Essays in Financial Economics
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Abstract

This dissertation consist of three essays.

In the first essay, I examine optimal dynamic contracting between risk-averse investors and firm insiders in a dynamic general-equilibrium model with heterogeneous firms. The equilibrium optimal contract features a higher rate of inefficient liquidations in aggregate states with low productivity and a reward-for-luck policy in high productivity states. I show that with realistic parameters, moral hazard has a first-order quantitative effect on equilibrium dynamics of macro-economic quantities and asset prices. The conditional dynamics depend on the time-varying cross-sectional distribution of financial slack of firms in the economy. Moral hazard improves upon the predictions of the friction-less economy in three ways. First, the sensitivity of risk-premia and key macro-economic quantities rises after successive negative aggregate shocks. Accumulation of small shocks results in a disproportionately large decline in aggregate quantities, and a rise in risk-premia. Second, inefficiencies resulting from second-best contractual arrangements amplify the effect of primitive shocks and make the economy more sensitive to negative than to positive shocks. Third, controlling for current aggregate productivity, firm exit rates contain incremental information about future output growth.

In the second essay I quantitatively analyze the cross-sectional implications of optimal dynamic contracting between risk-averse investors and firm insiders in a dynamic general equilibrium model. I make two changes compared to essay one. First, I allow for firm-level investment. Second, I model firms to use a decreasing returns to scale technology instead of a linear one. My model makes two predictions on steady-state capital accumulation which are consistent with empirical evidence. First, I show that conditional on survival, younger firms are smaller and have higher expected growth rates. Second, investment rates in small firms are more volatile and more sensitive to realized cash flows than large firms. In the presence of aggregate shocks, my model predicts that the conditional aggregate dynamics of key macro-economic quantities depends on the cross-sectional distribution of firm-level investment rates. Controlling for aggregate productivity, states of the economy in which a higher fraction of firms have lower investment rates are characterized by low aggregate output, investment,
and consumption, and higher risk-premium. I provide quantitative estimates in a calibrated model.

In the third essay, we propose a simulation-based procedure for evaluating approximation accuracy of numerical solutions of general equilibrium models with heterogeneous agents. We measure the approximation accuracy by the magnitude of the loss suffered by the agents as a result of following sub-optimal policies. Our procedure allows agents to have knowledge of the future paths of the economy under suitably imposed costs of such foresight. This method is very general, straightforward to implement, and can be used in conjunction with various solution algorithms. We illustrate our method in the context of the incomplete-markets model of Krusell and Smith, where we apply it to two widely used approximation techniques: cross-sectional moment truncation and history truncation.

Thesis Supervisor: Leonid Kogan
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2.2 Unconditional Moments: This table reports unconditional moments of annual growth rates of consumption (\( \Delta C = C_{t+1}/C_t - 1 \)), output (\( \Delta Y = Y_{t+1}/Y_t - 1 \)), and investment (\( \Delta I = I_{t+1}/I_t - 1 \)). All data is from the period 1929 – 1998. The entries for \( \Delta C, \Delta C/\Delta Y \), and asset pricing moments are from Table 2 of Kaltenbrunner and Lochstoer (2010). Investment data is from BEA. The exit rate data is the issuer-weighted corporate default rate from Moody's over the same period. Model results are generated by 1000 independent panels of \( N = 10^5 \) firms over a time period of 300 quarters. Results are averages across independent panels. All values, except the investment to output ratio and the maximum Sharpe ratio are in percentages.
Chapter 1

Aggregate Dynamics in an Economy with Optimal Long-term Financing

1.1 Abstract

In this chapter I examine optimal dynamic contracting between risk-averse investors and firm insiders in a dynamic general-equilibrium model with heterogeneous firms. The equilibrium optimal contract features a higher rate of inefficient liquidations in aggregate states with low productivity and a reward-for-luck policy in high productivity states. I show that with realistic parameters, moral hazard has a first-order quantitative effect on equilibrium dynamics of macro-economic quantities and asset prices. The conditional dynamics depend on the time-varying cross-sectional distribution of financial slack of firms in the economy. Moral hazard improves upon the predictions of the frictionless economy in three ways. First, the sensitivity of risk-premia and key macro-economic quantities rises after successive negative aggregate shocks. Accumulation of small shocks results in a disproportionately large decline in aggregate quantities, and a rise in risk-premia. Second, inefficiencies resulting from second-best contractual arrangements amplify the effect of primitive shocks and make the economy more sensitive to negative than to positive shocks. Third, controlling for current aggregate productivity, firm exit rates contain incremental information about future output growth.
1.2 Introduction

Under the assumption of the Modigliani-Miller theorem, firms' investment decisions in frictionless markets are disconnected from their financing choices. Empirically, financing decisions matter and interact with investment decisions because of the presence of various frictions. In this chapter I examine the aggregate implications of a moral hazard problem between investors and firm insiders in a general-equilibrium model of investment. I explore the effects of the resulting financing friction on the aggregate dynamics of prices and macroeconomic quantities. I also analyze the optimal contracts between investors and firms in a general-equilibrium environment with aggregate productivity shocks.

A large literature, both theoretical and empirical, explores the effect of financing frictions on firm investment. A robust empirical property of financial constraints is that they bind more during recessions. A recent evidence of this is provided by Campello et al. (2009) who surveyed chief financial officers of US firms. They document that 86% of constrained firms cut back on investing in attractive projects in 2008, compared to only 44% during normal times. Recessions are also periods during which firm exit rates increase. My model generates such patterns. In contrast, prior studies that emphasize free cash flow problems or imperfect enforcement generate tighter constraints during booms.

My model describes a continuum of firms exposed to aggregate and firm-specific productivity shocks. The Modigliani-Miller theorem does not hold because of a moral hazard problem – insiders’ effort is not observable by investors. Insiders can shirk and lower expected output. In equilibrium, investors offer insiders an incentive contract which is subject to constraints induced by the moral hazard friction. Under the optimal contract, firms can sometimes be liquidated. This is necessary to provide

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1 Insiders in my model are individuals or groups with significantly higher control rights than cash-flow rights.

2 Brunnermeier et al. (2012) provide a nice survey of the theoretical developments. Fazzari et al. (1988), Rauh (2006), and others provide empirical evidence.

3 Dow et al. (2005) study the effect of free cash flow problems on investment, while Cooley et al. (2004) study an economy with imperfect enforcement of contracts.
ex ante incentives to the insiders, but liquidations are inefficient ex post – investors recover only a fraction of the firm value while insiders receive their outside option. My model also features risk-averse investors. In general equilibrium, variation of the investor's marginal utility across aggregate states impacts the cost of providing incentives. The possibility of inefficient liquidations in states with high state prices has a first-order effect on contract valuation (and therefore investment), and also on equilibrium policies.

I show that the moral hazard problem has a first-order quantitative effect on equilibrium dynamics in my model. With the agency problem, the conditional dynamics depend on the cross-sectional distribution of financial slack of firms in the economy. This is in contrast to the frictionless economy, in which future growth rates and risk-premia are completely determined by the current level of aggregate productivity. This result generates three predictions which improves upon the empirical fit of the frictionless benchmark. First, the conditional dynamics depends on the entire past sequence of aggregate shocks. Large drops in aggregate quantities and sharp increases in risk-premia are the result of the accumulation of a sequence of small negative shocks. Second, the amplification of primitive shocks is asymmetric – the economy displays a higher sensitivity to negative shocks. Third, firm exit rates contain incremental information about future output growth.

Prior research on the real effects of debt-financing (for instance the literature following the seminal works of Kiyotaki and Moore (1997) and Bernanke et al. (1999)) shows that financial frictions can amplify productivity shocks and increase their propagation. Allowing the firms and investors to choose the form of financing endogenously has several important implications for equilibrium dynamics. First, models that assume debt-financing prevent investors and firms from dynamically managing financial slack in response to productivity shocks. As a result, inside equity holders are forced to absorb all risk. A potential worry is that if investors and firms were allowed to optimally respond to the financing friction by using a better contract, counter-cyclical defaults and the accompanying amplification would not be an equilibrium outcome.

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4One measure of financial slack of a firm is the unused portion of its credit line.
Moreover, allowing for alternative forms of financing helps us better connect the models to empirical observations, since in reality firms widely use lines of credit, debt of multiple maturity, interest rate derivatives and other forms of state-contingent financial securities\(^5\).

The second advantage of using an optimal contracting framework is that it provides insight into the effect of aggregate shocks on contract features. I obtain two results on this front. First, the optimal contracts in my model feature a reward-for-luck policy in high productivity states, and a higher rate of inefficient liquidations in aggregate states with low productivity. The former arises due to persistent aggregate productivity shocks that make it more profitable for investors to lower the probability of firm liquidation during booms. Since terminations are necessary to provide incentives, investors optimally liquidate the firm when expected future profitability is low. This feature of optimal contracts generates counter-cyclical firm exit. The second result is a cross-sectional prediction. When the aggregate state transitions from a boom to a recession, investors optimally increase the financial slack of firms which have recently experienced high cash flows. This increase in slack is accompanied by lower cash pay-outs. On the other hand, firms with low cash flow realizations have their slack reduced. This redistribution of financial slack away from poorly performing firms towards those with recent high cash flows is the cheapest way to provide incentives. This prediction of my model is testable using a measure of slack such as unused line of credit.

This chapter relates to two existing strands of the literature. The first examines the effect of financial frictions on dynamics of aggregate macro-economic quantities and asset prices. Following the influential work of Bernanke and Gertler (1989), Bernanke et al. (1999), Kiyotaki and Moore (1997), and more recent results by Brunnermeier and Sannikov (2014), this literature examines the impact of aggregate shocks on an assumed form of financing, predominantly single-period debt contracts. Firm heterogeneity plays no role in these models (an important exception is Khan and

\(^5\)Sufi (2009) reports that 85\% of firms in his sample obtained a line of credit. This included fully equity financed firms which held no debt.
Thomas (2013)). Gomes and Schmid (2009) features financing using infinite maturity debt. Gomes et al. (2006) consider the impact of a reduced form specification of financing constraints on asset prices. While most prior research in this area restricts available forms of financing by assumption, I allow financing to be chosen endogenously and optimally in response to an agency problem. The sharply contrasting effects of debt financing versus optimal contracts has been shown by Krishnamurthy (2003), and more recently in Tella (2013). The key difference between these papers and the setting here is the presence of inefficient terminations in equilibrium.

The second area of research to which this chapter relates is optimal dynamic contracting. The financial contracts in my model are related to the discrete-time models of DeMarzo and Fishman (2007a), DeMarzo and Fishman (2007b), and Biais et al. (2007) (see also DeMarzo and Sannikov (2006) for a characterization of the contract in a continuous time setting.) DeMarzo et al. (2012), Hoffmann and Pfeil (2010), and Piskorski and Tchistyi (2010) consider the effects of persistent, publicly observable, productivity shocks on the optimal contract in a partial equilibrium setting. The key difference of my essay with these studies is that while these papers examine the optimal contract in a partial equilibrium setting with an exogenously assumed discount rate, I provide a general equilibrium perspective where the discount rate investors use to value future cash flows is an endogenous equilibrium outcome. The latter arises from consumption smoothing motives of the risk-averse representative investor.

Prior literature considers optimal contracts for other sources of financing frictions in a general equilibrium setting. Cooley et al. (2004) analyze the implications of limited contract enforcement, while Dow et al. (2005) model the effects of a free cash-flow problem on investment and state prices. In these models, equilibrium policies and prices deviate from the frictionless setting only in good states of the economy, and, the risk premium is lower in the economy with a free cash-flow problem compared to the frictionless economy. In my model, a stronger moral hazard problem increases consumption risk and therefore risk premia. A second difference is that in my model key features of dynamics arises from the heterogeneous cross-section, while the above studies consider a representative firm. Albuquerue and Wang (2008) also consider
an agency problem similar to Dow et al. (2005) but introduce investment-specific shocks rather than productivity shocks. My essay differs from these papers in the source of financing friction. Moreover, none of these models feature terminations in equilibrium. Finally, following the classic work of Kehoe and Levine (1993) and Alvarez and Jermann (2000), there is a literature which examines the effect of limited commitment on consumption insurance among individuals.

To summarize, I provide a general equilibrium framework to study the effect of aggregate shocks on the dynamics of macro-economic quantities and asset prices. This framework provides insight into contract features under aggregate uncertainty, such as reward-for-luck, that are missed by models with an exogenously imposed form of financing such as debt. An appealing feature of my framework is that it allows me to model firms with a small fraction of inside-equity. This allows me to study the effect of aggregate shocks in an economy populated by large, public firms as opposed to manager-owned, debt-financed firms which are more representative of small, entrepreneurial firms.

The rest of this essay is organized as follows. In Section 1.3, I describe the model. In Section 1.4, I describe quantitative properties of the solution using realistic parameters. Section 1.5 concludes.

1.3 The Model

In this section I describe a general equilibrium with a continuum of firms who finance production by entering into long-term contracts with outside investors. I begin by describing the production sector with technology and investment, and then describe firm entry and details of the financial contract. Finally, I close the model with a description of the household sector.
1.3.1 The Environment

Production Sector

There is a continuum of ex-ante identical firms. Each firm is operated by insiders who rely on external financing to operate the technology. From now on, I will refer to the group of insiders as the “manager”, and use the terms inter-changeably. The production technology is linear. Firms produce a single homogenous good, which can be used both for consumption and investment

\[ y(x, z, k) = (x + z)k. \]  

The variable \( k \) is the firm’s capital/capacity which is installed when the firm begins to operate and is held fixed for the life of the firm. Each period, the capital requires a maintenance cost \( \delta k \). For simplicity, the parameter \( \delta \) is assumed to be constant across firms. \( x \) is an aggregate shock common to all firms and \( z \) is a firm-specific shock. \( x \) is a discrete Markov process with transition matrix \( \Gamma \)

\[ \Pr(x_{t+1}|x_t) = \Gamma. \]  

For simplicity, I assume that the aggregate shock \( x \) takes only two values \( x = \{x_G, x_B\} \). The firm-specific shock also takes on two values \( z_+ > z_- \), and is independently distributed across time and in the cross-section. The probability of a higher realization depends on the effort exerted by the firm manager

\[ \Pr(z_+) = \begin{cases} p, & \text{if effort } = 1 \\ p - \Delta p, & \text{if effort } = 0 \end{cases} \]  

Conditional on high effort by the manager, \( z \) is normalized to have zero mean\(^6\). Providing high effort is costly – shirking (effort= 0) provides the manager with constant private benefit \( Bk \). Although realizations of both shocks \( x \) and \( z \) are publicly observ-

---

\(^6\)The mean is absorbed in \( x \).
able, investors do not observe the manager's effort choice. They design a long-term contract which provides incentives to the manager to supply high effort.

Prospective managers have no initial wealth. Starting a firm requires an initial installation cost \( \tilde{c}k \) where \( \tilde{c} \) is randomly drawn from a uniform measure \( H \), and is revealed to the manager at the beginning of the period. I describe the support of \( H \) and its justification in more detail below. If the installation cost is sufficiently low, investors offer a prospective manager a contract specifying: (a) per-period payment that he will receive, and (ii) the probability of termination of the contract \( \zeta \) each period. Both of these policies are functions of the history of the individual manager's output together with the history of output of all the managers in the economy. I assume that the law of large numbers holds, so that the total output in any period is determined by the aggregate shock \( x_t \). Contract policies depend on the discount rate that the investor uses to value future cash flows. This is determined by the representative household's equilibrium consumption process. The possibility of exits makes the household's future consumption depend on the entire cross-sectional distribution of firms (not just the mean). The aggregate state, which I denote by \( s \), is captured by the aggregate shock \( x \) and its entire past history. The contract assumes two-sided commitment in which both the manager and the investor agree to abide by the terms of the contract in all possible contingencies, with no possibility of renegotiation. While this is a restrictive assumption, it provides a useful benchmark and can be thought of as a limiting case where renegotiation is extremely costly. This would be the case if the investors consist of a large dispersed pool of individuals.

For simplicity, I make four additional assumptions about the manager: (a) he values a consumption stream \( \{c_t\} \) as \( \sum_t \beta^t c_t \) with time preference parameter much smaller than that of the outside investor \( \beta_e \ll \beta_t \), (b) he has limited liability, (c) he does not participate in financial markets, and, (d) he has a time-invariant outside option normalized to zero\(^7 \).

As the following proposition shows, it is optimal for the manager to provide high

\[^7\text{This assumption could be relaxed in a richer model where the manager has an outside option of becoming a worker. The lower bound of } V \text{ is then determined by the rental rate of labor. I leave this extension for the future.}\]
effort, if the gain in expected output from working, is sufficiently large compared to the private benefit $B$.

**Proposition 1** If firm specific shocks satisfy $\Delta p(z_+ - z_-) >> B$, then there exists an optimal contract in which it is optimal for the manager to exert high effort.

**Proof.** Same as in Appendix 1, Proposition 13 of Biais et al. (2004). ■

**The Investor's problem and timing**

The financial contract between the outside investor and the firm manager serves three purposes: it provides for initial installation costs, covers maintenance costs (in the event of low realizations of $z$), and also provides for the manager's consumption. Following Spear and Srivastava (1987) and Green (1987), I use the dynamic programming approach to solve for the optimal contract. In this recursive formulation, the present discounted value of the future payments to the manager is a sufficient statistic for the entire past history of firm-specific realizations of $z$. I denote the manager's continuation value by $V_k$, where $k$ is the firm's capital. $V_k$ is the present value of future promised payments, and may be interpreted as the financial slack of this firm. The representative household's consumption and hence his discount rate process depends on the aggregate output net of payment to all managers in the economy. This dependence implies that individual contract policies depend not only on the individual manager's output but also on the output of all managers. The assumption of the law of large numbers implies that current productivity fully determines aggregate output. Manager payments in the cross-section is determined by the cross-sectional distribution of continuation values of managers. To sum up, individual contract policies depend on the manager's continuation value and the aggregate state of the economy which includes current productivity, and the cross-sectional measure of continuation values of all managers in the economy. In the presence of aggregate risk, i.e. time varying $\mu$, the cross-sectional distribution varies over time. I denote the cross-sectional distribution by $\mu$. This is the distribution before any exit or entry has taken place in that period. The aggregate state $s = \{x, \mu\}$ therefore includes both the current
aggregate shock $x$ and the cross-sectional distribution $\mu$. In addition to depending on $V$, contracts are conditioned on the entire aggregate state $s$ – in particular, each contract implicitly depends on the entire history of aggregate shocks, and also on the history of output of all existing firms through $\mu$. The assumption of linear technology allows firm capital $k$ to be scaled out. The investor’s problem is to offer a contract to the manager that maximizes the present discounted value of future cash flows

$$
\pi_t(s)F_t(V; s) = \max_{\zeta(s'), d_{\pm}(s'), V_{\pm}(s')} E^\Gamma \left[ \pi_{t+1}(s') \left( \zeta(s') \left[ (1 - \chi)(1 - \delta) \right] 
+ (1 - \zeta(s)) E^p \left( -\delta + x' + z_\pm - d_\pm(s') + F_{t+1}(V_{\pm}(s'; s')) \right) \right] 
\right]
V = E^\Gamma \left[ (1 - \zeta(s')) E^p (d_{\pm}(s') + \beta^\varepsilon V_{\pm}'(s')) \right], \forall s',
B/\Delta p \leq (d_{\pm}(s') + \beta^\varepsilon V_{\pm}'(s')) - (d_{\pm}(s') + \beta^\varepsilon V_{\pm}'(s'))], \forall s',
s' = \{x', \mu'\}, \quad x' = \Gamma x, \quad \mu' = \mathcal{H}(x', x, \mu), \quad \left\{ \zeta, d_{\pm}(s'), V_{\pm}(s') \right\} \in [0, 1] \times R^4_+.
$$

In the above equations, the firm’s capital $k$ has been explicitly scaled out. $F_t$ is the investor’s value of the contract (scaled by $k$). The choice variables are the termination probability of the manager $\zeta$, the manager’s payments $d_{\pm}$ and continuation value $V_{\pm}$. Each of these depend on the manager’s continuation value $V$, on the realized aggregate state $s'$, and the firm-specific shock $z_{\pm}$. The expectation $E^\Gamma[\cdot]$ is computed assuming the transition probability matrix $\Gamma$ for aggregate shocks $x$, while the expectation $E^p[\cdot]$ is computed assuming that the agent exerts high effort. This assumes that $z_+$ is realized with probability $p$.

