Squaring-Up Method for Relative Degree Two Plants

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Abstract

Non-square multi-input-multi-output (MIMO) plants are becoming increasingly common, as the addition of multiple sensors is becoming prevalent. However, square systems are needed sometimes as an leverage when it comes to design and analysis, as they possess desirable properties such as strict positive realness. This paper presents a squaring up method that adds artificial inputs to a class of MIMO plants with relative degree two and stable transmission zeros, where number of outputs exceeds number of inputs. The proposed method is able to produce a square plant that has stable transmission zeros and uniform/nonuniform relative degree, and is used to carry out adaptive control of this class of plants and shown to lead to satisfactory performance in a numerical study.

I. Introductions

Square systems play a key role in control theory development because of some unique properties they may possess such as left/right invertibility [1]. Additionally, in order for a system to be strictly positive real (SPR) it must necessarily be square [2]. The SPR property is essential for prescribing the direction of parameter adaptation and guarantees stability through KYP lemma [3]. Therefore, in adaptation design of multivariable parametric uncertainties [3], [4], square minimum-phase systems are commonly assumed. To extend these results to non-square systems, a squaring-up (or down) method is usually needed, which effectively produces a minimum-phase square system through addition (or deletion) of suitable inputs or outputs.

The squaring-down method is first attempted in 1970s [5], [6] and its zero placement was observed to be equivalent to pole-placement using output feedback in a transformed space. Since pole-placement using output feedback can be achieved only under some specific conditions, the squaring-down method can be restrictive. Literature on squaring-up methods were rather sparse until the work by [7], [8]. It has been shown the zero-placement in the square-up method is equivalent to pole-placement using state feedback in a transformed coordinate and therefore is much more feasible. On the other hand, squaring-up methods involve the addition of pseudo inputs or outputs and therefore can only be used as a preliminary step in the overall control design.

Recently, the squaring-up method has gained increasing interest in adaptive control design [9]–[11]. One key finding is that the pseudo-inputs (or outputs) can be used for feedback gain design which yields good properties that usually only exists in a square system. The first procedure in these papers is to perform squaring-up, then design a feedback compensator so that an underlying sub-system becomes SPR. The design has been proved plausible [12] but only in the "lifted" design space, which then is fulfilled by a squaring-up method proposed in [13]. The proposed squaring-up method [13] preserves the SPR properties of the plant, which enables the design of adaptive output feedback control for general non-square MIMO systems [10], [11]. Both the adaptive control designs and the squaring-up method in these literature are subject to a restrictive assumption that the underlying plant models have uniform relative degree one, which prevents the design to be applicable to plants that have actuator dynamics. Although the restriction has been lifted in [14] where an adaptive controller for a relative degree two plant model is proposed, such control design currently can only be applied on a square plants. In this paper, we propose a squaring-up method for relative degree two plants and extend the adaptive control design to such plants.

This paper is organized as follows. With some preliminaries in Section II, we formulate in Section III the squaring-up problem. We then first present the squaring-up method for relative degree one plants in Section IV, and extend it to a squaring-up method for relative degree two plants in Section V. Adaptive control of a non square plant with relative degree two is shown in section VI. Simulations on a linearized very flexible aircraft (VFA) model is shown in Section VII.

II. Preliminaries

We use the notation $\Sigma_{p\times m} : \{A, B, C, D\}$ to denote a transfer function matrix

$$\Sigma_{p\times m} : \{A, B, C, D\} = G(s) = C(sI - A)^{-1}B + D. \quad (1)$$

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with a realization $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ as in

$$\dot{x} = Ax + \underbrace{\begin{bmatrix} b_1, & b_2, & \cdots, & b_m \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}}_{u}$$
$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} c^1 \\ c^2 \\ \vdots \\ c^p \end{bmatrix}}_{C} x + Du \tag{2}$$

We assume $D \equiv 0$ unless otherwise specified and denote the plant model as $\Sigma_{p\times m} : \{A, B, C\}$. The plant model $\Sigma_{p\times m}$ is square if $m = p$, tall if $m < p$, and fat if $m > p$. Square-up is the process by which a non-square plant model is made square through the addition of more inputs or outputs until $m = p$. Square-down is a similar process where a square plant model is reached through the removal of inputs or outputs. We define Markov parameter series as follows.

Definition 1. The Markov parameter series for ith input of $\Sigma_{p\times m} : \{A, B, C, D\}$ is defined as $\mathscr{M}_i = \left\{M_i^1, M_i^2, \cdots, M_i^k, \cdots\right\}$ for $k \in \mathbb{N}^+$, where

$$M_i^k = \begin{cases} d_i & k = 0 \\ CA^{k-1}b_i & k \geq 1 \end{cases} \tag{3}$$

and d_i is the ith column of D.

The input relative degree of the plant model is defined as follows.

Definition 2. A linear plant model $\{A, B, C\}$ has

a) input relative degree $\mathfrak{r} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m \end{bmatrix}^T \in \mathbb{N}^{m\times 1}$ if and only if

$$i) \quad \forall j \in \{1, \cdots, m\},\ \forall k \in \{0, \cdots, r_j - 1\} \ : \ M_i^k = 0_{p\times 1}, \quad \text{and} \tag{4}$$
$$ii) \quad rank \begin{bmatrix} M_1^{r_1}, & M_2^{r_2}, & \cdots, & M_m^{r_m} \end{bmatrix} = m \tag{5}$$

b) uniform input relative degree $r \in \mathbb{N}$ if and only if it has input relative degree $\mathfrak{r} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m \end{bmatrix}^T$ and $r = r_1 = r_2 = \cdots = r_m$.

c) nonuniform input relative degree $\mathfrak{r} \in \mathbb{N}^m$ if and only if it has input relative degree $\mathfrak{r} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m \end{bmatrix}^T$ and $r_i \neq r_j$ for some $i, j \in 1, 2, \cdots, m$ and $i \neq j$.

