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Absolutely maximally entangled states, combinatorial designs, and multiunitary matrices

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Absolutely maximally entangled (AME) states are those multipartite quantum states that carry absolute maximum entanglement in all possible bipartitions. AME states are known to play a relevant role in multipartite teleportation, in quantum secret sharing, and they provide the basis novel tensor networks related to holography. We present alternative constructions of AME states and show their link with combinatorial designs. We also analyze a key property of AME states, namely, their relation to tensors, which can be understood as unitary transformations in all of their bipartitions. We call this property multiunitarity.

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I. INTRODUCTION

A complete characterization, classification, and quantification of entanglement for quantum states remains an unfinished long-term goal in quantum information theory (see Ref. [1] for a review). Nevertheless, a large number of relevant results related to entanglement are well known. It is, for instance, known that generic condensed-matter physical systems characterized by homogeneous nearest-neighbor interactions carry an amount of entropy bounded by an area law [2]. This fact opens the possibility of applying tensor network techniques to describe the ground state of relevant physical systems [3], including such phenomena as quantum phase transitions [4].

However, there are many situations where entanglement entropy is maximal or nearly maximal, that is, it scales as the volume of the quantum system. This is the case of generic time evolution of local Hamiltonians [5], of evolution of a quantum computer addressing a quantum Merlin-Arthur problem, and of random states [6]. The latter example is useful for our discussion, since it is known that the average entanglement of a random pure state $|\psi\rangle$ of N qubits is close to maximal. Indeed, making use of the von Neumann entropy

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho), \tag{1}$$

it is possible to show [7,8] that the average entropy of the reduced state $\sigma = \mathrm{Tr}_{N/2}|\psi\rangle\langle\psi|$ to $N/2$ qubits reads

$$S(\sigma) = \frac{N}{2}\log 2 - c + (\text{subleading terms}), \tag{2}$$

where the constant c depends on the choice of the ensemble used to generate random states [6,9]. For convenience we consider the natural logarithm throughout the work. In the case where the average is defined with respect to the unitarily invariant Haar measure on the space of unit vectors of size 2^N, the constant reads $c = 1/2$ [7]. Although random states are *almost* maximally entangled (ME), their entropy differs from the maximal value by a negative constant. One may then investigate, when truly ME states appear, what properties they have and what they are useful for. A relevant motivation for this study is the role that absolute maximal entanglement may play in the context of holography, as we discuss later.

The aim of the present paper is to extend the relation of absolutely maximally entangled (AME) states to a different branch of mathematics, so-called combinatorial designs. On one hand, new families of multipartite maximally entangled (MME) states are presented. On the other hand, we demonstrate a direct link between multipartite entanglement and combinatorial structures, including mutually orthogonal Latin squares and Latin cubes and symmetric sudoku designs. Furthermore, we introduce the concept of *multiunitary matrices*, which exist in power prime dimensions, show their usage in constructing AME states and present a small catalog of such matrices in small dimensions.

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This work is organized as follows. In Sec. II we present some important aspects of entanglement theory, AME states, and their link with holography. In Sec. III we review the current state of the art of AME states for multipartite systems. In Sec. IV we consider the particular case of AME states having minimal support and its relation to classical codes and orthogonal arrays (OAs) of index unity. A link among AME states, mutually orthogonal Latin squares and Latin cubes, and hypercubes is explored in Sec. V. In Sec. VI we introduce the concept of multiunitary matrices and demonstrate that they are one-to-one connected with AME states. Additionally, we provide some constructions of multiunitary matrices and their associated AME states. In Sec. VII we resume and discuss the most important results obtained in our work and conclude. In Appendix A we discuss the nonexistence of AME states of four qubits and present the most entangled known states. In Appendix B we explain in detail the simplest case of two-unitary matrices of order $D = d^2$, which are associated with AME states of four qu*d*its. Some examples of multiunitary matrices are presented. In Appendix C we establish a relationship between AME states and a special class of sudoku. Finally, in Appendix D we present a minicatalog of multiunitary matrices existing in low dimensions.

II. AME STATES: DEFINITION AND BASIC PROPERTIES

A. Definition

A lot of attention has recently been paid to the identification of entangled states of N-party systems, such that tracing out arbitrary $N-k$ subsystems, the remaining k subsystems have associated a maximally mixed state [10–15]. Such states are often called k-uniform [16,17], and by construction the integer number k cannot exceed $N/2$. In this paper, we focus on the extremal case, $k = \lfloor N/2 \rfloor$ (we put the floor function $\lfloor\rfloor$ to include cases of N even and odd), and analyze properties of states called AME (see [18,19]). Also, such states were previously known as perfect MME states [20].

The definition of AME states corresponds to those quantum states that carry maximum entropy in *all* their bipartitions. It is a remarkable fact that the existence of such states is not at all trivial and deepens into several branches of mathematics. Let us be more precise and define an AME(N, d) state $|\psi\rangle \in \mathcal{H}$, made with N qudits of local dimension d, $\mathcal{H} = (C^d)^{\otimes N}$ as a state such that its reduced density matrices in any subspace $\mathcal{A} = (C^d)^{\otimes \frac{N}{2}}$, $\mathcal{H} = \mathcal{A} \otimes \bar{\mathcal{A}}$, carry maximal entropy

$$S(\rho) = \frac{N}{2} \log d \quad \forall \mathcal{A}. \tag{3}$$

This is tantamount to asking whether every reduced density matrix to k qudits, that is, ρ, can be proportional to the identity

$$\rho = \frac{1}{d^k} I_{d^k} \quad \forall k \leqslant \frac{N}{2}. \tag{4}$$

Let us note the fact that a k-uniform state is also k'-uniform for any $0 < k' < k$.

There is an obstruction to a state's reaching maximal entanglement in all bipartitions due to the concept of monogamy of entanglement [21,22]. Every local degree of freedom that tries to get maximally entangled (ME) with another one is, then, forced to disentangle from any third party. Therefore, entanglement can be seen as a resource to be shared with other parties. If two local degrees of freedom get largely entangled among themselves, then they are less able to be entangled with the rest of the system. But this rule is not always fulfilled. There are cases where the values of the local dimensions d and the total number of qudits N are such that AME states exist. For a given N, there is always a large enough d for which there exists an AME state [18]. However, the lowest value of d such that an AME state exists is not known in general.

Let us mention that AME states are useful and necessary for accomplishing certain classes of multipartite protocols. In particular, in Ref. [18], it was shown that AME states are needed to implement two categories of protocols. First, they are needed to achieve perfect multipartite teleportation. Second, they provide the resource needed for quantum secret sharing. These connections hint at further relations between AME states and different branches of mMathematics. For instance, AME states are related to Reed-Solomon codes [23]; also, AME states (and k-uniform states in general) are deeply linked to error correction codes [16].

There is yet another surprising connection between AME states and holography [24,25]. It can be seen that AME states provide the basis for a tensor network structure that distributes entanglement in a most efficient and isotropic way. This tensor network can be proven to deliver holographic codes, which may be useful as quantum memories and as microscopic models for quantum gravity. A key property for these new developments is related to the properties of multiunitarity that are explored in Sec. VI B.

B. Local unitary equivalence

Entanglement is invariant under choices of local basis. It is then natural to introduce the concept of local unitary (LU) equivalence among AME states. Two quantum states, $|\Phi\rangle$ and $|\Psi\rangle$, are called LU-equivalent if there exist N LU matrices $U_1, \ldots U_N$ such that

$$|\Phi\rangle = U_1 \otimes U_2 \otimes \ldots \otimes U_N |\Psi\rangle. \tag{5}$$

If $|\Psi\rangle$ is an ME state, any other state LU-equivalent to it is also ME. We define AME(N,d) as the set of all AME states in the Hilbert space $\mathcal{H}(N,d)$ and denote their elements by a Greek letter, e.g., $|\Omega_{4,3}\rangle \in$ AME(4,3) is an AME of four qutrits.

LU transformations introduce equivalence classes of states. The question naturally arises which state should be chosen as the representative of the class, which is denoted the *canonical form* of an AME state. It is possible to argue in two directions. On one hand, we may consider that a natural representative may carry all the elements of the computational basis. It would then be necessary to establish theorems and a criterion to fix the coefficients. On the other hand, an alternative possibility is to choose the element of the class with minimal support on a computational basis. Results in both directions are presented in Sec. IV.

It is not known in general how many LU classes there are in the set AME(N,d) for every N and d. This question can be tackled by the construction of LU invariants. A few examples are at hand for few qubits. For three qubits, it is known how to obtain a canonical form of any state using LU and that all states are classified by five invariants [26]; only one of

them is genuinely multipartite, the tangle. For four qubits, there are ways to construct a canonical form and to find the hyperdeterminant as well. Yet, it is unknown how to proceed to larger local dimensions and numbers of parties. It is arguable that the subset of AME states is characterized by several LU invariants, probably related to distinct physical tasks. In this case, there would be different AME states not related to each other by LU.

C. AME and holography

Quantum holography amounts to the fact that the information content of a quantum system is that of its boundary. It follows that the information present in the system is far less than the maximum allowed. Degrees of freedom in the bulk will not carry maximal correlations, nor will the von Neumann entropy of any subpart of the system scale as its volume.

To gain insight into quantum holography, it is natural to investigate the bulk and boundary correspondence of the operator content of the theory [27]. On the other hand, quantum information brings a new point of view on this issue, since it focuses on the properties of states rather than on the dynamics that generates them. In this novel context, we may ask, What is the structure of quantum states that display holographic properties? That is, we aim to find which is the detailed entanglement scaffolding that guarantees that information flows from the boundary to the bulk of a system in a perfect way.

