Robust Estimation and Failure Detection for Reentry Vehicle Attitude Control Systems

by

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B.S., Mechanical Engineering (1995)
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Submitted to the Department of Mechanical Engineering in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1998

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Vehicle Attitude Control Systems

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Abstract

This thesis presents a robust failure detection methodology for the attitude control systems of Reusable Launch Vehicles (RLVs). In particular, we evaluate the residual according to a threshold criterion to determine anomalies in the control aerosurfaces. Only elevon failures are considered since the rudder is relatively ineffective as a control surface for high angles of attack during the initial stages of reentry. We also consider the related problem of estimating the thrust from multiple jets firing in the Reaction Control System (RCS). The thrust estimates may be used for monitoring the health of the RCS, i.e. detecting jet failures. For accurately known state-space dynamics, the Kalman filter provides the optimal estimate in the least-squared error sense. Because of its optimality, the Kalman filter may be used as a residual generator for fault detection as well as an estimator for jet thrust. During reentry, however, plant model uncertainties present tremendous difficulties for the filter as the vehicle’s aerodynamics vary widely with rapidly changing Mach Number, angle of attack, and altitude. Consequently, the Kalman filter’s estimates degrade severely in the presence of uncertainty. Filters based on robust $H_{\infty}$ or game-theoretic estimation are shown to give promising estimation and detection performance results for a wide range of Mach Numbers and angles of attack when tried on a simulation of the space shuttle Orbiter’s attitude control system.

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Acknowledgments

My favorite sight is the Boston skyline about twenty minutes after sunset. Standing on the northern side of Harvard Bridge, I often look eastward to see a cloudless, dark blue sky and Boston's red and golden lights reflected in the still waters of the Charles River. It is a particularly beautiful sight during the winter, and yet despite the cold, I am often filled with a feeling of warmth and contentment. I will miss these moments the most.

A little longer than two years have passed since I came to M.I.T. from California. It has been a long and bumpy road, and I would like to thank the people who have helped me along the way. There have been many who deserve mention for their assistance in my graduate studies. I would like to extend my gratitude to the Charles Stark Draper Laboratory for providing me a research project and funding. Without their support, this thesis would not have been possible. In particular, I would like to thank Roger Hain and Rami Mangoubi for serving as my advisors at Draper. Their insights, knowledge, and patience have been invaluable to me. I would also like to thank my friends and labmates Chris Dever, Nhu Ho, Rudy Boehmer, Carla Haroz, Tony Giustino, Chad Tilbury, and Beau Lintereur. I enjoyed tremendously their camaraderie both inside and outside the lab. Thanks also to Christian Jacquemont who opened his home to me and provided me lunches whenever we talked about classwork or research on robust estimation.

Thanks to Professor Frank Feng in the Mechanical Engineering Department for his eleventh hour rescue in serving as my M.I.T. thesis supervisor.

My warmest thanks go out to the friends from UC Davis with whom I have been able to keep in contact—Alex Rueff, Dave Wiprut, Nganha Vu, Brad Winsor, Jennifer Jue, and Annie Koo. Thanks to Sandra Uesugi for the banana bread recipe that everyone loves. Thanks to my old roommate Roberto Almanzan who shares a number of my hobbies—your various reviews and speculations are always entertaining. Particular thanks to one of my closest and dearest friends, Tomoko Takano. We always managed to talk at least once a week despite our schedules. Thanks for all the recipes, gifts, and care packages that wound their way from Michigan to Massachusetts.

Many thanks to the friends I made in Ashdown House over time. Thanks to those I met first and made me feel welcome—Victor Luchangco, Tony Falcone, Payman Kassaei, Sanjay Pahuja, Ioanid Rosu, Chris Spohr, and Matt Secor. Those who have already graduated include Tim Derksen, Hing Tan, Rosanna Tse, and Celia Huey. Thanks for Joel Guzman, another alumnus of UC Davis, for conversations about computers and traveling, billiard games, and many dinners around town. From Ashdown desk, I would like to thank Emy Chen,
Wendy Yu, Kari Bingen, Carrie Metzger, and William Moyne. Thanks also to Christine Butts, Steph Zielinski, and Steph Sharo—my other “bosses.” Thanks to my friends on AHEC—Tom Lee, Rebecca Xiong, Megan Hepler, and Debbie Hyams. Thanks to Pat Walton whom I saw bright and early every Sunday morning. Thanks to John Matz and Kathy Liu who always greeted me warmly. Thanks to Marc Johnson for our talks about favorite authors. Thanks to Tom Burbine for keeping me up to date on Seinfeld even when I no longer had time to watch it. Tommy-Jo Galletti is also from California and entered M.I.T. the same semester I did—what a coincidence that we graduate at the same time.

Thanks to Nana Lee for her patience every time I stuck my camera in her face. Thanks to Nancy Hsiung for our discussions on photography, music, and art. Thanks to Gaston Tudury for his insights on photography. Thanks also to Professor Vernon Ingram for encouraging me to exhibit my photos. There are so many of you to remember—my apologies for any omission.

I would like to thank my family for their enduring love and support. Thanks for always believing I could meet the goals I set out to accomplish. In closing, I would like to express my sincerest gratitude to Elaine Yee. Despite my shortcomings, you still accepted me and encouraged me to become better than I am. You have been my best friend and confidante. I love you very, very much. You’re the one I want to spend the rest of my life with, and I look forward to a long life of companionship with you. For you, I hereby dedicate this document.

This thesis was prepared at the Charles Stark Draper Laboratory, Inc., under Independent Research and Development Project Number 13246.

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Chapter 1

Introduction

A major consideration in the design and development of future aerospace vehicles is the reduction of life cycle costs. A key enabling technology that will help aerospace systems to do this is Vehicle Health Management (VHM). In particular, VHM systems for Reusable Launch Vehicles (RLVs) would monitor off-nominal performance and possibly reconfigure vehicle subsystems to correct for degraded or failed components. At the heart of such a system would be a Fault Detection and Isolation (FDI) algorithm.

