY OF SATURATION, SATURATION SIGNAL RATIO AND ERROR RATIO AS FUNCTIONS OF DRIVE INTENSITY AT SATURATING ELEMENT.
possible the determination of a mean value of $\sigma_e^1/\sigma_{ip}^1$ in the cases of negligible saturation. This mean value is $\sigma_e^1/\sigma_{ip}^1 = 0.050$ within a probable error of 5 per cent.

If the actual autocorrelation function of the input signal were the theoretical curve of Fig. 10.5, the $\sigma_e^1/\sigma_{ip}^1$ ratio for negligible saturation would be expected to equal the theoretical ratio $\sigma_e/\sigma_{ip} = 0.0995$ (see discussion following Eq. 10.6). The fact that the observed ratio is approximately one half of the theoretical ratio is attributed to the deviation of the actual autocorrelation function of the input signal from the theoretical one—especially in the neighborhood of zero $\tau$. The actual auto-correlation function would be expected to have zero slope at $\tau = 0$ since the noise signal filtered by the R-C network to form the input signal $\Theta_i$ is flat over only a finite frequency band rather than the infinite band postulated in deriving the theoretical curve. 10.1

10.1. If it is assumed that the effect of the band-pass filter contained in the Noise Generator of Fig. 10.3 may be represented by an equivalent simple R-C low-pass filter placed after the Demodulator, one may calculate the time constant which this equivalent filter must have in order to explain the observed ratio $\sigma_e^1/\sigma_{ip}^1$ of the rms error to the rms input under linear operating conditions. Such a calculation shows that this equivalent filter must have a time constant equal to 0.0296 times that of the R-C Low-pass Filter used to shape the noise signal in Fig. 10.3. That is, the equivalent filter must have a time constant of 0.0118 seconds corresponding to a pass band of +13.5 cycles per second. This agrees reasonably well with the pass band of the Noise Generator's band-pass filter which is approximately ±20 cycles per second. The equivalent R-C filter would be expected to have a smaller pass band (measured to the half-power point) than the sharper band-pass filter actually used.
Unfortunately, the precision to which the actual autocorrelation function may be determined does not permit a definite confirmation of this expectation.

Several observations can be made on the basis of Fig. 10.11. First, the actual system behaves so much like the fictitious linear system up to $\frac{\sigma_s}{\Theta_{SL}} = 0.3$ that no departures from linear behavior can be detected. Second, at $\frac{\sigma_s}{\Theta_{SL}} = 0.5$ the error ratio begins to increase above unity but has not deviated enough to invalidate the use of the fictitious linear system for approximately calculating the rms error. At this point the probability of saturation is about 0.10. Third, the probability of saturation and the error ratio both vary linearly (as nearly as can be determined on the basis of the few points) for large values of $\frac{\sigma_s}{\Theta_{SL}}$. Fourth, even for values of $\frac{\sigma_s}{\Theta_{SL}}$ where the probability of saturation is of the order of 0.70, the error ratio does not exceed 5.0. Fifth, for large values of $\frac{\sigma_s}{\Theta_{SL}}$ the actual rms saturation signal $\sigma_s^I$ tends to approach the linear range limit $\Theta_{SL}$. (This is evident from the fact that the product $(\sigma_s^I/\sigma_s)(\sigma_s/\Theta_{SL})$ tends to approach unity as $\sigma_s/\Theta_{SL}$ becomes large.) This last observation is in agreement with common sense—as the input level is raised, the saturation signal tends to jump from one extreme to the other without spending much time in between.

The next four figures (Figs. 10.12 through 10.15) show the measured probability-distribution functions for input $\Theta_i$, output $\Theta_o$, error $\Theta_e$, and saturation signal $\Theta_s$
with the drive intensity $\sigma_s/\theta_{SL}$ as a parameter. The ordinate of each plot is the probability $P$ of the given variable being below the amplitude corresponding to the abscissa. The abscissae of each plot are the ratios formed by dividing the amplitudes of the given variable by the measured rms value of that variable.

Figure 10.12 compares the measured points of the probability distribution of the input $\theta_1$ with the normal (Gaussian) distribution. The measured distribution is independent of signal level so only one set of points is shown. In view of the method used to generate the input signal, a normal distribution would be anticipated. The observed points fit the normal curve as closely as can be expected considering the limited observation period used (30 seconds) and the accuracy of measurement. The horizontal shift of the measured points to the right is not deemed very significant since it corresponds to a bias of 9 per cent of full scale. A bias up to 6 per cent of full scale could easily be introduced by a combination of unbalance in the oscillograph and displacement of the pen operating point caused by nonlinearity in the oscillograph amplifier. Another 3 per cent of bias could be expected because of the limited sample.