A concept separate from holography turns out to be very useful to address the analysis of holography from this new quantum information perspective, that is, tensor networks of the kind of matrix product states, projected entangled pair states, and the multiscale entanglement renormalization ansatz. Indeed, tensor networks provide a frame in which to analyze how correlations get distributed in quantum states and, thus, to understand holography at the level of quantum states. Each connection among ancillary indices quantifies the amount of entanglement which links parts of the system. Holography must necessarily rely on some very peculiar entanglement structure [28].

The first attempt to understand the basic property behind holography of quantum states was presented in [24]. There, it was proposed to create a quantum state on a triangular lattice based on a tensor network that uses as ancillary states an AME state. To be precise, the state $|\Omega\rangle \in \mathrm{AME}(4,3)$ [see Eq. (11)] was defined on tetrahedrons, in such a way that the vertices in its basis connect the tensor network and the tip of the tetrahedron corresponds to a physical index.

In a subsequent work [25], another construction was based on the five-qubit, $|\Upsilon_{5,2}\rangle \in \mathrm{AME}(5,2)$, state 8. Again, the fact that the internal construction of the state is based on isometries is at the heart of the holographic property.

There are two obvious observations on the surprising relation between AME states and holography. The first is related to the natural link between AME states and error correction codes. It is arguable, then, that the essence of holography is error correction, which limits the amount of information in the system. The second is that the very property responsible for holography is multiunitarity, which is analyzed in depth in Sec. VI. It is further arguable that multiunitarity is the building block of symmetries, since the sense of direction is lost and can be defined at will. These ideas deserve a much deeper analysis.

D. Related definitions

1. *Maximally entangled sets*

In the context of the convertibility of states via local operations and classical communication (LOCC), multipartite entanglement is significantly different from the bipartite case. While in the bipartite case, there is a single ME state (up to local unitaries) that can be transformed into any other state by LOCC (and cannot be obtained from any other), in the multipartite scenario this is no longer true. In Ref. [29], the notion of the ME set of N-partite states is introduced as the set of states from which any state outside of it can be obtained via LOCC from one of the states within the set and no state in the set can be obtained from any other state via LOCC. Note that this notion of maximal entanglement is strictly weaker than the AME, in the sense that most (or all) states in the ME set will not be AME states, but any AME state will be in its corresponding ME set. In Ref. [29], the ME set is characterized for the case of three and four qubits. It is interesting to point out that, unlike the three-qubit case, deterministic LOCC transformations are almost never possible among fully entangled four-partite states. As a consequence, while the ME set is of measure 0 for three-qubit states, almost all states are in the four-qubit ME set. This suggests the following picture: given a fixed local dimension and for an increasing number of parties, AME states become more and more rare at the same time that more and more states need to be included in the ME set. In other words, while ME states defined from an operational point of view become typical when the number of parties increases, AME states are exotic.

The issue of interconvertibility between quantum states has been addressed under a larger set of operations than LOCC: stochastic local operations and classical communication (SLOCC). SLOCC identify states that can be interconverted by LOCC in a nondeterministic way, but with a nonzero probability of success. In this respect, a systematic classification of multipartite entanglement in terms of equivalence classes of states under SLOCC is presented in Ref. [30]. In particular, it is shown that the SLOCC equivalence class of multipartite states is characterized by ratios of homogeneous polynomials that are invariant under local action of the special linear group. This work generalizes for an arbitrary number of parties and finite local dimension the complete classification made for four qubits in Ref. [31].

2. *Multipartite maximally entangled states*

In Ref. [12], MME states are introduced as those states that maximize the average entanglement (measured in terms of purity) where the average is taken over all the balanced bipartitions, i.e., $|\mu| = \lfloor N/2 \rfloor$. More specifically, MME states are defined as the minimizers of the *potential of multipartite entanglement*,

$$\pi_{\mathrm{ME}} = \binom{N}{\lfloor N/2 \rfloor} \sum_{|\mu|=\lfloor N/2 \rfloor} \pi_\mu, \tag{6}$$

where $\pi_\mu = \mathrm{Tr}(\rho_{(\mu)}{}^2)$ is the purity of the partition μ. Note that the above potential is bounded by $1/d^{\lfloor N/2 \rfloor} \leqslant \pi_{\mathrm{ME}} \leqslant 1$ and its lower bound is only saturated by AME states.

By minimizing the multipartite entanglement potential, explicit examples of AME states of five and six qubits are presented in Ref. [12]. It is remarkable that even for a relatively small number of qubits ($N \geqslant 7$), this minimization problem has a landscape of a parameter space with a large number of local minima, which implies a very slow convergence. The reason for this is frustration. The condition that purity saturates its minimum can be satisfied for some but not all the bipartitions (see [13] for details). In this respect, in [32] the minimization of the multipartite entanglement potential is mapped into a classical statistical mechanics problem. The multipartite entanglement potential is seen there as a Hamiltonian which is minimized by simulated annealing techniques.

III. EXAMPLES OF AME STATES

A. Qubit AMEs

Let us consider states made out of qubits, that is, the dimension of the local Hilbert space is $d = 2$. The simplest cases of AME states are any of the Bell states. There is a unique partition of two qubits and it is possible to entangle both parties maximally. It is easy to argue that there is a unique quantity that describes the amount of entanglement in the system. This can be chosen to be the first eigenvalue of the Schmidt decomposition of the state or some other quantity derived from it such as the von Neumann entropy. All two-particle states, whatever their local dimension, can be entangled maximally. These states are of no interest for our present discussion, which is genuinely centered in multipartite entanglement.

In the case of three qubits the well-known GHZ state [33] is an AME, but it is known that there is no four-qubit AME [11]. The amount of degrees of freedom in the definition of the state is insufficient to fulfill all the constraints stemming from the requirement of maximum entanglement.

For five and six qubits, there are AME states. In particular, a five-qubit state, $|\Upsilon_{5,2}\rangle \in \mathrm{AME}(5,2)$, can be defined by the coefficients of the superposition of basis states that form it,

$$|\Upsilon_{5,2}\rangle = \frac{1}{2^{5/2}} \sum_{i=0}^{2^5-1} c_i^{(\Upsilon)} |i\rangle, \tag{7}$$

where we have used the usual shorthand notation for the elements in the computational basis and the coefficients have the same modulus and signs given by [12]

$$\begin{aligned} c^{(\Upsilon)} = \{&1,1,1,1,1,-1,-1,1,1,-1,-1,\\ &1,1,1,1,1,1,1,-1,-1,1,-1,1,\\ &-1,-1,1,-1,1,-1,-1,1,1\}. \end{aligned} \tag{8}$$

The state AME(5,2) found in [15] was proven to be useful for a number of multipartite tasks in Ref. [34]. It can also be found as the superposition of a perfect error correcting code as presented in Ref. [35] as well as from OAs [17].

For the sake of completeness, let us also provide an AME six-qubit state,

$$|\Xi_{6,2}\rangle = \frac{1}{\sqrt{2}^6} \sum_{i=0}^{2^6-1} c_i^{(\Xi)} |i\rangle, \tag{9}$$

with

$$\begin{aligned} c^{(\Xi)} = \{&-1,-1,-1,+1,-1,1,1,1,\\ &-1,-1,-1,1,1,-1,-1,-1,\\ &-1,-1,1,-1,-1,1,-1,-1,\\ &1,1,-1,1,-1,1,-1,-1,\\ &-1,1,-1,-1,-1,-1,1,-1,\\ &1,-1,1,1,-1,-1,1,-1,\\ &1,-1,-1,-1,1,1,1,-1,\\ &1,-1,-1,-1,-1,-1,-1,1\}. \end{aligned} \tag{10}$$

The case of seven qubits remains unsolved but numerical evidence hints at the impossibility of finding such an AME state [36]. For eight qubits or more, there are no AME states [16].

B. A central example: AME(4,3)

The first nontrivial example of AME for larger local dimensions corresponds to a state made of four qutrits, AME(4,3). Its explicit construction is

$$|\Omega_{4,3}\rangle = \frac{1}{3} \sum_{i,j,=0,1,2} |i\rangle|j\rangle|i+j\rangle|i+2j\rangle, \tag{11}$$

where all qutrit indices are computed mod(3). It is easy to verify that all the reduced density matrices to two qutrits are equal to $\rho = \frac{1}{9} I_9$, so that this state carries entropy $S = 2 \log 3$ for every one of its bipartitions.

The state $|\Omega_{4,3}\rangle$ can be viewed as a map of a two-qutrit product basis into a second one. That is,

$$|\Omega_{4,3}\rangle = \sum_{i=0}^{8} |u_i\rangle|v_i\rangle = \sum_{i,j=0}^{8} |u_i\rangle U_{ij} |u_j\rangle, \tag{12}$$

where $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ are product bases for two qutrits, and $|v_i\rangle = U_{ij}|u_j\rangle$. The matrix U_{ij} is not only unitary (as it must be, as a consequence of multiunitarity) but also a bijective map between the sets of words of length 2 over an alphabet $\mathbb{Z}_3 = \{0,1,2\}$. In other words, the entries of U_{ij} are 0, with a single 1 per row or column. This property remains the same whatever partition is analyzed, though the unitary will vary.

A second feature of the state $|\Omega_{4,3}\rangle$ is that the Hamming distance between any pair of elements in the state is three; $D_H = 3$. As all the sequences in $|\Omega_{4,3}\rangle$ differ in three elements, any single-qutrit error can be corrected. Hence, maximal entanglement is related to error correction codes. Both of these properties are related to the fact that $|\Omega_{4,3}\rangle$ is an AME of minimal support, a concept we are going to explore in Sec. IV.