The first-line in Equation 2.4 is the investor’s Bellman equation. The investor values all cash-flows using the representative household’s discount rate $\pi(s)$. The first term is the investor’s payment if the contract is terminated. His proceeds from liquidation is a fraction $(1 - \chi)$ of the physical assets of the firm after depreciation. The second term is his claim on firm output, and consists of current output net of maintenance cost and payment to the manager. The second line of Equation 2.4 is the promise-keeping constraint which ensures that the manager’s present continua-
tion value is the discounted sum of his future payments. In this line, I have implicitly assumed that if the contract is terminated, the manager receives his outside option of zero (his future consumption is set to zero). The third line is the incentive compatibility constraint. The law of motion of the aggregate state $s$ includes the law of motion of the cross-sectional distribution of manager continuation values, $\mathcal{H}(x', x, \mu)$, and depends both on $\Gamma$ and on contract policies $V'$. In solving the contracting problem, agents take this law of motion as given. In the rational expectations equilibrium, the law of motion used by agents is consistent with the realized law of motion. Likewise, investors and the manager take the discount rate $\pi(s')$ as given. In general equilibrium, the latter is the marginal utility of the representative household evaluated at the equilibrium aggregate consumption level. In the model, the household has constant relative risk-aversion $\gamma$, therefore, $\pi(s')/\pi(s) = \beta' \left( C^*_t(s')/C^*_t(s) \right)^{-\gamma}$. The quantity, $C^*$, is the equilibrium aggregate consumption of the household, and is determined by the household’s consumption and savings decision.  

Timing in this economy is shown in Fig 2-1. At the beginning of each period, the aggregate shock $x$ is observed. In accordance with the contractual agreement, manager continuation values (discounted utility of future payments) are adjusted depending on the realized aggregate state. As will be shown below, the aggregate state depends not only on the realized value of the aggregate shock $x$, but also on the cross-sectional distribution of the continuation values of all managers in the economy. A public lottery for termination of the contract is held in which firms with low continuation values might exit. Maintenance is paid for continuing firms. Production takes place next, i.e. firms realize $z$ and produce output. Finally, managers are paid,

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8In valuing contracts and determining optimal policies, the only stochastic component of cash flows priced by investors are those associated with systematic market-wide risk factors. We can either view the representative household as holding a well-diversified portfolio consisting of all the contracts and a risky-free asset in zero net supply, or picture identical, individual investors each entering into a long-term contract with a single manager. In the latter case, individual investors act as pass-throughs – they collect payments and pass them on to the representative household. Since they belong to the same risk-sharing household, all investors use the same discount rate to value cash flows. Both these pictures have identical pricing implications: the idiosyncratic component of cash flows of individual contracts can be completely diversified away, and are therefore not priced. The household’s aggregate consumption process is the only systematic risk factor used in pricing risky cash flows.
Figure 1-1: Intra-period time-line of the model. The aggregate shock $x$ is revealed at the beginning of the period, after which the lottery for termination is held. A manager whose contract is terminated receives nothing, while the investor recovers a fraction, $(1 - \chi)$, of the (depreciated) physical assets of the firm. If the manager continues, he receives maintenance cost $\delta k$. Production takes place when the firm-specific shock $z$ is realized. The output $y$ is divided between the manager and investors. The period ends.

new contracts are initiated, and the representative household consumes.

Firm entry and investment

Aggregate investment takes place through entry of new firms\(^9\). Each period, there is a mass of potential managers with projects ready to enter the economy. To begin production, a one-time fixed cost $\tilde{\varepsilon}k$ has to be paid. This random cost is drawn from a uniform measure $H$, and is revealed to potential managers at the beginning

\(^9\)I borrow this modeling technique from Gomes et al. (2003)
of each period. To ensure balanced growth, I assume the mass of potential entrants is proportional to the measure of existing firms: \( H = hN_t \), where \( N_t = \int d\mu_t \) is the total number of firms in existence at the beginning of period \( t \) before any entry and exit in that period. The constant of proportionality \( h \) is a measure of the investment opportunity set. At the aggregate level, this entry cost mimics adjustment costs and has the effect of making asset prices more volatile.

A new firm becomes operational if the manager is able to secure financing by entering into a long-term financial contract with an outside investor. I assume perfectly competitive lending markets so that investors break even. Managers have all the bargaining power and choose the highest possible payoff subject to the investor’s participation constraint

\[
V_0 = \sup \{ V : F(V) \geq \bar{e}k \}. \tag{1.5}
\]

**Figure 1-2:** Entry condition of new managers into the economy. Panel A shows an example. The solid curve shows the investor’s value (scaled by \( k \)) of a new contract, \( F(V) \), as a function of promised utility, \( V \), to the manager. The initial installation cost for this firm is \( ek \). This contract is initiated with the manager starting with promised utility \( V_0 \). Panel B shows that only those managers who draw an installation cost less than the maximum contract value to the investor begin operations, while those with more expensive projects are rationed out. The random installation cost \( \bar{e} \), is drawn from a uniform distribution. The mass of managers waiting to enter each period is assumed to be proportional to the number of existing firms in the economy, with \( h \) as the proportionality constant.
installation cost, and in subsequent periods, conditional on continuation, the investor commits to paying the maintenance cost $\delta k$, and compensates the manager according to his past and present performance. Production begins from the period subsequent to entry. Projects with high entry costs are unable to secure funding and expire worthless. The right panel of Figure 1-2 shows this graphically.

Households

I assume that the economy is populated by a single representative household. Individual investors are assumed to be members of this single risk-sharing household and therefore, they all share the same stochastic discount factor. This household derives utility from consumption of the single good $C_t$ and has standard time-separable power utility

$$E_0 \sum_{t=0}^{\infty} \beta_t^t C_t^{1-\gamma_t} \frac{1}{1-\gamma_t},$$

where $\gamma_t$ is the household's risk-aversion, and $\beta_t$ is the time preference parameter. The household derives income from accumulated wealth, and makes consumption and investment decisions to maximize expected lifetime utility subject to the budget constraint

$$W_{t+1} = (W_t - C_t) \left[ (1 - \sum_i w_t^i) R_{t+1}^i + \sum_i w_t^i R_{t+1}^i \right],$$

where $w_t^i$ is the fraction of household's wealth invested in contract $i$, $R_{t+1}^i$ is the gross risk-free rate, and the return from contract $i$ is $R_{t+1}^i = (F_{t+1}^i + \tau_{t+1}^i) / F_t^i$. By assumption, the household is more patient than managers ($\beta_e < \beta_l$). In addition to investing in firms through the financial contracts, the household also invests in a single risk-free asset which is in zero net supply. The household takes the risk-free rate and the market price of each of the financial contracts as given, and chooses consumption, and his financial portfolio to maximize utility subject to the budget constraint.
1.3.2 General Equilibrium and Aggregation

The equilibrium is a Markov, perfect, competitive equilibrium. The state space of this problem includes the cross-sectional distribution of manager continuation values, which is infinite dimensional. The formal definition of the equilibrium is given below:

**Definition 1** Recursive equilibrium – A recursive competitive equilibrium is defined as a set of functions for (i) initial contract state \( V_0(x, \mu) \), (ii) contract policies \( \Phi(k, V, x, \mu) = \{V_\pm(k, V, x, \mu), d_\pm(k, V, x, \mu), \zeta(k, V, x, \mu)\} \), (iii) consumption policies of the representative household \( C(x, \mu) \), and (iv) law of motion of states \( s' \sim \{x', \mu'\} \), such that (i) individual contracts are optimal, (ii) the initial state is such that the investor breaks even (Eq. 2.5), (iii) the representative household’s policies are optimal (Eq. 2.4) subject to the household’s budget constraint, Eq. 1.7

(iv) the goods market clears

\[
C^*(s) = \int [xk - d(s)] d\mu - I(s) - L(s). \tag{1.8}
\]

where aggregate investment \( I \) and loss from liquidation \( L \) are given by

\[
I(s) = \int_0^{\bar{\varepsilon}(s)} ekdH + \int \delta kd\mu - D(s)k = \left[ \frac{h}{2} \bar{\varepsilon}^2(s) + \delta - D(s) \right] k \int d\mu, \tag{1.9}
\]

\[
L(s) = \chi(1 - \delta)kD(s) \int d\mu. \tag{1.10}
\]

\( D(s) \) is the rate of termination of contracts,

(v) the market for contracts clears

\[
W^*_c = \int F_t(V) d\mu_t \tag{1.11}
\]

where \( W^*_c \) is the household’s wealth invested in financial contracts, (vi) the bond market clears

\[
W^*_b = 0 \tag{1.12}
\]
where $W^*_t$ is the household’s wealth invested in the risk-free asset, and (vi) the law of motion of the cross-sectional distribution of continuation values, $\mathcal{H}(x', \mu)$ is consistent with individual contract policies and the stochastic process for $z$.

The following proposition summarizes properties of asset returns and the risk-free rate in the economy.

**Proposition 2** The equilibrium stochastic discount factor in this economy is defined by $\pi'_{t+1}/\pi_t = \beta_t (\frac{C^*_{t+1}}{C^*_t})^{-\eta}$, where $C^*_t$ is the equilibrium consumption of the representative household. All gross returns $R^i_t$ in this economy, satisfy the no-arbitrage relation $E_t[\frac{\pi_{t+1}}{\pi_t} R^i_{t+1}] = 1$. The risk-free rate, in particular is given by $R^f_t = 1 + r_t = 1/E_t[\frac{\pi_{t+1}}{\pi_t}]$.

**Proof.** The household takes the risk-free rate, $r_t$, and the ex-dividend contract prices $F^i_t$ as given. The household’s Bellman equation is

$$U_t(g, b, s) = \max_{C_t \geq 0, g', b'} \left[ u(C_t) + E_t[\beta^i U_{t+1}(g', b', s')] \right], \quad s = \{x, \mu\},$$

subject to the budget constraint

$$g_t + \sum_{i \in \text{Continue}} b^i_t (F^i_t + y^i_t - d^i_t - \delta)k + \sum_{i \in \text{Exits}} (1 - \chi)(1 - \delta)k$$

$$- \sum_{i \in \text{Entry}} b^i_t k = \frac{g_{t+1}}{1 + r_t} + \sum_{i \in \text{Continue}} b^i_{t+1} F^i_t + C_t$$

where $g$ and $b$ are the household’s holding of risk-free asset and the long-term contracts, and $d^i_t$ is the payment received by the manager from contract $i$ in period $t$. The aggregate state $s$ explicitly depends on aggregate shock $x$ and also on the distribution of continuation values of surviving managers. The expectation is over realizations of $x'$ given that the current shock is $x$, with exogenously specified transition probability matrix $\Gamma$. By “continue” I mean only those firms which are still in existence. This excludes new entrants. Finally, $1 - \chi$ is the recovery rate given liquidation has occurred. The law of large numbers is assumed to hold, so that $C_t$ does not depend on individual realizations of firm-specific shock $z$. 

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Market clearing: In equilibrium, the bond market and the market for financial contracts clear
\[ g_t = g_{t+1} = 0, \quad b^i_t = b^i_{t+1} = 1, \quad (1.15) \]
for continuing firms \( i \). Substituting the market clearing conditions into the budget constraint Eq.1.14, prices \( (r_t \text{ and } F_t^i) \) drop out, and the goods market clearing condition becomes
\[ C_t + \sum_{i \in \text{Continue}} d^i_t = Y_t - I_t - L_t, \]
where
\[ I_t = \sum_{i \in \text{Alt}} \delta k + \sum_{i \in \text{Entrants}} \hat{c}^i k - \sum_{i \in \text{Exits}} k, \]
\[ L_t = \chi(1 - \delta) \sum_{i \in \text{Exits}} k, \]
where the fractional loss \( 0 \geq \chi \geq 1 \). This coincides with Eq. 1.9 with aggregate investment \( I \) and loss from liquidation \( L_t \). I assume that entering firms start production in the period subsequent to their entry.

Optimality conditions: Prices \( F_t^i \) and \( r_t \) get determined by the household’s first-order conditions
\[ u'(C_t)F_t^i = \beta^l E\left[u'(C_{t+1})(\tau^i_{t+1} + F^i_{t+1})\right], \quad 1 = E\left[\beta^l \frac{u'(C_{t+1})}{u'(C_t)}(1 + r_t)\right], \]
where \( \tau^i_t = -\delta + y^i_{t+1} - d^i_{t+1} \) is the investor’s cash flow (scaled by \( k \)) after paying for maintenance and paying the manager. The stochastic discount factor used to discount cash flows is, therefore, \( \pi_{t+\delta}/\pi_t = \beta^l u'(C_{t+1})/u'(C_t) \).

With aggregate shocks, the distribution of continuation values (and therefore manager payments), changes over time. There is no steady-state distribution to which the economy settles into, and for this reason, proving the existence of an equilibrium is difficult. This is not unique to the set-up here and is common to incomplete-market models with aggregate uncertainty. The resulting dynamics is quite rich, aggregate.
Figure 1-3: Intra-period notation used in the appendix. $F^1_t(V^1_t; s)$ is the investor’s value function immediately after the aggregate shock $x_t$ is realized, but before the lottery for termination, as a function of the manager’s continuation value $V^1_t$. $F^2_t(V^2_t; s)$ is the investor’s value function after the lottery, but before the firm-specific productivity shock, $z_t$, is realized. Finally, $F^3_t(V^3_t; s)$ is the investor’s value function after the firm-specific shock is realized.

$F^1_t(V^1_t, s)$ $F^2_t(V^2_t, s)$ $F^3_t(V^3_t, s)$

Nature draws $x_t$ Lottery for termination Nature draws $z_t$

Quantities and risk premia depend both on the current aggregate shock, and on the cross-sectional distribution of manager continuation values. Assuming an equilibrium exists, the following proposition proves that the investor’s value function is concave, and the marginal cost of providing incentives is bounded from above.

**Proposition 3** The investor’s value function $F(V; s)$ is a (weakly) concave function of the manager's promised utility $V$. The slope of $F(V; s)$ is bounded from below by $F'(V; s) \geq -1$ for all $s$.

**Proof.**

The bound on $F'(V; s)$ follows from the argument that paying the manager a dollar of cash costs at most a dollar. Therefore, the marginal cost $-F'(V; s) \leq 1$.

To prove concavity of $F$, decompose into 3 sub-problems as shown in Figure 1-3. Starting from the end of each period, first, consider the subproblem of maximizing
the investor’s value conditional on continuing (not exiting):

**Firm-specific shock:**

\[
F^2_t(V^2_t, s) = \max_{d_{\pm} \geq 0, V^{\pm}_t \geq 0} \left[ -\delta + x + p \left( z_+ - d_+ + F^3_t(V^3_+, s) \right) + (1 - p) \left( z_- - d_- + F^3_t(V^3_-, s) \right) \right],
\]

\[
V = E^p \left[ d_+(s) + V'_+(s) \right], \quad \forall s
\]

\[
B \leq \Delta p \left[ \left( d_+(s) + V'_+(s) \right) - \left( d_-(s) + V'_-(s) \right) \right], \quad \forall s.
\]

Let \( T \) be the Bellman operator (i.e. right side of first line). Let \( C(R_+) \) be the set of bounded, continuous, weakly concave functions that equal a constant at 0. Since, \( C'(R_+) \subset C(R_+) \) of the complete metric space \( C(R_+) \), it is sufficient to show that \( T[C'(R_+)] \) maps this set into itself. To verify this, let \( F^3 \in C'(R_+) \), and let

\[
V^0 \neq V^1, \quad \theta \in (0, 1), \quad \text{and} \quad V^0 = \theta V^0 + (1 - \theta)V^1.
\]

Let \( (V^i_\pm) \) attain \( (TF^3)(V^i) \) for \( i = 0, 1 \). Next, I show that for \( V^\theta \), the controls \((d^*_\pm, V^*_\pm)\) are feasible, where they are chosen to be \( V^*_\pm = \theta V^0_\pm + (1 - \theta)V^1_\pm \) and manager payments \( d^*_\pm \) defined so that \( d^*_\pm = \theta d^0_\pm + (1 - \theta)d^1_\pm \) is a feasible choice. This follows because both the promise-keeping constraint and the IC constraints are linear. Since \( F^3 \) is concave, it follows that \( TF^3(V^\theta) \geq \theta TF^3(V^0) + (1 - \theta)TF^3(V^1) \), and thus \( TF^3 \) is concave.

**Public randomization:**

\[
F^1_t(V^1_t, s) = \max_{\zeta \in (0,1), V^2_t \geq 0} \left[ (1 - \zeta)F^2_t(V^2_t, s) + \zeta (1 - \chi)(1 - \delta)k \right],
\]

\[
V^1_t = (1 - \zeta)V^2_t.
\]

Concavity of \( F^1_t \) follows from the fact that \( F^2_t \) is concave, and the constraint is linear.
Aggregate shock/state change:

\[ \pi_t(s)F^3_t(V^3_t, s) = \max_{\varepsilon_t^3, \varepsilon_t^\beta \geq 0} E_t^\Gamma [\pi_{t+1}(s')F^1_{t+1}(V^1_{t+1}, s')] , \]

\[ V^3_t = \beta E_t^\Gamma [V^1_{t+1}] . \]

Once again, the constraint is linear, and concavity of \( F^3 \) follows from concavity of \( F^1 \).

I close this section by characterizing the equilibrium in the frictionless economy. Without the agency problem, current aggregate productivity and the total number of firms are sufficient statistics to characterize the equilibrium.

### 1.3.3 No agency problem

When the manager receives no benefit from shirking (\( B = 0 \)), the agency problem disappears. All positive NPV projects are funded. By assumption, the manager is sufficiently impatient so that he values the firm less than outside investors. Managers with positive NPV projects immediately sell all of firm equity to outside investors. Investors pay for the initial set-up cost \( \delta k \) and, thereafter, pay per-period maintenance cost \( \delta k \) and receive all cash future flows.

**Proposition 4** *(Equilibrium allocations)* The competitive equilibrium is characterized by aggregate consumption \( C(x_t) = c(x_t) \int d\mu_t \) where \( \int d\mu_t \) is the total number of firms in the economy. Aggregate output \( Y(x_t) = x_t \int d\mu_t \), and investment \( I(x_t) = \left[ \frac{\delta}{2} \bar{c}^2(x_t) + \delta \right] k \int d\mu_t \) where

\[
\bar{c}(x_t) = c(x_t)^\gamma f(x_t), \quad c(x_t) = x_t - \delta \bar{c}^2(x_t) - \delta, \quad (1.16)
\]

and where \( f(x_i) \) is

\[
f(x_i) = E_0 \left[ \sum_{s=1}^{\infty} \beta^s \left( c(x_s) \prod_{j=1}^{s} (1 + h\bar{c}(x_j)) \right)^{-\gamma} x_s | x_0 \right] .
\]
The manager is assumed to be sufficiently impatient so that $\beta_c$ satisfies the condition in Equation 1.18 of the Appendix.

**Proof.** In equilibrium, the firm which just breaks even (zero net present value) has present value of cash flows equal to cost. For this firm, scaling out $k$,

$$
\bar{e}_t(x_t) = E_t \left[ \sum_{s=1}^{\infty} \beta^s \left( \frac{C_{t+s}}{C_t} \right)^{-\gamma} x_{t+s} | x_t \right].
$$

Conjecture that aggregate household consumption $C_t = c(x_t)N_t$ where the total number of firms in the economy $N_t = \int d\mu_t$, and grows at the rate $\bar{e}(x_t)$.

$$
\bar{e}_t(x_t) = E_t \left[ \sum_{s=1}^{\infty} \beta^s \left( \frac{c(x_{t+s})N_{t+s}}{c(x_t)N_t} \right)^{-\gamma} x_{t+s} | x_t \right],
$$

$$
= c(x_t)^\gamma E_t \left[ \sum_{s=1}^{\infty} \beta^s \left( c(x_{t+s}) \prod_{j=1}^{s} (1 + \bar{e}(x_j)) \right)^{-\gamma} x_{t+s} | x_t \right],
$$

which is of the form conjectured in the first of Eq. 1.16 with

$$
f(x_t) = E_0 \left[ \sum_{s=1}^{\infty} \beta^s \left( c(x_s) \prod_{j=1}^{s} (1 + \bar{e}(x_j)) \right)^{-\gamma} x_s | x_0 \right]. \tag{1.17}
$$

The second equation in Eq. 1.16 follows from the goods market clearing condition

$$
C_t(x_t) = c(x_t) \int d\mu_t = (x_t - \delta) k \int d\mu_t - \bar{e}^2 \int d\mu_t.
$$

where the last term is the consumption of managers who have positive net present value projects and have just entered the economy.