In general, many engineering systems rely on fault detection and isolation methodologies to detect anomalies and isolate a failed component. One of the most commonly used methods is simply hardware redundancy where multiple sensors are used, and failures are recognized through some voting mechanism. In many practical situations, however, hardware redundancy may not be feasible or desirable because of the greater weight and costs incurred by additional hardware. Furthermore, a sensor may not be available to measure a state of interest directly, or the sensor may be too noisy to be reliable. After all, any real engineering application involving signals to be processed will include some unwanted noisy component. To alleviate these problems, an analytic observer or estimator may be constructed that uses the measurements and a priori knowledge of the system dynamics to provide the necessary information.

The first model-based estimator for linear systems, the Wiener filter, was
developed by Norbert Wiener shortly after World War II. This filter gives an estimate with the minimum mean-squared error between a signal and its estimate when the plant dynamics are expressed in terms of known auto- and cross-correlation functions for the signal and noise. The Wiener filter also assumed wide-sense stationary signals, i.e., random sequences whose mean and covariance statistics do not change over time. In 1960, Kalman made an important breakthrough by developing a state-space equivalent to Wiener’s frequency domain filter. Assuming an accurate description of the plant dynamics as well as Gaussian probability densities for the process and sensor noises, the Kalman filter also achieves an estimate with the minimum mean-squared error. Furthermore, Kalman showed that this filter is stable for parametric uncertainties of the plant, provided that the plant dynamics are stable [12]. On the other hand, its performance degrades considerably with increased uncertainty. In other words, modeling errors between the true dynamics and the filter design model will cause the Kalman filter estimate to diverge from the actual system state.

Returning to the concept of FDI systems for RLVs, our particular objective in this thesis is to detect and isolate failures in the space shuttle Orbiter’s attitude control system. The Orbiter’s attitude control system is composed of control aerosurfaces and a Reaction Control System (RCS) that provide complete rotational authority and limited three-axis maneuvering capability. The challenge for the FDI system would be to distinguish between failures in the aerosurface actuators and failures in the RCS jets during atmospheric reentry. For the purposes of this thesis, we choose the reentry flight phase since rapid changes in atmospheric properties cause large aerodynamic uncertainty. This uncertainty makes it difficult to rely on a single accurate linear state-space model for the Orbiter and a single Kalman filter design based on that model. Specifically, the Kalman filter design would rely on models that are in part a function of the vehicle’s altitude, angle of attack, and Mach number. All of these quantities vary rapidly over a large range during the short reentry phase of the Orbiter’s flight.
Even if altitude, angle of attack, and Mach number were accurately known, rapid gain scheduling may not be desirable or even possible due to the large data storage requirements it entails. Several Kalman filters based on a number of different state-space models would need to be run in parallel. Furthermore, some blending logic would need to be formulated to incorporate all the measurements from the various filters. This only adds to the computational complexity of the system.

Recent advancements in estimation theory include robust game-theoretic $H_\infty$ filter designs based on the small gain theorem [1], [14],[15],[16]. Such filters are robust to a general class of noise and plant model uncertainties. The steady-state robust filter derived in [1] is used in [20] to estimate the space shuttle Orbiter's RCS jet thrust. The results in [20] demonstrate that steady-state robust filters can yield improved thrust estimates for a wide range of Mach numbers, although the filter response is unacceptably slow.

The goal of this thesis is to design a filter architecture based on transient, discrete-time, robust filters that can distinguish between failures in the control aerosurfaces and failures in the RCS. The architecture consists of two robust filters operating in parallel. One filter is designed to detect failures in the aerosurfaces. The other filter is tuned to estimate jet thrust in the presence of aerodynamic model uncertainty. Moreover, if a failure in the aerosurfaces is detected, then it may be possible to reconfigure the jet thrust estimator in order to obtain an accurate estimate of the jet thrust.

The thesis is organized as follows. Chapter 2 provides a brief summary of the space shuttle Orbiter's atmospheric reentry trajectory. We then give an overview of the Orbiter's attitude control system which is comprised of the control aerosurfaces and RCS jets. It is primarily the RCS that controls and maintains rotational stability during the initial stages of reentry. The general operation of the RCS is described, and the location and operation of the various aerosurfaces are also explained.

In Chapter 3, we formulate our estimation problem. The Orbiter's rota-
tional vehicle dynamics are developed. Linear, state-space matrix representations of the dynamics at different operating conditions are considered. In particular, the different models come from the equilibrium operating conditions for a given altitude, angle of attack, and Mach number. Process noise models in the discrete-time state evolution equations are formulated, and sensor noise models are introduced. We initially take the differences between a pair of state-space matrices to define our uncertainty in the rotation dynamics. A Gauss-Markov model of a random variable is used to represent the jet thrust estimate.

In Chapter 4, different filtering methods are examined. The discrete-time Kalman filter is the $H_2$ optimal estimator. In other words, it provides the best estimate in terms of the least-squared error, provided that the dynamics of a system are accurately known. An $H_\infty$ or game-theoretic minimax estimator is often used in the case of unknown noise model to guard against worst-case measurement disturbances. For our problem, however, an extension of these filters in the form of a robust $H_\infty$ filter provides better results. The robust filter performance exceeds that of other “robustified” filters such as the overdesigned Kalman filter [10] and the exponentially-forgetting Kalman filter. This is easily seen in some example cases.

In Chapter 5, the complete Fault Detection and Isolation (FDI) architecture is presented to distinguish between failures in the aerosurfaces and failures in the RCS. This filter architecture assumes there are no simultaneous failures. Results are presented to demonstrate its effectiveness.