It is also natural to expect AME states to accommodate easily to some magic-square-like relations. A simple example is as follows. Write the coefficients of $|\Omega_{4,3}\rangle$ as a matrix of row i and column j, giving the composed value a of the remaining

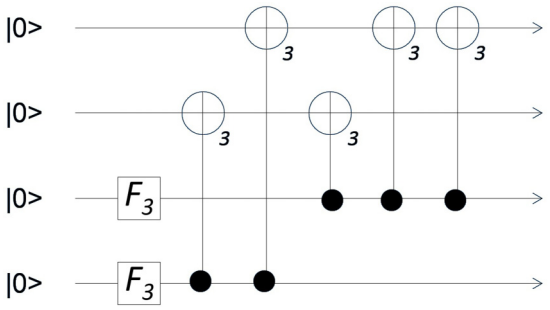

FIG. 1. (Color online) Quantum circuit required to generate the state AME $\Omega_{4,3}$ (4 qutrits) based on the Fourier gate F_3 and control-adders gate C_3-adder.

two qutrits from 0 to 8, that is, $a = 3(i + j)\text{mod}(3) + (i + 2j)\text{mod}(3)$. The square reads

$$\begin{array}{ccc} 0 & 5 & 7 \\ 4 & 6 & 2 \\ 8 & 1 & 3, \end{array}$$

where all rows and columns add up to 12. The same properties are maintain if we interchange the indices in the state. These kinds of combinatorial designs are going to be explored in Sec. V.

The state $|\Omega_{4,3}\rangle$ can be created by a quantum circuit composed by the following two gates:

$$\begin{aligned} \text{Fourier:} \quad & F_3|0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle). \\ C_3\text{-adder:} \quad & U_{C\text{-adder}}|i\rangle|j\rangle = |i\rangle|(i + j)\text{mod}3\rangle. \end{aligned} \tag{13}$$

The gate C_3 generalizes the CNOT gate for qubits and it is represented in the circuit using the usual symbol for CNOT with the subscript 3. The $|\Omega_{4,3}\rangle$ state can be constructed as a sequence of a Fourier F_3 and C_3-adders following the circuit depicted in Fig. 1 acting on the initial state $|0000\rangle$ of four qutrits.

C. General expression for AME states

States AME(5,2) and AME(6,2) have the maximal number of terms, whereas AME(4,3) has the minimal possible number. Therefore the following question arises: Is there a general expression for AME(N,d) having the maximal number of d^N terms? Facchi partially solved this issue for qubits in [20]. It can be shown that a state expressed as

$$|\Psi\rangle = \sum_{k=0}^{2^N-1} z_k|k\rangle, \quad z_k = r_k\xi_k, \tag{14}$$

where the $|k\rangle$ span the whole computational basis and r_k and ξ_k are, respectively, the modulus and the phase of the complex coefficients z_k, is an AME if it satisfies the equations

$$\sum_m r_{l\oplus_m r_{l'}\oplus_m \xi_{l\oplus_m \xi_{l'}\oplus_m}} = 0, \tag{15}$$

where l, l', and m stand for both parts of a certain balanced bipartition (of $\lfloor N/2 \rfloor$ particles) and $\bigoplus$ means sum modulo 2^N.

The general form of the squared modulus of the coefficients will be

$$r_k^2 = 2^{-N} + \sum_{\frac{N}{2}<n\leqslant N} \sum_{j\in S^n} c_j^{(n)} \prod_{1\leqslant h\leqslant n} \left(2k_h^{(j,n)} - 1\right), \tag{16}$$

where S^n stands for the set of bipartitions of the system into groups of n and $N - n$ particles and j is an index for the bipartitions of this set. h is an index for particles contained in the bipartition j, and $k_h^{(j,n)}$ stands for the value (0 or 1) of particle h in ket k, in the case of a certain bipartition j of set S^n. The real coefficients $c_j^{(n)}$ are free as long as they satisfy Eq. (15) and the normalization condition.

With this expressions all AMEs of a small number of qubits are classified. It is also noteworthy that it is possible to define AME states with maximal support and uniform amplitudes $r_k = 1/\sqrt{d^N}\ \ \forall k$ [by just setting all c_j to 0 in Eq. (16)] [20].

IV. AME STATES OF MINIMAL SUPPORT AND CLASSICAL CODES

In Ref. [37], a subclass of AME(N,d) states is shown to be constructed by means of classical maximum-distance-separable (MDS) codes. In this section we show that such a subclass corresponds to the set of AME states of minimal support and exploit these ideas to get conditions for their existence.

A. Support of AME states

From the explicit examples we have presented in Sec. III, AME states appear to need different numbers of elements to be written. For instance, AME(4,3) is made of the superposition of nine states, all weighted with the same coefficient. Yet AME(6,2) as written using the 64 basis states with coefficients either 1 or -1.

Let us define the *support* of a state $|\psi\rangle$ as the number of nonzero coefficients when $|\psi\rangle$ is written in the computational basis. The *support of a class* is defined as the support of the state inside the class with minimal support. Note that the support of a class defines, in turn, another equivalence class. Two states are support-equivalent if they belong to LU classes with equal support.

In this sense, it is interesting to point out that state AME(6,2), defined in Eq. (9) with the maximal support of 2^6 states, is LU-equivalent to a state of support 16 when some Hadamard gates on its basis are applied. It can be proven that 8 is the minimal support that this state could have but it is not attainable. Also, in [15] a state AME(5,2) is built using eight elements, while the theoretical minimum would be four. We may wonder why the naive minimum possible number of $2^{\lfloor N/2 \rfloor}$ elements is not always attained. This question is answered in Ref. [17], where a one-to-one relationship is proven between k-uniform states having minimal support and a kind of combinatorial arrangement known as the OA. Therefore, the nonexistence of such states having minimal support is due to the nonexistence of some classes of OAs (those having strength 1). In the following subsections we study in detail AME states having minimal support.

B. Equivalence between AME states of minimal support and MDS codes

An ME state in AME(N,d) belongs to the class of minimal support iff it is LU-equivalent to a state $|\Psi\rangle$ with support $d^{\lfloor N/2\rfloor}$, i.e.,

$$|\Psi\rangle = \frac{1}{\sqrt{d^{\lfloor N/2\rfloor}}} \sum_{k=1}^{d^{\lfloor N/2\rfloor}} r_k e^{i\theta_k} |x_k\rangle, \tag{17}$$

where $x_k \in \mathbb{Z}_d^N$ are words of length N over the alphabet $\mathbb{Z}_d = \{0,\ldots,d-1\}$, and $r_k > 0$ and $\theta_k \in [0,2\pi)$ are their modulus and phases, respectively.

Given a bipartition $A = \{a_1,\ldots,a_n\}$, it is useful to introduce the subword $x_k[A]$ of the word x_k for partition A as the concatenation of the a_1-th, a_2-th, $\ldots$,a_n-th letters of x_k; that is, $x_k[A] = x_k[a_1]x_k[a_2]\ldots x_k[a_n]$. Let us also denote by $X_\Psi = \{x_k, k = 1,\ldots,d^{\lfloor N/2\rfloor}\}$ the set of words which $|\Psi\rangle$ has support on, and by $X_\Psi[A] = \{x_k[A], k = 1,\ldots,d^{\lfloor N/2\rfloor}\}$ the set of all subwords $x_k[A]$ corresponding to bipartition A. With this notation, the reduced density matrix of partition A can be written as

$$\rho = \sum_{k'} \langle x_{k'}[\bar{A}]||\Psi\rangle\langle\Psi||x_{k'}[\bar{A}]\rangle, \tag{18}$$

where

$$|\Psi\rangle = \frac{1}{\sqrt{d^{\lfloor N/2\rfloor}}} \sum_{k=1}^{d^{\lfloor N/2\rfloor}} r_k e^{i\theta_k} |x_k[A]\rangle|x_k[\bar{A}]\rangle. \tag{19}$$

In order for $|\Psi\rangle$ to be an AME, the reduced density matrix of any bipartition $A = \{a_1,\ldots,a_{\lfloor N/2\rfloor}\}$ needs to be the completely mixed state. It is easy to see that this has the following implications:

(1) The modulus $r_k = 1$ for all k.

(2) The phases θ_k are arbitrary.

(3) For any balanced bipartition A, with $|A| = \lfloor N/2\rfloor$ and $|\bar{A}| = \lceil N/2\rceil$, two words $x_i, x_j \in X_\Psi$ have subwords $x_i[\bar{A}] = x_j[\bar{A}]$ if and only if $i = j$.

(4) For any balanced bipartition A, the set of subwords $X_\Psi[A]$ contains all the words of length $\lfloor N/2\rfloor$, i.e., $X_\Psi[A] = \mathbb{Z}_d^{\lfloor N/2\rfloor}$.

Conditions 3 and 4 imply that any pair of different words $x_i, x_j \in X_\Psi$ has *Hamming distance*

$$D_H(x_i,x_j) \geqslant \lceil N/2\rceil + 1, \tag{20}$$

where the Hamming distance between two code words is defined as the number of positions at which they differ, e.g., $d_H(00010,10000) = 2$. To see this, note that otherwise there would exist a balanced bipartition A' for which $x_i[A'] = x_j[A']$ for $i \neq j$, and consequently, the set $X_\Psi[A']$ would not contain all the possible words of length $\lfloor N/2\rfloor$, that is, $X_\Psi[A'] \subset \mathbb{Z}_d^{\lfloor N/2\rfloor}$.

A set of M words of length N over an alphabet of size d that differ pairwise by at least a Hamming distance δ is called a *classical code*. What we have shown above is that the existence of AME(N,d) states of minimal support implies the existence of classical codes of $M = |X_\Psi| = d^{\lfloor N/2\rfloor}$ words with Hamming distance $\delta = \lceil N/2\rceil + 1$. The codes produced by AME states are special in the sense that they saturate the *Singleton bound* [38],

$$M \leqslant d^{N-\delta+1}. \tag{21}$$

This type of code, which saturates the Singleton bound, is called an MDS code.