Denote $r_s = \sum_{i=1}^m r_i$ as the total relative degree of a plant model. Not every MIMO plant model has input relative degree. It is noted that

$$y^{(r_i)} = CA^{r_i}x + CA^{r_i-1}b_1u_1 + \cdots + CA^{r_i-1}b_mu_m + \cdots + Cb_1u_1^{(r_i-1)} + \cdots + Cb_mu_m^{(r_i-1)} = CA^{r_i}x + CA^{r_i-1}b_i^{(n)}u_i \tag{6}$$

if $G(s)$ have input relative degree $r = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m \end{bmatrix}^T$; it shows that u_i should start to have nonzero (and linearly independent) contribution towards the r_ith derivative of at least one output in y. Generically, any MIMO plant model has input relative degree since condition $i)$ and $ii)$ are generically satisfied.

From (5) in Definition 2, for a non square plant model to have input relative degree, there must be more (or equal) number of outputs than inputs, i.e. $p \geq m$. We define the transmission zeros of a MIMO plant model as follows.

Definition 3. [15] For a non-degenerate m-input and p-output linear plant model with minimal realization $\Sigma_{p\times m} : \{A, B, C, D\}$, the transmission zeros are defined as the finite values of s such that $rank[R(s)] < n + \min[m, p]$, where

$$R(s) = \begin{bmatrix} sI - A & B \\ C & D \end{bmatrix}. \tag{7}$$

We denote the set of transmission zeros as $Z[\Sigma_{p\times m}]$ or $Z[R(s)]$ for $\Sigma_{p\times m}$ or $R(s)$, respectively. Without loss of generality, suppose $D = 0$ and we have squared up $\Sigma_{p\times m} : \{A, B, C\}$, which has n_z transmission zeros and input relative degree $r = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m \end{bmatrix}^T$ with $r_s = \sum_{i=1}^m r_i$, and produce a square plant model $\overline{\Sigma}_{p\times p} : \{A, \overline{B}, C\}$ by appending $\overline{B} = [B, B_a]$. For a square plant model, the number of transmission zeros satisfies the following proposition.

Proposition 4. *For a square plant model $\overline{\Sigma}_{p\times p} : \{A, \overline{B}, C\}$ with input relative degree $\mathfrak{r} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_p \end{bmatrix}^T$, the number of transmission zeros $\overline{n}_z$ is exactly*

$$\overline{n}_z = n - \overline{r}_s \geq 0 \tag{8}$$

where $\overline{r}_s = \sum_{i=1}^{p} r_i$ *is its total relative degree.*

The rank condition (7) in Definition 3 implies that the squaring-up procedure cannot change or remove any existing transmission zeros in $\Sigma_{p\times m}$ [13], i.e.

$$\overline{n}_z \geq n_z \tag{9}$$

Also, the minimum relative degree for each added inputs is 1 since $D = \overline{D} = 0$, i.e.

$$\overline{r}_s - r_s = \sum_{i=m+1}^{p} r_i \geq m - p \tag{10}$$

Combining inequality (8)(9) and (10), we derive an upbound on the number of transmission zeros in $\Sigma_{p\times m} : \{A, B, C\}$.

Proposition 5. *For a tall plant model* $\Sigma_{p\times m} : \{A, B, C\}$ *with* $p \geq m$ *and input relative degree* $\mathfrak{r} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m \end{bmatrix}^T$*,* $r_s = \sum_1^m r_i$*, the number of transmission zeros* n_z *satisfies*

$$0 \leq n_z \leq (n - r_s - (p - m)) \tag{11}$$

where the equality holds when $m = p$*.*

III. Problem Statement

Consider an non-square strictly proper plant model $\Sigma_{p\times m} : \{A, B, C\}$ as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{12}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are known matrices. Without loss of generality, this paper only considers the case when the plant model is tall, i.e. $p \geq m$. In addition, we assume $\Sigma_{p\times m}$ satisfies following assumptions:

Assumption 1. $\{A, B, C\}$ *is a minimal realization;*

Assumption 2. B *has full column rank, i.e.* $rank(B) = m$ *and* C *has full row rank, i.e.* $rank(C) = p$*;*

Assumption 3. Σ_p *has input relative degree* $\mathfrak{r} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m \end{bmatrix}^T$

We denote

$$r_s = \sum_{i=1}^{p} r_i \leq n \tag{13}$$

as the total relative degree of Σ_p. For feasible squaring-up, we assume that

Assumption 4. *All of* Σ_p*'s* n_z *transmission zeros are stable, and satisfies* $(n - r_s - n_z) \geq (m - p)$

From Proposition 5, Assumption 4 guarantees that we can add at least $(m-p)$ relative degree one inputs, or $(m-p)$ relative degree zero inputs (or some combinations of relative degree one and relative zero inputs).

Given the tall plant model $\Sigma_{p\times m} : \{A, B, C\}$, the goal is to find an augmentation $\hat{B} \in \mathbb{R}^{n\times(p-m)}$ such that the plant model $\overline{\Sigma}_{p\times p} : \{A, \overline{B}, C\}$, where $\overline{B} = [B, \hat{B}]$, is square, has stable transmission zeros and input relative degree $\overline{\mathfrak{r}} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m, \cdots, r_p \end{bmatrix}^T$ with

$$\overline{r}_s = \sum_{i=1}^{m} r_i. \tag{14}$$

We will approach the problem by first introducing a squaring-up method for relative degree one plants, and extend it to relative degree two plants.

IV. Squaring-Up Method: Relative Degree One Cases

This section discusses a squaring up method when $\Sigma_{p\times m}$ has uniform input relative degree one, i.e. $r = 1$, or equivalently CB is full rank. To state the squaring-up problem, we examine closely plant model's Rosenbrock matrix and interpret the goal mathematically [7]. The Rosenbrock matrix $R(s)$ of the given plant model Σ_p can be written in the observer canonical form $\tilde{R}(s)$ as

$$R(s) \xrightarrow{T} \tilde{R}(s) = \left[\begin{array}{cc|c} sI_p - A_{11} & -A_{12} & B_{11} \\ -A_{21} & sI_{n-p} - A_{22} & B_{21} \\ \hline I_p & 0 & 0 \end{array}\right] \tag{15}$$