For a final analysis, Chapter 6 summarizes the results and insights from the previous chapters. Recommendations are made for the direction of future research.
Chapter 2

Space Shuttle Orbiter Attitude Control Effectors

2.1 Operation During Atmospheric Reentry

Reusable Launch Vehicles rely on their aerosurfaces and Reaction Control Systems for attitude control. During on-orbit operation, only the RCS may be used since aerosurfaces require atmospheric dynamic pressure to be effective. As an example, the space shuttle Orbiter’s aerosurfaces consist of the following:

- Elevons, for pitch and roll axes control
- Rudder, for yaw axis control
- Speedbrake, for lateral stability
- Bodyflap, for pitch trim

Its Reaction Control System consists of bi-propellant jets that are fired in the roll and yaw directions to provide desired thrust and augment the aerosurfaces. Shuttle reentry begins with a deorbit burn where the Orbiter is oriented in a tail-first position and jets are fired to slow the vehicle and allow capture by the earth’s gravity and atmosphere.
The RCS jets then reorient the Orbiter to a high angle of attack, nose-first position. During the first part of reentry, the Orbiter is oriented to a 40° angle of attack, allowing the heat shield on the underside of the Orbiter's fuselage to protect the crew and cargo compartments. Descent of the vehicle is controlled by changing bank angle in a series of slow roll reversal maneuvers known as S-turns. The high angle of attack is maintained until the vehicle descends and decelerates to below Mach 10. After slowing to below Mach 10, the Orbiter gradually reduces its angle of attack to 10°. The RCS are the sole attitude effects until enough atmospheric dynamic pressure accumulates for the aerosurfaces to become useful. At an attitude of approximately 40,000 feet and about 30 minutes before the nominal landing time, the control surfaces are activated and attitude control is provided by both the RCS and the aerosurfaces. During the last stages of reentry, attitude control is achieved by the aerosurfaces acting alone.

The elevons are the first surfaces to become effective. At dynamic pressures of 10 psf, the RCS rolls jets are deactivated, and roll control is provided solely by the elevons. When dynamic pressure reaches 20 psf, pitch thrusters are disabled, and pitch attitude is controlled by the elevons as well. Because the Orbiter spends much of its reentry flight phase at a relatively high angle of attack, yaw is the last axis to be controlled by the aerosurfaces. Adverse aerodynamics caused by the Orbiter's body and wings shadowing the rudder render it relatively ineffective for yaw control. The rudder does not become effective until the angle of attack decreases to below 10°. Yaw jets remain active for almost the entire reentry flight phase until an altitude of approximately 45,000 feet.

Figure 2-1 shows the nominal atmospheric reentry profile. The actual flight dynamics may be approximated by a nonlinear model. However, linear state-space representations of the rotational dynamics may be found by linearizing about equilibrium flight conditions described by altitude, angle of attack, and Mach number. The circles on the graph represent the equilibrium flight conditions for linear models incorporated into this thesis [20].
2.2 Orbiter Control Aerosurfaces

In this section, we describe the operation of the Orbiter's different control aerosurfaces. More exhaustive information may be found in [23].

The space shuttle Orbiter does not have the same control surfaces as a conventional airplane. Instead, the Orbiter has five movable control surfaces. These are the inboard and outboard elevons on its large delta wings, a vertical rudder, a speedbrake (split rudder), and a body flap. Table 2.1 gives the maximum deflection and deflection rate of each of the aerosurfaces. In Figure 2-2, the location and direction of positive aerosurface deflection are defined.

2.2.1 Elevons

Conventional aircraft have separate elevators and ailerons to provide pitch and roll control authority. The Orbiter controls pitch and roll through a pair of
Figure 2-2: Location and Direction of Orbiter Aerosurface Deflections

<table>
<thead>
<tr>
<th>Control Surface Actuator</th>
<th>Positive Deflection</th>
<th>Maximum Displacement (degrees)</th>
<th>Maximum Rate (degrees/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elevons</td>
<td>$\delta_{eL}, \delta_{eR}$</td>
<td>$-35$ to $+20$</td>
<td>20</td>
</tr>
<tr>
<td>Rudder</td>
<td>$\delta_{R}$</td>
<td>$\pm 22.8$</td>
<td>10</td>
</tr>
<tr>
<td>Speedbrake</td>
<td>$\delta_{SB}$</td>
<td>0 to $+87.2$</td>
<td>5</td>
</tr>
<tr>
<td>Bodyflap</td>
<td>$\delta_{BF}$</td>
<td>$-11.7$ to $+22.55$</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Table 2.1: Maximum Aerosurface Deflections and Deflection Rates
elevons. Each elevon is split into an inboard and outboard section, so that there are four segments that may be operated independently. However, the elevons are usually operated in such a way as to recreate the effects of conventional elevator and aileron deflections. Furthermore, during reentry, the inboard and outboard sections deflect in unison, effectively reducing the number of surfaces from four to two. The elevator deflection is defined as the average of the port and starboard elevon deflections. A positive elevator deflection causes a negative pitching moment.

\[ \delta_e = \frac{1}{2} (\delta_{eL} + \delta_{eR}) \quad (2.1) \]

Aileron deflection is defined to be half the difference between port and starboard elevon deflection. Positive aileron deflection initially results in a positive roll torque.

\[ \delta_u = \frac{1}{2} (\delta_{eL} - \delta_{eR}) \quad (2.2) \]

Neither the equivalent elevator nor the equivalent aileron deflection may be at their maximum value simultaneously since their definitions both depend on the elevon position. Figure 2-3 shows how the elevator and aileron deflections are related. The elevons have a range of $-35^\circ$ to $+20^\circ$ and are rate limited to $20^\circ$/sec.

![Graph showing the relationship between aileron and elevator deflections](image)

Figure 2-3: Maximum Simultaneous Aileron and Elevator Deflections
2.2.2 Rudder

The rudder aerosurface on the Orbiter is very similar in structure to the conventional aircraft rudder and is used to control yaw axis rotation. It is located on the aft portion of the vertical stabilizer. The rudder has a range of ±22.8° with no speedbrake deflection and is rate limited to 10°/sec.

2.2.3 Speedbrake

The Orbiter's rudder is actually composed of two panels that open symmetrically to form the speedbrake. The speedbrake helps provide lateral stability during the later stages of reentry. It has a range of 0° to +87.2° and is rate limited to 5.0°/sec. Just as the elevator and aileron deflections are coupled, so are the rudder and speedbrake deflections. If one surface is at partially deflected, the corresponding maximum deflection of the other surface is reduced. Figures 2-4 and 2-5 show how the deflections for the two surface are related.