The converse statement is also true. That is, the existence of an MDS code also implies the existence of an AME state with minimal support [37]. The argument is the following: a code of $M = d^{\lfloor N/2\rfloor}$ words of length N and Hamming distance $\delta = \lceil N/2\rceil + 1$ has all its subwords associated with any balanced bipartition $\bar{A}$ of size $|\bar{A}| = \lceil N/2\rceil$ different, which implies condition 3 above. Thus, *AME states of minimal support are equivalent to classical MDS codes*.

This equivalence can be exploited to see that a necessary condition for the existence of AME(N,d) states with minimal support (and, equivalently, of MDS codes) is that the local dimension d and the number of parties $N > 3$ fulfill

$$d \geqslant \lceil N/2\rceil + 1. \tag{22}$$

We can prove it as follows: let us try to construct an AME state by building an MDS code. Due to the relabeling freedom, the first $d+1$ words of the code can be chosen as

$$d+1 \text{ code words} \begin{cases} 0 \quad \ldots\ldots\ldots 00, & 0\ldots\ldots\ldots 0, \\ 0 \quad \ldots\ldots\ldots 01, & 1\ldots\ldots\ldots 1, \\ \vdots \qquad\qquad\quad \vdots & \qquad\qquad\quad \vdots \\ 0 \quad \ldots 0(d-1), & (d-1)\ldots(d-1), \\ 0 \quad \ldots\ldots\ldots 10, & x_d[\lceil N/2\rceil]\ldots x_d[N], \end{cases}$$

where the letters $x_d[i]$, for $\lceil N/2\rceil \leqslant i \leqslant N$, are still unknown and every word is written in two subwords of lengths $\lfloor N/2\rfloor$ and $\lceil N/2\rceil$, respectively.

Note that (i) none of the unknown letters $x_d[i]$ can be 0 in order for the word x_d to have Hamming distance $\lceil N/2\rceil + 1$ with the first word $0\ldots00\ldots0$, and (ii) none of the unknown letters $x_d[i]$ can be repeated in order for x_d to have Hamming distance $\lceil N/2\rceil + 1$ from the other $d-1$ words. Therefore, if $\lceil N/2\rceil$ variables, $x_d[i]$, must take $\lceil N/2\rceil$ different values, and all of them must be different from 0, it is necessary to extend the alphabet, forcing that $d \geqslant \lceil N/2\rceil + 1$.

Interestingly, Eq. (22) forbids the existence of AME(N,2) states having minimal support for $N > 3$. However, this does not represent a proof that AME(4,2) does not exist. Also, this inequality is saturated for states AME(4,3) and AME(6,4). The existence of the cases AME(8,5) and AME(8,6) is still an

open question, whereas states AME(8,7) and AME(8,8) are known.

C. A less trivial example: AME(6,4)

An application of the above connection between AME states and MDS codes is the construction of AME states by exploring the set of all words and selecting those which differ in at least $\lfloor N/2 \rfloor + 1$ elements. For the case of AME(6,4), such a search gives a state with an equal superposition of the following entries:

$$\{000000,001111,002222,003333,010123,011032,$$
$$012301,013210,020231,021320,022013,023102,$$
$$030312,031203,032130,033021,100132,101023,$$
$$102310,103201,110011,111100,112233,113322,$$
$$120303,121212,122121,123030,130220,131331,$$
$$132002,133113,200213,201302,202031,203120,$$
$$210330,211221,212112,213003,220022,221133,$$
$$222200,223311,230101,231010,232323,233232,$$
$$300321,301230,302103,303012,310202,311313,$$
$$312020,313131,320110,321001,322332,323223,$$
$$330033,331122,332211,333300\}. \quad (23)$$

Such an AME state carries the minimum possible support.

D. Construction of AMEs with minimal support

Finding AME states by exploring the set of all words is highly inefficient and, in practice, becomes unfeasible for a relatively small number of parties. In this context, the Reed-Solomon codes [23] can be a useful tool to produce systematic construction of MDS codes and, equivalently, AME states.

Let us review here the particular case of d prime and $N = d + 1$. Let us refer to the elements of the superposition in the quantum states as words x_i and the word of a half-partition as u_i. The code words are obtained using the action of a generator G, $x_i = u_i \cdot G$. The problem is then reduced to fixing G. It can be shown that a family of valid generators is given by

$$G = \begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ g_0 & g_1 & \dots & g_{d-1} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ g_0^k & g_1^k & \dots & g_{d-1}^k & 1 \end{pmatrix}, \quad (24)$$

where $d \in$ prime, $N = d + 1$, and $k = \lfloor N/2 \rfloor$.

The case of AME(4,3) can be reobtained using

$$G = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}. \quad (25)$$

Another concrete example corresponds to $g_0 = 1, g_1 = 1, \dots g_6 = 6$, which corresponds to AME(8,7), with $N = 8$, $d =$ 7-dits. Then a total of 7^4 code words is obtained that differ by a minimum Hamming distance $d_H = 5$.

As the above construction can only be accomplished for d being prime and $N = d + 1$, it is interesting to address the question whether, given some AME(N,d), it is possible to construct another AME(N',d'). In this context, the following result can be useful, as it allows us to construct an AME(N',d) for any $N' < N$: if there exists an AME state with minimal support in $\mathcal{H}(N,d)$, with N being even, then there exist other AME states with minimal support in the Hilbert spaces $\mathcal{H}(N',d)$ for any $N' \leqslant N$.

To prove the latter statement, we consider separately the transitions $N \to N-1$ and $N-1 \to N-2$.

a. Transition $N \to N-1$: The existence of an AME state with minimal support implies the existence of a code of $d^{N/2}$ words of length N with Hamming distance $d_H \geqslant N/2+1$. Let us order the words in the code in increasing order and take the subset of the first $d^{N/2-1}$ words which start with 0. Note that by suppressing such a 0, we get a code of $d^{N/2-1}$ words of length $N-1$ with Hamming distance $N/2+1$, forming an AME.

b. Transition $N-1 \to N-2$: From the previous step, we are left with a code of $d^{N/2-1}$ words of length $N-1$ and Hamming distance $N/2+1$. Note that by suppressing an arbitrary letter from all the words of the code, one is left with a set of $d^{(N-2)/2}$ words of length $N-2$ with Hamming distance $N/2+1-1 = (N-2)/2+1$, which is an MDS code. By iterating the previous procedure, we obtain MDS codes (and AME states of minimal support) for any $2 \leqslant N' \leqslant N$. Equivalently, MDS codes are constructed by considering OAs of index unity [17].

Note that examples of this type of AME states have been introduced above. For $d = 2$ we have the GHZ state (support 2); for $d = 3$ it corresponds to the AME(4,3) state defined in Eq. (11), which has support 9; and for $d = 4$, the AME(5,4) state that is defined in Eq. (33), which has support 16.

E. Nonminimal support AMEs and perfect quantum error correcting codes

AME states are deeply related to classical error correction codes and compression [16,35,39]. This is somewhat intuitive since maximal entropy is related to maximally mixed subsets. The measure of any local degree of freedom delivers an output which is completely random. This is, in turn, the basic element to correct errors. Hence, a relation between the elements superimposed to form an AME state and error corrections codes is expected.

Let us illustrate the connection of an AME(5,2) state with the well-known five-qubit code [35]. It is easy to see that by applying some Hadamard gates on local qubits, the AME(5,2) state, defined through the 32 coefficients given in Eq. (8), is LU-equivalent to a state with fewer nonzero coefficients. Actually, a representative of the same AME(5,2) class is found to have only eight coefficients. That state corresponds to a superposition of the two logical states in the error correcting codes found in [35],

$$|\Omega_{5,2}\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L), \quad (26)$$

where the logical qubits are defined as

$$|0\rangle_L = \tfrac{1}{2}(|00000\rangle + |00011\rangle + |01100\rangle - |01111\rangle),$$
$$|1\rangle_L = \tfrac{1}{2}(|11010\rangle + |11001\rangle + |10110\rangle - |10101\rangle).$$

Note the fact that the coefficients carry both plus and minus signs as is the case for nonminimal-support AMEs.

V. COMBINATORIAL DESIGNS AND *k* UNIFORMITY

Combinatorial designs are arrangements of elements satisfying some specific properties and so called *balanced* [40]. Such elements are restricted to a finite set, typically considered as subsets of integer numbers. Some remarkable examples are block designs, t-designs, orthogonal Latin squares, and OAs (see [41], and references therein). Combinatorial designs have important applications in quantum physics [42]. Indeed, a connection between genuinely MME states and OAs has recently been found [17]. Furthermore, they are a fundamental tool in the optimal design of experiments [43,44]. The existence of some combinatorial designs can be extremely difficult to prove. For example, the existence of Hadamard matrices in every dimension multiple of 4 (i.e., the *Hadamard conjecture*) has been debated since 1893 [45] and it represents one of the most important open problems in combinatorics.

Let us introduce some important definitions: A *Latin square* is a square arrangement of size d having d symbols ordered in such a way that every row and every column contain a different symbol [41]. Two Latin squares of the same size are *orthogonal* if, when superimposed, all the formed pairs of symbols are different. Additionally, M Latin squares are *mutually orthogonal* (MOLS) if they are pairwise orthogonal. Furthermore, an OA(r,N,d,k) is a square arrangement having N columns, r rows, and d symbols such that every combination of k symbols is repeated the same number of times along the rows. This number, λ, is called the index of the OA and it satisfies $\lambda = r/d^k$. Finally, an OA is *irredundant* if every collection of $N-k$ symbols of the OA is not repeated along the rows. For example, OAs of index unity, $\lambda = 1$, are irredundant [17].