Figure 10.13 shows how the measured probability distribution of the error $\theta_e$ varies with drive intensity $\sigma_s/\theta_{SL}$. For the low drive intensity, corresponding to a probability of saturation $P_s$ of substantially zero, this
Fig. 10.12: Probability distribution of input signal $\theta_i$. Normal distribution shown.

Amplitude of input relative to its RMS value $\theta_i$.

Observed points at various $\theta_i$. Graph plotted on log-linear paper.
FIG. 10:13 PROBABILITY DISTRIBUTION OF ERROR $\theta_e$
FIG. 10.14

PROBABILITY DISTRIBUTION OF OUTPUT $\theta_0$

For $\frac{\sigma_5}{\theta_{5L}} = 0.165$

For $\frac{\sigma_5}{\theta_{5L}} = 0.576$

For $\frac{\sigma_5}{\theta_{5L}} = 1.65$

AMPLITUDE OF OUTPUT RELATIVE TO ITS RMS VALUE $\theta_0$
Fig. 10.15 Probability distribution of saturation signal $\theta_s$.
distribution is nearly normal (except for bias). For the higher $\sigma_s/\theta_{SL}$ corresponding to probabilities of saturation of 0.12 and 0.71, respectively, the error shows increasing departure from the normal distribution in the direction of greater probability of errors of large magnitude.

Figure 10.14 gives the measured probability distribution of the output $\theta_0$ for the same three drive intensities as in the preceding figure. There appears to be no significant departure from the normal distribution as the drive intensity increases. This agrees with the common sense prediction for this experiment that the output distribution will be far less sensitive to saturation than the error distribution.

Figure 10.15 displays the measured probability distributions of the saturation signal $\theta_s$. For the low drive intensity, corresponding to negligible probability of saturation, the distribution of the saturation signal should be identical to that of the error, and on this basis the measured curve is plotted through the poorly spaced points. (The poor spacing is the result of the scale factor used, which in turn is the result of the desire to have all the oscillograms of saturation signals on such a scale that the saturation limit would appear at 75 per cent of full scale.) Interesting departures from normal distributions are exhibited by the saturation signal at values of drive intensity which cause appreciable saturation. The tendency toward almost linear variation of probability in the nonsaturated region with
increased probability of saturation is especially noteworthy.

10.6. CONCLUSIONS

The preceding subsection has given results which describe the measured behavior of the experimental system in considerable detail. It is believed that tests made and the results obtained for this simple system make clear what was meant in Subsection 10-1 by experimental tests designed to check the saturation assumption. No general conclusions concerning the saturation assumption are possible since only one very special system has been tested under one set of very special conditions. The reader may extrapolate at his own risk from the special conclusions given in the next paragraph.

For the special system tested, which exhibited output velocity saturation, and for the particular input signal used, certain conclusions may be drawn:

One—Limitation of the rms saturation signal that would exist in the fictitious linear system which is the linear model of the actual system is effective in controlling the probability of saturation in the actual system.

Two--The rms saturation signal of the fictitious linear system may be allowed to go as high as one half of the maximum linear response of the saturating element in the actual system without invalidating error calculations based on the fictitious linear system.
Three--The probability of saturation is approximately 0.10 when the rms saturation signal of the fictitious linear system is 0.50 of the maximum linear response of the saturating element in the actual system, and this probability decreases with decreasing signal level.

Four--The probability distribution of the output of the actual system is substantially the same as the probability distribution of the input for ratios of the rms saturation signal of the fictitious linear system to the maximum linear response of the saturating element of the actual system which range from zero to much greater than one half.

The first conclusion tends to verify the saturation assumption used in the analysis presented in the main body of this paper insofar as a single experiment is able to. The second, third, and fourth conclusions represent design data for systems of the type used in this experiment.

It is hoped that other investigators will perform experiments similar to the one recorded here on an assortment of systems resulting from a large variety of problems. In this way a catalogue of design data eventually can be compiled.

As experience is gained in the design of non-linear systems, it is very probable that more effective compensating network designs will be evolved than those obtained by the application of the design theory of this
paper. Undoubtedly these better designs will be more specialized than those of this paper and will be arrived at by a combination of intuition, analysis, and experiment. The experimental work necessary to accumulate the catalogue of design data referred to above may very well be the stepping stone to better designs. In the meantime, until better designs are available, it is hoped that results based upon the method of this paper will serve as a useful guide post.
11.1. GENERAL COMMENTS ON EQUIPMENT FOR STATISTICAL EXPERIMENTS IN SERVOMECHANISM FIELD

11-1. SIGNAL GENERATION

The purpose of this section is to point out some of the equipment difficulties that confront the experimentalist when attempting to assess the performance of servomechanisms and similar systems when they are subjected to stochastic signals. One of the first problems encountered is the generation of suitable test signals. Test signal generation forms the subject of this subsection.