where $T^{-1} = \begin{bmatrix} C^{-R} & C^{\perp} \end{bmatrix}$ is an invertible coordinate transformation matrix satisfying $CC^{-R} = I_p$ and $CC^{\perp} = 0_{p\times(n-p)}$. Since transmission zeros are invariant under coordinate transformation, $Z[R(s)]$ coincide $Z[\tilde{R}(s)]$. The geometrical goal then is to design $\hat{B}_{12} \in \mathbb{R}^{p\times(p-m)}$, $\hat{B}_{22} \in \mathbb{R}^{(n-p)\times(p-m)}$ and $\hat{D}_2 \in \mathbb{R}^{p\times(p-m)}$ such that the squared-up plant model $\hat{R}(s)$ as

$$\overline{R}(s) = \left[\begin{array}{cc|cc} sI_p - A_{11} & -A_{12} & B_{11} & \hat{B}_{12} \\ -A_{21} & sI_{n-p} - A_{22} & B_{21} & \hat{B}_{22} \\ \hline I_p & 0 & 0 & \hat{D}_2 \end{array}\right] \tag{16}$$

satisfies

Condition 1. *$\overline{R}(s)$ only loses rank at a set of finite s that lie in the open left haft of the complex plane, and*

Condition 2. *Eq.(4) and (5) holds for some $\mathfrak{r} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m, \cdots, r_p \end{bmatrix}^T$.*

The use of $\hat{D}_2$ depends on the choice of the relative degree of the added inputs, which will be discussed separately in the following subsections.

A. Mode 0: Adding Inputs of Relative Degree Zero

This mode introduces new inputs with relative degree zero, which requires $\hat{D}_2 \neq 0$. This case does not fit the problem definition in Section III, and therefore only introduced here as a reference for the use in relative degree two cases. Since B has rank m, with some permutations we can put all independent rows of B in B_{11} and perform row elimination on (16) as

$$\overline{R}_1(s) = \left[\begin{array}{cc|cc} sI_m - A_{11} & -A_{12} & B_{11} & \hat{B}_{12} \\ \times & sI_{n-m} - \tilde{A}_{22} & 0 & \tilde{B}_{22} \\ \hline I_m & 0 & 0 & 0 \\ 0 & C_{22} & 0 & \hat{D}_{22} \end{array}\right] \tag{17}$$

where $\hat{D}_2 = \begin{bmatrix} \hat{D}_{21} \\ \hat{D}_{22} \end{bmatrix}$, $\hat{D}_{21} = 0$, and $\begin{bmatrix} I_p & 0 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & C_{22} \end{bmatrix}$ where $C_{22} = \begin{bmatrix} I_{p-m} & 0 \end{bmatrix}$; $\tilde{A}_{22} = A_{22} - B_{21}B_{11}^{-1}A_{12}$ and $\tilde{B}_{22} = \hat{B}_{22} - B_{21}B_{11}^{-1}$. It is noted that this mode only requires $rank[C] = m < p$. It follows that $Z[\overline{R}(s)] = Z[\overline{R}_1(s)] = Z[\overline{R}_1^s(s)]$ where

$$\overline{R}_1^s(s) = \left[\begin{array}{c|c} sI_{n-p} - \tilde{A}_{22} & \tilde{B}_{22} \\ \hline C_{22} & \hat{D}_{22} \end{array}\right]. \tag{18}$$

is a submatrix of $\overline{R}_1(s)$. With an invertible $\hat{D}_{22}$, the transmission zeros of $\overline{R}_1^s(s)$ is the eigenvalues of $\tilde{A}_{22} - \tilde{B}_{22}\hat{D}_{22}^{-1}C_{22}$. It can be shown that $(\tilde{A}_{22}, C_{22})$ must be a detectable pair and its unobservable mode is the pre-existing transmission zeros of $R(s)$ (following the same argument presented in [13]). The complete procedure of squaring-up for Mode 0 is as follows:

$$\text{pickany } \hat{B}_{12} \in \mathbb{R}^{p\times(p-m)} \tag{19}$$

$$\text{pick}\hat{D}_{22} \text{ s.t. } rank\begin{bmatrix} B_{11} & \begin{bmatrix} 0 \\ \hat{D}_{22} \end{bmatrix} \end{bmatrix} = p \tag{20}$$

$$\tilde{A}_{22} = A_{22} - B_{21}B_{11}^{-1}A_{12} \tag{21}$$

$$W^T = \text{lqr}(\tilde{A}_{22}^T, C_{22}^T) \tag{22}$$

$$\tilde{B}_{22} = W\hat{D}_{22} \tag{23}$$

$$\hat{B}_{22} = \tilde{B}_{22} + B_{21}B_{11}^{-1} \tag{24}$$

It is noted that (21)-(24) are used to satisfy Condition 1, while (20) guarantees Condition 2 with $r = [1, 1, 1, \cdots, 0, 0, 0]$, where r includes $m + r$ relative degree one. In some extreme cases that a $\hat{D}_{21} = 0$ cannot satisfy (20), design $\hat{B}_{12}$, $\hat{B}_{22}$ and $\hat{D}_{22}$ with $\hat{D}_{21} = 0$ using (21)-(24), and then introduce a $\hat{D}_{21} \neq 0$ of a small magnitude such that (20) holds. The continuity of transmission zero function $Z[\cdot]$ determines that if $\left\|\hat{D}_{21}\right\|$ is small enough, $Z[R(s)]$ are still stable.