A. Relation to mutually orthogonal Latin squares

Let us consider the explicit expression of the AME(4,3) state defined in Eq. (11):

$$\begin{aligned}|\Omega_{4,3}\rangle = \tfrac{1}{3}(&|0000\rangle + |0112\rangle + |0221\rangle \\ &+ |1011\rangle + |1120\rangle + |1202\rangle \\ &+ |2022\rangle + |2101\rangle + |2210\rangle).\end{aligned} \tag{27}$$

Here, the third and fourth symbols appearing in every term of the state can be arranged into two Greco-Latin squares of size 3:

$$\begin{array}{ccc} A\alpha & B\gamma & C\beta \\ B\beta & C\alpha & A\gamma \\ C\gamma & A\beta & B\alpha \end{array} = \begin{array}{ccc} A\spadesuit & K\clubsuit & Q\diamondsuit \\ K\diamondsuit & Q\spadesuit & A\clubsuit \\ Q\clubsuit & A\diamondsuit & K\spadesuit \end{array}, \tag{28}$$

where every symbol of the sets $\{A,\alpha\}$, $\{B,\beta\}$, and $\{C,\gamma\}$ is associated with 0, 1, and 2, respectively. Hence a pair of symbols may represent a card of a given rank and suit. Note that the first two digits in Eq. (27) may be interpreted as addresses determining the position of a symbol in the square. Furthermore, by considering four MOLSs [46] of size 5,

$$\begin{array}{ccccc} 0000 & 4321 & 3142 & 2413 & 1234, \\ 1111 & 0432 & 4203 & 3024 & 2340, \\ 2222 & 1043 & 0314 & 4130 & 3401, \\ 3333 & 2104 & 1420 & 0241 & 4012, \\ 4444 & 3210 & 2031 & 1302 & 0123, \end{array} \tag{29}$$

we define a 2-uniform state of six subsystems with five levels each:

$$\begin{aligned}|\Phi_5\rangle = \tfrac{1}{5}(&|000000\rangle + |104321\rangle + |203142\rangle + |302413\rangle \\ &+ |401234\rangle + |011111\rangle + |110432\rangle + |214203\rangle \\ &+ |313024\rangle + |412340\rangle + |022222\rangle + |121043\rangle \\ &+ |220314\rangle + |324130\rangle + |423401\rangle + |033333\rangle \\ &+ |132104\rangle + |231420\rangle + |330241\rangle + |434012\rangle \\ &+ |044444\rangle + |143210\rangle + |242031\rangle + |341302\rangle \\ &+ |440123\rangle).\end{aligned} \tag{30}$$

A state locally equivalent to $|\Phi_5\rangle$ has been found in [17] but its connection to MOLS is first given here. By considering the standard construction of maximal sets of $d-1$ MOLS of prime size d we can generalize the above construction for quantum states of a prime number of levels d and $N = d+1$ parties as follows:

$$|\Phi_{d+1,d}\rangle = \frac{1}{d}\sum_{i,j=0}^{d-1} |i,j\rangle \bigotimes_{m=1}^{d-1} |i+jm\rangle. \tag{31}$$

It is well known that a maximal set of $d-1$ MOLSs of size d exist for every prime power $d = p^m$ [47]. This means that the above general expression can be extended to the case of prime power level systems. For instance, the maximal set of three MOLSs of order $d = 4$ can be represented by a figure (color online);

$$\begin{array}{cccc} A\spadesuit & K\diamondsuit & Q\heartsuit & J\clubsuit \\ J\diamondsuit & Q\spadesuit & K\clubsuit & A\heartsuit \\ Q\clubsuit & J\heartsuit & A\diamondsuit & K\spadesuit \\ K\heartsuit & A\clubsuit & J\spadesuit & Q\diamondsuit \end{array}. \tag{32}$$

This design determines an AME(5,4) given by

$$\begin{aligned}|\Omega_{5,4}\rangle = &|00000\rangle + |10312\rangle + |20231\rangle + |30123\rangle \\ &+ |01111\rangle + |11203\rangle + |21320\rangle + |31032\rangle \\ &+ |02222\rangle + |12130\rangle + |22013\rangle + |32301\rangle \\ &+ |03333\rangle + |13021\rangle + |23102\rangle + |33210\rangle,\end{aligned}$$

where every symbol of the sets $\{A,\spadesuit,\text{blue}\}$, $\{J,\diamondsuit,\text{orange}\}$, $\{Q,\clubsuit,\text{green}\}$ and $\{K,\heartsuit,\text{red}\}$, is associated with 0, 1, 2, and 3, respectively, while the first two digits of every term label the position of a symbol in the pattern. In the above expression a normalization factor is required. Note that this state, or a state

equivalent with respect to LU transformations, arises from the Red-Solomon code of length 5 [48].

Furthermore, the construction can be extended to any dimension d in the following way:

$$|\Phi_{d+1,d}\rangle = \frac{1}{d}\sum_{i,j=0}^{d-1} |i,j\rangle \bigotimes_{m=1}^{\mathcal{N}(d)} |\lambda_m[i,j]\rangle, \tag{33}$$

where $\mathcal{N}(d)$ denotes the maximal number of MOLSs of size d. Here $\lambda_m[ij]$ denotes the entries of the mth Latin square, so the above expression can be considered a direct generalization of Eq. (2). It is worth adding that, for dimensions $d \geqslant 12$ not equal to a prime power number, only lower bounds for the function $\mathcal{N}(d)$ are known [46]. The problem is solved only for smaller dimensions, as $\mathcal{N}(6) = 1$, in agreement with the unsolvability of the famous Euler problem of 36 officers, while $\mathcal{N}(10) = 2$ (see [47], where the explicit form of a pair of MOLSs of size 10 is derived). Thus for $d = 10$, expression (33) describes a 2-uniform state of four subsystems with 10 levels each.

In general, the problem of constructing $N-2$ MOLSs of size d is equivalent to constructing a 2-uniform state of N subsystems of d levels each having d^2 positive terms. This comes from a fundamental property of MOLSs: the existence of $N-2$ MOLSs of size d based on d different symbols is equivalent to an OA$(d^2,N,d,2)$ which has index unity [41]. In other words, any subset of 2 symbols occurs the same number of times along the rows. In our previous work, we have proven that this kind of OA produces 2-uniform states of N qudits [17].

OAs of index unity have the form OA(d^k,N,d,k) and they are one-to-one connected to k-uniform states of N qudits which have the minimal number of d^k terms [17]. The fact that every reduction to k qudits is maximally mixed relies on the fact that every combination of k symbols of the OA makes only a single appearance along the rows. This implies that every reduction to k qudits of the state has a constant diagonal. Additionally, the fact that every set of $N-k$ symbols is never repeated along the rows implies that every reduction to k qudits is a diagonal matrix. In the case of $k = \lfloor N/2 \rfloor$ the state is AME(N,d) by definition. If the number of terms of the state is lower than d^k, then at least one diagonal entry of every reduction to k qudits is 0, which contradicts the definition of the k-uniform state.

B. AME states and hypercubes

In the previous section we considered maximal sets of MOLSs to construct 2-uniform states of qudits. However, this construction is not useful to find AME states for $d > 4$. The aim of this section is to consider combinatorial arrangements for constructing AME states in such cases. The main result is inspired in a generalization of the AME(4,3) state $|\Omega\rangle$ given in Eq. (27) and the AME state of six ququarts presented in Eq. (23). In Ref. [17] it was shown that this state can be derived from the irredundant OA IrOA(64,6,4,3). Furthermore, this OA can be interpreted as a set of three mutually orthogonal Latin cubes (see Fig. 2). Also note that the AME(4,3) state $|\Omega\rangle$ arises from IrOA(9,4,3,2) (see Eq. (B1) in Ref. [17]). Thus, if k mutually orthogonal hypercubes of dimension k having

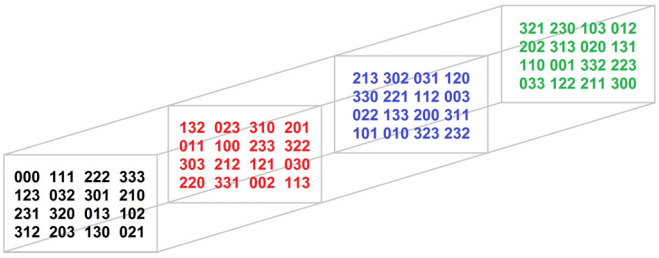

FIG. 2. (Color online) Three mutually orthogonal Latin cubes of dimension 3 and size 4. This arrangement allows us to generate a state AME(6,4) of six ququarts. Each of the 12 planes (4 horizontal, 4 vertical, and 4 oblique) contains a set of three MOLSs of size 4.

$k+1$ symbols exist, they are one-to-one connected with IrOA$((k+1)^k,2k,k+1,k)$, and therefore, it would produce an AME$(2k,k+1)$ state. This family of states would saturate the bound

$$d \geqslant \frac{N}{2} + 1, \tag{34}$$

already defined in Eq. (22) Note that for $k = 1$ one obtains the standard Bell state, $|\Phi_1\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ (1 Latin square of size 2 with 2 symbols). Taking $k = 2$ and using two MOLSs of size 3 and three symbols, we arrive at the AME(4,3) state $|\Omega_{4,3}\rangle$ of Eq. (11), which, from this point of view, can be considered a generalization of the Bell state. Furthermore, the state (23), corresponding to $k = 3$ also belongs to this family and it is associated with three mutually orthogonal Latin cubes of dimension 3 and size 4. It is interesting to check whether there exist other states with $k \geqslant 5$ belonging to this family.