Two basic approaches to the signal generation problem are possible. One is to use suitably "shaped" noise from a noise generator to actuate the system under test. This scheme is the one used in the experimental work reported in the preceding section. The second scheme is to generate a test signal by means of a reproducer mechanism from some suitable original recording. Magnetic wire (or tape) recorders and photoelectric "players" are two types of reproducers that could be used. The original recording could be a shaped noise function or a mathematically generated function.

The advantage of the first approach to the signal generation problem is simplicity. The advantage of the second approach is repeatability of results. With the first approach the experimentalist may never be certain that his signal has the same properties from one time to another. This is not a serious matter if the equipment tested uses
signals of relatively high frequencies so that checks, such as monitoring the rms value of the signal, can be conveniently employed. But with servomechanisms the frequencies commonly encountered are so low that the averaging period of an rms indicator must be of the order of minutes. At the present time no indicator with such a long averaging period exists. The second approach eliminates the necessity for monitoring equipment by using a pseudo-stochastic signal which is actually periodic with a very long period. Once the characteristics of this signal are established, they are effectively "frozen" and cannot change with time.

At the time the experimental work reported in Section 10 was undertaken, no suitable reproducers were available for "playing back" a recording of a signal in the servomechanism frequency range. Wire recorders require modulation and demodulation of signals, and these additional complications offset to a large extent the advantages to be gained by the second approach to signal generation. Photoelectric "oscillogram readers" have been conceived, but no satisfactory operating model existed. At the time of the experimental work, the only feasible way for playing back a recording was to manually track an oscillogram. This necessarily limited the frequency range to the extremely low servomechanism frequencies. Some experimentation in this direction was attempted using the Rockefeller Differential Analyzer at the Massachusetts Institute of Technology but was dropped.
when it became evident that certain modifications of this machine would be needed to bring its speed up to an economical point.

A small amount of work was done on the mathematical generation of suitable input signals. An attempt was made to create a function possessing the exponential autocorrelation function of Eq. 5.1. The plan was to set up a random sequence of numbers +1 and -1, using a table of random sampling numbers (Reference 11.3), by assigning even numbers the value +1 and odd numbers -1 or by some similar scheme. Use of these numbers as amplitude factors in sequence of exponential pulses $e^{-ft}$ spaced uniformly in time theoretically yields a signal having the desired autocorrelation function. In practice the length of sample of this signal necessary to get reasonable convergence of the autocorrelation function proved to be much longer than had been experienced with natural data. This phenomenon was not fully investigated because of the decision to switch to the first method of signal generation on account of the reproduction difficulties. However, it does indicate that the problem of mathematically generating test signals may be a very interesting one for further investigation.

Besides the analytical problem of the generation of test signals by mathematical means, there is the interesting practical problem of signal reproduction which is largely unsolved in the servomechanism frequency
range. In addition to the magnetic wire or tape recorders and the photoelectric reproducers which require blacking in one side of an oscillogram, many possibilities exist such as automatic curve followers, automatic interpolators working from digital information on tapes, etc. Much room for invention exists in this area.

11-2. INSTRUMENTATION

The principal limitation on the accuracy of the results of the experimental work reported in Section 10 was the inaccuracy of the oscillograph used. The instrument made by the Brush Development Company is primarily a qualitative indicator and is not to be considered as a precision instrument. Other direct-writing instruments exist, but no known instrument has the low (1 or 2\(^{-1}\)) static error which is necessary for even relatively crude engineering work. The dynamic performance of all direct-writing instruments is so poor that dynamic error ratings are seldom given.

There seems to be no reason why much more accurate direct-writing instruments cannot be made, particularly in the servomechanism frequency range. A very interesting engineering problem would be the design of a direct-writing instrument having the following rating for each channel:

- Maximum writing-point acceleration - 300 meters sec\(^{-2}\)
- Maximum writing-point amplitude - 0.040 meters
- Maximum static error - 1% of full scale
Maximum dynamic error (to 20 cps sine-wave test signal producing an amplitude of 0.010 meters) - 0.001 meters

Maximum dynamic error (to 10 cps sine-wave test signal producing an amplitude of 0.040 meters) - 0.001 meters

An instrument of these specifications would be extremely useful in the servomechanism field for both stochastic and ordinary testing. Use of a closed-loop feedback scheme, rather than the spring-restrained galvanometer principle so commonly used at present, should make the achievement of such ratings not at all impossible.