The manager's time preference parameter is chosen so that he is sufficiently impatient, and values the project less than investors. This requires satisfying both the
With no moral hazard, there are no exits and as the above Proposition shows, the current aggregate state $x$ and the total number of firms are sufficient statistics for aggregate output, investment, consumption, and asset prices. Details of the cross-sectional distribution do not matter for quantities and prices. Primitive shocks do not propagate and there is no history dependence.\(^{10}\)

\section{1.4 Model Solution}

In the presence of moral hazard, aggregate quantities and asset prices depend on the cross-sectional distribution of manager continuation values. This is because continuation values do not scale linearly with the aggregate shock, and as a result, the shape of the distribution changes over time, never settling into a steady-state distribution. The household’s consumption policy depends on the shape of the distribution. The resulting curse of dimensionality makes it infeasible to solve for the exact equilibrium\(^{11}\). The stochastic discount rate of the household depends directly on the growth rate of aggregate consumption, which is influenced by the rate of firm liquidations each period. Any approximation scheme, therefore, needs to be able to track both the shape and dynamics of the left-tail sufficiently accurately. In the Appendix, I outline a new numerical scheme which approximates the cross-sectional distribution by a mixture of two normal distributions. This approach is of independent use in any

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\(^{10}\)Note that although there is no additional propagation of primitive aggregate shocks, consumption growth is persistent because the aggregate productivity shocks are persistent.

\(^{11}\)The situation is similar to incomplete-market models with aggregate risk, where non-linearity of policies makes it impossible for the heterogenous cross-section to be aggregated in a tractable manner (for example Krusell and Smith (1998)).
model of default where it is crucial to track the shape of the distribution near the left-tail.

1.4.1 Parameters and Unconditional Moments

The values of the 12 parameters in the model are listed in Table 1.1. All parameters are quarterly values. The technology parameters $x_G, x_B, \delta, h$, and the representative household’s preference parameters $\beta_l, \gamma_l$ are chosen to match the first two moments of key aggregate quantities and asset returns. The manager’s time-preference parameter $\beta_e = 0.92$ is set to a value much lower than $\beta_l = 0.99$. The transition matrix $\Gamma$, is estimated from NBER quarterly recession data over the period 1929–1998, using the method of maximum likelihood. The estimated transition intensity out of the state $x_G$ (labelled $G$ in Table 1.1) is 0.05. The transition intensity out of the state $x_B$ (labelled $G$ in Table 1.1) is 0.30. The firm-specific shocks $z_\pm$ are chosen to occur with equal probability ($p = 0.5$), and the size of $|z_\pm|$ corresponds to an annual volatility of 31.6%. The fraction of capital lost from termination of contracts $\chi$ is chosen to be $\chi = 0.15$. The strength of the moral hazard problem depends on the ratio of the insider’s private benefit to average loss in output from shirking, $B/\Delta p(z_+ - z_-)$. This ratio corresponds to insiders’ share of firm equity. I set this value to 15%. This is higher than the average CEO ownership in the US of 5.5% reported by Jenter and Lewellen (2013), because insiders in my model includes senior management and others with disproportionate control rights compared to cash flow rights.

Table 1.2 show unconditional moments of aggregate macroeconomic quantities and asset prices. To generate the model moments, I simulate 1000 independent panels of $N = 10^8$ firms over a time period of 280 quarters, and average over these 1000 independent simulations. The top panel reports the first two (unconditional) moments of key macro quantities. The volatility of consumption growth is close to the data, and is higher than in the friction-less model. The difference is due to exits in high marginal utility states, which lowers investment and increases consumption risk. The volatility of output growth in the model is close to the data, as is the level of investment as a fraction of output. Investment is higher in the frictionless economy because
there are no (inefficient) liquidations. Although investment volatility is higher than in the frictionless economy, it is smaller than in the data. This could be remedied by allowing the investment opportunity set, measured by the constant $h$, to co-vary positively with the aggregate shock.

Firm liquidations in my model are counter-cyclical, as in the data. For the data counterpart of firm exits, I proxy the exit rate by the corporate default rate data from Moody's. The mean annual default rate in the data of 1.2% is close to the model implied liquidation rate of 1.4%. The model produces a higher volatility of exit rates, and a higher average correlation of exits with output growth than in the data.

The lower panel of Table 1.2 compares asset pricing moments implied by the model to the empirical counterparts. The level and volatility of the risk-free rate is low in the model with moral hazard because of the household's precautionary savings motive due to persistent shocks to consumption growth. Exits increase consumption risk, which helps lower the risk-free rate in the model with moral hazard compared to the frictionless setting. I define the model implied equity-premium as the expected excess return over the risk-free rate to a claim on aggregate consumption. The high expected return on aggregate consumption is because of an increased precautionary savings motive due to exits and a high value of risk-aversion ($\gamma_l = 7$). It is higher than in the frictionless model because costly termination in states in which the investor's marginal utility is high raises the expected return. The comparison between equity-premium in the data and the model is not exact. Since there is no exact data counterpart, the number reported in the data column is the usual equity premium, which is the excess return to a claim on the aggregate dividend stream. The latter is much more volatile than aggregate consumption and therefore expected to have a higher expected return. The model implied volatility of the stochastic discount factor is close to the data.

These results show that the first two moments of macro-economic quantities and asset prices are reasonably close to the data counterparts. This implies that the choice of parameters is reasonable. Next, I discuss aggregate dynamics and the behavior of contract policies. First, without aggregate shocks, and then with aggregate risk.

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12To be clear, bond default rates are an imperfect proxy for firm liquidations.
1.4.2 No aggregate uncertainty

With positive private benefit to the manager from shirking, investors need to induce high effort by providing incentives using the contract described in Section 1.3.1. Without aggregate uncertainty $x_G = x_B$, the investor’s consumption growth rate is a constant in the steady-state, and therefore, so is the investor’s discount rate $eta_t(C_{t+1}/C_t)^{-\eta}$. This is the setting considered in DeMarzo and Fishman (2007a), DeMarzo and Fishman (2007b), and also in Biais et al. (2007). I quickly review the features of the optimal contract in this familiar case. The reader is referred to these papers for more details.

The form in which incentives are provided depend on the manager’s continuation utility. The left panel of Figure 1-2 shows the investor’s value of future cash flows from the contract as a function of the manager’s continuation value. It is weakly concave. At very low values, increasing the manager’s continuation value lowers the probability of inefficient termination of the firm. This leads to an initial increase in the value (to the investor). At very high continuation values, increasing the manager’s continuation value leads to a decrease in value to the investor because it constitutes a wealth transfer from the investor to the manager. Figure 1-4 shows optimal policies. The panel on the left shows the manager’s continuation value after realization of firm output. His continuation value changes from $V$ to $V_+$ for high output (solid, blue curve), and to $V_-$ (dashed, red curve) for low output. The spread provides incentives for him to provide high effort. At very low $V$, this spread cannot be provided, and incentives are provided by threatening termination. The panel on the right, shows cash policies of this contract. From the figure, we see that cash is paid only after a sufficiently high continuation value is achieved by the manager. The solid, blue curve shows payments received by the manager after a high cash flow realization, while the dashed, red curve shows his cash compensation for low output. To sum up, for very high continuation values, the manager is compensated by cash payments. At intermediate values, he is compensated by future promises. At very low continuation values, incentives are provided through the threat of terminating the contract. Figure
Figure 1-4: Incentive provision by the optimal contract. The left panel shows the manager’s continuation value after realization of output. His continuation value changes from $V$ to $V_+$ for high output (solid, blue curve), and to $V_-$ (dashed, red curve) for low output. The spread provides incentives for him to provide high effort. At very low levels of manager’s continuation value, this spread cannot be provided, and incentives are provided by the threat of termination. The right panel shows payments received by the manager as a function of his continuation value. The solid, blue curve shows his payment $d_+$, for high output. The dashed, red curve shows his payment $d_-$ for low output.

Figure 1-5 shows the fractional loss in firm value due to the borrowing constraints arising from the moral hazard problem. The solid, black curve shows firm value defined as the sum of the investor’s value and the manager’s continuation utility. Firm value is lower than the first-best value because of the cost of providing incentives to the manager and because of the possibility of inefficient termination. The dotted, blue curve plots the stationary distribution of manager continuation values. In this example, about half of the firms suffer a loss greater than 10%.

Implementation of the optimal contract is not unique. DeMarzo and Fishman (2007b) show how to implement this contract using a combination of inside and outside equity, long-term debt, and a line of credit. In this implementation, the manager keeps $B/\Delta p(z_+ - z_-)$ of firm equity, while outsiders hold the remaining fraction of equity, together with long term debt with fixed coupon payments, and a line of credit. None of the results presented in this essay depend on the particular form
Figure 1-5: Loss in value of match due to borrowing constraints. The solid, black curve shows the ratio of the value of match between the manager and outside investors, with moral hazard, as a fraction of the frictionless value. The value of the match is defined as the sum of the investor’s value of the contract and the manager’s promised utility. The dotted, blue curve shows the cross-sectional distribution of manager continuation values in the steady-state of the economy without aggregate uncertainty ($x_G = x_B$).

of implementation. Therefore, I do not discuss this further, and refer the interested reader instead to DeMarzo and Fishman (2007b) and Biais et al. (2007).

The discussion above focussed on the contract under no aggregate uncertainty. Next I discuss dynamics in the presence of aggregate shocks.

1.4.3 Aggregate Uncertainty

The economy with the moral hazard problem differs from the frictionless benchmark both in terms of contract policies and also in terms of aggregate dynamics of quantities and risk-premia. First I discuss features of the optimal contract which arise as a consequence of aggregate shocks. Next, I discuss dynamics of aggregate quantities and risk-premia when optimal contracts are used for financing.
Contract Features

Persistent aggregate shocks introduces time variation in the discount rate that investors use to value contract cash flows. The optimal contract depends on the aggregate state, and the optimal adjustment to each manager's continuation value is made in the cheapest possible way while still providing incentives for high manager effort. The investor achieves this by equating the marginal cost of increasing future promised payments to the manager across aggregate states. This generates two new contract features compared to the environment with a constant discount rate discussed above.

First, in my model, a reward-for-luck policy towards firm managers is optimal, as are counter-cyclical firm termination rates. When the aggregate state changes from $x_B$ to $x_G$, investors optimally increase the continuation values of managers with very low continuation values. The intuition for this reward-for-luck feature is the following. The assumption of persistent aggregate productivity shocks makes it more profitable for investors to lower the probability of inefficient liquidation of the firm during booms. Therefore, when the aggregate state switches from $x_B$ to $x_G$, firm managers with very low continuation values, and at significant risk of termination have their continuation values increased. This is reward-for-luck, since the aggregate state is not under the control of the firm manager. However, because terminations are necessary to provide incentives, investors optimally liquidate the firm when expected future profitability is low.

The left panel of Figure 1-6 shows the adjustment to the manager's continuation value when the aggregate state changes from $x_G$ to $x_B$. The initial continuation value is plotted along the x-axis, while the adjusted continuation value is plotted along the y-axis. The solid, red curve shows the new continuation value after the shock is realized. The black, dotted curve is the 45° line. Note that managers with a very low initial value of $V$ have their continuation values set to zero after $x_B$ is realized, and exit the economy. Managers whose continuation values are low, as a result of recent poor performance, have a higher probability of being terminated. In fact, those with continuation values below a threshold, have their continuation values
Figure 1-6: Change in continuation value and its effect on the distribution when the aggregate state changes from boom to recession. The left panel shows the adjustment to the manager’s continuation value after the low aggregate shock \( x_B \) is realized. The initial continuation value is plotted along the x-axis, while the adjusted continuation value is plotted along the y-axis. The solid, red curve shows the new continuation value after the shock is realized. The black, dotted curve is the 45° line. Note that managers with a very low initial value of \( V \) have their continuation values set to zero after \( x_B \) is realized, and exit the economy. The right panel shows the effect of this change in continuation values on the cross-sectional distribution. The dotted, blue curve shows the distribution before the aggregate shock is realized. The current productivity is \( x_G \). The solid, red curve shows the cross-sectional distribution of continuation values of managers after realization of the low aggregate productivity shock, \( x_B \). The mass of firms accumulated at \( V = 0 \) exit immediately.

set to zero resulting in certain termination. This sharp increase in termination rates of poorly performing managers is reminiscent of the sharp non-linear drop in the refinancing probability for fully drawn down loans during the crisis documented by Mian and Santos (2011). The effect on the distribution as a result of this adjustment of continuation values immediately after the aggregate state changes is shown in the right panel of Figure 1-6. The dotted, blue curve is the initial distribution, which in this example, is the stochastic steady-state distribution. The solid, red curve is the resulting distribution after realization of \( x_B \). Note the accumulation of firms at \( V = 0 \). The upshot of this is that, in equilibrium, there is an increase in firm exit rates during recessions as shown by the left panel of Figure 1-7. The result of several bad shocks is more severe. With each successive realization of \( x_B \), more firms are
pulled closer towards the termination boundary. This causes the exit rate to increase sharply as shown in the left panel of Figure 1-8.

The second prediction of my model is a cross-sectional difference in contract policy when the aggregate state changes. Figure 1-6 illustrates this graphically when the aggregate state changes from $x_G$ to $x_B$. This results in a redistribution of financial slack (measured by $V$) in the cross-section of firms when the economy transitions to a different aggregate state. The intuition for the adjustment in continuation values is the following. For managers with relatively high continuation values (as a result of recent, good performance), the likelihood of inefficient firm liquidation is very low. The main cost is from postponing the manager's consumption, and this cost varies across aggregate states because of variation in the marginal utility of the representative household. In the low productivity state, the household's marginal utility is higher than in the state $x_G$, and therefore, relatively speaking, the manager is more patient. This makes it cheaper to back-load manager payments in this aggregate state. Investors increase the manager's continuation value in exchange for lower cash payments this period. The left panel of Figure 1-6 illustrates this behavior for high values of $V$. Since $V$ measures the financial slack of the firm, the model predicts that in a recession, it is optimal for firms with high recent cash flows, to decrease cash payments to managers and increase financial slack. This is a testable prediction using measures of financial slack such as cash or unused lines of credit. Note that this increase in continuation values for high $V$, when the aggregate state changes from $x_G$ to $x_B$, is exactly the opposite of firm managers with very low values of $V$. The latter have their continuation values decreased.

**Dynamics**

The presence of financial frictions introduces an important qualitative change to the dynamics of aggregate quantities and risk premia. With agency problems, both the conditional dynamics of aggregate output, investment, and consumption, depend on the past history of aggregate shocks. Risk premia and future growth rates of aggregate quantities depend strongly on the past sequence of shocks. This is in contrast
Figure 1-7: Dynamics in a short recession. In both panels, time is plotted on the x-axis. The left panel shows the increase in exit rates after a short recession. The economy starts at the stochastic steady-state and experiences two successive low aggregate shocks $x_B$ at $t = \{1, 2\}$. Exit rates in the figure are annualized and expressed in percent. The right panel shows the increase in the ratio of the volatility of the stochastic discount rate to its mean $\sigma(\pi')/E[\pi']$, and corresponds to the maximum Sharpe ratio in this economy.

The dependence on past shock sequences is due to the dependence of the exit rate on past history. Figure 1-6 shows that when the aggregate state changes from $x_G$ to $x_B$, firms below a threshold on the left-tail are liquidated immediately. Those with slightly higher continuation values have their continuation values lowered. This increases the density of firms near the termination threshold, and if another low aggregate shock is realized next period, the exit rate increases. The left panel of Figure 1-7 shows the increase in exit rate for two successive realizations of $x_B$, while the left panel of Figure 1-8 shows the sharp increase following five successive shocks. The increase in exit rate in states in which the representative household’s marginal utility is high lowers the value of new investment. This, in turn, increases consumption risk, leading to an increase in the market risk-premium, as shown in the right panels of Figures 1-7 and 1-8.
Figure 1-8: Dynamics in a long recession. In both panels, time is plotted on the x-axis. The left panel shows the increase in exit rates after a short recession. The economy starts at the stochastic steady-state and experiences five successive low aggregate shocks $x_B$ at $t = \{1, 2, 3, 4, 5\}$. Exit rates in the figure are annualized and expressed in percent. The right panel shows the increase in the ratio of the volatility of the stochastic discount rate to its mean $\sigma(\pi')/E[\pi']$, and corresponds to the maximum Sharpe ratio in this economy.

This channel leads to three predictions. The first implication is that, with moral hazard, risk-premia depend on the sequence of aggregate shocks. The sharp dependence of past shock sequence is seen by comparing the right panels of Figure 1-7 and Figure 1-8. As the right panel of Figure 1-8 shows, the maximum Sharpe ratio in the economy\textsuperscript{13} increase from 0.26 to only 0.28. However, after four successive realizations of $x_B$, the increase in risk premia is much larger. A fifth negative shock increases the maximum Sharpe ratio from 0.39 to 0.45. The frictionless economy does not exhibit such dependence on past aggregate shocks. With no moral hazard, risk premia and growth rates of aggregate quantities take on only two values.

Second, the economy with moral hazard shows a higher sensitivity to negative shocks than positive ones, and this sensitivity increases with a successive sequence of negative shocks\textsuperscript{14}. Figure 1-9 shows this asymmetry for a single shock. The solid,

\textsuperscript{13}This quantity is the ratio of the conditional volatility of the stochastic discount factor to the conditional mean.

\textsuperscript{14}Aggregate investment is pro-cyclical both with and without the agency problem, because of increased exits during recessions and lower levels of entry.
Figure 1-9: Asymmetric response of investment and output to a one standard deviation aggregate shock. The left panel shows the impulse response of aggregate investment to a single shock at time zero. In the initial state, aggregate productivity is high ($x = x_G$) and the cross-sectional distribution of continuation values is the stochastic steady-state of the economy. The dashed red curve shows the response to a negative shock, while the solid blue curve is the response to a positive shock. The right panel shows the same for output. Notice the asymmetric response.

The blue curve shows aggregate investment response to a one standard deviation, single, positive shock. The initial state of the economy is the stochastic steady-state with high current productivity. The shock produces an immediate increase in investment of 4.1%. The lower dotted, red curve shows that a negative shock of the same magnitude produces a 5.5% drop (an asymmetry of 1.35). The asymmetric response is due to terminations occurring in states in which the investors’s marginal utility is very high. With a risk-averse investor, a negative realization of the aggregate shock has two effects. First, it lowers this period’s output because of low productivity. Second, because the exit rate goes up precisely when marginal utility is high, the value of entering goes down. This leads to even lower investment. Non-zero risk-premia causes investment to respond asymmetrically to positive and negative shocks.

The asymmetry is much more pronounced for five successive low shocks compared to five successive high shocks as seen from Figure 1-10. This is because of much higher consumption volatility after five successive $x_B$, which lowers the value to entry, and
Figure 1-10: Asymmetric response of investment and output to five successive one standard deviation aggregate shocks. The left panel shows the impulse response of aggregate investment to five successive shocks at \( t = \{-4, -3, -2, -1, 0\} \). In the initial state at \( t = -4 \), aggregate productivity is high \( (x = x_G) \) and the cross-sectional distribution of continuation values is the stochastic steady-state of the economy. The dashed red curve shows the response to negative shocks, while the solid blue curve is the response to positive shocks. The right panel shows the same for output. Notice the larger asymmetric response compared to Figure 1-9 and the much longer recovery time.

Therefore aggregate investment. The time it takes investment to revert back to its normal rate is more than five times longer than the recovery time for a single shock. The qualitative behavior is similar for investment (left panel) and output (right panel), except that the effect on output is more long lasting because of the lingering effect of low investment in past periods.

Third, exit rates predict future growth rates after controlling for current aggregate productivity. Table 1.3 shows this for one period ahead growth rates of output and investment. This is because current aggregate productivity is not a sufficient statistic for the state of the economy. The exit rate, or any other observable which depends on the cross-sectional distribution of continuation values, provides incremental information. Campbell (1998) finds that exit rates predict future growth rates of output at both short and long horizons.
1.5 Conclusion

In this essay, I show that moral hazard produces quantitatively large distortions on key aggregate quantities and asset prices. The conditional dynamics of the economy with this agency problem depends strongly on the cross-sectional distribution of financial slack of firms in the economy. In contrast to the frictionless economy, aggregate quantities and risk premia depend on the sequence of past aggregate shocks, in particular, showing an increasing sensitivity to a long sequence of negative shocks. In my model, deep recessions or rare disasters are the result of a series of successive small negative shocks, rather than a single large shock.