![Diagram](image)

Figure 2-4: Relationship Between Rudder and Speedbrake Deflections

2.2.4 Bodyflap

The fifth control aerosurface is a horizontal surface mounted ventrally on the centerline of the Orbiter, beneath the main engines. The bodyflap is essentially a pitch trim surface in that if the Orbiter's center-of-mass is fore or aft of its nominal position, an appropriate bodyflap deflection restores the vehicle's
Figure 2-5: Maximum Simultaneous Rudder and Speedbrake Deflections

dynamics. The bodyflap’s trimming helps prevent saturation of the elevons. The bodyflap is capable of deflecting from $-11.7^\circ$ to $+22.55^\circ$ and is rate limited to $1.3^\circ$/sec.

2.3 Reaction Control System

In this section, we look at the operation of the space shuttle Orbiter’s Reaction Control System (RCS). More detailed information regarding the system subcomponents may be found in [23].

The RCS consists of forty-four bi-propellant jets that, together with the control aerosurfaces, provide attitude control and limited three-axis maneuvering capability. The RCS operates during all phases of flight, from orbital ascent through atmospheric reentry. During on-orbit operation, the RCS is used for the following:

- Small $\Delta V$ burns,
- Trim of large burns,
- Attitude hold,
- Attitude maneuvering

During reentry, the RCS is used as well to achieve pitch control during entry interface and yaw control during high angle of attack flight. Thirty-eight of the
jets are primary jets, each capable of providing 870 lb. of thrust in vacuum. The other six jets are smaller, vernier jets capable of providing only 25 lb. of thrust and are used for precise attitude control.

Figure 2-6: Location of Reaction Control System Pods

Three separate pods comprise the RCS. One pod is located in the forward nose section. The other two pods are located to port and to starboard in the aft section of the Orbiter. Figure 2-6 is a detailed schematic of the directions in which RCS jets fire. Figure 2-7 shows some of the essential components present in an RCS thruster module. The components of an RCS pod include high pressure helium tanks, pressure regulation, relief, and isolation valves, propellant tanks, thrusters, and an electrical heating system.

As mentioned previously in Chapter 1, the RCS jets are the sole attitude effectors for the initial part of reentry since there exists insufficient dynamic pressure in the thin atmosphere for the control surfaces to be effective. As the Orbiter loses altitude, falling into the increasingly denser atmosphere, dynamic pressure increases and the aerosurfaces become activated. The aerosurfaces deflect in order to augment the jets until there exists enough dynamic pressure to control the Orbiter's attitude with the aerosurfaces alone.
2.3.1 Jet Propulsion Subsystems

Figure 2-8 shows the network of pipes and valves in the forward and rear RCS pods. Two helium tanks provide high pressure gaseous helium to fuel and oxidizer tanks in each pod. The high pressure helium is used to convey the propellants to the thrusters. In the propellant tanks, the gaseous helium and liquid propellants coexist as a random mixture. Consequently, each propellant tanks contains an acquisition device that operates on the surface tension properties of the propellant. The acquisition device ensures that propellant is properly expelled from the tank over a wide range of orientations and accelerations. In particular, the acquisition devices for the forward RCS pod are designed for use in low gravity environments, while those in the rear RCS pods are designed for both low and high gravity environments. From the tanks, the propellants are fed to the individual thrusters via a network of valves and manifolds.

The primary manifolds (numbered 1/2/3 in Figure 2-8) on the forward RCS pod supply four of the primary jets. Manifold (4) four leads to only two primary jets. The two forward vernier RCS jets are supplied by manifold (5) five. On the rear RCS pods, each of the primary manifolds (1/2/3/4) connects to three primary jets. The vernier manifold (5) supplies two jets on each of the rear pods.
Figure 2-8: Plumbing for the Forward and Rear RCS Pods
The various components of the RCS are separated by isolation valves located through the propulsion subsystems. For example, four helium isolation valves in each kind of RCS pod, two for the fuel and two for the oxidizer, separate the helium and propellant tanks. Below each of the helium isolation valves is a helium regulator, used to step down the helium pressure from approximately 3,000 psi. to 256 psi. After the regulators, the helium lines in all three RCS pods rejoin and enter the quad check valves. Quad check valves are used to keep the propellant from flowing back upstream into the helium supply tanks. Furthermore, pressure relief valves, burst discs, and manually operated isolation valves are present at the output of the quad check valves to prevent the propellant tanks from becoming overpressurized.

Farther past the pressure relief components are the propellant tanks. There are two propellant tanks in each pod, one for the monomethyl hydrazine fuel and one for the nitrogen tetroxide oxidizer. The fuel and oxidizer are hypergolic compounds, combusting spontaneously when they come into contact with one another.

### 2.3.2 Thruster Layout

Figure 2-9 shows the essential components of an RCS thruster module. On the forward RCS pod, there are housed fourteen primary jets and two vernier jets. The two aft RCS pods equally divide the remaining twenty-four primary jets and four vernier jets. Figure 2-9 shows the general locations of the RCS pods on the Orbiter. Figure 2-9 also shows jet placement on the forward RCS pod. Figure 2-10 shows some of the different jets housed in an aft pod.

The RCS jets are commanded to fire through an electronically-controlled injection system. Thrust-on commands are interpreted by the Orbiter’s attitude control system which then sends electrical signals to solenoids that activate the fuel and oxidizer valves. Propellants flow separately through the injector assemblies and into the combustion chamber where they come into contact and
Figure 2-9: Location of the RCS pods

Figure 2-10: Jet Placement on an Aft RCS Pod
ignite. Thrust is then produced by the rapid expansion of the combustion gases as they pass through the nozzle. When a thruster is commanded off, the solenoids are de-energized and closed through spring and pressure loads.