B. Mode 1: Adding Inputs of Relative Degree One

This mode introduces new inputs with relative degree one, which requires $\hat{D}_2 = 0$ and then can be used to solve the squaring-up problem defined in Section III for the relative degree one case. The dual form of this case has been solved in the Ref. [7], [13] and is adopted here by performing a transpose on all system matrices. With $\hat{D}_2 = 0$ and some row elimination, $Z[\overline{R}(s)] = Z[\overline{R}_1(s)]$ where

$$\hat{R}_1(s) = \left[\begin{array}{cc|cc} sI_p - A_{11} & -A_{12} & B_{11} & \hat{B}_{12} \\ \times & sI_{n-p} - \tilde{A}_{22} & 0 & 0 \\ \hline I_p & 0 & 0 & 0 \end{array}\right] \tag{25}$$

where $\tilde{A}_{22} = A_{22} - B_2 B_1^{-1} A_{12}$, $B_2 = \begin{bmatrix} B_{21} & \hat{B}_{22} \end{bmatrix}$, and $B_1 = \begin{bmatrix} B_{11} & \hat{B}_{12} \end{bmatrix}$. It is noted that this mode requires $rank[C] = p$. B_1 is invertible since $\hat{B}_{12} = (null(B_{11}))^T$ where $null(\cdot)$ stands for the null space of $(\cdot)$. Then it is clear that $Z[\overline{R}_1(s)]$ are the eigenvalues of $\tilde{A}_{22}$. As a result, the complete procedure for Mode 1 is as follows:

$$\hat{B}_{12} = (null(B_{11}))^T \text{ s.t. } B_1 = \begin{bmatrix} B_{11} & \hat{B}_{12} \end{bmatrix} \text{ is invertible} \tag{26}$$

$$\tilde{A}_{22}^* = A_{22} - \begin{bmatrix} B_{21} & 0 \end{bmatrix} B_1^{-1} A_{12} \tag{27}$$

$$\begin{bmatrix} \times \\ E^* \end{bmatrix} = B_1^{-1} A_{12}, \ E^* \in \mathbb{R}^{(p-m)\times(n-p)} \tag{28}$$

$$\hat{B}_{22}^T = \text{lqr}(\tilde{A}_{22}^{*T}, E^{*T}) \tag{29}$$

It is noted that $(\tilde{A}_{22}^*, E^*)$ must be a detectable pair and its unobservable mode is the pre-existing transmission zeros of $R(s)$ (see [13] for a proof). (26) guarantees that Condition 2 is satisfied with $\mathfrak{r} = 1$, and (27)-(29) guarantees that Condition 1 is satisfied.

C. Mode H-0-1: Adding Inputs with Both Relative Degree One and Zero

Mode 0 adds all $(p - m)$ relative degree zero inputs. Mode 1 adds all $(p - m)$ relative degree one inputs. This section introduces Mode H-0-1, which is a hybrid of Mode 0 and Mode 1, i.e. adds n_1 relative degree one inputs and n_2 relative degree zero inputs, so that $n_1 + n_2 = p - m$. The upbound of n_1 depends on the rank of C.

Suppose $rank[C] = r_c \geq m$. We can part $C = \begin{bmatrix} C_{r_c} \\ C_{p-r} \end{bmatrix}$ where C_r includes all independent rows of C. Then we can use Mode 1 to add $(r_c - m)$ relative degree one inputs $\hat{B}_1$ such that

$$\hat{R}_r(s) = \left[\begin{array}{c|cc} sI - A & B & \hat{B}_1 \\ \hline C_{r_c} & 0 & 0 \end{array}\right] \tag{30}$$

has stable transmission zeros and uniform input relative degree one. Then we can apply Mode 0 on

$$\overline{R}_1(s) = \left[\begin{array}{c|ccc} sI - A & B & \hat{B}_1 & \hat{B}_2 \\ \hline C_{r_c} & 0 & 0 & 0 \\ C_{p-r_c} & 0 & 0 & \hat{D}_2 \end{array}\right] \tag{31}$$

to obtain a $\hat{B}_2$ and a $\hat{D}_2$ such that $\overline{R}_1(s)$ has stable transmission zeros and input relative degree. The above results for relative degree one plants are summarized in the following Lemma.

Lemma 1. *Given a plant model $\Sigma_{p\times m} : \{A, B, C, 0\}$ that satisfies assumptions 1 and 4, in addition, has $rank(B) = m$, $rank(C) = r_c \geq m$ and uniform relative degree one, there exists a $\hat{B} \in \mathbb{R}^{n\times(p-m)}$ and $\hat{D} \in \mathbb{R}^{p\times(p-m)}$ such that the squared-up plant $\overline{\Sigma}_{p\times m} : \{A, \overline{B}, C, \overline{D}\}$, where $\overline{B} = [B, \hat{B}]$ and $\overline{D} = [D, \hat{D}]$, has all stable transmission zeros, and nonuniform relative degree, i.e. $r_i = 1$ for $i = 1, 2, \cdots, r_c$, $r_j = 0$ for $j = r_c + 1, \cdots, p$.*

Since the problem definition in Section III prohibits us to use D in squaring-up, this mode is only introduced to be used in relative degree two cases.

V. Squaring-up Method: Relative Degree Two Cases

Previous section introduces different squaring-up methods for plant model with relative degree one. It is the focus of this paper to develop a squaring-up method for plant models with relative degree two. This section focuses on the plant model with input relative degree $\mathfrak{r} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m \end{bmatrix}^T$ and $\max_i [r_i] = 2$. We will separate the case of uniform relative degree two and nonuniform relative degree two.

A. Uniform Relative Degree Two

This case assume that the plant has uniformly relative degree two, i.e. $r = 2$, which implies that in the observer canonical form (15), $B_{12} = 0$ and $A_{12}B_{22}$ is full rank, i.e.

$$\overline{R}(s) = \left[\begin{array}{cc|cc} sI_p - A_{11} & -A_{12} & 0 & \hat{B}_{12} \\ -A_{21} & sI_{n-p} - A_{22} & B_{21} & \hat{B}_{22} \\ \hline I_p & 0 & 0 & 0 \end{array}\right]. \tag{32}$$

To solve the squaring-up problem, we use $\hat{D}_2 = 0$. It is noted that $Z[\overline{R}(s)] = Z[\overline{R}_s(s)]$ where

$$\overline{R}_s(s) = \left[\begin{array}{c|cc} sI_{n-p} - A_{22} & B_{21} & \hat{B}_{22} \\ \hline -A_{12} & 0 & \hat{B}_{12} \end{array}\right] \tag{33}$$

is a submatrix of $\hat{R}(s)$. Since $A_{12}B_{21}$ is full rank, the problem is reduced to the case presented in Section IV: squaring up a plant model $\Sigma_{p\times m} : \{A_{22}, B_{21}, A_{12}\}$ with relative degree one.