There are several interesting avenues for future research. In the next chapter I extend the model to allow for investment at the firm level along the lines of Clementi and Hopenhayn (2006). That model makes predictions about the joint behavior of investment and financial slack in the cross-section in the presence of aggregate uncertainty. The model quantitatively analyzes the impact of moral hazard on capital accumulation. Another important extension would be to relax the assumption of two-sided commitment assumed here.
Table 1.1: Parameter values: Values of all parameters used in simulations. The model is calibrated at quarterly frequency. The manager's time preference parameter is $\beta_e$, private benefit from low effort is $B$, and $\Delta p$ is the amount by which the probability of drawing the high firm-specific shock $z_+$ is reduced when the manager shirks. The investor's time preference parameter is $\beta_i$ and has power utility with risk-aversion $\gamma_i$. The technology parameters are: firm-specific productivity $z_{\pm}$, with $p$ the probability of drawing $z_+$, and $|z_+| = |z_-|$; aggregate productivity $x_{G,B}$ with Markov transition intensities $\zeta_{G,B}$. $\delta$ is the maintenance cost of capital each quarter. $\chi$ is the fraction of capital that is lost on firm termination.

| $\beta_e$ | $B/\Delta p$ | $\beta_i$ | $\gamma_i$ | $|z_{\pm}|$ | $p$ | $x_G$ | $x_B$ | $\zeta_G$ | $\zeta_{B}$ | $\delta$ | $\chi$ |
|-----------|--------------|-----------|-----------|-------------|----|--------|--------|------------|-------------|--------|------|
| 0.92      | 0.03         | 0.99      | 7         | 0.20        | 0.5| 0.031  | 0.030  | 0.05       | 0.30        | 0.01   | 0.15 |
Table 1.2: Unconditional Moments: This table reports unconditional moments of annual growth rates of consumption ($ΔC = C_{t+1}/C_t - 1$), output ($ΔY = Y_{t+1}/Y_t - 1$), and investment ($ΔI = I_{t+1}/I_t - 1$). All data is from the period 1929 – 1998. The entries for $ΔC$, $ΔC/ΔY$, and asset pricing moments are from Table 2 of Kaltenbrunner and Lochstoer (2010). Investment data is from BEA. The exit rate data is the issuer-weighted corporate default rate from Moody’s over the same period. Model results are generated by 1000 independent panels of $N = 10^5$ firms over a time period of 300 quarters. Results are averages across independent panels. Equity return is the return for a claim on aggregate household’s consumption. All values, except the investment to output ratio and the maximum Sharpe ratio are in percentages.

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
<th>Frictionless</th>
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<tbody>
<tr>
<td>$σ(ΔC)$</td>
<td>2.70</td>
<td>2.47</td>
<td>1.30</td>
</tr>
<tr>
<td>$\frac{σ(ΔC)}{σ(ΔY)}$</td>
<td>0.52</td>
<td>0.63</td>
<td>0.86</td>
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<tr>
<td>$I/Y$</td>
<td>0.19</td>
<td>0.21</td>
<td>0.29</td>
</tr>
<tr>
<td>$\frac{σ(ΔI)}{σ(ΔY)}$</td>
<td>3.32</td>
<td>2.70</td>
<td>1.30</td>
</tr>
<tr>
<td>$ρ(ΔI, ΔY)$</td>
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<td>0.87</td>
<td>1.00</td>
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<tr>
<td>Exit rate</td>
<td>1.2</td>
<td>1.4</td>
<td>0</td>
</tr>
<tr>
<td>$σ$(Exits)</td>
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<td>2.8</td>
<td>0</td>
</tr>
<tr>
<td>$ρ$(Exits, $ΔY$)</td>
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<td>-0.81</td>
<td>-</td>
</tr>
<tr>
<td>$E[r']$</td>
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<td>1.9</td>
<td>8.0</td>
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<tr>
<td>$E[r^c - r']$</td>
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<td>3.6</td>
<td>0.7</td>
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<tr>
<td>Sharpe ratio</td>
<td>0.33</td>
<td>0.27</td>
<td>0.09</td>
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</table>
Table 1.3: Predicting output and investment growth: This table reports simulation results showing additional information about the aggregate state beyond current realized productivity. Model results are generated by 1000 independent panels of $N = 10^5$ firms over a time period of 280 quarters. Results are averages across independent panels. t-stats are computed using Newey-West standard errors with 5 lags, and reported in brackets.

<table>
<thead>
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<th>Exit rate($t$)</th>
<th>$x_t$</th>
<th>$R^2$</th>
</tr>
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<tbody>
<tr>
<td>log($Y_{t+1}/Y_t$)</td>
<td>0.27</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>(2.1)</td>
<td>(3.1)</td>
</tr>
<tr>
<td>log($I_{t+1}/I_t$)</td>
<td>0.76</td>
<td>3.7</td>
</tr>
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<td></td>
<td>(2.4)</td>
<td>(4.2)</td>
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</table>
Appendix: Computational Algorithm

Frictionless economy

The unknowns are $c(x_i)$, $f(x_i)$, and $\bar{e}(x_i)$. First, eliminate $c(x_i)$ using $c(x_i) = x_i - \frac{h}{2} \bar{e}^2(x_i) - \delta$ and $\bar{e}(x_i) = c(x_i)^n f(x_i)$ and from the definition of $f(x_i)$ we have the system

$$c(x_i) = x_i - \frac{h}{2} \bar{e}^2(x_i) - \delta,$$

$$f(x_i) = \bar{e}(x_i) \left[ x_i - \frac{h}{2} \bar{e}^2(x_i) - \delta \right]^{-n},$$

$$f(x_i) = E_0 \left[ \sum_{s=1}^{\infty} \beta^s \left( c(x_s) \prod_{j=1}^{s} \left( 1 + h\bar{e}(x_j) \right) \right)^{-n} x_s | x_0 \right].$$

The model has two aggregate states $x = \{x_1, x_2\}$ on a Markov chain with transition probability $\Gamma$. I compute the two constants $f_1$ and $f_2$ by solving the system using a standard root-search technique.

With moral hazard

In the presence of aggregate shocks, the cross-sectional distribution of manager continuation values in this economy varies with time, and is therefore, a state-variable. I approximate the distribution as a mixture of two normal distributions, $N(m_1, \sigma_1)$ and $N(m_2, \sigma_2)$, where $m_1 < m_2$ by convention. Apart from the two means and standard-deviations, the relative mixture of the normal distribution with lower mean, $m_1$, is also relevant. I denote the mixture by $\eta$. The five parameters, $(\eta, m_1, \sigma_1, m_2, \sigma_2)$, is a low-dimensional proxy for the distribution $\mu(V)$. I computed the equilibrium fixed point in two different ways. First, by allowing all five parameters to vary, and second by fixing the values of $\sigma_1$ and $\sigma_2$. In the latter case, the values $\sigma_1$ and $\sigma_2$ were set equal to the values obtained by fitting a mixture of normals to the stochastic steady-state distribution. There was no substantial difference in dynamics of aggregate quantities and the pricing kernel between the two solutions. Since there is a tremendous loss in
computational speed with no substantial gain in accuracy from treating \( \sigma_1 \) and \( \sigma_2 \) as
dynamic variables, I fixed them at the stochastic steady-state values. The shape of
the distribution \( \mu(V) \) with this approximation is given by the parameters \((\eta, m_1, m_2)\),
with the three parameters defined on a discrete grid.

The equilibrium is computed iteratively:

1. I start with a guess for the law of motion of the three parameters approximating
the distribution:

\[
\eta' = \mathcal{H}_\eta(x', x, \eta, m_1, m_2) \\
m_1' = \mathcal{H}_{m_1}(x', x, \eta, m_1, m_2) \\
m_2' = \mathcal{H}_{m_2}(x', x, \eta, m_1, m_2),
\]

with a similar equation for the pricing kernel \( \pi'(s')/\pi(s) \). Each of the above
functions is defined on a discrete grid. For intermediate points, I use linear
interpolation.

2. Next, I solve for the optimal contract policies Eq. 2.4. The above set of equa-
tions, is used for the law of motion of the aggregate state \( \mu' = \mathcal{H}(x', x, \mu) \).

3. For each point on the discrete grid over \((\eta, m_1, m_2)\), I compute next period’s
distribution using the policy functions in step 2, together with entry and exit.
Note that this depends also on the current aggregate shock \( x \) and next period’s
shock \( x' \). I fit the new distribution with a mixture of normals using maximum
likelihood. This updates the guess for \( \mathcal{H} \) above. I also compute the ratio of
marginal utilities \( \beta_i(C(s')/C(s))^{-\gamma_i} \), which provides an updated guess for the
pricing kernel at the current grid-point.

I iterate these steps till both the pricing kernel and the law of motion of \((\eta, \mu_1, \mu_2)\)
satisfy a convergence criterion.
Bibliography


Chapter 2

Capital Accumulation with Optimal Long-term Financing

2.1 Abstract

In this chapter I quantitatively analyze the cross-sectional implications of optimal dynamic contracting between risk-averse investors and firm insiders in a dynamic general equilibrium model. I make two changes compared to chapter one. First, I allow for firm-level investment. Second, I model firms to use a decreasing returns to scale technology instead of a linear one. My model makes two predictions on steady-state capital accumulation which are consistent with empirical evidence. First, I show that conditional on survival, younger firms are smaller and have higher expected growth rates. Second, investment rates in small firms are more volatile and more sensitive to realized cash flows than large firms. In the presence of aggregate shocks, my model predicts that the conditional aggregate dynamics of key macro-economic quantities depends on the cross-sectional distribution of firm-level investment rates. Controlling for aggregate productivity, states of the economy in which a higher fraction of firms have lower financial slack are characterized by low aggregate output, investment, and consumption, and higher risk-premium. I provide quantitative estimates in a calibrated model.
2.2 Introduction

Investment and financing are two of the most important choices made by firms. In this chapter, I quantitatively examine the interaction of firms’ investment and financing decisions in the presence of financial frictions. Prior research on the effect of borrowing constraints on firm level investment fixes the form of financing, usually single period-debt, between investors and firms. In reality, financing choices are an equilibrium outcome of the interaction between investors and firms. The two parties choose an arrangement which allows them to best achieve their objectives, subject to constraints induced by frictions such as agency problems or information asymmetry. I therefore motivate financial constraints to arise endogenously as a result of an agency problem between outside investors and firm insiders. The financial contract in my model is long-term and optimal given the agency problem. This approach provides a natural benchmark for quantifying the effect of financial frictions on firm dynamics that is free of ad-hoc assumptions on the financial arrangement between firms and investors.

My model has four main ingredients. First, I consider a heterogenous cross-section of firms which use a decreasing returns to scale technology for production. They are exposed to productivity risk which consist of both aggregate and firm specific shocks. Second, financing constraints arise endogenously because of a moral hazard problem: firm insiders can divert a fraction of firm output and consume it privately. To provide incentives to insiders to share firm output, investors enter into a long-term financial contract with insiders. Third, under the optimal contract firms can sometimes be liquidated. Although ex-post inefficient for investors and insiders, liquidations are necessary to provide incentives. When a firm is liquidated, investors recover a fraction of firm value and insiders get their outside option. Fourth, the investor uses a stochastic discount rate to discount cash flows to value the contract. This discount rate is endogenous, and determined in equilibrium by the investor’s consumption and savings decision. In short, I integrate a dynamic principal agent problem within a stochastic growth model.

I show that moral hazard has a first-order quantitative effect on cross-sectional
investment rates. My model makes two predictions on steady-state capital accumulation which are consistent with empirical evidence. First, I show that conditional on survival, younger firms are smaller and have higher expected growth rates. Second, investment rates in small firms is more volatile and is more sensitive to realized cash flows than large firms. Both of these predictions are in line with empirical evidence documented in Evans (1987) and Hall (1987). The intuition for these results is simple. Older firms have survived due to good performance. The optimal contract rewards firm insiders for good performance by promising a bigger share of future firm output. This relaxes the agency problem, making investment rates less sensitive to firm performance. As a result, investment volatility decreases and firm growth rates drop. My model also predicts economies with more severe agency problems to have higher cross-sectional dispersion in firm-level investment rates.

In the presence of aggregate shocks, my model predicts that controlling for aggregate productivity, states of the economy in which a higher fraction of firms have lower financial slack are characterized by low aggregate output, investment, and consumption, and higher risk-premium\(^1\). I provide quantitative estimates in a calibrated model. The intuition for these results is similar to that in chapter 1. With low aggregate shocks, more firms lose financial slack leading to a higher rate of firm liquidations and riskier consumption. The optimal contract is also similar and exhibits a reward-for-luck feature in which insiders are rewarded for aggregate luck.

This chapter is related to two strands of existing literatures. The first examines the effect of financial frictions on macroeconomic quantities and asset prices. The second is optimal dynamic contracting. Chapter 1 contains a list of relevant references. The recent work of Khan et al. (2014) shares the same goal as this chapter. The key difference is that while these authors assume that firms are financed by single period debt, I allow contracts to be long-term and optimal. The optimal contract in my model is related to the partial equilibrium model of Clementi and Hopenhayn (2006). My contribution to this literature is that I examine the effect of contract policies

\(^1\)Intuitively, financial slack is a measure of the resilience of firms to survive negative cash-flow shocks. In my model it is measured by the continuation value of firm insiders.
in an environment in which the discount rate that the investor uses to value future cash flows from the contract is stochastic and determined in general equilibrium from consumption smoothing motives.

The rest of this essay is organized as follows. In Section 2.3, I describe the model. In Section 2.4, I describe quantitative properties of the solution using realistic parameters. Section 2.5 concludes.

2.3 Model

In this section I describe a general equilibrium with a continuum of firms who finance production by entering into long-term contracts with outside investors. I begin by describing the production sector with technology and investment, describe details of the financial contract, and finally, close the model with a description of the household sector.

2.3.1 The Environment

Production Sector

There is a continuum of firms in the \([0, 1]\) interval which produce a single homogenous good which can be used both for consumption and investment. Firms are operated by insiders (the manager) who use a decreasing returns to scale technology

\[
y_t^i = (x_t + z_t^i)(k_t^\nu n_t^{1-\nu})^\alpha,
\]

where \(x_t\) is an aggregate shock common to all firms and \(z_t^i\) is a firm-specific shock. \(x\) is a discrete Markov process with transition matrix \(\Gamma\)

\[
Pr(x_{t+1}|x_t) = \Gamma.
\]

For simplicity, I assume that the aggregate shock \(x\) takes two values \(x = \{x_G, x_B\}\). The firm-specific shock also takes on two values \(z_+ > z_-\), is normalized to have zero
mean, and is independently distributed across time and in the cross-section.

Each firm needs capital $I_0$ to be installed before operations can begin. Firm output is privately observable by the manager. The moral hazard problem is that the manager could divert firm output for private consumption (not observed by investors). Diversion is not perfectly efficient and only a fraction $\lambda$ of diverted resources is available for the manager’s consumption. The parameter $\lambda$ measures the severity of the agency problem. To provide incentives, investors will sometimes liquidate a firm. Liquidation is inefficient to both the manager and investors. When a firm is liquidated, the contract with the existing manager is terminated and his consumption permanently drops to zero. The firm undergoes costly restructuring which results in a loss $\chi I_0$ of firm assets\(^2\). After restructuring, investors hire a new manager who is randomly drawn from the pool of workers. A new contract is initiated such that investors break even.

The financial contract between the investor and the manager specifies: (a) per-period payment that the manager will receive, (b) the amount of working capital the investor will provide to the firm, and (c) the probability of termination of the contract each period. All of these policies are functions of the history of the individual manager’s output together with the history of output of all the managers in the economy. In addition, investors pay for the initial installation sunk cost $I_0$. I assume that the law of large numbers holds, so that the total output in any period is determined by the aggregate shock $x_t$. Contract policies depend on the discount rate that the investor uses to value future cash-flows. This is determined by the representative household’s equilibrium consumption process. Non-linear payout policies and the possibility of exits makes the household’s future consumption depend on the entire cross-sectional distribution of firms (not just the mean). The aggregate state, which I denote by $s$, is captured by the aggregate shock $x$ and its entire past history. The contract assumes two-sided commitment in which both the manager and the investor agree to abide by the terms of the contract in all possible contingencies, with no possibility of renegotiation. While this is a restrictive assumption, it provides a useful benchmark.

\(^2\)The measure of firms remains invariant over time.
and can be thought of as a limiting case in which renegotiation is extremely costly. This would be the case if the investors consist of a large dispersed pool of individuals.

For simplicity, I make three additional assumptions about the manager: (a) he values a consumption stream \( \{c_t\} \) as \( \sum_t \beta^t c_t \) with time preference parameter much smaller than that of the outside investor \( \beta_a << \beta_t \), (b) he has limited liability, and (c) he does not participate in financial markets.

Firms rent labor in the spot market. Labor choice is static. Each firm takes wages as given, and demands labor

\[
\arg \max_{n_t^i} y_t^i - w_t n_t^i ,
\]  

(2.3)

where \( w_t \) is the wage rate. Capital installed in each period depreciates fully after production in the following period. This is a simplifying assumption because it avoids the need to track the cross-sectional distribution of capital.

The Investor’s problem and timing

The financial contract between the outside investor and the firm manager serves three purposes: it provides for initial installation/restructuring costs, provides the firm with working capital (investment), and also provides for the manager's consumption. Following Spear and Srivastava (1987) and Green (1987), I use the dynamic programming approach to solve for the optimal contract. In this recursive formulation, the present discounted value of the future payments to the manager is a sufficient statistic for the entire past history of firm-specific realizations of \( z \). I denote the manager's continuation value by \( V \). It is the present value of future promised payments, and may be interpreted as the financial slack of this firm. Investors value a contract using the representative household’s marginal utility process. The household’s consumption depends on the aggregate output net of payment to all managers in the economy and loss from firm liquidations. Contract policies therefore depend not only on the manager’s continuation value, but also on the cross-sectional distribution of continuation values of all managers in the economy.
In the presence of aggregate risk, i.e. time varying $x$, the cross-sectional distribution of manager continuation values varies over time. I denote the cross-sectional distribution by $\mu$. This is the distribution at the beginning of a period before any restructuring has taken place. The aggregate state $s = \{x, \mu\}$ therefore includes both the current aggregate shock $x$ and the cross-sectional distribution $\mu$. In addition to depending on $V$, contracts are conditioned on the entire aggregate state $s$. The investor’s problem is to offer a contract to the manager that maximizes the present discounted value of future cash flows

$$F_i(k, V; s) = \max_{\zeta, k', d_\pm, V_\pm} E^\Gamma \left[ \zeta \left[ (1 - \chi) I_0 \right] \right. $$
$$\left. + \left( 1 - \zeta \right) \left( -k' + R k^\alpha \right) \right. $$
$$\left. - p d_+ - (1 - p) d_- + \frac{\pi_{t+1}(s')}{\pi_t(s)} F_{t+1}(k', V_\pm(s'; s')) \right]$$

$$V = E^\Gamma \left[ \left( 1 - \zeta \right) \left( p (d_+ + \beta^\sigma V_+') + (1 - p) (d_- + \beta^\sigma V_-') \right) \right],$$

$$d_+ + \beta^\sigma V_+' \geq d_- + \beta^\sigma V_-' + \lambda(R_+ - R_-) k^\tilde{\alpha}, \forall s,$$

$$s' = \{x', \mu'\}, \quad x' = \Gamma x, \quad \mu' = \mathcal{H}(x', x, \mu),$$

$$\left\{ \zeta, k', d_\pm, V_\pm \right\} \in [0, 1] \times R^3_+, \quad (2.4)$$

where $R(w_t) = \left[ \frac{\alpha (1 - \nu) \omega t}{\omega t^{(1 - \nu)}} \right]^{\frac{1 - \alpha(1 - \nu)}{1 - \alpha}}$ and the constant $\tilde{\alpha} = \alpha \nu / \left[ 1 - \alpha (1 - \nu) \right]$. For compactness, I have suppressed the dependence of all choice variable $\zeta, k', d_\pm, V_\pm$ on the aggregate state. In the above equations, $F_i$ is the investor’s value of the contract. The choice variables are the termination probability of the manager $\zeta$, investment $i = k'$, the manager’s payments $d_\pm$ and continuation value $V_\pm$. Each of these depend on the manager’s continuation value $V$, on the realized aggregate state $s'$, and the firm-specific shock $z_\pm$. The expectation $E^\Gamma[\cdot]$ is computed assuming the transition probability matrix $\Gamma$ for aggregate shocks $x$.

The first-line in Equation 2.4 is the investor’s Bellman equation. Future cash-flows are discounted using the representative household’s discount rate $\pi(s)$. The first term is the investor’s payment if the contract is terminated. The investor’s proceeds from
liquidation is a fraction \((1 - \chi)I_0\) of firm assets installed when the firm began operations. The second term is his claim on firm output when the firm is not liquidated. It consists of current output net of investment and payment to the manager. The second line of Equation 2.4 is the promise-keeping constraint which ensures that the manager's present continuation value is the discounted sum of his future payments. In this line, I have implicitly assumed that if the contract is terminated, the manager receives his outside option of zero (his future consumption is set to zero). The third line is the incentive compatibility constraint. The law of motion of the aggregate state \(s\) includes the law of motion of the cross-sectional distribution of manager continuation values, \(H(x', x, \mu)\), and depends both on \(\Gamma\) and on contract policies \(V_x^s\).