In addition to good oscillographic equipment, there is a definite need for some instrument to replace the function the rms signal-level indicators used in ordinary test work. This equipment should permit the experimentalist to gauge the rms levels of stochastic signals with sufficient precision to make system adjustments without making a large number of unnecessary oscillograms. Long-averaging-period rms indicators are not satisfactory because of the waiting time required to get accurate readings. Better would be an instrument which would record the integral of the square of the signal from a starting time established by pushing a button. The average slope of the resultant curve could be determined by eye in less time than the settling time of a long-period rms indicator.
The major apparatus used in processing the oscillograms in the experimental work of Section 10 was the Servomechanisms Laboratory Correlator (see Fig. 10.4). This device proved to be quite satisfactory, as far as its basic accuracy is concerned, for the purposes of determining correlation functions and rms values. It leaves something to be desired in the way of operating convenience, and its dependence upon manual tracking sets the principal limitation on its effective accuracy. Operator fatigue is such that the average individual can be expected to work with it only three or four hours a day. For this reason it would be desirable to incorporate automatic tracking if the use of the correlator is likely to become extensive. Pending installation of automatic tracking, many minor improvements could be made such as more convenient control of the tracking indices, arrangement of the tracking indices to permit less parallax, and better paper speed control.

A minor addition to the correlator which would increase its usefulness is a device to facilitate measurement of the fraction of the time an oscillogram is below (or above) a given level. Such measurements are necessary to determine probability distribution functions. This could be simply a conveniently operated clutch which an operator could engage whenever the oscillogram was below (or above) the given level. Engagement of the clutch would cause the paper motion to turn a register. Operation of the
clutch could simultaneously cause a pencil line to be drawn on the oscillogram to serve as a check on the operator's accuracy.

In conclusion it should be mentioned that the need for high-accuracy oscillographic equipment and machinery for correlating oscillograms could be largely eliminated if devices for determining correlation functions directly from the signals in question could be had. In a higher frequency range, a directly actuated correlator has been built by Mr. H. E. Singleton of the Research Laboratory for Electronics of the Massachusetts Institute of Technology (see Reference 19.5). In the servomechanism frequency range, however, no directly actuated correlator has been built, and the problem of building one appears rather difficult because of the long averaging periods required if sampling methods are used.
12. GLOSSARY OF SYMBOLS

The following list includes the principal symbols used in the main body of this report. Signals in systems are usually denoted by "θ" with subscripts to specify a particular quantity (e.g., "i" for input, "o" for output, "e" for error, and "d" for desired output). "H" is the usual symbol for transfer function in a filter, and "h" for the impulse response. The subscript "c" ordinarily refers to compensating network, and "f" to fixed network. A correlation function and its transform are indicated, respectively, by "ϕ" and "Φ". Universally, "g" means rms value with subscripts indicating of what quantity.

Additional symbols are used in certain sections of the appendix. These are defined in each instance when they are first introduced. In Section 10 primes are used on certain symbols to indicate actual or measured values as opposed to values obtained by analysis of a linear model.

A(ω) abbreviation for
\[
\left( \overline{H_f(ω)} H_f(ω) \left[ \tilde{ϕ}_{11}(ω) + \sum_{k=1}^{K} e_{ck} I_{11k}(ω) \right] \right. \\
\left. + \sum_{n=1}^{N} \rho_{sn} \overline{H_{sn}(ω)} H_{sn}(ω) \tilde{ϕ}_{11}(ω) \right)
\]

B(ω) abbreviation for
\[
\overline{H_f(ω)} \left[ \tilde{ϕ}_{1d}(ω) + \sum_{k=1}^{K} e_{ck} I_{1dk}(ω) \right]
\]

C_{dk} kth transient desired output signal
C_{ek} kth transient error
C_{1k} kth transient input signal
C_{ok} kth transient output signal
D  denominator polynomial; also, disturbing signal

\(d_i\)  \(i\)th error coefficient (see p. 2.14, footnote 2.4)