1) Adding Inputs of Relative Degree One: Since Assumption 4 holds, we can always use Mode 0 to find a $\hat{B}_{12}$ and a $\hat{B}_{22}$ such that $Z[\overline{R}_s(s)]$ are stable and $\begin{bmatrix} A_{12}B_{21} & \hat{B}_{12} \end{bmatrix}$ is full rank. In this case, we have added $(p-m)$ inputs of relative degree one and the squared-up plant model has $[n-r_s-(m-p)]$ stable transmission zeros.

2) Adding Inputs of Relative Degree Two: If the dimension of the plant model $\Sigma_{p\times m}$ satisfies that $n-n_z-r_s=2(p-m)$ and $rank[A_{12}]=p$, we can add all inputs of relative degree two. By applying Mode 1 on (33), we can find a $\hat{B}_{22}$ and $\hat{B}_{12}=0$ such that $\begin{bmatrix} A_{12}B_{22} & A_{12}\hat{B}_{22} \end{bmatrix}$ is full rank and the squared-up plant model has $[n-r_s-2(m-p)]$ stable transmission zeros.

3) Adding Inputs Mixed of Relative Degree One and Two: If the dimension of the plant model $\Sigma_{p\times m}$ is bounded by $(p-m)<n-n_z-r_s<2(p-m)$ or $rank[A_{12}]=r_a<p$, we can choose to add inputs mixed of relative degree one and two. By applying Mode H-0-1 on (33), we can append r_a-m relative degree two inputs and $p-r_a$ relative degree one inputs.

B. Non-Uniform Relative Degree Two

This sub-section consider the case that some of $r_i=1$ and some of $r_i=2$. Without loss of generality, we assume the first n_1 inputs of B are relative degree one, and the following n_2 inputs are relative degree two, where $n_1+n_2=m$. Then $\hat{R}(s)$ can be rewritten as

$$\hat{R}(s)=\left[\begin{array}{cc|ccc} sI_p-A_{11} & -A_{12} & B_{11} & 0 & \hat{B}_{13} \\ -A_{21} & sI_{n-p}-A_{22} & B_{21} & B_{22} & \hat{B}_{23} \\ \hline I_p & 0 & 0 & 0 & 0 \end{array}\right] \tag{34}$$

where $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$ corresponds to n_1 relative degree one inputs, and $\begin{bmatrix} 0 \\ B_{22} \end{bmatrix}$ corresponds to n_2 relative degree two inputs. The goal is to find $(p-m)$ inputs as $\begin{bmatrix} \hat{B}_{13} \\ \hat{B}_{23} \end{bmatrix}$ to achieve the squaring-up goals. Performing row elimination yields $Z[\overline{R}(s)]=Z[\overline{R}_s(s)]$ where

$$\overline{R}_s(s)=\begin{bmatrix} sI_{n-p}-A_{22} & B_{21} & B_{22} & \hat{B}_{23} \\ -A_{12} & B_{11} & 0 & \hat{B}_{13} \end{bmatrix} \tag{35}$$

is a sub-matrix of $\overline{R}(s)$. It is noted that B_{11} must have rank n_1, then performing row permutations so that the first n_1 rows of B_{11}, as B_{111} in $B_{11}=\begin{bmatrix} B_{111} \\ B_{112} \end{bmatrix}$, has full rank. Performing suitable partition will transform $\overline{R}(s)$ into

$$\overline{R}_s(s)=\begin{bmatrix} sI_{n-p}-A_{22} & B_{21} & B_{22} & \hat{B}_{23} \\ -A_{121} & B_{111} & 0 & \hat{B}_{131} \\ -A_{122} & B_{112} & 0 & \hat{B}_{132} \end{bmatrix}. \tag{36}$$

Then performing row elimination using the row of B_{111} will yield

$$\overline{R}_{s1}(s)=\begin{bmatrix} sI_{n-p}-\tilde{A}_{22} & 0 & B_{22} & \tilde{B}_{23} \\ -A_{121} & B_{111} & 0 & \hat{B}_{131} \\ -\tilde{A}_{122} & 0 & 0 & \tilde{B}_{132} \end{bmatrix} \tag{37}$$

where $\tilde{A}_{22}=A_{22}-B_{21}B_{111}^{-1}A_{121}$, $\tilde{A}_{122}=A_{122}-B_{112}B_{111}^{-1}A_{121}$, $\tilde{B}_{23}=\hat{B}_{23}-B_{21}B_{111}^{-1}\hat{B}_{131}$ and $\tilde{B}_{132}=\hat{B}_{132}-B_{112}B_{111}^{-1}\hat{B}_{131}$. Then column and row elimination indicates that $Z[\overline{R}_{s1}(s)]=Z[\overline{R}_{s2}(s)]$ where

$$\overline{R}_{s2}(s)=\begin{bmatrix} sI_{n-p}-\tilde{A}_{22} & B_{22} & \tilde{B}_{23} \\ -\tilde{A}_{122} & 0 & \tilde{B}_{132} \end{bmatrix}. \tag{38}$$

Since B_{22} correspond to relative degree two inputs, $\tilde{A}_{122}B_{22}$ has full rank. The problem is reduced to the case presented in Section IV: squaring up the relative degree one plant model $\Sigma_{p\times m}:\{\tilde{A}_{22},\tilde{A}_{122},B_{22}\}$.

1) Adding Inputs of Relative Degree One: Since Assumption 4 holds, we can always add $(p-m)$ inputs of relative degree one. Applying Mode 0 to $\Sigma_{p\times m}:\{\tilde{A}_{22},\tilde{A}_{122},B_{22}\}$ yields a $\tilde{B}_{23}$ and a $\tilde{B}_{132}$. The added inputs $\hat{B}_{13}$ and $\hat{B}_{23}$ can be calculated using an iterative procedure:

$$\text{while} rank[B_{11}, A_{12}B_{22}, \hat{B}_{13}] < p \tag{39}$$

$$\text{pick} \hat{B}_{131} \in \mathbb{R}^{n_1 \times (p-m)} \tag{40}$$

$$\tilde{A}_{22} = A_{22} - B_{21}B_{111}^{-1}A_{121} \tag{41}$$

$$\tilde{A}_{122} = A_{122} - B_{112}B_{111}^{-1}A_{121} \tag{42}$$

$$\text{Mode 0on } \{\tilde{A}_{22}, \tilde{A}_{122}, B_{22}\} \Rightarrow (\tilde{B}_{23}, \tilde{B}_{132}) \tag{43}$$

$$\hat{B}_{23} = \tilde{B}_{23} + B_{21}B_{111}^{-1}\hat{B}_{131} \tag{44}$$

$$\hat{B}_{132} = \tilde{B}_{132} + B_{112}B_{111}^{-1}\hat{B}_{131} \tag{45}$$

$$\hat{B}_{13} = \begin{bmatrix} \hat{B}_{131} \\ \hat{B}_{132} \end{bmatrix} \tag{46}$$