In solving the contracting problem, agents take this law of motion as given. In the rational expectations equilibrium, the law of motion used by agents is consistent with the realized law of motion. Likewise, investors and the manager take the discount rate \(\pi(s')\) as given. In general equilibrium, the latter is the marginal utility of the representative household evaluated at the equilibrium aggregate consumption level. In the model the household has constant relative risk-aversion \(\gamma\), therefore, \(\pi(s')/\pi(s) = \beta^t(C^*_{t+1}(s')/C^*_t(s))^{-\gamma^t}\). The quantity, \(C^*\) is the equilibrium aggregate consumption of the household, and is determined by the household's consumption and savings decision. \(^3\)

Timing in this economy is shown in Fig 2-1. At the beginning of each period, the aggregate shock \(x\) is observed. In accordance with the contractual agreement, manager continuation values (discounted utility of future payments) are adjusted depending on the realized aggregate state. The aggregate state depends not only on

\(^3\)In valuing contracts and determining optimal policies, the only stochastic component of cash flows priced by investors are those associated with systematic market-wide risk factors. We can either view the representative household as holding a well-diversified portfolio consisting of all the contracts and a risky-free asset in zero net supply, or picture identical, individual investors each entering into a long-term contract with a single manager. In the latter case, individual investors act as pass-throughs – they collect payments and pass them on to the representative household. Since they belong to the same risk-sharing household, all investors use the same discount rate to value cash flows. Both these pictures have identical pricing implications: the idiosyncratic component of cash flows of individual contracts can be completely diversified away, and are therefore not priced. The household's aggregate consumption process is the only systematic risk factor used in pricing risky cash flows.
the realized value of the aggregate shock $x$, but also on the cross-sectional distribution of the continuation values of all managers in the economy. A public lottery for termination of the contract is held in which firms with low continuation values might exit. Production takes place next, i.e. firms realize $z$ and produce output. Finally, managers are paid, working capital for the next period is provided to the firm. Firms which were liquidated undergo restructuring (new contracts are initiated), and the representative household consumes.

I assume perfectly competitive lending markets so that investors break even. Managers have all the bargaining power and choose the highest possible payoff subject to
the investor’s participation constraint

\[ V_0 = \sup \{ V : F(V) \geq I_0 \}. \] (2.5)

**Households**

I assume a continuum of workers of unit measure. Each worker supplies \( N \) units of labor inelastically in the competitive labor market. Workers are members of a single risk-sharing household and therefore, they share the same stochastic discount factor. This household derives utility from consumption of a single good \( C_t \) and maximizes

\[ E_0 \sum_{i=0}^{\infty} \beta_i^t \frac{C_t^{1-\gamma}}{1-\gamma}. \]

The household derives income from accumulated wealth and wages, and makes consumption and investment decisions to maximize expected lifetime utility subject to the household’s budget constraint.

**2.3.2 General Equilibrium**

The state-space of the Markov, perfect, competitive equilibrium includes the cross-sectional distribution of manager continuation values, which is infinite dimensional. The formal definition of the equilibrium is given below:

**Definition 2**Recursive equilibrium – A recursive competitive equilibrium is defined as a set of functions for (i) the initial contract state \( V_0(x,\mu) \), (ii) contract policies \( \Phi(k,V,x,\mu) = \{k'(k,V,x,\mu), V_+(k,V,x,\mu), d_+(k,V,x,\mu), \zeta(k,V,x,\mu)\} \), (iii) consumption policies of the representative household \( C(x,\mu) \), and (iv) law of motion of states \( s' \sim \{x',\mu'\} \), such that (i) individual contracts are optimal, (ii) the initial state is such that the investor breaks even (Eq. 2.5), (iii) the representative household’s policies are optimal (Eq. 2.4) subject to the household’s budget constraint,
(iv) the labor market clears

\[ N = \int n_t d\mu_t, \quad (2.6) \]

(v) the goods market clears

\[ C^*(s) = \int [x(k^n n^{1-\nu})^\alpha - d(s)] d\mu - I(s) - L(s), \quad (2.7) \]

where aggregate investment \( I \) and loss from liquidation \( L \) are given by

\[ I(s) = \int k'd\mu, \quad (2.8) \]

\[ L(s) = (1 - \chi)I_0D(s) \int d\mu, \quad (2.9) \]

\( D(s) \) is the rate of termination of contracts\(^4\),

(vi) the market for contracts clears

\[ W^*_c = \int F_c(V)d\mu_t \quad (2.10) \]

where \( W^*_c \) is the household’s wealth invested in financial contracts, (vii) the bond market clears

\[ W^*_b = 0 \quad (2.11) \]

where \( W^*_b \) is the household’s wealth invested in the risk-free asset, and (viii) the law of motion of the cross-sectional distribution of continuation values, \( \mathcal{H}(x', \mu) \) is consistent with individual contract policies and the stochastic process for \( z \).

The following proposition summarizes properties of asset returns and the risk-free rate in the economy.

**Proposition 5** The equilibrium stochastic discount factor in this economy is defined by \( \frac{\pi_{t+1}}{\pi_t} = \beta_t \left( \frac{C^*_{t+1}}{C^*_t} \right)^{-\eta} \), where \( C^*_t \) is the equilibrium consumption of the representative household. All gross returns \( R^i \) in this economy, satisfy the no-arbitrage rela-

\(^4\)The assumption of full depreciation implies investment \( i = k' \).
tion \( E_t \left[ \frac{R_{t+1}}{\pi_t} R_t \right] = 1 \). The risk-free rate, in particular is given by \( R_t = 1 + r_t = 1/E_t \left[ \frac{R_{t+1}}{\pi_t} \right] \).

**Proof.**

The household takes the risk-free rate, \( r_t \), and the ex-dividend contract prices \( F_t \) as given. The household's Bellman equation is

\[
U_t(g, \tilde{b}, s) = \max_{C_t \geq 0, g', \tilde{b}'} \left[ u(C_t) + E[\beta U_{t+1}(g', \tilde{b}', s')] \right], \quad s = \{x, \mu\},
\]

subject to the budget constraint

\[
w_tN + g_t + \sum_{j \in \text{Continue}} b_t^j (F_t^j + y_t^j - d_t^j - \tilde{b}_t^j) - \sum_{j \in \text{Restructure}} (1 - x) I_0 = \frac{g_{t+1}}{1 + r_t} + \sum_{j \in \text{Continue}} b_{t+1}^j F_t^j + C_t
\]

where \( g \) and \( \tilde{b} \) are the household's holding of risk-free asset and the long-term contracts, \( d_t^j \) is the payment received by the manager from contract \( i \) in period \( t \), \( \tilde{b}_t^j \) is working capital advanced to firm \( j \), and \( w_t \) is the wage rate. The aggregate state \( s \) explicitly depends on the aggregate shock \( x \) and also on the distribution of continuation values of surviving managers. The expectation is over realizations of \( x' \) given that the current shock is \( x \), with exogenously specified transition probability matrix \( \Gamma \).

By "continue" I mean only those firms which were not restructured this period. The law of large numbers is assumed to hold, so that \( C_t \) does not depend on individual realizations of firm-specific shock \( z \).

**Market clearing:** In equilibrium, the bond market, the market for financial contracts, and the labor market clear

\[
g_t = g_{t+1} = 0, \quad b_t^j = b_{t+1}^j = 1, \quad N = \sum_{j \in \text{Continue}} n_t^j.
\]
Substituting the market clearing conditions into the budget constraint Eq. 2.13, prices \((r_t, w_t, \text{ and } F^i_t)\) drop out, and the goods market clearing condition becomes

\[ C_t + \sum_{i \in \text{Continue}} d_t^i = Y_t - I_t - L_t, \]

where

\[ I_t = \sum_{i \in \text{Continue}} i_t^i, \]
\[ L_t = \sum_{i \in \text{Restructure}} (1 - \chi) I_0, \]

where the fractional loss \(0 \geq \chi \geq 1\). This coincides with the relation in the main text Eq. 2.7 with aggregate investment \(I_t\) and loss from liquidation \(L_t\). I assume that entering firms start production in the period subsequent to their entry.

**Optimality conditions:** Prices \(F^i\) and \(r_t\) get determined by the household’s first-order conditions

\[ u'(C_t) F^i_t = \beta^i E\left[u'(C_{t+1})(r_{t+1}^i + F_{t+1}^i)\right], \quad 1 = E\left[\beta^i u'(C_{t+1})/u'(C_t)(1 + r_t)\right], \]

where \(r_{t+1}^i = y_{t+1}^i - d_{t+1}^i - i_{t+1}^i\) is the investor’s cash flow after investment and paying the manager. The stochastic discount factor used to discount cash flows is, therefore, \(\pi_t = \beta^i u'(C_{t+1})/u'(C_t).\)

With aggregate shocks the distribution of continuation values (and therefore manager payments), changes over time. There is no steady-state distribution to which the economy settles into, and for this reason, proving the existence of an equilibrium is difficult. This is not unique to the set-up here and is common to incomplete-market models with aggregate uncertainty. The resulting dynamics is quite rich, aggregate quantities and risk premia depend both on the current aggregate shock, and on the cross-sectional distribution of manager continuation values. Assuming an equilibrium exists, Proposition 6 proves that the investor’s value function is concave, and the
marginal cost of providing incentives is bounded from above.

**Proposition 6** The investor's value function $F(V; s)$ is a (weakly) concave function of the manager's promised utility $V$. The slope of $F(V; s)$ is bounded from below by $F'(V; s) \geq -1$ for all $s$.

**Proof.** The proof is exactly the same as in chapter 1. ■

I close this section by characterizing the equilibrium in the frictionless economy. Without the agency problem, current aggregate productivity is a sufficient statistic to characterize the equilibrium.

### 2.3.3 Full Information Benchmark

If none of the diverted output can be used by the manager $\lambda = 0$, the agency problem disappears. Since managers are more impatient than investors, they immediately sell their equity to investors. They are paid their share (which they consume immediately) and operate the technology. With no financial frictions, all firms receive the first-best amount of capital.

**Proposition 7** (Equilibrium allocations) The competitive equilibrium is completely characterized by prices and aggregate quantities which depend only on current aggregate productivity $x_t$.

**Proof.**

In the absence of agency problems, all firms make identical decisions. They take wages $w_t$ as given, and choose labor $n_t$ to maximize operating profit $\pi_t = y_t - w_n_t$. Labor is chosen after observing the aggregate shock $x_t$, but before realization of the firm-specific shock

$$n_t^* = \arg \max [x_t(k^{\nu}n_t^{1-\nu})^\alpha - wn_t] = [R(w_t)/w_t]k_t^{\alpha_1},$$

where $R(w_t) = \left[\frac{\alpha_{1-\nu}(1-\nu)}{\alpha_{1-\nu}(1-\nu)}\right]^{\frac{1}{1-\alpha(1-\nu)}}$ and the constant $\alpha_1 = \alpha \nu / [1 - \alpha (1 - \nu)]$. The firm's operating profit $\pi_t = R(w_t)k_t^{\alpha_1}$. Each period, firms choose working capital $i_t$,
to maximize firm value

\[
\max_{\{i_s\}} E_t \left[ \sum_{s=0}^{\infty} \beta_t^{t+s} \left( \frac{C_t^{*+s}}{C_t^{*}} \right)^{-\gamma_t} \left( R(w_{t+s})k_t^{\alpha_1} - i_s \right) \right].
\]

where I used the household’s equilibrium consumption \( C^* \) for the pricing kernel. Since capital depreciates fully after the next period, \( i_t = k_{t+1}^t \). The optimal capital choice condition is

\[
E_t \left[ \beta_t \left( \frac{C_{t+1}^*}{C_t^*} \right)^{-\gamma_t} \left( R(w_{t+1})\alpha_1 k_t^{\alpha_1-1} \right) \right] = 1. \tag{2.15}
\]

Finally, the goods market clearing

\[
Y_t = x_t \left( k_t^{\nu} n_t^{1-\nu} \right)^{\alpha} = C_t^* + i_t^*, \tag{2.16}
\]

where I have used the fact that there is a unit measure of identical firms. Equilibrium wages \( w_t \) are determined by the labor market clearing condition

\[
n_t^* = \left[ R(w_t)/w_t \right] k^{\alpha_1} = N. \tag{2.17}
\]

Equations 2.15 - 2.17 characterize the frictionless competitive equilibrium.

\[\blacksquare\]

### 2.4 Model Solution

With moral hazard, the dynamics of aggregate quantities and asset prices are richer. Even though firms are ex-ante identical, they differ ex-post because of realized profits. Unlike a static model where incentives are provided by cash payments, in a dynamic setting insiders are provided incentives through promises of higher share of future firm output. Panel A of Figure 2-2 shows this. The solid, blue curve shows the insider’s continuation value when they report high current period output, while the lower dotted, red curve shows continuation value for a report of low current output.

In my model, firms with recent good performance have higher investment rates.
Figure 2.2: Panel A shows the manager's continuation value after realization of output. The solid, blue curve shows continuation value for high output while the dotted, red curve shows continuation value for low output. Panel B shows investment policy as a function of the firm's financial slack $V$. Investors provide capital $k_{t+1}$ at the end of the previous period $t$.

This is because investors reward good performance by committing to share a higher fraction of future firm output with insiders (in other words by increasing the continuation value). This aligns the incentives of the insiders with investors and lowers the loss arising from the agency problem. With higher returns to investment, insiders increase investment to firms with recent good performance. Panel B of Figure 2.2 shows this.

The non-linear dependence of investment on the continuation values of insiders makes it challenging to solve for the equilibrium. It prevents aggregating all firms into a single representative firm. Unlike the frictionless setting where all firms make identical investment decisions, the aggregate state of the economy depends on the fraction of firms that are financially constrained and by how much. In other words, key aggregate macro quantities and risk premium depends on the cross-sectional distribution of financial slack of all firms. I solve the model using the approach used in chapter 1. The parametric family I use for this model is the scaled $\beta$ distribution. I omit computational details because they are very similar to those in chapter 1.
2.4.1 Parameters

Table 2.1 lists all parameter values. The model is simulated at quarterly frequency and all values in the table are quarterly. The aggregate shock process $x_g$, $x_b$, and the investor's preference parameters $\gamma_l, \beta_l$ are chosen to match the first two moments of key aggregate quantities and asset returns. The manager's time preference parameter $\beta_e = 0.92$ is chosen to be lower than the investor's $\beta_l = 0.99$. The transition matrix $\Gamma$, is estimated from NBER quarterly recession data over the period 1929 - 1998, using the method of maximum likelihood. The estimated transition intensity out of the state $x_G$ (labelled $\zeta_G$ in Table 2.1) is 0.05. The transition intensity out of the state $x_B$ (labelled $\zeta_G$ in Table 2.1) is 0.30. The firm-specific shocks $z_{\pm}$ are chosen to occur with equal probability $p = 0.5$, and the magnitude of $|z_{\pm}|$ is chosen to match the average volatility of investment rates $i/k$ in the data of 0.34. A key parameter of my model is $\chi$. It measures the loss incurred by investors when a firm is restructured after liquidation. I set $\chi = 0.15$ and this choice corresponds to a loss of 40% of the value of an unconstrained firm. I set the parameter which measures the efficiency with which insiders can divert resources to be $\lambda = 0.15$. To provide incentives, under the optimal contract, $\lambda$ corresponds to insiders' share of firm equity. I set this value to 15%. This is higher than the average CEO ownership in the US of 5.5% reported by Jenter and Lewellen (2013) because in my model insiders include senior management and others with disproportionate control rights compared to cash-flow rights.

Table 2.2 shows the unconditional moments of aggregate macroeconomic quantities and asset prices. To generate the model moments, I simulate 1000 independent panels of $N = 10^5$ firms over a time period of 280 quarters, and average over these 1000 independent simulations. The volatility of consumption growth is close to the data, and is higher than in the frictionless model. The intuition is similar to chapter 1. Higher consumption risk arises from the sharp decline in investment in bad aggregate states which hinders the ability to smooth aggregate productivity shocks. The drop in investment is due to poor investment opportunities and also due to a higher

---

5The aggregate shock is persistent.
rate of contract terminations in high marginal utility states.

Firm liquidations in my model are counter-cyclical, as in the data and the intuition is straightforward. Although liquidations are inefficient, they are necessary for incentive provisions. In the presence of aggregate shocks, investors find it profitable to keep the firm alive in states with high productivity and liquidate with a higher probability in bad times when current and future expected cash flows are lower. My model is able to generate aggregate liquidation rate of 2.3%, which is comparable to the observed default rate of 1.2% in the data\(^6\).

In my model, firms give the investor a consumption profile which is smoother than output by adjusting investment. Their ability to do so is constrained when productivity is low. This is because of two effects. First, investment is less profitable. Second, the optimal contract features a higher rate of firm liquidations in states in which the marginal utility of the investor is high. This makes investment even less profitable in these states and investment is only partially irreversibility. The level of investment drops sharply in these states, and hinders the ability of firms to smooth aggregate consumption. The volatility of consumption approaches that of output. The model implied volatility of aggregate investment is, however, lower than in the data.

The presence of moral hazard improves the predictions of the frictionless model in regard to the asset pricing moments. The risk-free rate drops due to precautionary savings motive of the investor and the Sharpe ratio increases. The latter is due to the possibility of higher rates of firm liquidations following a sequence of successive bad aggregate shocks. My model therefore endogenously generates rare disasters characterized by very low levels of aggregate output, investment, and consumption, and high levels of risk-premia.

These results show that the first two moments of macro-economic quantities and asset prices are reasonably close to the data counterparts. This implies that the choice of parameters is reasonable. Next, I discuss aggregate dynamics and the behavior of

\(^6\)For the data counterpart of firm exits, I proxy the exit rate by the corporate default rate data from Moody's.
contract policies. First, without aggregate shocks, and then with aggregate risk.

2.4.2 No aggregate uncertainty

Without aggregate uncertainty $x_G = x_B$, the investor's consumption growth rate is a constant. This is the setting considered in Clementi and Hopenhayn (2006), although in a partial equilibrium setting.

My model makes two cross-sectional predictions about firm growth rates which are broadly consistent with data. First, it predicts that, on average, younger firms have higher average investment rates so that they grow faster than older firms. The panel on the left in Figure 2-4 shows average investment rates as a function of firm-age\(^{7}\). The intuition for this is that, on average, investors increase financial slack to prevent costly liquidation. A second prediction of my model is that younger firms have more volatile investment rates. The panel on the right of Fig 2-4 shows this. The intuition for this is that older firms have survived due to good performance. This had the effect of relaxing the agency problem. Investment in older firms is therefore closer to the first-best and less volatile. Both of these predictions are in line with empirical evidence presented in Evans (1987) and Hall (1987).

My model generates a non-degenerate size distribution as a result of the financing friction. Figure 2-3 shows the steady state distribution of firm size. In the model generated size distribution, there is a much higher proportion of medium and large firms compared to the data. This is not surprising, since in reality, firm size distribution is shaped by other factors, not just financial constraints. However, it is interesting that financial frictions can play a quantitatively important role in shaping the distribution.

My model predicts that economies with a more severe agency problem between investors and firm insiders would have a larger cross-sectional dispersion in firm-level investment rates. The intuition for this is the following. Consider two firms in two different economies with the same amount of financial slack $V$. The firm in the

\(^{7}\)In my model, firm age is measured from the time a firm begins operations after restructuring a previously liquidated firm.
Figure 2-3: Firm size distribution in steady-state. Firm size is measured by the amount of working capital \( k_t \). All parameters as in Table 2.1.

The economy with a more severe agency problem (higher value of \( \lambda \)) invests less than its counterpart. As a result, in the economy with a more severe agency problem, the demand for labor is lower which results in lower wages. This enhances the operating revenue of unconstrained firms enabling them to invest more. My model predicts that economies with a higher value of \( \lambda \) have greater cross-sectional dispersion in investment rates.

2.4.3 Aggregate Uncertainty

The qualitative behavior of the economy in the presence of aggregate shocks is qualitatively very similar to chapter 1. I include results for completeness. Figure 2-5 shows the increase in both the level and the sensitivity of the Sharpe ratio with successive low aggregate shocks \( x_B \). Panel A of this figure shows the increase after two successive \( x_B \) shocks, while panel B shows the much larger increase after five such successive shocks. The increase is due to a higher rate of firm exits after a long sequence of low aggregate shocks.

Primitive shocks are amplified more during recessions than booms. Figure 2-6 shows the response of aggregate investment and output to a single aggregate shock.
Figure 2-4: Firm growth rates. The left panel shows the decline in average investment rate of firms with firm age. The right panel shows the decline in volatility of investment rates with firm age. All parameters as in Table 2.1.

The left panel shows the impulse response of aggregate investment to a single shock at time zero. In the initial state, aggregate productivity is high \( x = x_G \) and the cross-sectional distribution of continuation values is the stochastic steady-state of the economy. The dashed red curve shows the response to a negative shock, while the solid blue curve is the response to a positive shock. The right panel shows the same for output. There is a small amount of asymmetry in both investment and output, but the magnitude of this asymmetry is not very big for a single shock.