E  see Eqs. 5.25, 5.48, 5.60, and 5.100

e  base of natural logarithms

\(\mathcal{E}\)  actuating signal

F  general symbol for function of a function

f  general symbol for function; also, constant appearing in Eq. 5.1

\(G_c\)  transfer function of compensating controller

\(\mathcal{G}_c\)  impulse response of compensating controller

\(G_f\)  transfer function of feedback network

\(\mathcal{G}_f\)  impulse response of feedback network

H  over-all system function

\(H_c\)  transfer function of compensating network

\(h_c\)  impulse response of compensating network

\(h_{cl}\)  arbitrary impulse response

\(H_{co}\)  transfer function of optimum compensating network

\(h_{co}\)  impulse response of optimum compensating network

\(H_f\)  transfer function of fixed network

\(h_f\)  impulse response of fixed network

\(H_o\)  optimum over-all system function

\(H_{sn}\)  transfer function of nth saturation signal computer

\(h_{sn}\)  impulse response of nth saturation signal computer

\(I_{ek}\)  integral square of \(k\)th transient error

\(I_{emk}\)  limit upon \(I_{ek}\)

\(I_{ldk}\)  cross-translation function (or transform thereof) of \(k\)th transient input and desired output signals
\( I_{1k} \) autotranslation function (or transform thereof) of kth transient input signal

\( L \) constant appearing in explicit solution formulae for \( H_{co} \)

\( t \) left side of Eq. 3.18 (or transform thereof)

\( P \) probability

\( P_s \) probability of saturation

\( Q_{ml} \) see Eq. 5.101

\( S_{ml} \) see Eqs. 5.24, 5.47, 5.59, and 5.99

\( T \) integration limit

\( t \) time

\( u \) integration variable dimensions of time; also, unit step function

\( V \) see Eq. 5.64

\( v \) integration variable (dimensions of time)

\( w \) integration variable (dimensions of time\(^{-1}\))

\( x \) integration variable (dimensions of time)

\( y \) integration variable (dimensions of time)

\( \alpha \) see Eqs. 5.46 and 5.52

\( \delta \) delta (impulse) function

\( \epsilon \) convenience variable used in application of variation principle

\( \iota \) see Eq. 5.49

\( \eta \) \( \eta/\pi \) is amplitude of flat ("white noise") power-density spectrum

\( \Theta_d \) desired output

\( \Theta_e \) error
\( \Theta_f \) input to fixed network
\( \Theta_i \) input signal
\( \Theta_o \) output signal
\( \Theta_r \) return signal
\( \Theta_{SLn} \) maximum linear response of nth saturating element
\( \Theta_{sn} \) nth saturation signal
\( \nu \) \( \sigma^2_{ip} \) as \( f \to 0 \) (see Eq. 5.53)
\( \pi \) ratio of circumference to diameter of circle
\( \rho_{ck} \) Lagrangian multiplier for kth integral square of transient error
\( \rho_{sn} \) Lagrangian multiplier for nth mean-square saturation signal
\( \sigma_e \) rms error
\( \sigma_{ip} \) rms value of input signal
\( \sigma_{sn} \) rms response of the linear model of the nth saturating element in the fictitious linear system
\( \sigma_{snn} \) limit upon \( \sigma_{sn} \)
\( \tau \) shift parameter appearing in correlation functions
\( \Phi_{id} \) cross-power-density spectrum of input and desired-output signals
\( \phi_{id} \) cross-correlation function of input and desired-output signals
\( \Phi_{ii} \) power-density spectrum of input signal
\( \phi_{ii} \) autocorrelation function of input signal
\( \omega \) angular frequency (real or complex)
\( \omega_n \) see Eq. 5.49
LHP  abbreviation for lower half plane
UHP  abbreviation for upper half plane
  +  as superscript indicates factor with poles and zeros in UHP
  +  as subscript indicates component with poles in UHP
  -  as superscript indicates factor with poles and zeros in LHP
  -  as subscript indicates component with poles in LHP
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14. BIOGRAPHICAL NOTE

On May 14, 1919, George Cheney Newton, Jr., was born in Milwaukee, Wisconsin. He received his precollege schooling in the public schools of Iron Mountain, Michigan. During the years 1937 to 1941 he attended the Massachusetts Institute of Technology and was awarded the degree Bachelor of Science in Electrical Engineering in June, 1941. During the following year he was employed as a Research Assistant by the Servomechanisms Laboratory at M.I.T. After a brief assignment with a special Naval Ordnance plant in York, Pennsylvania, he transferred to the Sperry Gyroscope Company as an Assistant Project Engineer in 1943. His activities with the Sperry Gyroscope Company were primarily devoted to the development of aircraft fire control equipment for bomber defense. In 1946 he left the Sperry organization as a Senior Project Engineer to become an Instructor in Electrical Engineering at M.I.T. Promotion to the rank of Assistant Professor occurred in 1949. From 1946 until the present, his teaching assignments have been principally to the graduate courses in servomechanisms.

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