2.3.3 Redundancy Management Control

The current Redundancy Management (RM) system aboard the Orbiter affords some measure of automatic fault detection for the RCS jets. The software tracks the available jets, processes the status of the manifolds, and performs the following tasks:

- **Chamber Pressure failed jet detection.** The RM compares the jet firing command and thruster chamber pressure to identify jets that have failed to fire when commanded to fire. When the chamber pressure signal does not indicate a thruster has been fired, yet the jet fire command is present for a predetermined number of consecutive computer cycles, the jet is considered failed off. The processing time for the RCS is 80 msec for all major modes of flight. Typically, the limiting number of computer cycles is three, although the maximum limit may be set to as high as six cycles. When a jet is failed-on, it is automatically deselected, and the jet selection is reconfigured.

- **Reaction Jet Driver failed jet detection.** To determine whether a thruster is firing although it has not been commanded to fire, the RM uses the jet firing command and the Reaction Jet Drivers (RJD) output. The RJD is used to process electrical signals to and from individual thrusters. As in the Chamber Pressure failed jet detection monitor, the jet is considered failed on when the firing command and the RJD output are different for a predetermined number of consecutive computer cycles. For most missions, the set number of consecutive cycles is three, although it may be configured to a maximum of six. If a jet is failed-on, warnings are sent to the crew. In this case, the jet must be manually deselected before the attitude control
system may reconfigure the jet firing patterns.

- *Monitor for degraded (leaking) jets.* A jet is considered leaking when either the fuel or the oxidizer injector temperature sensor falls below a preset limit for a given number of consecutive cycles. When propellant leaks without combusting, there is evaporative cooling of the injector. The critical temperatures for the RCS are 20°F for the fuel and 30°F for the oxidizer. The vernier jets have as their limits 30°F for both fuel and oxidizer except during on-orbit flight and during operations prior to ascent. At these times, the limits are set to 130°F. For the jet-leak-monitor, the cycling time is 1.92 seconds. When a jet is failed as leaking, warnings are sent to the crew. The jet is automatically deselected, and the attitude control system reconfigures jet selection.

### 2.3.4 RCS Sensor Data

The status of the Reaction Control System may be monitored through a variety of sensors. The Orbiter's Inertial Measurement Units (IMUs) and rate gyros provide attitude and angular rate measurements. These global sensors may be used to check the actual effect of a commanded RCS maneuver on the Orbiter's rotational dynamics. The thruster modules themselves have a suite of local sensors, specific to almost every component of the RCS. Such sensors include ones to measure chamber pressure, manifold pressure, and fuel and oxidizer temperatures. In addition, there are pressure and temperature sensors for the helium, fuel, and oxidizer tanks. Figure 2-11 shows some of the various sensors present among the RCS subsystems.

### 2.3.5 Typical Failures in the RCS

Due to its complexity, the RCS has been prone to numerous minor problems since space shuttles were first launched. For example, during the flight of STS-51, 40% of the recorded anomalies involves problems with the thrusters. Most
Figure 2-11: RCS Sensor Locations
<table>
<thead>
<tr>
<th>Anomaly</th>
<th>Description</th>
<th>Cause</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thruster Leak</td>
<td>Propellant escapes from jet</td>
<td>Injector valves fail to close</td>
<td>▶ Small disturbance torque</td>
</tr>
<tr>
<td>Helium Regulator</td>
<td>Regulator does not regulate pressure</td>
<td>Leaking seal in regulator</td>
<td>▶ Loss of He pressurant</td>
</tr>
<tr>
<td>Failure</td>
<td></td>
<td></td>
<td>▶ Reliance on one regulator</td>
</tr>
<tr>
<td>Sensor Failure</td>
<td>Any of the following:</td>
<td>Any sensor failure</td>
<td>▶ False alarms</td>
</tr>
<tr>
<td></td>
<td>▶ Valve position</td>
<td></td>
<td>▶ False deselects</td>
</tr>
<tr>
<td></td>
<td>▶ Manifold status</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>▶ Tank Pressure</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>▶ Tank temperature</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>▶ Injector temperature</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>▶ Chamber pressure incorrect</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Injector Heater</td>
<td>Electrical heaters do not heat injector rings</td>
<td>▶ Failed heater element</td>
<td>▶ Possible deselect as leaking</td>
</tr>
<tr>
<td></td>
<td></td>
<td>▶ Failed thermostat</td>
<td></td>
</tr>
<tr>
<td>Helium Ingestion</td>
<td>Helium gas in propellant lines</td>
<td>▶ Large flow demand</td>
<td>▶ Interrupted thrust</td>
</tr>
<tr>
<td></td>
<td></td>
<td>▶ Multiple jets firing simultaneously</td>
<td>▶ Possible deselect by RM</td>
</tr>
</tbody>
</table>

Table 2.2: Summary of Typical RCS Failures
of these anomalies involved leaking jets. Leaks often occur at the injectors. Both the fuel and oxidizer are prone to leaking, as well as the helium regulators. Other problems include valves becoming stuck or propellant feed lines suffering from helium ingestion. Table 2.2 summarizes some of the most common failures, including their causes and effects.

One potentially dangerous type of failure is helium ingestion which occurs when gaseous helium enters the propellant acquisition device. When helium bubbles pass through the system, combustion is temporarily interrupted. For the primary jets, this is not a problem. For the vernier jets, on the other hand, the problem may be persistent enough for the thruster to be come unreliable. The flight software will then fail the jet and deactivate it for the remainder of the mission. There does, however, exist the minute possibility that a helium bubble could cause combustion instability and catastrophic explosion of the vernier jet. Because of this possibility, the number of jets firing simultaneously from the same pod is limited to four [22].
Chapter 3

Problem Formulation

A Fault Detection and Isolation (FDI) algorithm is generally used to troubleshoot anomalies present in complex engineering systems. An analytic estimator or filter is often implemented rather than hardware redundancy because of potential restrictions on size, weight, and costs. The filter works by processing the sensor measurements, combined with a priori knowledge of the plant dynamics to provide necessary information regarding the states of the system.