VI. AME STATES AND MULTIUNITARY MATRICES

A. Unitary matrices and bipartite systems

Let us illustrate the connection between unitary matrices and AME states for the simplest case of two qubits. Let us assume that the state of the system is given by

$$|\phi\rangle = \frac{1}{\sqrt{2}}(U_{0,0}|00\rangle + U_{0,1}|01\rangle + U_{1,0}|10\rangle + U_{1,1}|11\rangle), \tag{35}$$

where $\rho^{(A)} = \frac{1}{2}UU^\dagger$, $\rho^{(B)} = \frac{1}{2}(U^T)(U^T)^\dagger$, and the superscript T denotes transposition. Any unitary matrix U of size 2 represents a Bell-like state. Furthermore, the Pauli set of four unitary matrices $\mathcal{U} = \{\mathbb{I},\sigma_x,\sigma_y,\sigma_z\}$, orthogonal in the sense of the Hilbert-Schmidt product, defines the ME Bell basis in $\mathcal{H}_2 \otimes \mathcal{H}_2$.

B. Multiunitary for AME(4,3)

We consider again the AME(4,3) state of four qutrits $|\Omega_{4,3}\rangle$ and represent its coefficients by a four-index tensor,

$$|\Omega_{4,3}\rangle = \sum_{i,j,k,l=0,1,2} t_{ijkl}\,|ijkl\rangle, \tag{36}$$

where the entries of the tensor t can be expressed as the product of the Kronecker delta functions

$$t_{ijkl} = \tfrac{1}{9}\delta_{k,i+j}\delta_{l,i+2j}. \tag{37}$$

Here, the addition operations are modulo 3. As discussed in the previous section, all nonzero coefficients are equal. The tensor t_{ijkl} consists of $3^4 = 81$ elements, which can be reshaped to form a square matrix of order 9. Note that there exist altogether $\binom{4}{2} = 6$ different ways of choosing a bipartition of the indices and forming a matrix $U_{\mu,\nu}$. That is,

$$(\mu,\nu) = \begin{cases} (i+2j,k+2l),(k+2l,i+2j), \\ (i+2k,j+2l),(j+2l,i+2k), \\ (i+2l,j+2k),(j+2k,i+2l). \end{cases} \tag{38}$$

The nontrivial property of an AME(4,3) tensor is that these six matrices are unitary. As the transposition of a unitary matrix remains unitary, it is sufficient, in this case, to check unitarity for the three cases appearing in the first column on the right-hand side of Eq. (38). That is, taking combined indices in the original tensor,

$$t_{ijkl} = U^{(1)}_{(ij)(kl)} = U^{(2)}_{(ik)(jl)} = U^{(3)}_{(ij)(kl)}, \tag{39}$$

absolute maximal entanglement is achieved if the matrices $U^{(1)}$, $U^{(2)}$, and $U^{(3)}$ correspond to different changes of bases, that is, unitary matrices.

We refer to this particular kind of unitary matrices as *multiunitary*. See Appendix B for further explanation of the reorderings of elements in matrices of square size $D \geqslant 4$.

C. General multiunitarity

Let us consider a more general case of pure states of N subsystems with d levels each. That is,

$$|\phi\rangle = \sum_{s_0,\ldots,s_{N-1}=0}^{d-1} t_{s_0,\ldots,s_{N-1}} |s_0,\ldots,s_{N-1}\rangle. \tag{40}$$

Let us assume here that the number of subsystems is even, $N = 2k$, so there exist $M = \binom{2k}{k}$ possible splittings of the system into two parts of the same size.

A necessary condition for $|\phi\rangle$ to be an AME state is that the tensor t with $2k$ indices, reshaped into a square matrix of size d^k, forms a unitary matrix U. This is so, as the reduction associated with the first k qudits, given by $\rho^{(k)} = UU^\dagger$, should be proportional to the identity. To arrive at an AME$(2k,k)$ state, similar conditions have to hold for all M different square matrices obtained from the tensor t by all possible ways of reshaping its entries into a square matrix. This observation provides clear motivation to introduce the notion of multiunitarity: A square matrix A of order d^k ($k \geqslant 2$), acting on a composed Hilbert space $\mathcal{H}_d^{\otimes k}$ and represented in a product basis by $A_{\substack{n_1,\ldots,n_k \\ \nu_1,\ldots,\nu_k}} := \langle n_1,\ldots,n_k|A|\nu_1,\ldots,\nu_k\rangle$, is k-unitary if it is unitary for $M = \binom{2k}{k}$ reorderings of its entries, corresponding to all possible choices of k indices out of $2k$.

In this way, we can establish the one-to-one connections

$$\text{AME}(2,d) \equiv \text{ unitary of order } d$$

for bipartite systems of two qudits having d levels each, and in general,

$$\text{AME}(2k,d) \equiv \ k\text{-unitary of order } d^k$$

for multipartite systems of $N = 2k$ qudits having d levels each. By construction, 1-unitarity reduces to standard unitarity. Any k-unitary matrix with $k > 1$ is called multiunitary. It is well-known that unitarity of matrices is invariant under multiplication. Multiunitarity imposes more restrictions on a given matrix U than unitarity. Therefore, the product of two multiunitary matrices in general is not multiunitary. For instance, the matrix O_8 [see Eq. (41)] is Hermitian and 3-unitary, but $O_8^2 = \mathbb{I}$ is only 1-unitary, as it represents a six-qubit quantum state equivalent to GHZ.

Similarly, the case of the AME(4,3) state $|\Omega_{4,3}\rangle$ reduces to analyzing the properties of the tensor t_{ijkl} in Eq. (36) and verifying the multiunitarity of U. Indeed, for this state we have $U = \text{Perm}(0,5,7,4,6,2,8,1,3)$, $U^{T_2} = \text{Perm}(0,5,7,1,3,8,2,4,6)$, and $U^R = \text{Perm}(0,2,1,4,3,5,8,7,6)$, where T_2 and R mean partial transposition and reshuffling (see Appendix B for further details). Here, Perm denotes a permutation matrix. A minicatalog of all the multiunitary matrices defined in this work is given in Appendix D.

D. AME and Hadamard matrices

For six qubits, the AME(6,2) state $|\Xi_{6,2}\rangle$ of Eq. (10) having a maximal number of terms arises from graph states [49]. Let us write it explicitly:

$$\begin{aligned}
|\Xi_{6,2}\rangle = \tfrac{1}{8}(&-|000000\rangle - |000001\rangle - |000010\rangle + |000011\rangle - |000100\rangle + |000101\rangle + |000110\rangle + |000111\rangle \\
&- |001000\rangle - |001001\rangle - |001010\rangle + |001011\rangle + |001100\rangle - |001101\rangle - |001110\rangle - |001111\rangle \\
&- |010000\rangle - |010001\rangle + |010010\rangle - |010011\rangle - |010100\rangle + |010101\rangle - |010110\rangle - |010111\rangle \\
&+ |011000\rangle + |011001\rangle - |011010\rangle + |011011\rangle - |011100\rangle + |011101\rangle - |011110\rangle - |011111\rangle \\
&- |100000\rangle + |100001\rangle - |100010\rangle - |100011\rangle - |100100\rangle - |100101\rangle + |100110\rangle - |100111\rangle \\
&+ |101000\rangle - |101001\rangle + |101010\rangle + |101011\rangle - |101100\rangle - |101101\rangle + |101110\rangle - |101111\rangle \\
&+ |110000\rangle - |110001\rangle - |110010\rangle - |110011\rangle + |110100\rangle + |110101\rangle + |110110\rangle - |110111\rangle \\
&+ |111000\rangle - |111001\rangle - |111010\rangle - |111011\rangle - |111100\rangle - |111101\rangle - |111110\rangle + |111111\rangle).
\end{aligned}$$

This state leads to the following orthogonal matrix of order $D = 2^3 = 8$ which is 3-unitary:

$$O_8 = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}. \tag{41}$$

Note that the entries of $|\Xi_{6,2}\rangle$ are given by the concatenation of the rows of O_8, up to normalization. This matrix is symmetric and equivalent up to enphasing and permutations [50] to the symmetric Hadamard matrix $H_8 = H_2^{\otimes 3}$.

Note that O_8 is 3-unitary but $H_2^{\otimes 3}$ is not, so permutation or enphasing of a unitary matrix can spoil its multiunitarity. Moreover, from the concatenation of the rows of $H_2^{\otimes 3}$ we generate only a 1-uniform state, which means that $H_2^{\otimes 3}$ is only 1-unitary (i.e., unitary).

We conjecture that for any AME state one can choose suitable LU operations such that it is related to a multiunitary complex Hadamard matrix. It represents an ME state with the maximal number of terms having all entries of the same amplitude. For example, AME states arising from coding theory [16,23] and graph states [49] are of this form. We recall that k-uniform states of N qudits with d levels having the minimum number of terms (d^k) are closely related to linear MDS codes (see Sec. 4.3 of Ref. [41]) and also to OAs of index unity [17]. This reasoning implies that any pure state of N subsystems with d levels each having at least one reduction to $N/2$ qudits maximally mixed can have all its entries of the form $\pm d^{-N/2}$ if d is even.