The asymmetric response is much more pronounced after a longer sequence of shocks. Figure 2-7 shows the distinct asymmetry. This asymmetry arises because of a large increase in consumption risk after five successive \( x_B \) which severely depresses the value of new investment leading to a sharp drop in aggregate investment. The decrease in the investor’s marginal utility from successive realizations of high shocks \( x_C \) is much smaller than the sharp drop in marginal utility from a similar sequence of low aggregate shocks \( x_B \). This makes the economy more sensitive to negative aggregate shocks.
2.5 Conclusion

In this chapter I show that borrowing constraints arising from moral hazard have a first order quantitative effect on firm-level investment dynamics. Conditional on survival, younger firms have investment rates which have higher mean and volatility. In the presence of the agency friction, the conditional aggregate dynamics depends on the cross-sectional distribution of firm-level investment rates. Controlling for aggregate productivity, states of the economy in which a higher fraction of firms have lower investment rates are characterized by low aggregate output, investment, and consumption, and higher risk-premium.

There are several interesting avenues for further research. In particular, it would interesting to relax the assumption of complete depreciation of capital each period. This would make the model more realistic and allow a more realistic calibration. Secondly, it would be interesting to relax the assumption of two-sided commitment. Finally, allowing for the outside option of the investors to vary with aggregate condi-
Figure 2-6: Response of aggregate investment and output. The left panel shows the impulse response of aggregate investment to a single shock at time zero. In the initial state, aggregate productivity is high ($x = x_G$) and the cross-sectional distribution of continuation values is the stochastic steady-state of the economy. The dashed red curve shows the response to a negative shock, while the solid blue curve is the response to a positive shock. The right panel shows the same for output.

tions would be an important extension.
Figure 2-7: Asymmetric response of investment and output to five successive one standard deviation aggregate shocks. The left panel shows the impulse response of aggregate investment to five successive shocks at $t = \{-4, -3, -2, -1, 0\}$. In the initial state at $t = -4$, aggregate productivity is high ($x = x_G$) and the cross-sectional distribution of continuation values is the stochastic steady-state of the economy. The dashed red curve shows the response to negative shocks, while the solid blue curve is the response to positive shocks. The right panel shows the same for output. Notice the larger asymmetric response compared to Figure 8 and the much longer recovery time.
Bibliography


Table 2.1: Parameter values: Values of all parameters used in simulations. The model is calibrated at quarterly frequency. The manager's time preference parameter is $\beta_e$, $\lambda$ is the efficiency of diversion by managers. The investor's time preference parameter is $\beta_l$ and has power utility with risk-aversion $\gamma_l$. The technology parameters are: firm-specific productivity $z_\pm$, with $p$ the probability of drawing $z_+$, and $|z_+| = |z_-|$; aggregate productivity $x_{G,B}$ with Markov transition intensities $\zeta_{G,B}$. $N$ is chosen so that each worker spends a third of his time endowment working. $\chi$ is the fraction of installed capital lost when a firm is liquidated.

| $\beta_e$ | $\lambda$ | $\beta_l$ | $\gamma_l$ | $|z_\pm|$ | $p$ | $x_B/x_G$ | $\zeta_G$ | $\zeta_B$ | $\alpha$ | $\nu$ | $N$ | $\chi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.92 | 0.15 | 0.99 | 4 | 0.20 | 0.5 | 0.99 | 0.05 | 0.30 | 0.85 | 0.29 | 0.33 | 0.15 |

Table 2.2: Unconditional Moments: This table reports unconditional moments of annual growth rates of consumption ($\Delta C = C_{t+1}/C_t - 1$), output ($\Delta Y = Y_{t+1}/Y_t - 1$), and investment ($\Delta I = I_{t+1}/I_t - 1$). All data is from the period 1929 – 1998. The entries for $\Delta C$, $\Delta C/\Delta Y$, and asset pricing moments are from Table 2 of Kaltenbrunner and Lochstoer (2010). Investment data is from BEA. The exit rate data is the issuer-weighted corporate default rate from Moody's over the same period. Model results are generated by 1000 independent panels of $N = 10^5$ firms over a time period of 300 quarters. Results are averages across independent panels. All values, except the investment to output ratio and the maximum Sharpe ratio are in percentages.

<table>
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<tr>
<th>Data</th>
<th>Model</th>
<th>Frictionless</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(\Delta C)$</td>
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<td>2.32</td>
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<tr>
<td>$\sigma(\Delta C)/\sigma(\Delta Y)$</td>
<td>0.52</td>
<td>0.68</td>
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<tr>
<td>$I/Y$</td>
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<td>0.22</td>
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<tr>
<td>$\sigma(\Delta I)/\sigma(\Delta Y)$</td>
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<td>2.18</td>
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<tr>
<td>Exit rate</td>
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<td>2.3</td>
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<tr>
<td>$\rho$(Exits, $\Delta Y$)</td>
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<td>-0.87</td>
</tr>
<tr>
<td>$E[r_f]$</td>
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</tr>
<tr>
<td>Sharpe ratio</td>
<td>0.33</td>
<td>0.21</td>
</tr>
</tbody>
</table>
Chapter 3

Accuracy Verification for Numerical Solutions of Equilibrium Models

This chapter is based on joint work with Leonid Kogan.

3.1 Abstract

We propose a simulation-based procedure for evaluating approximation accuracy of numerical solutions of general equilibrium models with heterogeneous agents. We measure the approximation accuracy by the magnitude of the loss suffered by the agents as a result of following suboptimal policies. Our procedure allows agents to have knowledge of the future paths of the economy under suitably imposed costs of such foresight. This method is very general, straightforward to implement, and can be used in conjunction with various solution algorithms. We illustrate our method in the context of the incomplete-markets model of Krusell and Smith (1998), where we apply it to two widely used approximation techniques: cross-sectional moment truncation and history truncation.
3.2 Introduction

Cross-sectional heterogeneity among households and firms is at the heart of many important economic phenomena. Tractable aggregation of dynamic heterogeneous-agent economies usually requires restrictive assumptions, such as complete financial markets. Financial frictions, such as credit constraints, or unhedgeable sources of risk, such as idiosyncratic labor income shocks, make aggregation impossible and give rise to equilibrium models with irreducible agent heterogeneity. In such models, the cross-sectional distribution of agent characteristics (e.g., household wealth or firm capital and productivity) is a high- or infinite-dimensional state variable.

In a highly influential paper, Krusell and Smith (1998) introduced an approximation technique for handling infinite-dimensional general-equilibrium models. Their method relies on approximating the state of the economy with a low-dimensional state vector, typically keeping track of only a few moments of the cross-sectional distribution of agent-specific variables, e.g., individual wealth. This approximation method has been applied to a number of models (Heathcote et al. (2009) and Guvenen (2011) survey solution methodology and applications). A significant practical limitation of this and alternative approximation methods is that currently there is no reliable general methodology for verifying the accuracy of approximate solutions.\footnote{Two heuristic approaches are commonly used to evaluate approximate solutions. The first approach examines the accuracy of forecasts of the future state of the economy made by the parsimonious model. Forecasts of aggregate states of the economy are compared with actual realizations from the simulation, and accuracy of the forecasts, e.g., their $R^2$, are used to judge the approximation quality. The limitation of this approach is that a high forecast $R^2$ does not guarantee low individual welfare loss. The second approach, due to den Haan and Marcet (1994), considers Euler equation errors under the approximate solution along the simulated path of the economy. Under the null hypothesis that the agent’s policies are optimal, the $L^2$ norm of the Euler equation errors is distributed as a $\chi^2$ random variable, and a standard hypothesis test can be carried out. The limitation of this method is that small Euler equation errors do not imply low welfare loss. As we show in the Appendix, Euler equation errors can be small while sub-optimal policies are infinitely costly in welfare terms.} In this essay we propose a simulation-based method for computing bounds on accuracy of solutions obtained by various approximation methods.

In an approximate equilibrium, agents follow relatively simple policies that avoid the burden of solving a dynamic optimization problem with an high-dimensional state...
space. This implies that their policies are sub-optimal. We measure the accuracy of
the approximate solution as the welfare loss suffered by the agents as a result of using
such sub-optimal policies. This concept of approximation quality has the appeal of
capturing the impact of agents’ errors in economically meaningful terms. It also fits
naturally with the interpretation of the approximate solutions suggested by Krusell
and Smith. They view the approximate solution of the original model as an exact
equilibrium in a near-rational economy, in which agents pursue suboptimal policies.
Such suboptimal policies are plausible as a description of near-rational behavior if
the welfare loss agents suffer because they fail to fully optimize is small, and thus
expanding further resources on improving the policies is economically unjustifiable.
This argument is in the spirit of modeling economic agents as satisficing rather than
optimizing, as in Simon (1978).

Our approach establishes an upper bound on the agents’ welfare loss without
computing the optimal policies, which is typically a prohibitively difficult task. In
particular, we alter the original problem of an agent by enlarging his information set
to allow for perfect knowledge of the future path of the aggregate state process of the
economy while simultaneously penalizing the agent’s objective for such foresight.\footnote{The basic idea of using information relaxations and martingale multipliers to formulate a dual stochastic optimization problem can be traced back to Bismut (1973) (in a continuous-time setting) and Rockafellar and Wets (1976) and Pliska (1982) (in the discrete-time finite horizon setting). Back and Pliska (1987) apply this technique to basic theoretical problems in financial economics. Most of the existing applications of information relaxations deal with the optimal stopping problems, typically in the context of pricing American or Bermudan options, e.g., Davis and Karatzas (1994), Rogers (2002), Haugh and Kogan (2004), and Andersen and Broadie (2004). Rogers (2007) and Brown et al. (2010) extend the information-relaxation idea to general dynamic optimization problems. We use the formulation in Brown et al. (2010), which incorporates both perfect and partial information relaxations and derives penalty processes from the value function of the original problem. This essay is the first to apply the information relaxation approach to approximate solutions of heterogeneous-agent equilibrium models.} The modified problem is much more tractable than the original problem because
the aggregate state of the economy in the modified problem follows a deterministic
process. Moreover, if the penalty for perfect foresight is chosen properly, which we
discuss in detail in the main text, the value function of the modified problem is higher,
in expectation, than the value function of the original problem. We thus obtain an
upper bound on the agent’s welfare, while the lower bound results from following the
sub-optimal policy prescribed by the approximate solution. The gap between the two bounds limits the agent’s welfare loss from above. A small gap indicates that the degree of sub-optimality is economically small, and the approximate equilibrium is indeed near-rational. A large gap, in contrast, does not immediately imply that the sub-optimal policy is grossly inefficient, as it may result from the value function of the modified problem being significantly higher than the value function of the original problem.

To illustrate the potential of our method, we apply it to the original model in Krusell and Smith (1998). It is a stochastic growth model in which individual agents face uninsurable labor income risk as well as aggregate shocks productivity of capital. Krusell and Smith compute an approximate equilibrium by summarizing the cross-sectional distribution of wealth among the agents using only the average per capita level of wealth. We also approximate the equilibrium in this model using history truncation, as in Veracierto (1997) and Chien and Lustig (2010), which entails agents keeping track of a finite history of the recent aggregate productivity shocks.

We quantify the accuracy of both methods. We establish, in particular, that both approximation methods imply relatively low individual welfare loss for most initial configurations of the economy – starting in an aggregate state of relatively high likelihood with the level of the agent’s capital stock within the bulk of the cross-sectional wealth distribution. Both of the approximation methods are designed to describe equilibrium dynamics when the economy is in stochastic steady state. Thus, for the calibrated model under consideration, our simulation method confirms that both approximation techniques are accurate when used for their intended purpose.

Next, we stress-test the performance of the above approximation algorithms by using them to describe the transition dynamics in an economy perturbed away from its steady state. Starting from the steady state of the Krusell-Smith model, we consider two experiments: (i) an unanticipated five-fold permanent increase in the volatility of aggregate productivity shocks; or (ii) an unanticipated 50% reduction in capital stock of all agents in the economy. In the first case, the economy transitions to a new steady state following a permanent regime shift. In the second case, the economy
reverts to the original steady state following a large transient shock.

It is a priori unclear how well the two approximation algorithms may perform under the above experiments. Both algorithms are designed to approximate the equilibrium when the economy is in steady state. The history truncation algorithm, in particular, relies on the assumption that a finite history of aggregate productivity shocks provides accurate information about the cross-sectional distribution, which does not apply to either of the two experiments. Accordingly, we find that in the first experiment the history truncation algorithm leads to welfare loss bounds that are an order of magnitude larger than those computed around steady state of the original model. However, in the second experiment, welfare loss bounds remain comparable to those around steady state.

The Krusell-Smith algorithm is applicable to both experiments in principle, since the average per capital capital stock may provide an accurate summary of the state of the economy even away from its steady state. However, there is no general guarantee of its approximation quality. Our algorithm allows us to quantify how well the Krusell-Smith algorithm performs in each case. We find that, in both experiments, welfare loss bounds increase. However they remain of modest magnitude—comparative to those obtained for the history truncation method around steady state of the original model.

Another contribution of this essay is to extend the information relaxation methodology to problems with recursive Epstein-Zin preferences. Non-separable preferences imply that the timing of the resolution of uncertainty affects agents's welfare, making the direct application of the existing information relaxation methodology infeasible—it is simply impossible to evaluate the agent's objective over a single future path of the economy. We overcome this difficulty using the variational characterization of Epstein-Zin preferences due to Geoffard (1996) and Dumas et al. (2000). Because of the wide-spread use of nonseparable preference in economic models, our analysis substantially expands applicability of information relaxation methods to analysis of general equilibrium models.

The rest of this essay is organized as follows. In Section 3.3 we formulate the
relaxed problem and outline the construction of penalty functions. To illustrate our
approach, we apply it to a model for which the optimal policy is known in closed form.
In Section 3.5 we extend our method to non-separable preferences. In Section 3.4, we
apply our method to the Krusell-Smith model. Section 3.6 concludes.

3.3 Information relaxation

3.3.1 The main idea

In our analysis of approximate equilibria, we apply the information relaxation method
proposed in Brown et al. (2010). We introduce the main ideas of this method in this
section, and refer the readers to Brown et al. (2010) for full technical details.

Consider a standard finite-horizon consumption-savings problem. Time is discrete,
t = \{0, \cdots, T\}. Each period the agent receives a random labor income which takes
two possible values \{y_H, y_L\}. The probability of receiving \(y_H\) is \(p\) each period. The
agent chooses consumption \(c_t\) and stores the rest in a risk-free asset with constant total
return \(R\). We denote the agent’s feasible consumption policy by \(C = (c_0, c_1, \ldots, c_T)\).

At each date \(t\), the agent observes the history of income shocks realized up to
and including this date, denoted by \(y^t = (y_0, y_1, \ldots, y_t)\), but not the future shocks.
All feasible consumption policies must be adapted to the information structure of
the agent, i.e., consumption choices are functions of the observed past histories
of income shocks. Thus, making the agent’s information structure explicit, \(C =
(c_0(y^0), c_1(y^1), \ldots, c_T(y^T))\).

The agent has a time-separable constant relative risk aversion utility function with
curvature \(\gamma\). Let \(w_t\) denote agent’s wealth at the beginning of period \(t\). The agent
solves the dynamic optimization problem

\[
\max \{C: c_t \leq w_t\} \quad \mathbb{E}_0 \left[ \sum_{t=0}^{T} \beta^t \frac{c_t^{1-\gamma}}{1 - \gamma} \right],
\]
where agent’s wealth and consumption satisfy the dynamic budget constraint

\[ w_t = (w_{t-1} - c_{t-1})R + y_t. \] (3.2)

We denote the value function of the above problem by \( V_t(w_t) \),

\[ V_t(w_t) = \mathbb{E}_t \left[ \sum_{t=0}^{T} \beta^t \left( c_t^* \right)^{1-\gamma} \right], \] (3.3)

where \( C^* \) is the optimal consumption policy. We denote the time-0 expected utility as \( V_0 \), suppressing the explicit dependence on initial wealth.

We formulate a relaxed problem by allowing the agent to have access to information about the future realizations of income shocks. The name “information relaxation” reflects the notion that this formulation relaxes information constraints placed on the agent. Specifically, consider a complete information relaxation, whereby we allow the agent to condition her consumption choices on the knowledge of the entire future sequence of income shocks. To distinguish the feasible policies of the relaxed problem from those of the original problem, we denote the former by \( C^R = (c_0^R(y^T), c_1^R(y^T), \ldots, c_T^R(y^T)) \). While providing the agent with such informational advantage over the original formulation, we impose a penalty on the objective function, designed to offset the effect of information relaxation. The penalty is a stochastic process \( \lambda_t \), which depends on the consumption policy and the entire path of income shocks, \( y^T: \lambda_t(C^R, y^T) \). The only requirement we impose on the penalty process is that if the consumption process is chosen to depend only on the information available to the agent, the resulting penalty is non-positive in expectation, i.e.,

\[ \mathbb{E}_0 [\lambda_t (C, y^T)] \leq 0. \] (3.4)

It is easy to see that the value function of the relaxed problem,

\[ V_0^R = \max_{\{C^R; c_t^R \leq w_t^R\}} \mathbb{E} \left[ \sum_{t=0}^{T} \beta^t \left( c_t^R \right)^{1-\gamma} \right], \] (3.5)
subject to the the dynamic budget constraint

\[ w_t^R = (w_{t-1}^R - c_{t-1}^R)R + y_t, \quad (3.6) \]

is at least as high as the value function of the original problem. The reason is that the consumption policy \( C^*, \) optimal under the agent's original problem (3.1-3.2), is also a feasible policy for the relaxed problem (3.5-3.6), and the expected penalty under such policy adds a non-negative term to the agent's expected utility, according to (3.4). Thus, we establish that

\[ V_0^R \geq V_0. \quad (3.7) \]

Next, consider a feasible but suboptimal consumption policy \( \tilde{C}. \) Under this suboptimal policy, the expected utility of the agent is given by

\[ \tilde{V}_0 = E_0 \left[ \sum_{t=0}^T \beta^t (\tilde{C}_t)^{1-\gamma} \right], \quad (3.8) \]

which results in a welfare loss of \( V_0 - \tilde{V}_0. \) To estimate the welfare loss resulting from a suboptimal strategy, we use the inequality (3.7) to conclude that the agent's welfare loss is bounded above by the difference between the value function of the relaxed problem (3.5-3.6) and the expected utility under the suboptimal policy \( \tilde{C}_t(y^t): \)

\[ V_0 - \tilde{V}_0 \leq V_0^R - \tilde{V}_0. \quad (3.9) \]

We thus have a framework for computing bounds on welfare loss resulting from suboptimal strategies: define a valid penalty process for the relaxed problem, and then compare the expected utility of the agent under information relaxation (this problem is deterministic and hence much easier to solve than the original stochastic dynamic optimization problem) with her expected utility under the suboptimal policy of interest. While this formulation is rather general, it is only useful as long as the resulting bound is relatively tight, i.e., as long as the value function of the relaxed problem \( V_0^R \) is not much higher than the expected utility of the agent under the
optimal consumption policy, $V_0$. Brown et al. show that it is possible to make the difference $V_0^R - V_0$ arbitrarily small. In particular, they show (using backwards induction) that under an ideal penalty, $V_0^R - V_0 = 0$.