The a priori knowledge used in modern estimation schemes typically takes the form of a state-space representation of the plant dynamics. In particular, a linear, time-invariant state-space representation is generally used because of the great body of theory that already exists for linear systems. One way to detect analytically the existence of a failure is to look for discrepancies between the actual plant’s output and a model-based estimate of that output. One may treat the error as a random sequence, so that when a failure is present, the frequency content shifts from white. Plant models, however, are usually incomplete and inaccurate, often ignoring high-frequency dynamics. Furthermore, FDI algorithms based on the quality of the measurement residuals assume a particular failure mode. Incorrect plant dynamics and failure mode modeling may lead to a high false alarm rate, making it difficult to distinguish failures properly.

Another method for analytic fault detection and isolation assumes that the failure mode may be modeled as a Gauss-Markov random variable. In other
words, the failure is treated as the output of a shaping filter. This Gauss-Markov random state variable can then be appended to the system's model representation. The filter is then comprised of an augmented linear system with plant states and failure states. In any case, a robust FDI test, i.e., one designed to be robust to plant model and failure mode uncertainties, must be able to differentiate between these kinds of errors so as to avoid excessive false alarms as well as missed detections.

As mentioned in Chapter 1, the goal of this thesis is to design a robust fault detection and isolation system for the space shuttle Orbiter's attitude control system. The robust FDI system would need to be able to distinguish between failures in the control aerosurfaces and failures in the RCS. As we will see in Chapter 5, it is convenient to design an FDI architecture using two robust filters. Using these two estimators in parallel allows us to perform the appropriate failure isolation in the presence of plant model uncertainties.

In this chapter, the model structure for the space shuttle Orbiter's rotational dynamics is formulated. This simplified, linear model was originally derived in [29], based upon the particular trim operating conditions for a given altitude, angle of attack, and Mach number. Process noise models to accommodate neglected high-frequency dynamics and measurement noise models for the Orbiter rate-gyro sensors used in this application are introduced. A discussion is made of the Gauss-Markov model for the RCS thrusters as well as possible uncertainty sources for the augmented system. Complete state-space representations for each of the various linearized models are given in Appendix A. Relevant vehicle parameters are given in Appendix B.
3.1 Model Identification

3.1.1 Attitude Coordinate Representations

The derivation in [29] treats the Orbiter as a rigid body and uses two coordinate frames to characterize the rotational motions. The relationship between external torque and angular acceleration (essentially Newton's Second Law) is found first in the body axis frame. These equations are then transformed to the stability axis frame where a more convenient set of angular coordinates are used.

Body-centered coordinates use a local, right-handed orthogonal frame aligned with the Orbiter body. Figure 3-1 illustrates the body axis frame. The $x$-body axis points through the Orbiter’s nose, positive in the forward direction. A rotation about the $x$-body axis, referred to as roll, is positive when the starboard wing is down. The positive $y$-body axis extends outward from the starboard wing. A positive rotation about the $y$-body axis, referred to as pitch, is defined when the nose of the Orbiter points up. The $z$-body axis extends down through the Orbiter body. Rotation about the $z$-axis, yaw, is positive when the Orbiter rotates clockwise when viewed from the top. Changes in the roll, pitch, yaw rotational angles, i.e., roll, pitch, and yaw rates, are also designated $p$, $q$, and $r$, respectively.

![Figure 3-1: Coordinates in the Body Axis Frame](image)

The stability axis frame uses another local, right-handed orthogonal frame
aligned with the Orbiter's freestream velocity. The Orbiter's attitude dynamics are more easily described using this frame. The stability axis is defined by two rotations, $\alpha$ and $\beta$. After rotating through these angles, the x-body axis and freestream velocity vector coincide. Figure 3-2 depicts these rotations between the body axis frame and the stability axis frame. The first rotation through an angle $\alpha$ is performed such that the rotated z-body axis (alternatively, the $z'$-axis) is orthogonal to the plane defined by the rotated x-body axis (alternatively referred to as the $x'$-axis) and the freestream velocity vector. The angle $\alpha$ is now designated as the angle of attack. A second rotation by an amount $\beta$ aligns the $x'$-axis with the freestream velocity. The angle $\beta$ is referred to as the sideslip angle. In general, the sideslip angle for Orbiter during reentry will remain small. In addition, another quantity known as the bank angle $\phi$ is defined as the amount of rotation from wings-level flight, about the freestream velocity.

![Figure 3-2: Rotations from the Body Axis Frame to the Stability Axis Frame](image)

The sign conventions for bank, angle of attack, and sideslip are defined with respect to wings-level flight. Abusing notation, we define a positive roll angle as increasing bank angle (at zero angle of attack and zero sideslip), a positive pitch rotation as increasing angle of attack (at zero sideslip), and a positive yaw rotation as decreasing sideslip (at zero angle of attack). Figure 3-3 show the angles and their relationships in the stability frame.
3.1.2 Application of Newton's Second Law

Applying Newton's Second Law to the rotational motion of the Orbiter, we find that the equations of motion are as follows:

$$\Sigma \tau_{\text{external}} = I \alpha + \omega \times I \omega$$  \hspace{2cm} (3.1)

where

$$\Sigma \tau_{\text{external}} = \text{sum of all external moments acting on the Orbiter}$$

$$\alpha = \text{angular accelerations in body coordinates}$$

$$\omega = \text{angular rates in body coordinates}$$

$$= \begin{bmatrix} p & q & r \end{bmatrix}^T$$

$$I = \text{inertia tensor}$$

$$= \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}$$

The products of inertia $I_{xy}$ and $I_{yz}$ are approximated to be zero since the $x$-$z$ plane is a plane of symmetry (see Figure 3-1). In what follows, we assume that the only external moments acting on the Orbiter are those due to aerodynamic forces and due to the thrusters. Consequently, we may separate these mo-
ments along each of the body axes. We use the notation \( \tau = [ L \quad M \quad N ]^T \), 
\( \dot{\tau} = [ \dot{L} \quad \dot{M} \quad \dot{N} ]^T \) to indicate those moments caused by the aerodynamic 
forces and RCS jet firings, respectively.