E. Further constructions of AME states

Note that the state AME(4,3) $|\Omega_{4,3}\rangle$ is not equivalent with respect to local unitaries to a real state having all its entries of the form $\pm 3^{-2}$. This is a consequence of the fact that a real Hadamard matrix of size 9 does not exist. However, the state $|\Omega_{4,3}\rangle$ is equivalent under LU operations to the state (See Eq. (3) in Ref. [51])

$$|\Omega'_{4,3}\rangle = \frac{1}{9} \sum_{i,j,k,l=0}^{2} \omega^{j(i-k)+l(i+k)} |i,j,k,l\rangle, \tag{42}$$

where $\omega = e^{2\pi i/3}$. This state is associated with the following 2-unitary complex Hadamard matrix:

$$U_P = \tfrac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 1 & w & w^2 & 1 & w^2 & w \\ 1 & 1 & 1 & w^2 & 1 & w & w & 1 & w^2 \\ 1 & 1 & 1 & w & w^2 & 1 & w^2 & w & 1 \\ 1 & w & w^2 & 1 & w^2 & w & 1 & 1 & 1 \\ w & w^2 & 1 & 1 & w^2 & w & w^2 & w^2 & w^2 \\ w^2 & 1 & w & 1 & w^2 & w & w & w & w \\ 1 & w^2 & w & 1 & 1 & 1 & 1 & w & w^2 \\ w^2 & w & 1 & w & w & w & 1 & w & w^2 \\ w & 1 & w^2 & w^2 & w^2 & w^2 & 1 & w & w^2 \end{pmatrix}. \tag{43}$$

Interestingly, every integer power $(U_P)^m$ is a complex Hadamard matrix for m nondivisible by 4 and $(U_P)^8 = \mathbb{I}$. Moreover, U_P is equivalent to the tensor product of Fourier matrices $F_3 \otimes F_3^\dagger$, that is,

$$U_P = \mathcal{D} F_3 \otimes F_3^\dagger \mathcal{P} \mathcal{D}, \tag{44}$$

where $\mathcal{D} = \mathrm{Diag}(1,1,1,1,\omega,\omega^2,1,\omega^2,\omega)$ is a diagonal unitary matrix, while $\mathcal{P}$ is a permutation matrix which changes the order of the columns from $\{1,\ldots,9\}$ to $\{1,4,7,2,5,8,3,6,9\}$.

In order to construct a 2-unitary matrix one has to take a unitary U such that its partially transpose U^{T_2} and the reshuffled matrix U^R are unitary (see Appendix B). In the case of a matrix U of size $D = 3^2$ this implies that the set of nine 3×3 unitary matrices appearing in the 3×3 blocks of Eq. (43) define an orthogonal basis for the Hilbert-Schmidt product. It is thus possible to obtain AME states by considering orthogonal bases of unitary operators. For instance, one can construct the $|\Omega_{4,3}\rangle$ state from the following matrix:

$$U'_P = \frac{1}{\sqrt{3}} \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \omega & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \omega^2 & 0 & 0 & \omega \\ \hline 0 & 0 & 1 & 0 & 0 & \omega & 0 & 0 & \omega^2 \\ 1 & 0 & 0 & \omega^2 & 0 & 0 & \omega & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & \omega^2 & 0 & 0 & \omega & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & \omega & 0 & 0 & \omega^2 & 0 & 0 \end{array}\right). \tag{45}$$

We applied here the orthogonal basis defined by the displacement operators of size $d = 3$,

$$D_{p_1,p_2} = \tau^{p_1 p_2} X^{p_1} Z^{p_2}, \tag{46}$$

where $p = (p_1, p_2) \in \mathbb{Z}_d^2$, $\tau = -e^{\pi i/d}$, $\omega = e^{2\pi i/d}$, $X|k\rangle = |k+1\rangle$, and $Z|k\rangle = \omega^k |k\rangle$. These operators define the discrete Weyl-Heisenberg group. This approach can be easily generalized to any prime $d > 2$. Indeed, for d prime every reordering of indices leads us to the same matrix up to permutation of columns and rows, and therefore it remains unitary. This shows a construction of AME(4, d) working for any prime number of levels d. Moreover, it is likely that this construction can be generalized for prime powers d by considering the theory of Galois fields. Observe that the above construction is essentially different from the construction of AME states used in coding theory. Indeed, the tensor products of N displacement operators bases of size d, i.e., the set $\{D_{p_1^1,p_2^1} \otimes \ldots \otimes D_{p_1^N,p_2^N}\}$, produce codes and states AME(N, d) [16].

VII. CONCLUSIONS AND OUTLOOK

In this work we have analyzed some new properties of Absolutely Maximally Entangled (AME) states in multipartite systems and studied their connections with other areas of mathematics such as classical error correction codes and combinatorial designs. More specifically, we have reviewed and extended methods for constructing AME states and

explored their relation to the field of combinatorial designs. For instance, a state AME(4,3) consisting of four ME qutrits is linked to the set of two mutually orthogonal Latin squares of order 3, while a state AME(6,4) made of six ququarts is related to the set of three mutually orthogonal Latin cubes of order 4. Furthermore, we introduced the notion of k-unitary matrices of order $D = d^k$ and demonstrated that they are one-to-one connected to AME($N = 2k$, d) states of N qudits having d levels each. Such matrices are, by definition, unitary after $M = \binom{2k}{k}$ specific rearrangements of its entries. By considering such matrices we have proven the existence of AME$(4,p)$ states for every prime $p > 2$ and this also implies the existence of AME$(3,p)$ states. We believe that this construction can be generalized to any prime power number of levels $d = p^r$ by considering Galois fields. We remark that k-unitary matrices are at the core of the use of AME states in holography [24,25].

We have seen that AME states of minimal support are in one-to-one correspondence with the classical MDS codes. In this sense, we have realized that all the systematic constructions found in the literature of AME states lie on the Reed-Solomon codes to generate MDS codes [37] which are valid only for prime d and $N = d + 1$. We have shown that these methods can be extended to any $N \leqslant d + 1$. The existence of systematic constructions of AME states beyond prime d and of nonminimal support AMEs remains open to question.

Let us, finally, mention some other open questions:

a. *Seven qubits*. It is still unknown whether there exists an AME state of seven qubits.

b. *Classification of AME and LU invariants*. It is not known whether there is a clear-cut classification of AME states which is related to LU invariants. The fact that some AME states carry different minimal supports or that some AME states are immediately related to Reed-Solomon codes hints at some unknown structure among AME states.

c. *Computation of invariants*. LU invariants grow exponentially with the size of the number of parties. An example is is the hyperdeterminant, which has only been computed up to four qubits. There are no computations of hyperdeterminants of four qutrits. Do AME states carry maximum values for some LU invariant? From the general theory of hyperdeterminants, we know that the rank of the tensor defining a four-qutrit state is 1269, which seems out of reach for any practical computation.

d. *Bell inequalities.* It is still not clear whether AME states are the best states to maximally violate multipartite Bell inequalities. A good example to start with is AME(4,3).

The small corner of the Hilbert space formed by AME states remains mainly unexplored and may be more complex than expected.

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APPENDIX A: FOUR-QUBIT ENTANGLED STATES

For completeness we present in this Appendix a discussion of ME states of four qubits. To see that no AME(4,2) states exist it is sufficient to analyze the purity of reduced density matrices. The purity $\mathrm{Tr}\rho^2$ serves as a measure of the degree of mixedness of the density matrix ρ, but also as a measure of entanglement of the initially pure state reduced to ρ by a partial trace.

The same argument works for multipartite systems. Let us denote the qubits A, B, C, and D. The purity of the bipartition AB of a state $|\psi\rangle$ is given by $\mathrm{Tr}(\rho_{AB}^2)$, with $\rho_{AB} = \mathrm{Tr}_{CD}|\psi\rangle\langle\psi|$ being the reduced density matrix of AB. The theoretical minimum for this quantity is $1/N$, so $1/4$ in the case of four qubits. However, one cannot attain this minimum for the three bipartitions at the same time, as proven analytically in Ref. [11]. The best one can have is $1/3$ in all bipartitions. One state that accomplishes this is given in that same paper:

$$|\mathrm{HS}\rangle = \frac{1}{\sqrt{6}}[|0011\rangle + |1100\rangle + \omega|0101\rangle + \omega|1010\rangle + \omega^2(|0110\rangle + |1001\rangle)], \quad \text{(A1)}$$

where $w = \exp(\frac{2i\pi}{3})$. This state also has the maximum entropy of entanglement. Another state that has minimum purity in all bipartitions is

$$|\mathrm{HD}\rangle = \frac{1}{\sqrt{6}}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle + \sqrt{2}|1111\rangle). \quad \text{(A2)}$$

This state was found by the authors and in Ref. [52] also to carry the maximum hyperdeterminant, an extension of the concept of determinant to higher dimensions [53], and an interesting entanglement measure that is intrinsically multipartite. An equivalent state appeared in Ref. [10] as an example of a "symmetric maximally entangled state" of four qubits.

Let us note that AME states are defined through their entanglement properties and thus can be transformed into any equivalent state under local unitaries. Gour and Wallach [54] found $|L\rangle$ and $|M\rangle$ states while searching, respectively, for the states that maximize the average Tsallis α entropy of entanglement for $\alpha > 2$ and for $0 < \alpha < 2$:

$$|L\rangle = \frac{1}{\sqrt{12}}((1+\omega)(|0000\rangle + |1111\rangle) + (1-\omega)(|0011\rangle + |1100\rangle) + \omega^2(|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle)), \quad \text{(A3)}$$

$$|M\rangle = \frac{1}{\sqrt{2}}\left(\left(\frac{i}{2} + \frac{1}{\sqrt{12}}\right)(|0000\rangle + |1111\rangle) + \left(\frac{i}{2} - \frac{1}{\sqrt{12}}\right)(|0011\rangle + |1100\rangle) + \frac{1}{\sqrt{3}}(|0101\rangle + |1010\rangle)\right), \quad \text{(A4)}$$

where $\omega = e^{2\pi i/3}$. Remarkably, states $|L\rangle$ and $|M\rangle$ can be transformed by SLOCC into states $|\mathrm{HD}\rangle$ and $|\mathrm{HS}\rangle$ respectively. They are called SLOCC equivalents.