2) Adding Inputs of Relative Degree Two: If the dimension of the plant model $\Sigma_{p\times m}$ satisfies that $n-n_z-r_s = 2(p-m)$ and $rank[\tilde{A}_{122}] = p$, we can add all inputs of relative degree two. By applying Mode 1 on $\Sigma_{p\times m} : \{\tilde{A}_{22}, \tilde{A}_{122}, B_{22}\}$, we can find a $\hat{B}_{22}$ and $\hat{B}_{12} = 0$ such that $\begin{bmatrix} B_{11} & A_{12}B_{22} & A_{12}\hat{B}_{23} \end{bmatrix}$ is full rank and the squared-up plant model has $[n-r_s-2(m-p)]$ stable transmission zeros.

3) Adding Inputs Mixed of Relative Degree Two and One: If the dimension of the plant model $\Sigma_{p\times m}$ is bounded by $(p-m) < n-n_z-r_s < 2(p-m)$ or $rank[\tilde{A}_{122}] = r_a < p$, similar iterative procedure can be employed here, only that applies Mode 2 on $\{\tilde{A}_{22}, \tilde{A}_{122}, B_{22}\}$. Mode 2 will then determine $(r_a - m)$ relative degree two inputs and $(p - r_a)$ relative degree one inputs. Some caution should apply to ensure that $\begin{bmatrix} B_{11} & A_{12}B_{22} & \hat{B}_{13,n_1} & A_{12}\hat{B}_{23,n_2} \end{bmatrix}$ is full rank. The squaring-up results for relative degree two plants are summarized in the following Theorem.

Theorem 1. *Given a plant model $\Sigma_{p\times m} : \{A, B, C\}$ that satisfies assumptions 1 to 4, in addition, has relative degree $\mathfrak{r} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m \end{bmatrix}^T$ with $\max_i [r_i] = 2$, there exists a $\hat{B} \in \mathbb{R}^{n\times(p-m)}$ such that the squared-up plant $\overline{\Sigma}_{p\times m} : \{A, \overline{B}, C\}$, where $\overline{B} = [B, \hat{B}]$, has all stable transmission zeros, and nonuniform relative degree $\bar{\mathfrak{r}} = \begin{bmatrix} r_1, & r_2, & \cdots, & r_m, \cdots, r_p \end{bmatrix}^T$ with some $r_i = 2$ for $i = m+1, \cdots, m+m_s$ and some $r_j = 1$ for $j = m_s+1, \cdots, p$, where $m \geq m_s \geq p$ and can be arbitrary, depending on the design of $\hat{B}$.*

This completes our squaring-up method for relative degree two plants.

VI. Applications to Adaptive Output-Feedback Control

This section incorporate the squaring up procedure with adaptive output-feedback control design and therefore extend the control design to non-square plant models. Collaboration between squaring-up method and adaptive control for relative degree one case has been solved in [9]–[11]. This paper focus such collaboration for relative degree two cases.

A. Problem Statement

Our starting point is a linear non-square plant model, which includes the integral action part of a baseline controller, and can be represented as (see [9]–[11] for a detail procedure)

$$\begin{aligned} \dot{x} &= Ax + B\Lambda^*(u + \Theta^{*T}x) + B_z z_{cmd} \\ y &= Cx \\ z &= C_z x + D_z\Lambda^*[u + \Theta^{*T}x] \end{aligned} \tag{47}$$

where $x \in \mathbb{R}^n$ are states, $u \in \mathbb{R}^m$ are control input, $y \in \mathbb{R}^p$ are measurement outputs and $z \in \mathbb{R}^r$ are tracking outputs. The plant model is non-square since there is more outputs than inputs, i.e. $p \geq m$. There is equal or less number of tracking outputs than inputs, i.e. $r \leq m$. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $B_z = \mathbb{R}^{n\times r}$, $C_z \in \mathbb{R}^{r\times n}$ and $D_z \in \mathbb{R}^{r\times m}$ are known, and $\Theta^* \in \mathbb{R}^{n\times m}$, $\Lambda^* \in \mathbb{R}^{m\times m}$ are unknown. Λ^* represents unknown actuator anomalies, and Θ^* represents unknown state-dependent input perturbations, such as flexible wing deformation in aircraft. Define $A^* = A + B\Lambda^*\Theta^{*T}$.

Assumptions on the plant model necessary for adaptive control can be found in [14]. An additional assumption that $(n - n_z - 2m) \geq (p-m)$, where n_z is the number of transmission zeros of $\{A, B_2C\}$, is satisfied for the squaring-up method to be feasible.

B. Control Design

We choose the control input u as

$$u = u_{bl} + u_{ad} \tag{48}$$

where u_{bl} is determined using a baseline observer-based controller and u_{ad} by an adaptive controller. The baseline control u_{bl} is chosen as

$$u_{bl} = -K^T x_m \tag{49}$$

where $K^T \in \mathbb{R}^{m\times n}$ is designed by the linear quadratic regulator (LQR) technique, and x_m is the state of a minimal observer as

$$\begin{array}{c} \dot{x}_m = Ax_m + Bu_{bl} + B(as+1)(\Psi_m^T \overline{e}_{ay}) \\ +B_z z_{cmd} + L(y - y_m) \\ y_m = Cx_m,\ z_m = C_z x_m + D_z u_{bl} \end{array} \tag{50}$$

where

$$e_{ay} := aR^{-1}Se_y, \quad \text{and } e_y := y - y_m \tag{51}$$

and signal $\overline{(\cdot)}$ is a filtered version of signal $(\cdot)$ as

$$\begin{array}{c} a\cdot \dot{\overline{u}}_{bl} + \overline{u}_{bl} = u_{bl} \\ a\cdot \dot{\overline{x}}_m + \overline{x}_m = x_m \\ a\cdot \dot{\overline{e}}_{ay} + \overline{e}_{ay} = e_{ay}. \end{array} \tag{52}$$