Substituting into Equation (3.1) and expanding terms, we get

\[
I_{xx}\dot{p} - I_{xx}\dot{r} + (I_{zz} - I_{yy})qr - I_{xz}pq - L = \hat{L} \tag{3.2}
\]
\[
I_{yy}\dot{q} + (I_{xx} - I_{zz})pr - I_{xz}(p^2 - r^2) - M = \hat{M} \tag{3.3}
\]
\[
I_{zz}\dot{r} - I_{xz}\dot{p} + (I_{yy} - I_{xx})pq - I_{zz}qr - N = \hat{N} \tag{3.4}
\]

The above equations are linearized about an equilibrium flight condition 
using small perturbation theory. In other words, each of the angular velocities 
p, q, and r is replaced by its equilibrium value (usually 0) plus small, nonzero 
perturbations. The equilibrium conditions are then subtracted from the system 
of equations, so that we are left with with dynamics for the perturbations 
alone. For further details, see [29] and [20]. The resulting system of equations 
is as follows, with lowercase letters now indicating perturbation moments which 
represent small deviations from equilibrium moments.

\[
I_{xx}\dot{p} - I_{xx}\dot{r} - l = \hat{l} \tag{3.5}
\]
\[
I_{yy}\dot{q} - m = \hat{m} \tag{3.6}
\]
\[
I_{zz}\dot{r} - I_{xz}\dot{p} - n = \hat{n} \tag{3.7}
\]

The perturbation aerodynamic moment terms \( l, m, \) and \( n \) are expanded into 
partial derivatives with respect to aerosurface deflection, vehicle attitude, and 
vehicle rates. These partial derivatives as also known as stability and control 
derivatives (see Appendix B). The system of equations is normalized with re-
spect to velocity and later transformed to the stability axis frame. As described 
in [20], the lateral and longitudinal modes of motion are decoupled from one an-
other. After simplification, we obtain equations for the simplified aerodynamics
model with the following definitions applying:

\[ \phi = \text{bank angle (rad)} \]
\[ \alpha_T = \text{trim angle of attack (rad)} \]
\[ \bar{\alpha} = \alpha - \alpha_T = \text{Perturbation from } \alpha_T \text{ (rad)} \]
\[ \beta = \text{sideslip angle (rad)} \]
\[ \gamma = \text{flight path angle (rad)} \]
\[ \bar{q} = \text{dynamic pressure (lb/ft}^2) \]
\[ S = \text{wing reference area (ft}^2) \]
\[ c = \text{reference chord (ft)} \]
\[ b = \text{wingspan (ft)} \]
\[ V_T = \text{trim velocity (ft/sec)} \]
\[ g = \text{acceleration due to gravity (ft/sec}^2) \]
\[ m = \text{mass of the Orbiter (slugs)} \]

The angle of attack (longitudinal mode) is governed by

\[ \frac{d^2 \bar{\alpha}}{dt^2} + 2\zeta_\alpha \omega_\alpha \frac{d\bar{\alpha}}{dt} + \omega_\alpha^2 \bar{\alpha} = K_{\delta e} \tau_{\delta e} \frac{d\delta_e}{dt} + K_{\delta e} \delta_e + u_y \quad (3.8) \]

where

\[ \omega_\alpha^2 = - \left( \frac{\bar{q}S_c}{I_{yy}} \right) C_{Ma} \]
\[ \zeta_\alpha = \frac{1}{2\omega_\alpha} \left( \frac{\bar{q}S}{V_T} \right) \left[ \frac{C_D + C_{La}}{m} - \left( \frac{c^2}{2I_{yy}} \right) C_{Mq} \right] \]
\[ K_{\delta e} = \left( \frac{\bar{q}S_c}{I_{yy}} \right) C_{Ms_e} \]
\[ \tau_{\delta e} = \left( \frac{I_{yy}}{mcV_T} \right) \left( \frac{C_{D\delta e \sin(\alpha_T)} - C_{L\delta e \cos(\alpha_T)}}{C_{Ms_e}} \right) \]
The lateral sideslip dynamics are as follows:

\[
\frac{d^2 \beta}{dt^2} + 2\zeta_\beta \omega_\beta \frac{d\beta}{dt} + \omega_\beta^2 \beta + K_\phi \frac{d\phi_s}{dt} = K_{\beta_{sr}} \dot{\delta}_r + K_{\beta_{sa}} \dot{\delta}_a + u_x \sin(\alpha_T) - u_z \cos(\alpha_T) \tag{3.9}
\]

where

\[
\omega_\beta^2 = \left( \frac{\bar{q}Sb}{I_{xx}} \right) C_{n\beta}'
\]

\[
\zeta_\beta = -\frac{1}{2\omega_\beta} \left( \frac{\bar{q}S}{mV_L} \right) \left[ C_{Y_b} + \left( \frac{mb^2}{2I_{zz}} \right) C_{I_b} \right]
\]

\[
K_\phi = \left( \frac{g}{V_L} \right) + \left( \frac{\bar{q}Sb^2}{2I_{zz}V_L} \right) C_{I_\phi}
\]

\[
K_{\beta_{sr}} = -\left( \frac{\bar{q}Sb}{I_{zz}} \right) C_{n_{sr}}'
\]

\[
K_{\beta_{sa}} = -\left( \frac{\bar{q}Sb}{I_{zz}} \right) C_{n_{sa}}'
\]