In fact Verstraete *et al.* gave a classification of all pure four-qubit states in nine SLOCC-inequivalent classes [31]. The most important class is called the generic class and is presented by Gour and Wallach [54] in the following compact form:

$$\mathcal{G} \equiv \{z_0|\phi^+\rangle|\phi^+\rangle + z_1|\phi^-\rangle|\phi^-\rangle + z_2|\psi^+\rangle|\psi^+\rangle + z_3|\psi^-\rangle|\psi^-\rangle | z_0,z_1,z_2,z_3 \in \mathbb{C}\}, \tag{A5}$$

where $|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$ are the Bell states.

The method to transform a state from the computational basis into the Bell basis (which means obtaining the values of the z_i) is explained in detail in the paper by Verstraete *et al.* [31]. Any two states that have the same four parameters z_0, z_1, z_2, and z_3 in this basis are SLOCC-equivalent. The SLOCC equivalence between state $|L\rangle$ and state $|\mathrm{HD}\rangle$ (as well as between state $|M\rangle$ and state $|\mathrm{HS}\rangle$) is proven by showing that they have precisely the same z_i coefficients.

Even though the AME(4,2) state does not exist it is interesting to note that four-qubit symmetric states having reductions maximally mixed in the *symmetric subspace* exist [55], and this holds for every bipartition.

APPENDIX B: PARTIAL TRANSPOSITION, RESHUFFLING, AND TWO-UNITARITY

For any matrix X of a square order $D = d^2$ represented in a product basis, $X_{\substack{mn\\ \mu,\nu}} := \langle m,n|X|\mu,\nu\rangle$, one defines [56] its partial transposition, $X^{T_2}_{\substack{mn\\ \mu\nu}} = X_{\substack{m\nu\\ \mu n}}$, and reshuffling, $X^{R}_{\substack{mn\\ \mu\nu}} = X_{\substack{m\mu\\ n\nu}}$. To get a better feeling for these particular reorderings of elements we consider a matrix X of order 4. Let us now switch to the standard, two-index notation and write its elements as X_{ij} with $i,j = 1,2,3,4$. Here, we have $\binom{4}{2} = 6$ different reorderings of two indices out of four. Two of these reorderings are particularly interesting: (a) the partially transposed matrix X^{T_2} is equivalent to the matrix with all four blocks of size 2 transposed, and (b) the reshuffled matrix X^R is obtained by taking lexicographically each 2×2 block of X, reshaping it into a vector of length 4, and putting into the reordered matrix X^R. That is,

$$X^{T_2} := \left[\begin{array}{cc|cc} X_{11} & \mathbf{X_{21}} & X_{13} & \mathbf{X_{23}} \\ \mathbf{X_{12}} & X_{22} & \mathbf{X_{14}} & X_{24} \\ \hline X_{31} & \mathbf{X_{42}} & X_{33} & \mathbf{X_{43}} \\ \mathbf{X_{24}} & X_{42} & \mathbf{X_{34}} & X_{44} \end{array}\right],$$
$$X^{R} := \left[\begin{array}{cc|cc} X_{11} & X_{12} & \mathbf{X_{21}} & \mathbf{X_{22}} \\ \mathbf{X_{13}} & \mathbf{X_{14}} & X_{23} & X_{24} \\ \hline X_{31} & X_{32} & \mathbf{X_{41}} & \mathbf{X_{42}} \\ \mathbf{X_{33}} & \mathbf{X_{34}} & X_{43} & X_{44} \end{array}\right]. \tag{B1}$$

Here, colored exchanged entries are set in boldface (colors are visible in the online version). The other three matrices are transpositions of U, U^{T_2}, and U^R. Therefore, a matrix U of size 4 is multiunitary if U, U^{T_2}, and U^R are unitary. The same restrictions hold for matrices of size d^2.

It is possible to demonstrate that 2-unitary matrices of size $D = 4$ do not exist [11]. The smallest 2-unitary matrix exists for order $D = 9$ [see Eq. (43)].

APPENDIX C: SYMMETRIC SUDOKU, GENERALIZED AME(4,3) STATES, AND TWO-UNITARY PERMUTATIONS

According to the usual sudoku rules all digits in a single row or column of the matrix or block of size 3 are different. Let us distinguish other sets of nine elements: *location*, which contains all digits from the same place in each block [e.g., nice centers of the blocks, set in red in matrix (C1) online]; *broken rows*, containing three rows of length 3 occurring in the same position of three blocks [blue example in matrix (C1) online]; and an analogous notion of *broken column*.

Note that the standard operation of matrix transpose, X^T, exchanges columns of the matrix with its rows. The partial transposition X^{T_2} exchanges rows with broken columns and columns with broken rows. Furthermore, the reshuffling operation, X^R, interchanges blocks with rows and columns with locations [see (B1)].

The following matrix shows an example of a *symmetric sudoku* pattern analyzed in [57]: all digits in each column, each row, each block, each location (red online), each broken row (blue online), and each broken column (boldface) are different.

$$S_9 := \left[\begin{array}{ccc|ccc|ccc} \mathbf{8} & 1 & 6 & \mathbf{2} & 4 & 9 & \mathbf{5} & 7 & 3 \\ \mathbf{3} & 5 & 7 & \mathbf{6} & 8 & 1 & \mathbf{9} & 2 & 4 \\ \mathbf{4} & 9 & 2 & \mathbf{7} & 3 & 5 & \mathbf{1} & 6 & 8 \\ \hline 7 & 3 & 5 & 1 & 6 & 8 & 4 & 9 & 2 \\ 2 & 4 & 9 & 5 & 7 & 3 & 8 & 1 & 6 \\ 6 & 8 & 1 & 9 & 2 & 4 & 3 & 5 & 7 \\ \hline 9 & 2 & 4 & 3 & 5 & 7 & 6 & 8 & 1 \\ 1 & 6 & 8 & 4 & 9 & 2 & 7 & 3 & 5 \\ 5 & 7 & 3 & 8 & 1 & 6 & 2 & 4 & 9 \end{array}\right]. \tag{C1}$$

Consider now a matrix $\mathcal{P}_1$ of size 9 with all entries equal to 0 except for nine elements placed in the positions of digit "1" in matrix (C1) equal to unity. As all nine digits in each location of S_9 are different, it is clear that this is a legitimate permutation matrix of size 9. As the address of each nonzero element can be interpreted as a pair of two ternary digits, it represents a Graeco-Latin square, (28), and determines the AME(4,3) state $|\Omega\rangle$ (27).

Thus permutation matrix $\mathcal{P}_1$ is 2-unitary, as its partial transpose and reshuffling remain unitary. The same property holds also for other permutation matrices $\mathcal{P}_m$ with $m = 2,\ldots,9$, obtained by placing nine 1s in the positions occupied by digits m in pattern (C1). The property of 2-unitarity is preserved if we enphase a permutation matrix by multiplying it by a diagonal unitary matrix $\mathcal{D}$ and take $\mathcal{P}'_m = \mathcal{D}\mathcal{P}_m$. Thus it is fair to say that the symmetric sudoku matrix, (C1), encodes nine families of enphased 2-unitary permutation matrices or families of AME(4,3) states.

APPENDIX D: MINICATALOG OF MULTIUNITARY MATRICES

Multiunitary matrices are defined for orders equal to powers of integers. As there are no 2-unitary matrices of size 4, the smallest interesting cases are $D = 8,9$ and $D = 16$. In general, multiunitary permutation matrices are one-to-one connected [17] to combinatorial arrangements called OAs of index unity [46]. Here, we present a summary of all the multiunitary matrices presented in this work and some additional ones.

(i) $D = 2^3 = 8$: The orthogonal matrix O_8 defined in (41) is 3-unitary. This matrix is equivalent with respect to permutations and enphasing to the three-qubit Hadamard matrix $H_8 = H_2^{\otimes 3}$ (which is 1-unitary). It is not possible to construct a 3-unitary permutation matrix of size $D = 8$ [17].

(ii) $D = 3^2 = 9$: There exist 2-unitary permutation matrices related to symmetric sudoku designs (see Appendix C). Also, permutation matrices defined at the end of Sec. VI B and the complex Hadamard matrix, (43), are 2-unitary. The latter is equivalent to the tensor product of two Fourier matrices $F_3 \otimes F_3$, but this product is 1-unitary only.

(iii) $D = 2^4 = 4^2 = 16$: In this case there are no 4-unitary matrices [16]. There exists a 2–unitary permutation matrix given by Perm(4,3,13,10,14,9,7,0,11,12,2,5,1,6,8,15) created from OA(16,4,4,2), which was generated using the Gendex Module NOA [58].

(iv) $D = p^2$ for a prime p: Although 2-unitary matrices of this size exist, they are neither complex Hadamard matrices (d^4 nonzero entries) nor permutations (d nonzero entries). Indeed they have p^2 nonzero entries which correspond to powers of the main root of the unity $e^{2\pi i/d}$.

(v) Construction: Let us assign (j,k) to the jth-row block of size p and the kth-column block of size p of U (note that U has d^2 blocks of size p). Each block (j,k) has to be filled with the displacement operator $D_{j,k}$ of size d. Displacement operators are defined in Eq. (46). The explicit U for $d = 3$ is provided in Eq. (45).

As we can see, for some dimensions there are no 2-unitary permutation matrices. Indeed, a 2-unitary matrix of size $d^{(d+1)/2}$ exists if and only if a projective plane of odd order d exists (see Theorem 8.43 in Ref. [41]). In particular, they exist for every prime (and odd prime power) d. For example, the existence of AME(4,3) states relies on the existence of a projective plane of order 3.

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