In (52), $a > 0$ is a free filter parameter. The adaptive part of control u_{ad} is chosen as

$$u_{ad} = -u_{bl} + (as+1)\left(\Lambda^T \overline{u}_{bl} - \Theta^T \overline{x}_m\right), \tag{53}$$

where parameter $\Lambda^T(t) \in \mathbb{R}^{m\times m}$, $\Theta^T(t) \in \mathbb{R}^{n\times m}$ and $\Psi_m(t) \in \mathbb{R}^{m\times m}$ will be adapted online by prescribing their derivative $\dot{\Lambda}^T$, $\dot{\Theta}^T$ and $\dot{\Psi}_m$, respectively, as

$$\begin{array}{c} \dot{\Theta}(t) = \Gamma_\theta \overline{x}_m e_y^T S_1^T sign(\Lambda^*) \\ \dot{\Lambda}^{-T}(t) = -\Gamma_\lambda \overline{u}_{bl} e_y^T S_1^T sign(\Lambda^*) \\ \dot{\Psi}_m = \Gamma_\psi \overline{e}_{ay} e_y^T S^T \end{array} \tag{54}$$

C. The Role of Squaring-Up

To design the control parameters L, R^{-1} and S, first we apply the squaring-up method in Section V to find a $\hat{B}$ such that $\overline{\Sigma}_{p\times p} : \{A, \overline{B}, C\}$, where $\overline{B} = [B, B_a]$ is square, and has stable transmission zeros and input relative degree, $\hat{r} = \left[\begin{array}{cccc} r_1, & r_2, & \cdots, & r_m, \cdots, r_p \end{array}\right]^T$, with $r_i = 1$ for $i = m+1, m+2, \cdots, p$. Then we add zeros into B by define

$$B_1 = aAB + B \tag{55}$$

which then formulate an relative degree one plant $\overline{\Sigma}_{p\times p} : \{A, \overline{B}_1, C\}$ with

$$\overline{B}_1 = \left[\begin{array}{cc} B_1 & B_a \end{array}\right]. \tag{56}$$

Then we apply the method in [11] to design L, R_{in}^{-1} and S as (see [11], [14] for a detail solution)

$$S = (C\overline{B}) \tag{57}$$

$$R^{-1} = R^{-1}(A_{in}, \overline{B}^1_{2,in}, \overline{C}_{in}, \overline{\Psi}_{max}) \tag{58}$$

$$L = \overline{B}_1 R_{in}^{-1} S, \tag{59}$$

which guarantees SPR properties of $\{(A + B\Psi^{*T} - L^*C), \overline{B}_1^*, SC\}$. $\overline{B}_1^*$ is an uncertain version of $\overline{B}_1$ satisfying $\overline{B}_1^* = \overline{B}_1 + Ba\Psi_m^{*T}$, and L^* is an uncertain version of L satisfying $L^* = L + Ba\Psi_m^{*T}R^{-1}S$. With suitable partition

$$S = \left[\begin{array}{c} S_1 \\ S_2 \end{array}\right], \tag{60}$$

where $S_1 \in \mathbb{R}^{m\times p}$, the fact that $\{(A + B\Psi^{*T} - L^*C), \overline{B}_1^*, SC\}$ is SPR implies that $\{(A + B\Psi^{*T} - L^*C), B_1^*, S_1C\}$ is also SPR. The SPR properties then are used to prove the stability of the adaptive control design (see [14]).

VII. Applications to VFA

This section applies the adaptive output feedback controller on a simplified very flexible aircraft (VFA) model (see [16] for model descriptions). The longitudinal and vertical dynamics of the VFA is coupled with the dynamics of rotational movement of outer wings with respect to the center wing about the axis of wing chord. The angle between the two adjacent wing planes is denoted as wing *dihedral* (η). A 7-state linear model has been developed including pitch mode, phugoid mode, and dihedral dynamics around a trim at 30ft/sec airspeed, 40,000 ft altitude and one dihedral [16].

Assuming that the airspeed is maintained by auto-thrust, we truncated the phugoid mode from the model and obtained a 4-state linear model with pitch mode and dihedral dynamics. Measurable states are vehicle vertical acceleration A_z and pitch rate q. Other states, such as angle of attack α, dihedral η and its rate $\dot{\eta}$ cannot be measured accurately and are not available for control. The goal is to use elevators δ_e to achieve the tracking of a vertical acceleration command on the center wing. We obtained a plant model around a trim of $\eta = 10^\circ$ as in (61):

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\eta} \\ \ddot{\eta} \\ \dot{\delta}_e \\ \dot{w}_{A_z} \end{bmatrix} = \underbrace{\begin{bmatrix} -4.104 & 1.013 & 0.193 & 0.100 & -0.795 & 0 \\ -54.04 & 0.255 & 1.845 & 21.41 & 5.991 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.044 & 0.819 & -0.075 & -6.518 & 0.195 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -123.12 & 0 & 0 & 0 & -23.84 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha \\ q \\ \eta \\ \dot{\eta} \\ \delta_e \\ w_{A_z} \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{B_2} \underbrace{\begin{bmatrix} u_e \end{bmatrix}}_{u} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{B_z} \underbrace{\begin{bmatrix} z_{A_z} \end{bmatrix}}_{z_{cmd}}$$

$$y = \begin{bmatrix} q \\ w_{A_z} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} x \tag{61}$$

where u_e are elevator commands input to the actuators. We also included integral tracking error term w_{A_z}. A first order actuator dynamics has been added.