Brown et al. define an ideal penalty as follows. The penalty is defined for each possible sequence of income shocks, and each possible sequence of consumption choices, without requiring the consumption policy to be non-anticipating. Specifically, consider an arbitrary path of income shocks $y^T$, and a budget-feasible positive sequence of consumption choices $c^T = (c_0, c_1, \cdots, c_T)$. Note that $c^T$ is not a consumption policy, it denotes a sequence of positive real numbers representing a particular path of consumption. The corresponding values of agent's wealth $(w_0, w_1, \cdots, w_T)$ satisfy the dynamic budget constraint (3.6). Given $w_t$, each term in the ideal penalty, $\lambda^*_t(c^T, y^T)$, is defined based on the value function of the original problem:

$$
\lambda^*_t(c^T, y^T) = V_{t+1}((w_t - c_t)R + y_{t+1}) - E[V_{t+1}((w_t - c_t)R + y_{t+1}) | c^T, y^T] \quad (3.10)
$$

$y_0, \cdots, y_{t+1}$ in the above expression are non-random, these are the values of income shocks from the particular path of shocks $y^T$ for which we are defining the ideal penalty, while $\tilde{y}_{t+1}$ is a random variable, the labor income shock at time $t + 1$. The second term on the right in (3.10) is an expectation of $V_{t+1}$ over the possible values of $\tilde{y}_{t+1}$, taking the realizations of income shocks $y_0, \cdots, y_t$, and consumption choices $c_0, \cdots, c_t$ as given. Thus, the second term depends only on $c_t$ and $y_t$, and so the penalty $\lambda^*_t$ depends on $c_t$ and $y^{t+1}$. In particular, $\lambda^*_t(c^T, y^T)$ depends on the contemporaneous consumption choice $c_t$ and the future income shock $y_{t+1}$ explicitly, and on the earlier consumption choices $c^{t-1}$ and income shocks $y^t$ implicitly, through $w^t$ and the dynamic budget constraint. Going forward, we use more concise notation for the penalty,

$$
\lambda^*_t(c^T, y^T) = V_{t+1}(w_{t+1}) - E[V_{t+1}(w_{t+1})]\quad (3.11)
$$

To develop some intuition for how the penalty affects the solution of the relaxed problem, consider the dependence of the ideal penalty on the contemporaneous consumption choice $c_t$, as shown in Figure 3-1. Consider the time-1 penalty $\lambda^*_1$, which
depends on $c_1, y_2,$ and $w_1$, where $w_1$ captures the dependence of $\lambda^*_1$ on prior consumption choices and income shocks. We consider two values of $w_1$, and for both plot the penalty as a function of $c_1$ and $y_2$. In both panels, the solid line plots the penalty as a function of the consumption choice $c_1$ for $y_2 = y_H$. The dash-dot line plots the penalty for $y_2 = y_L$. Consider a relaxed problem, with the agent observing the time-2 income shock in advance and using this information in his time-1 consumption decision. Without the penalty, the agent can take full advantage of his knowledge of the future. In particular, if the agent knows that the time-2 income shock is high, $y_2 = y_H$, it is optimal to choose higher time-1 consumption than if $y_2 = y_L$. The ideal penalty discourages such behavior. As long as the consumption choice is non-anticipating, i.e., $c_1$ does not depend on $y_2$, the expected value of the penalty is zero (shown by the dash line), and welfare of the agent is not impaired by the penalty. However, if the agent chooses higher consumption in the $y_2 = y_H$ state relative to the $y_2 = y_L$ state, the expected penalty is positive and lowers agent’s welfare. The ideal penalty is chosen so that the benefit of perfect foresight is offset by the negative effect of the penalty, and the agent finds it optimal to chose a non-anticipative consumption policy while knowing future realizations of income shocks. Comparing the left and the right panel, we observe the effect of the prior consumption choice on the time-1 penalty. For both realizations of $y_2$, the penalty $\lambda^*_1$ is larger in absolute value for $w_1 = 4$ than for $w_1 = 5$. This illustrates that the penalty is designed to discourage the agent from conditioning time-0 consumption choices on $y_2$. Selecting higher $c_0$ in the $y_2 = y_H$ state relative to the $y_2 = y_L$ state raises the expected penalty term $\lambda^*_1$, making it positive even if the consumption choice at time 1 is non-anticipative. This illustrates the inter-temporal connections between various penalty terms and consumption choices.

An ideal penalty is as difficult to compute as the solution of the original problem. We therefore define the penalty based on an approximation to the value function:

$$\lambda_t(C^R, y^T) = \hat{V}_{t+1}(w^R_{t+1}) - E_t \left[ \hat{V}_{t+1}(w^R_{t+1}) \right].$$  \hspace{1cm} (3.12)
Figure 3-2 shows an estimate of the upper bound on welfare loss of an agent using sub-optimal policies. The agent uses a consumption policy based on the optimal solution of the model with the probability of the high state equal to \( \hat{p} \), whereas the true probability is \( p = 0.9 \). We define \( \hat{V}_{t} \) to be the value function resulting from the agent’s consumption policy. The dashed line in Figure 3-2 shows the upper bound on welfare loss of the agent computed using the information relaxation approach, while the solid line is the actual welfare loss. In panel A, \( \hat{p} = 0.895 \), which results in welfare loss relative to the optimal policy of less than 2.5% in certainty equivalent terms. Information relaxation bounds maximum welfare loss at less than 4%. We repeat the same analysis for a higher quality sub-optimal policy corresponding to \( \tilde{p} = 0.899 \). Panel B shows the results. The actual welfare loss is now lower, at most 0.5%, and the upper bound is tighter as well, limiting the maximum loss to less than 2%.

The general information relaxation approach follows the same logic as the basic example above, with a multivariate state vector replacing the wealth of the agent as an argument in the value function, and allowing for multiple choice variables. In addition, the general approach allows for partial information relaxations, where the agent receives some but not complete information about the future. Formally, we describe the structure of the agent’s information as filtration \( F = \{ F_0, F_1, \ldots F_T \} \), and the information set of the relaxed problem as a finer filtration \( G = \{ G_0, G_1, \ldots G_T \} \), where \( F_t \subseteq G_t \subseteq F_T \). Then, we define the relaxed problem under the information structure \( G \), and we define the penalty process as

\[
\lambda_t = E_t[V(x_{t+1})|G_t] - E_t[V(x_{t+1})|F_t],
\]

where the two expectation operators above are conditional on the corresponding information sets, and \( x_{t+1} \) denotes the time-\((t + 1)\) state vector (to avoid introducing more notation, we suppress the dependence of the penalty process on the choice variables and the exogenous shocks).
3.4 Application: imperfect insurance with aggregate uncertainty

We demonstrate the potential of the information relaxation methodology by computing bounds on welfare loss in the incomplete market model of Krusell and Smith (1998). This model is a canonical example of a model with an infinite dimensional state space. We review the model and equilibrium concept briefly and refer the reader to the original paper Krusell and Smith (1998) for details. In this section, we compute bounds on welfare loss under two alternative solution approaches. The first one, which we call moment truncation, is due to Krusell and Smith (1998) and is based on using a small number of moments as a reduced summary of the cross-sectional distribution of agents. The second approach is based on truncating the history of aggregate shocks, and was applied to different problems by Veracierto (1997) and Chien and Lustig (2010).

3.4.1 The model

This is an Aiyagari model (Huggett (1993), Aiyagari (1994)) with aggregate uncertainty. Time is discrete, \( t \in \{0, 1, \cdots, \infty\} \). There is a continuum of agents of unit measure with identical constant relative risk aversion preferences:

\[
E_0 \left[ \sum_{t=0}^{\infty} \beta^t \frac{c_{t}^{1-\gamma}}{1-\gamma} \right].
\]  

(3.14)

There is a single consumption good produced using a Cobb-Douglas production function

\[
y_t = z_t k^{\alpha} l^{1-\alpha}, \quad \alpha \in [0, 1],
\]

(3.15)

where \( k \) and \( l \) are capital and labor inputs. Each period's output is partly used for consumption and partly added to the next-period capital stock, resulting in the
capital accumulation constraint

\[ k_t = (1 - \delta)k_{t-1} + y_{t-1} - c_{t-1}. \]  

(3.16)

All agents are exposed to the common aggregate productivity shock \( z_t \), which takes one of two values \( z = \{z_h, z_l\}, z_h > z_l \). Productivity shocks follow a Markov chain.

Households collect capital rent and labor income each period. Individual labor income is exposed to idiosyncratic employment shocks, \( \varepsilon_t \). We assume that each agent supplies \( \bar{I} \) units of labor if employed (\( \varepsilon_t = 1 \)), and zero units if unemployed (\( \varepsilon_t = 0 \)). Employment shocks are cross-sectionally independent, conditionally on the aggregate productivity shock. Thus, based on the law of large numbers, the unemployed fraction of the population depends only on the aggregate state. We denote the equilibrium unemployment rate conditional on \( z_h \) and \( z_l \) by \( u_h \) and \( u_l \). Then the aggregate labor supply in the two states is given by \( L_h = (1 - u_h)\bar{I} \) and \( L_l = (1 - u_l)\bar{I} \) respectively.

We look for a competitive recursive equilibrium. Let \( \psi_t(k, \varepsilon) \) denote the cross-sectional distribution function at time \( t \), defined over the individual capital stock and employment status. Aggregate output depends on the aggregate capital stock, \( K_t = \int \psi(k, \varepsilon)dkde \), and the aggregate supply of labor. Input prices in competitive equilibrium are determined by their marginal product, hence capital rent and wage rate are given by

\[ r(K_t, L_t, z_t) = \alpha z_t \left( \frac{K_t}{L_t} \right)^{\alpha-1}, \quad w(K_t, L_t, z_t) = (1 - \alpha)z_t \left( \frac{K_t}{L_t} \right)^\alpha. \]  

(3.17)

Individuals optimize their consumption-investment policies under rational expectations about market prices, i.e., we assume that they correctly forecast the law of motion of the equilibrium cross-sectional distribution of agents, denoted by

\[ \psi_t = H(\psi_{t-1}, z_{t-1}). \]  

(3.18)

Thus, optimal individual policies depend on the cross-sectional distribution of capital.
The value function of the agents satisfies the Bellman equation:

\[
V_t(k_t, \epsilon_t, z_t, \psi_t) = \max_{c_t \geq 0, k_{t+1} \geq 0} \left[ \frac{c_t^{1-\gamma}}{1-\gamma} + \beta E_t [V_{t+1}(k_{t+1}, \epsilon_{t+1}, z_{t+1}, \psi_{t+1})] \right]
\]

where

\[
k_{t+1} = (1 - \delta + \sigma_t)k_t + \omega_t \tilde{\epsilon}_t - c_t,
\]

\[
\psi' = H(\psi, z),
\]

The main difficulty in computing the competitive equilibrium arises because of the dependence of equilibrium prices on the cross-sectional distribution of agents. Thus, to solve for equilibrium, we must determine the law of motion in (3.18). To make the problem tractable, it is common to use a low-dimensional approximation to the infinite-dimensional cross-sectional distribution \( \psi_t \). Below we outline the two leading approximation strategies.

**Moment truncation**

This method, introduced in Krusell and Smith (1998), approximately captures all relevant information about the cross-sectional distribution of capital by its first \( K \) moments, \( \{m_1, m_2, \ldots, m_K\} \). In particular, in their analysis Krusell and Smith restrict their attention to the cross-sectional mean \( m_1 \) (we omit the sub-script and denote the distribution mean simply by \( m \)). To speed up computation further, Krusell and Smith posit an approximate log-linear law of motion for \( m \):

\[
\hat{H} : \log m' = a^z + b^z \log m, \quad z = \{z_g, z_b\}.
\]

In an approximate equilibrium, the dynamics of the cross-sectional distribution of capital in the economy must conform closely to the assumed law of motion (3.20), which results in a low-dimensional fixed-point problem.\(^3\)

---

\(^3\)To solve the fixed-point problem, we start with an initial guess for the four constants \( \{a^z, b^z\} \). The individual optimization problem is solved for optimal policies \( k'(k, \epsilon, z, m) \). With these policies, we simulate a long time series of the cross-sectional distribution using a sample cross-section of \( N = 10^4 \) agents. Least squares regression estimates of \( \{a^z, b^z\} \) are computed based on Eq. 3.20. We re-solve the individual problem with these update estimates of \( \{a^z, b^z\} \), and the new optimal policies are used to simulate a new time-series of \( m \). This is used to update \( \{a^z, b^z\} \), and this process is continued till the system converges. We stop when the maximum change in the policy function
History truncation

Another popular approximation strategy makes the problem tractable by truncating the history of aggregate shocks. The idea behind this approximation scheme is that the equilibrium distribution of capital at any time $t$ is a function of the initial distribution and the realized sequence of aggregate shocks $\{z_0, \cdots, z_t\}$. The key approximating assumption is that the effect of the initial distribution of capital on the equilibrium is transient, and the state of the economy depends essentially on a small number ($p$) of recent aggregate shocks. In our analysis, we set $p$ to three.$^4$

Under both approximation schemes, agents form their forecast for future prices based on a low dimensional proxy to the cross-sectional distribution, while the actual distribution of future prices depends on the true high-dimensional cross-sectional distribution. Thus, individual policies are sub-optimal and result in welfare loss for the agents.

3.4.2 The relaxed problem

We compute an upper bound on individual welfare loss by relaxing the information set of the agents, as we describe in Section 3.3. In particular, we start with an initial cross-sectional distribution of capital across agents and draw a sequence of aggregate shocks $z_0, \cdots, z_{T-1}$ and a panel of idiosyncratic employment shocks. Next, we aggregate the approximate equilibrium policies of the agents and compute the prices – capital returns $r_t$ and wages $w_t$ – corresponding to the realized sequence $k'$ between successive iterations is less than $10^{-8}$, and the change in the norm of the vector $\{a^*, b^*\}$ between successive iterations is less than $10^{-8}$.

$^4$In our simulations we use $T = 10^3$ and a finite cross-section of $N = 10^4$ agents. To solve for individual policies and the transition function $H$, we start with an initial guess for aggregate capital $K((Z_{t-3}, Z_{t-2}, Z_{t-1}, Z_t)$, from which we compute $r_t$ and $w_t$ using (3.17). Since the aggregate shock $z$ takes on only two values, $r_t$ and $w_t$ are defined over 16 points. Individual policies additionally depend on the agent's current capital and employment status. We then solve the individual decision problem (3.19) by backward induction. To update the function $K$ from one iteration to the next, we simulate a long path of the cross-sectional distribution using the individual policies and obtain the average $K$ over each of the 16 aggregate states. This provides us with an updated estimate of the prices $r_t$ and $w_t$ in each of the 16 states. We then re-solve the individual problem with these updated estimates, and continue the iterative process until the norm of the difference of $r$ between two successive iterations is less than $10^{-8}$. 

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of aggregate shocks. To minimize the gap between the value function of the relaxed problem and the value function of the original problem of the agent, we apply a partial relaxation, revealing only future aggregate shocks but not the agent’s idiosyncratic employment shocks.

We define the penalty according to (3.13), thus

$$
\lambda_t = E_t[\hat{V}_{t+1}(k_{t+1}, \varepsilon_{t+1}, z_{t+1}, \hat{\psi}_{t+1})|G_t] - E_t[\hat{V}_{t+1}(k_{t+1}, \varepsilon_{t+1}, z_{t}', \hat{\psi}_{t+1})|F_t],
$$

(3.21)

where $G_t$ denotes the information set of the agent and $\hat{\psi}$ is the low-dimensional proxy for $\psi$ used to obtain the approximate equilibrium solution. For the approximate equilibrium derived using moment truncation, $\hat{\psi}$ represents the cross-sectional mean of capital holdings by agent; for the solution using history truncation, $\hat{\psi}$ represents the truncated recent history of aggregate shocks $\{z_t, z_{t-1}, \ldots, z_{t-3}\}$. In (3.21), $G_t$ contains $z_{t+1}$, but not $\varepsilon_{t+1}$ and therefore we average over possible employed and unemployed future states. Knowledge of the transition probabilities for $z$ and $\varepsilon$ allows us to compute both the terms in (3.21) above explicitly as a function of the decision variable of the relaxed problem, $(k^R_{t+1}, c^R_t)$. Finally, we use the budget constraint to eliminate $k^R_{t+1}$. Along a particular path, the penalty $\lambda_t$ is then a function of consumption $c^R_t$ only.

### 3.4.3 Results

We carry out our baseline analysis using the same parameters as in Krusell and Smith (1998). The preference parameters are $\beta = 0.99$, and $\gamma = 1$. On the production side, the parameters are: the capital share $\alpha = 0.36$, the depreciation rate $\delta = 0.025$, aggregate productivity shocks take values $z_h = 1.01$, $z_l = 0.99$, and the corresponding aggregate unemployment rates are $u_h = 0.04$, $u_l = 0.10$. The transition probability matrix for the Markov chain describing the dynamics of aggregate shocks is the same as in Krusell and Smith (1998).\(^5\)

\(^5\)When solving the individual Bellman equation, we replace the continuous variable $0 \leq k \leq \infty$ by a finite grid with $n_k = 40$ points between 0 and $\tilde{k} = 500$. Since the policy is more non-linear near $k = 0$, for better accuracy, we place more points for lower values of $k$ using a density which is a
All of our simulation results use a sample cross-section of $N = 10^4$ agents. Sample paths are $T = 10^3$ long, and we average over 100 paths to compute unbiased estimates of the value functions $\hat{V}$ and $V^R$. Throughout, we report the welfare loss as a fractional certainty equivalent $\eta$

$$\eta = \frac{k_0' - k_0}{k_0}, \quad \text{where} \quad k_0' = \hat{V}_0^{-1}(V_0^R(k_0)).$$  \hspace{1cm} (3.22)

In the equation above, $\hat{V}$ is the expected utility under the sub-optimal policy of the agent in the approximate equilibrium, and $V^R$ is the value function of the relaxed problem. The numerator of $\eta$ is the extra capital needed by an agent following a sub-optimal policy to obtain the level of expected utility equal to the value function of the relaxed problem with initial capital $k_0^R$, keeping all other state variables the same.

**Baseline results**

The welfare loss of an agent depends on the current state—the agent’s capital stock, employment status, and the state of the aggregate economy—the distribution of capital across the agents. Our baseline results report welfare loss for an agent in a typical state of the economy, drawn from the stochastic steady state. Specifically, we initiate the model with a cross-sectional distribution $\psi_{-\infty}(k) \sim \mathcal{N}(k_{ss}, 3)$, where $k_{ss}$ is the steady-state level of capital in the absence of aggregate shocks and with $L = 1$ (our results are not sensitive to the choice of the initial distribution $\psi_{-\infty}(k)$). We then draw 100 independent random paths of aggregate and individual shocks. We choose each path to be 1,000 periods long to allow the economy to reach its stochastic steady state. We then estimate an equal-weighted average of welfare losses across the simulated paths, which represent the expected welfare loss for an agent conditional on starting in stochastic steady state. Figure 3-3 summarizes the results. Individual triple exponential function. The policy function for $k > \bar{k}$ is obtained by linear extrapolation. The upper limit $\bar{k}$ is chosen with the following consideration. In a model without aggregate uncertainty ($z_h = z_l = 1$), and $L = 1$, the steady state value of capital is $k_{ss} = (\frac{a^{-1} - (1 - \delta)}{\alpha})^{\frac{1}{\delta-1}}$, which for our parameter choice is $k_{ss} = 37.99$. $\bar{k}$ is chosen to be well above this value.
welfare losses are small, especially for high levels of initial capital. Figure 3-4 shows corresponding results for the approximate solution based on history truncation. Again we find that the approximate solution implies small welfare losses for most agents in the economy. Thus, our results verify that both the original approximation of Krusell and Smith based on moment truncation, and the alternative solution using history truncation, produce approximate equilibria in which agents come very close to fully optimizing their objectives when the economy is in a typical initial state.

Transitional dynamics

Next, we consider how accurately the approximate solutions describe the transitional dynamics of the economy following an unanticipated aggregate shock. Transitional dynamics of equilibrium models is often of great interest, yet the standard solution methods are not intended to approximate equilibrium dynamics accurately when the economy is away from its steady state. It is therefore unclear a priory how well the two approximation algorithms may perform under the above experiments, and it is essential to quantify approximation accuracy when drawing conclusions about transitional dynamics of the economy based on numerical solutions.

We compute transitional dynamics of the economy following two kinds of unanticipated shocks. The first shock is a permanent regime change: the economy gradually transitions from its baseline equilibrium to the new stochastic steady state following a five-fold increase in the volatility of aggregate shocks $z$. We compute agents policies under the new parameter values using both moment truncation and history truncation. Under the moment truncation approximation, we assume that agents immediately switch to new policies following a regime change. For history truncation, agents switch after three periods. In our second experiment, the economy experiences a large transitory shock: a sudden loss of 50% of capital stock of every agent.

We first use the information-relaxation algorithm to quantify how well the Krusell-Smith algorithm performs in each case. Figure 3-5 and Figure 3-6 show the results of the two experiments. We find that in both experiments the bounds on individual welfare loss increase from the baseline case. The increase is particularly significant in
the second case, with the bound on certainty equivalent loss increasing by an order of magnitude. The absolute magnitude of welfare loss under transitional dynamics is modest in the first experiment and is larger in the second experiment. Thus, our information relaxation method establishes that, following an increase in productivity variance, the moment truncation approximation generates relatively low individual welfare loss. For the transitional dynamics following a large loss in capital, the welfare loss due to using Krusell-Smith policies is potentially larger.

History truncation relies on the assumption that the past few aggregate productivity shocks provide accurate information about the cross-sectional distribution in the economy. Therefore, history truncation may perform poorly under transitional dynamics. We summarize the results in Figure 3-7 and Figure 3-9. In line with our intuition, history truncation algorithm leads to very large welfare loss bounds for the case of permanent shock to output variance. As we discuss in Section 3.3, these bounds are one-sided, and potential welfare loss suggested by our bounds may not necessarily imply that the true welfare loss is large. In this case, we confirm that agents in the model do suffer significant welfare losses when following the suboptimal strategies based on history truncation. We consider an alternative feasible suboptimal policy, which we computed as a part of the equilibrium using the Krusell-Smith algorithm. Such a policy leads to significant welfare gains for the agents relative to the native policies prescribed by the history-truncation equilibrium. Figure 3-8 shows the results.

For the case of transitory shock to the capital stock, welfare loss bounds increase but remain comparable to those for the economy in stochastic steady state. Thus, while not designed for this type of problem, history truncation approximation performs relatively well.