VFA aims at high-altitude and long-endurance flight in a hostile environment, which might induce model uncertainties. First, there might be a control surface damage that reduce control effectiveness by up to 90%, which is modeled as $\Lambda^* u$. Second, unsteady aerodynamics can couple with actuator dynamics, and induces aeroelastic effects, which can be modeled as a function of q and η [17]; moreover, the true eigenvalues of actuator dynamics might be different than the nominal values; these two effects are modeled as $B\Psi^{*T}x$. The overall uncertain model is

$$\begin{aligned} \dot{x} &= (A + B\Psi^{*T})x + B_2\Lambda^* u + B_z z_{cmd} \\ \Lambda^* &= 0.1 \\ \Psi^{*T} &= \begin{bmatrix} 0.1 & -0.1 & -1.0 & 0.22 & \Delta_a & 0 \end{bmatrix} \end{aligned} \tag{62}$$

The pitch mode of (62) is unstable and therefore loosing control effectiveness challenges stability.

We proceed to the control design. Using squaring-up procedures find

$$B_a = \begin{bmatrix} 0 & -0.0021 & -0.0001 & 0.0004 & 0.1655 & 1.000 \end{bmatrix}^T. \tag{63}$$

Using (59) and (57) yields a SPR pair of L and S:

$$L = \begin{bmatrix} -7.53 & 3.53 \\ 56.48 & -26.64 \\ -0.008 & -0.001 \\ 1.90 & -0.86 \\ 58.53 & -14.38 \\ -101.1 & 126.3 \end{bmatrix}, \quad S = \begin{bmatrix} 0.419 & -0.908 \\ 0.908 & 0.419 \end{bmatrix}. \tag{64}$$

Simulation results are shown below. For the baseline controller without adaptation terms, the resulting controller is an observer-based linear controller (referred as the baseline controller) and the CRM acts as an observer. Performing frequency domain analysis [9, Chapter 5], as shown in the Figure 1 for $\eta = 10^\circ$, indicates that the baseline controller has adequate stability margins and small output sensitivity; the gain margin is $[-35.7,\ 77.3]dB$ and the phase margin is $\pm 59.3^\circ$.

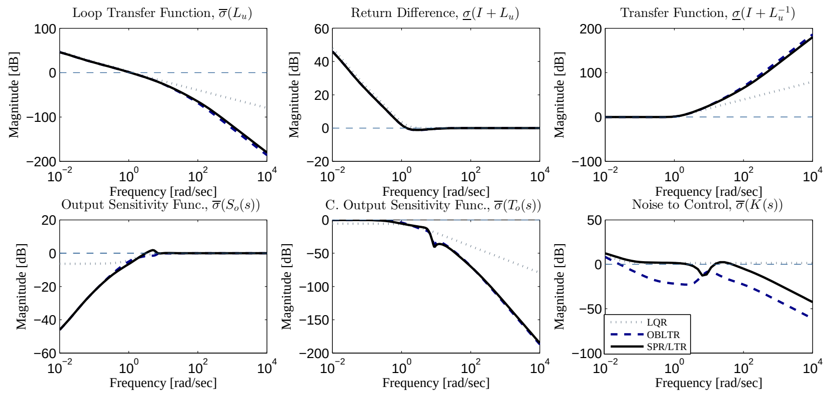
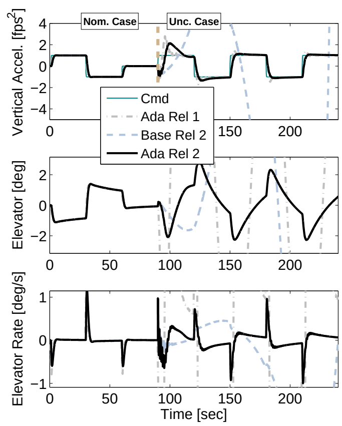
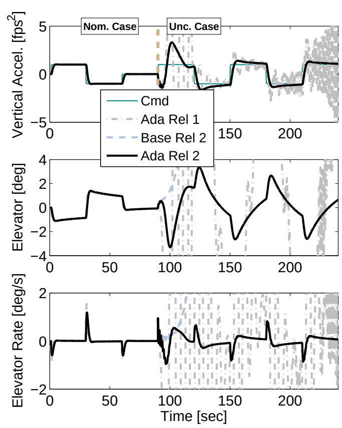


Figure 1: The frequency domain analysis of the baseline observer-based controller indicates it has adequate stability margins



(a) Actuators with time constant of 1.5 sec



(b) Actuators with time constant of 10 second

Figure 2: The tracking vertical acceleration A_z using the relative degree two adaptive controller, compared with the relative degree one adaptive controller; the baseline controller (without adaptation) is also shown; actuator uncertainties and dihedral drift effects kick in after $t = 90$ sec

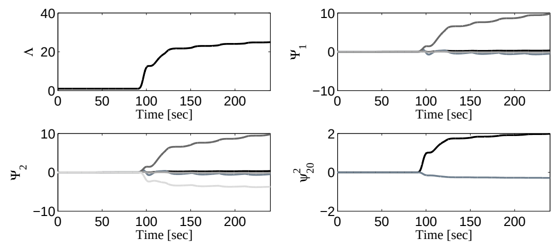


Figure 3: The parameter trajectories of the relative degree two adaptive controller in the simulation with actuators of time constant 10 sec, much slower than the modeled ones; actuator uncertainties and dihedral drift effects kick in after $t = 90$ sec

he time domain simulation results with the nonlinear VFA model are shown in Figure 2. Two actuator models were simulated, one with a time constant of 1.5 second, and the other 10 second. Two adaptive controllers were tested: one is relative degree one as developed in Ref. [11], which pretends the actuator dynamics is not present; the other is the relative degree two shown in Section VI based on a nominal actuator model as in (61). The baseline controller was also tested. With fast actuators, both adaptive controllers were able to achieve tracking goals while the baseline controller failed to do so, as shown in Figure 2a. When actuator dynamics was slow as shown in Figure 2, only relative degree two adaptive controller can achieve stable command tracking after the uncertainties are present. The parameter trajectories of the relative degree two adaptive controller are shown in Figure 3.

VIII. Conclusions

This paper presents an extension to the square-up method proposed when the underlying system has relative degree two. The resulting augmentation matrix is applied to adaptive control of a non-square plant to produce successful tracking under parametric uncertainties. Both the squaring-up procedure and the overall output-feedback based MIMO adaptive controller are numerically validated using a linear model of the very flexible aircraft with unknown actuator faults.

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