3.5 Non-separable preferences

In many models, preferences of agents are non-separable across time, a popular formulation being the Epstein-Zin model of preferences. Non-separable preferences pose a
challenge for the information relaxation approach in its original form. Because agents with non-separable preferences care about the timing of resolution of uncertainty, their utility depends directly on the structure of the information set, and hence it is not straightforward to define the appropriate objective function for the relaxed problem. To extend the information relaxation approach to Epstein-Zin preferences, we use the time-separable dual formulation of Geoffard (1996) and Dumas et al. (2000).

We outline the general approach. The standard formulation uses a non-linear time aggregator \( W \) to define an agent’s preference recursively:

\[
V(x_t) = W(c_t, E_t[V_{t+1}(x_{t+1})]).
\]

The agent maximizes the utility index over the set of feasible consumption policies, which we denote by \( A^F \) to emphasize that the agent’s policies must be adapted to filtration \( F \). The dual formulation of Geoffard (1996) and Dumas et al. (2000) casts this recursive problem into a min-max time-separable form

\[
T^*V(x_0) = \max_{E_0} \min_{\nu_t} E_0 \left[ \sum_{t=0}^{T} (1 - \nu_t)^t F(c_t, \nu_t) \right], \quad x_{t+1} = f_t(x_t, \phi_t, \omega),
\]

where \( x_t \) is the time \( t \) state vector, \( \omega \) is an element of the underlying probability space, \( \phi_t \) is the vector of decision variables, and \( f \) is the law of motion of the state vector. The felicity function \( F \) is the Legendre transform of the time aggregator \( W \)

\[
F(c, \nu) = \max_u \left[ W(c, u) - (1 - \nu)u \right],
\]

where \( u_t = E_t[V_{t+1}(x_{t+1})] \). The stochastic discount rate process \( \nu_t \) is chosen to minimize the discounted sum of current and future felicities

\[
\nu_t^*(c_t) = \arg \min_{\nu_t} \left[ F(c_t, \nu_t) + (1 - \nu_t)u_t \right].
\]

An upper bound on welfare \( V_0^R \) is obtained by setting the discount rate process to a feasible, adapted process \( \widehat{\nu}_t(c_t) \), and relaxing the agent’s information set from the agent’s information set \( F = \{ \mathcal{F}_0, \mathcal{F}_1, ... \mathcal{F}_T \} \) to a finer filtration \( G = \{ \mathcal{G}_0, \mathcal{G}_1, ... \mathcal{G}_T \} \),
where $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_T$. The value function of the relaxed problem is

$$V_0^R(x_0) = \max_{c \in A^0} \mathbb{E}_0 \left[ \sum_{t=0}^{T} (1 - \tilde{v}_t(c_t))^t \left\{ F(c_t, \tilde{v}_t(c_t)) - \lambda_t(c_t) \right\} \right],$$

where the penalties $\lambda_t$ are chosen from the feasible set

$$\mathcal{L}_F = \{ \lambda \in \mathcal{L} : \mathbb{E}_0[\lambda(C, \omega)] \leq 0, \ \forall C \in A^F \},$$

**Theorem 8** The value function of the relaxed problem is an upper bound on the value function of the original problem,

$$V_0(x_0) \leq V_0^R(x_0),$$

where

$$V_0^R(x_0) = \max_{c \in A^0} \mathbb{E}_0 \left[ \sum_{t=0}^{T} (1 - \tilde{v}_t(c_t))^t \left\{ F(c_t, \tilde{v}_t(c_t)) - \lambda_t(c_t) \right\} \right],$$

and the penalties $\lambda_t$ are chosen from the feasible set

$$\mathcal{L}_F = \{ \lambda \in \mathcal{L} : \mathbb{E}_0[\lambda(C, \omega)] \leq 0, \ \forall C \in A^F \}.$$

**Proof.**

$$V(x_0) = \max_{c \in A^F} \min_{\nu_t} \mathbb{E}_0 \left[ \sum_{t=0}^{T} (1 - \nu_t(c_t))^t F(c_t, \nu_t) \right]$$

$$= \max_{c \in A^F} \mathbb{E}_0 \left[ \sum_{t=0}^{T} (1 - \nu_t^*(c_t))^t F(c_t, \nu_t^*(c_t)) \right]$$

$$\leq \max_{c \in A^F} \mathbb{E}_0 \left[ \sum_{t=0}^{T} (1 - \tilde{v}_t(c_t))^t F(c_t, \tilde{v}_t(c_t)) \right]$$

$$\leq \max_{c \in A^F} \mathbb{E}_0 \left[ \sum_{t=0}^{T} (1 - \tilde{v}_t(c_t))^t \left\{ F(c_t, \tilde{v}_t(c_t)) - \lambda_t(c_t) \right\} \right]$$

$$\leq \max_{c \in A^F} \mathbb{E}_0 \left[ \sum_{t=0}^{T} (1 - \tilde{v}_t(c_t))^t \left\{ F(c_t, \tilde{v}_t(c_t)) - \lambda_t(c_t) \right\} \right] = V^R(x_0).$$
The first inequality arises from a sub-optimal choice of the discount rate process, the second because of the super-martingale property of $\lambda_t$, and the final one because $A^F \subseteq A^G$. The value function of the relaxed problem equals the value function of the original problem if $\hat{\nu} = \nu^*$ and the penalty is chosen ideally: $\lambda_t^* = E[V_t(x_{t+1})|G_t] - E[V_t(x_{t+1})|\mathcal{F}_t]$. The proof is similar to the one for separable preferences (see e.g. Brown et al. (2010)) and is omitted.

The expected utility from adoption of a heuristic policy $\hat{c}_t(x_t)$ provides a lower bound on the agent’s value function. This is computed recursively starting with the boundary condition $\hat{V}_T(x_T) = u_T(c_T)$, and proceeding backwards.

We illustrate our approach for the consumption-savings problem introduced in Section 3.3. Instead of the time-separable preference specification, we assume that the agent has Epstein-Zin preferences and solves the recursive maximization problem

$$V_t(w_t) = \max_{c_t \in (0, w_t)} \left[ \frac{1}{\gamma} \left[ c_t^{\rho} + \beta \left( \gamma E_0[V_{t+1}(w_{t+1})] \right) \right]^{\frac{1}{\beta}} \right] , \quad (3.26)$$

where $W$ is the time aggregator. The parameter $\rho$ is related to the elasticity of inter-temporal substitution by $\text{EIS} = 1/(1 - \rho)$, and the parameter $\gamma$ is related to the coefficient of relative risk-aversion by $\text{RRA} = 1 - \gamma$. $\beta$ is the time-preference parameter. The agent’s consumption and savings decision is subject to the budget constraint

$$w_{t+1} = (w_t - c_t)R + y_{t+1} . \quad (3.27)$$

The dual time-separable formulation of Geoffard (1996) and Dumas et al. (2000) converts (3.26) to a time-separable min-max problem

$$V(w_0) = \max_{\{C_{ct} \leq w_0\}} \min_{\nu_t} E_0 \left[ \sum_{t=0}^{T} (1 - \nu_t)^t F(c_t, \nu_t) \right] , \quad (3.28)$$

where the felicity function $F$ is the Legendre transform of the time aggregator in
\[(3.26)^6\]
\[
F(c, \nu) = \frac{c^\gamma}{\gamma} \left[ 1 - \beta^{-\frac{1}{\gamma}} \left( 1 - \nu \right)^\frac{\rho}{\rho - \gamma} \right]^{\frac{\rho - \gamma}{\rho}}.
\]

\(\nu_t\) is a stochastic discount rate process chosen to minimize the discounted sum of current and future felicities.

To produce an upper bound on welfare \(V_0^R\), the first step is to choose any feasible discount rate process \(\hat{\nu}_t(c_t)\), which is adapted to the information structure of the agent. Once this choice has been made, the relaxed problem reduces to the one in Section 3.3, and we follow exactly the same steps. In the notation of Section 3.3, the value function of the relaxed problem is

\[
V_0^R(w_0) = \max_{\{C^R, \hat{C}^R \leq w_0^R\}} \mathbb{E} \left[ \sum_{t=0}^{T} (1 - \hat{\nu}_t(c_t^R)) \left( F(c_t^R, \hat{\nu}_t(c_t^R)) - \lambda_t(C^R, y^T) \right) \right], \tag{3.29}
\]

where \(C^R = (c_0^R(y^T), c_1^R(y^T), ..., c_T^R(y^T))\) denote feasible consumption policies of the relaxed problem. The penalties \(\lambda_t\) satisfy the same condition as in Section 3.3

\[
\mathbb{E}_0 \left[ \lambda_t \left( C, y^T \right) \right] \leq 0. \tag{3.30}
\]

We solve the deterministic problem Eq. 3.29 for each simulated path of income shock, and average over many paths to obtain an unbiased estimate of \(V_0^R\).

As an explicit example of a sub-optimal policy, we consider policies \(\hat{\nu}_t(w_t)\) that would be optimal for an agent assigning incorrect probabilities to income shocks, i.e. using \(\hat{p}\) instead of \(p\). Figure 3-10 shows welfare loss in certainty equivalent units for \(\hat{p} = 0.895\), while Figure 3-11 shows welfare loss for \(\hat{p} = 0.899\). The true probability of the high state is \(p = 0.9\). In both plots, the dashed curve represents an upper bound to welfare loss, while the solid curve shows the actual welfare loss. The difference between the solid and the dashed curves is the difference between the value function of the relaxed problem and the true value function of the agent. This difference is

\[\text{For Epstein-Zin preferences, the time aggregator } W(c, u) = \frac{1}{\gamma} \left[ c^\rho + \beta(\gamma u)^{\rho/\gamma} \right]^{\gamma/\rho} \text{ from (3.26). The Legendre transform is defined as } F(c, \nu) = \max_u \left[ W(c, u) - (1 - \nu)u \right].\]
smaller for $\hat{p}$ close to $p$ because the penalty for perfect foresight is closer to the ideal penalty, and the discount rate $\hat{\nu}$ is closer to being optimal.

### 3.6 Conclusion

We apply a duality-based approach based on enlarging the information set of the agent to put bounds on his welfare loss from adopting heuristic policies. Unlike ad hoc approaches to test optimality such as the one based on Euler residuals, our method produces a provable upper bound on welfare loss resulting from suboptimal policies.

Because it avoids computing exact optimal policies, which is often impractical, the information relaxation approach has wide applicability. For instance, this framework could be used to bound welfare loss for approximate equilibria computed using perturbation techniques. The latter approach is widely used for DSGE models (for a recent application, see Mertens and Judd (2012) and Mertens (2011)), because of computational speed and is supported by the computational software Dynare.

In many models of economies with heterogeneous information, higher-order beliefs play an important role in defining equilibrium dynamics, and, as a result, agents face high-dimensional inference and optimization problems. Specifically, in such models the full history of prices is a part of the agents’ state space. A common technique for computing approximate equilibria relies on truncating the path of past prices, and keeping track of only a short portion of their history. The information relaxation approach could be used to shed light on the accuracy of such approximate equilibrium solutions. Yet another natural application is to evaluating welfare loss resulting from heuristic policies motivated by behavioral biases of the agents.

Our analysis leaves open an important and challenging question of how closely the equilibrium policies and prices in the bounded-rationality economy correspond to those in an exact equilibrium, in which all agents optimize their objectives fully. It is well know that in general, small mistakes by individual agents can lead to large differences in equilibrium outcomes (e.g., Akerlof and Yellen (1985), Jones and Stock
(1987), Naish (1993), Hassan and Mertens (2011)). Going forward, it would be important to develop quantitative tools for evaluating the effect of small deviations from individual rationality on equilibrium price dynamics.
Bibliography


Appendix: Euler equations and welfare loss: an example

The following example highlights the distinction between Euler equation errors and welfare loss. Consider the consumption and portfolio choice problem of an infinitely lived agent with CRRA preferences (Samuelson (1969) and Merton (1971)).
agent has access to a single risk-free asset and a single risky asset with independent
and identically distributed returns. Optimal consumption and portfolio policies solve
the Bellman equation

\[ V(W_0) = \max_{C_t, \phi_t} E_0 \left[ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right], \quad W_{t+1} = (W_t - C_t) \left( R_f + \phi_t(\tilde{R}_{t+1} - R_f) \right), \]

where \( \beta \) is the time-preference parameter, \( \gamma \) is the coefficient of relative risk aversion,
\( R_f \) and \( \tilde{R} \) are gross returns of the risk-free and risky asset, respectively. The optimal
portfolio policy in this setting is a constant \( \phi \) which maximizes the certainty equivalent
of next period wealth (see Samuelson (1969)):

\[ B^{1-\gamma} = \max_{\phi} E_t \left[ (R_f + \phi(\tilde{R}_{t+1} - R_f))^{1-\gamma} \right]. \]

Optimal consumption is a constant fraction of wealth \( C_t = \zeta^* W_t \).

As suboptimal policies, consider budget-feasible policies with a constant consumption-
wealth ratio \( \zeta = C_t/W_t \). Assume \( \gamma > 1 \). The expected utility under such policies,
indexed by \( \zeta \), is given by

\[ U(W_0, \zeta) = E_0 \left[ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right] = \frac{W_0^{1-\gamma} \zeta^{1-\gamma}}{1-\gamma} \sum_{t=0}^{\infty} \beta^t (1 - \zeta)^{(1-\gamma)t} B^{(1-\gamma)t}. \]

While the optimal choice is \( \zeta^* = 1 - \left( \beta B^{1-\gamma} \right)^{1/\gamma} \), there is a critical value \( \zeta_{\text{crit}} = 1 - \frac{\beta^{\gamma+1}}{B} \), such that for any \( \zeta > \zeta^* \) the expected utility of the investor is infinitely
negative.

The Euler residuals remain finite even with infinite loss in utility. The Euler
equation errors (for \( i = 1, 2 \) are)

\[ \varepsilon_t = 1 - E_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1}^i \right] = 1 - E_t \left[ \beta \left( (1 - \zeta)(R_f + \phi(\tilde{R}_{t+1} - R_f)) \right)^{-\gamma} R_{t+1}^i \right], \]

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where $R^1_{t+1} = R_f$, and $R^2_{t+1} = \hat{R}_{t+1}$ is the return on the risky asset. Since

$$E_t \left[ \beta \left( (1 - \zeta^*)(R_f + \phi(\hat{R}_{t+1} - R_f)) \right)^{-\gamma} R^i_{t+1} \right] = 1,$$

we conclude that

$$\varepsilon_t = 1 - \left( \frac{1 - \zeta}{1 - \zeta^*} \right)^\gamma.$$

Thus, the Euler equation errors are finite for $\zeta^* < \zeta < 1$, while the utility loss is infinite.
Figure 3-1: Ideal penalty $\lambda_1^*$ for the consumption-saving problem of Section 3.3.1, plotted as a function of time-1 consumption choice $c_1$. The risk-free interest rate $R = 1.02$. Labor income follows a two-state Markov chain with $y_L = 1.0$, and $y_H = 4.0$ with independent draws each period. The probability of $y_H$ is $p = 0.9$. The agent has power utility with relative risk aversion of 5, and time preference parameter $\beta = 0.9$. Time horizon is $T = 100$. The left panel corresponds to the time-0 wealth $w_0 = 4$, the right panel corresponds to $w_0 = 5$. The solid line plots the value of the penalty in state $y_2 = y_H$, while the dash-dot line plots $\lambda_1^*$ for $y_2 = y_L$. The dash line shows the expectation of the penalty over the two possible realizations of $y_2$, which is identically equal to zero.

$w_1 = 4$

$w_1 = 5$
Figure 3-2: Fractional certainty equivalent loss as a function of wealth of an agent using sub-optimal policies to smooth labor income risk using a single risk-free asset with constant interest rate $R = 1.02$. Labor income follows a two-state Markov chain with $y_L = 1.0$, and $y_H = 4.0$ with independent draws each period. The probability of $y_H$ is $p = 0.9$. In panel A, the agent uses a policy that is optimal for $\hat{\rho} = 0.895$, while in panel B he uses a policy optimal for $\hat{\rho} = 0.899$. The dashed line in both panels show the upper bound, while the solid line shows the actual welfare loss. The agent has power utility with relative risk aversion of 5, and time preference parameter $\beta = 0.9$. We use 500 paths to estimate the value function of the relaxed problem.
Figure 3-3: Upper bound on welfare loss for the incomplete markets model of Krusell and Smith where the approximate equilibrium is computed using moment truncation. All parameters values are identical to those in Krusell and Smith (1998). Fractional certainty equivalent loss is defined in (3.22). The value function of the relaxed problem is estimated by averaging over 100 paths of aggregate shocks. The agent is unemployed and the aggregate state of the economy is low. The cross-sectional distribution of capital shown by the area under the shaded curve is the stochastic steady-state distribution.
Figure 3-4: Upper bound on welfare loss for the incomplete markets model of Krusell and Smith where the approximate equilibrium is computed using history truncation. All parameters values are identical to those in Krusell and Smith (1998). Fractional certainty equivalent loss is defined in (3.22). The value function of the relaxed problem is estimated by averaging over 100 paths of aggregate shocks. The agent is unemployed and the aggregate state of the economy is low. The cross-sectional distribution of capital shown by the area under the shaded curve is the stochastic steady-state distribution.
Figure 3-5: Upper bound on welfare loss for the incomplete markets model of Krusell and Smith where the approximate equilibrium is computed using moment truncation. Starting from the steady-state, aggregate productivity experiences a one-time five-fold permanent increase in volatility. We compute agent policies under the new parameter values. Agents immediately switch to new policies following the regime change. Fractional certainty equivalent loss is defined in (3.22). The value function of the relaxed problem is estimated by averaging over 100 paths of aggregate shocks. The agent is unemployed and the aggregate state of the economy is low. The area under the shaded curve shows the cross-sectional distribution of capital, and corresponds to the stochastic steady state of the economy before the permanent shock is realized.
Figure 3-6: Upper bound on welfare loss for the transitional dynamics following a 50% loss in capital stock of all agents in the incomplete markets model of Krusell and Smith. The approximate equilibrium and policies are computed using moment truncation. Fractional certainty equivalent loss is defined in (3.22). The value function of the relaxed problem is estimated by averaging over 100 paths of aggregate shocks. The agent is unemployed and the aggregate state of the economy is low. The area under the shaded curve shows the cross-sectional distribution of capital immediately after the economy-wide capital loss is realized.
Figure 3-7: Upper bound on welfare loss for the incomplete markets model of Krusell and Smith where the approximate equilibrium is computed using history truncation. Starting from the steady-state, aggregate productivity experiences a one-time five-fold permanent increase in volatility. We compute agent policies under the new parameter values. Agents switch to new policies three periods after the regime change. Fractional certainty equivalent loss is defined in (3.22). The value function of the relaxed problem is estimated by averaging over 100 paths of aggregate shocks. The agent is unemployed and the aggregate state of the economy is low. The area under the shaded curve shows the cross-sectional distribution of capital, and corresponds to the stochastic steady state of the economy before the permanent shock is realized.
Figure 3-8: Approximate equilibrium in the incomplete markets model of Krusell and Smith, computed using history truncation, following a five-fold permanent increase in volatility of aggregate productivity. Fractional certainty equivalent loss is defined in (3.22). The dotted curve plots welfare loss relative to the feasible policy constructed using the Krusell-Smith moment truncation algorithm. The solid curve is the same curve as in Figure 3-7.
Figure 3-9: Upper bound on welfare loss for the transitional dynamics following a 50% loss in capital stock of all agents in the incomplete markets model of Krusell and Smith. The approximate equilibrium and policies are computed using history truncation. Fractional certainty equivalent loss is defined in (3.22). The value function of the relaxed problem is estimated by averaging over 100 paths of aggregate shocks. The agent is unemployed and the aggregate state of the economy is low. The area under the shaded curve shows the cross-sectional distribution of capital immediately after the economy-wide capital loss is realized.
Figure 3-10: Fractional certainty equivalent loss as a function of wealth for an agent using sub-optimal policies to smooth labor income risk using a single risk-free asset with constant interest rate $R = 1.02$. Labor income follows a two-state Markov chain with $e_L = 1.0$, and $e_H = 4.0$ with independent draws each period. The probability of $e_H$ is $p = 0.9$. The problem horizon is $T = 10$. The agent uses a policy that is optimal for $\hat{\gamma} = 0.895$. The dashed curve shows the upper bound, while the solid curve shows the actual welfare loss. The agent has Epstein-Zin utility with EIS equal to 1.5, relative risk aversion of 5, and time preference parameter $\beta = 0.9$. We use 500 simulated paths to estimated the value function of the relaxed problem.
Figure 3-11: Fractional certainty equivalent loss as a function of wealth for the same model as in Figure 3-10. The agent uses a policy that is optimal for $\hat{p} = 0.899$. The dashed curve shows the upper bound, while the solid curve shows the actual welfare loss. Parameters used are the same as in Figure 3-10.