Finally, the lateral bank angle dynamics are given by

\[
K_\beta + \frac{d^2 \phi_s}{dt^2} + \frac{1}{\tau_\phi} \frac{d\phi_s}{dt} = K_{\phi_{sr}} \dot{\delta}_r + K_{\phi_{sa}} \dot{\delta}_a + u_x \cos(\alpha_T) + u_z \sin(\alpha_T) \tag{3.10}
\]

where

\[
K_\beta = \left( \frac{\bar{q}Sb}{I_{xx}} \right) C_{n\beta}'
\]

\[
\frac{1}{\tau_\phi} = \left( \frac{g}{V_L} \right) \left[ \sin(\gamma_o) - \left( \frac{I_{xx}}{I_{zz}} \right) \sin(\alpha_T) \sin(\theta_o) + \left( \frac{I_{xx}}{I_{zz}} \right) \cos(\alpha_T) \cos(\theta_o) \right]
\]

\[
-\left( \frac{\bar{q}Sb^2}{2I_{xx}V_L} \right) C_{2_\phi}
\]

\[
K_{\phi_{sr}} = \left( \frac{\bar{q}Sb}{I_{xx}} \right) C_{I_{sr}}'
\]

\[
K_{\phi_{sa}} = \left( \frac{\bar{q}Sb}{I_{xx}} \right) C_{I_{sa}}'
\]

Equations (3.8)-(3.10) represent the Orbiter's linearized rotational dynamics at a prescribed Mach Number, angle of attack, and altitude. Figure 3-4 shows a block diagram of the simplified equations for aid in visualization. Furthermore,
the equations are combined and written in the state-space representation of
Equation (3.11).

\[ \begin{align*}
\begin{bmatrix} \dot{\phi} \\ \ddot{\alpha} \\ \dot{\beta} \\ \ddot{\phi} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -K_\beta & -\frac{1}{\tau_\phi} & 0 & 1 \\ 0 & -\omega_\alpha^2 & 0 & 0 & -2\zeta_\alpha \omega_\alpha & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \ddot{\alpha} \\ \beta \\ \dot{\phi} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_\beta^2 & -K_\phi & 0 \\ K_\phi \delta_a & 0 & K_\phi \delta_r & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \\ \delta_e \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_{\theta_\phi} \\ b_{\theta_\alpha} \\ b_{\theta_\beta} \end{bmatrix} \theta 
\end{align*} \] (3.11)

Figure 3-4: Block Diagram of Simplified Dynamics Model

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In the above equation, we have grouped all the terms related to the body frame thrust inputs \( u_x, u_y, \) and \( u_z \) into one multiple jet thrust state \( \theta \). Equation (3.11) provides us with a continuous-time linear state-space model of the form:

\[
\dot{x} = Ax + Bu + T\theta
\]  

(3.12)

Usually, we would want complete state measurements available to the attitude controller and fault detection system. For the purposes of this thesis, however, we will limit ourselves only to measurements involving the angular rates. We do this since we assume only IMU rate sensors are available which would allow our algorithms to be implemented eventually on current Orbiter hardware. Therefore, the state-space output equation is given by:

\[
y = \begin{bmatrix}
0 & 0 & 0 & 0.8192 & 0 & 0.5736 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0.5736 & 0 & -0.8192 \\
\end{bmatrix}
\]

\[
x = Cx
\]  

(3.13)

The state-space equations are then discretized (using a sampling period of \( T = 0.005 \) seconds) for actual filter design and implementation. Consequently, we will have a discrete-time linear state-space representation of the Orbiter’s dynamics in the following form:

\[
x_{k+1} = A_k x_k + B_k u_k + T_k \theta_k
\]

\[
y_k = C_k x_k
\]  

(3.14)

The control and stability derivatives encountered in Equations (3.8)-(3.10) are non-dimensional parameters that relate aerosurface deflections and vehicle properties, respectively, to the forces and moments acting on the Orbiter via appropriate scaling factors. As these three parameters change, so do the aerodynamic coefficients.
3.2 Disturbance Models

The dynamics represented by Equations (3.14) are defined for the open-loop and do not account for any disturbances in either the state evolution or measurement equations. This is not realistic since all practical engineering systems contain unwanted noise to some degree. In fact, the dynamics of the space shuttle Orbiter are more realistically modeled by the following:

\[
x_{k+1} = A_k x_k + B_k u_k + \tau_k \theta_k + G_k d_k \\
y_k = C_k x_k + E_k d_k
\]  \hspace{1cm} (3.15)

The vector \( d_k \) represents some input noise to the system. Therefore, in addition to control inputs (\( u_k \) and \( \theta_k \)), the states are now affected by process noise (\( G_k d_k \)) that arises from several different sources. For example, input disturbances, unmodeled, and incorrectly modeled dynamics may all be modeled as process noise disturbances to the system. Furthermore, the outputs of the system are corrupted by measurement noise (\( E_k d_k \)) that may be due to faulty or degraded sensors. A particular case of interest that simplifies the analysis is when the process noise and sensor noise inputs are uncorrelated such that \( G_k E_k^T = 0 \).

The process noise input matrix \( G_k \) is nominally taken as the discretization of \( \frac{1}{20} \text{diag}(B) \) from the continuous-time representation. This indicates a ±5% uncertainty in the aerosurface deflections. To add some measure of robustness to the estimator, the matrix \( G_k \) may be multiplied by a constant \( a > 1 \) to increase the design process noise covariance. However, there are limitations for this technique as is shown in [10]. In particular, it is shown that for small perturbations an overdesigned Kalman filter may provide adequate performance. When the perturbations are larger, however, an alternate approach is necessary.

The sensor noise input matrix \( E_k \) represents the quantization level of the rate gyroscopes and IMUs aboard the Orbiter. The level of quantization refers
to the finest resolution that can be measured for the angular rates and is driven by analog to digital conversion. No nonlinearities in the attitude rate measuring sensors were included since quantization level is the dominant factor affecting the introduction of measurement noise into the system output. Other such sources include gyroscope bias, angle random walk, factor error, nonorthogonality error, and misalignment error. In particular, for bias stability and angle random walk, there is insufficient time during atmospheric reentry for these sources to become significant enough to affect the quality of measurement. See [20] for additional discussion.

3.3 Gauss-Markov Model for the Thrust

Since we are concerned with failures in the Reaction Control System, we will estimate the multiple jet firing command and include this as an additional state in our filter design. For this purpose, we use a simple, scalar, high-bandwith Gauss-Markov model to represent the multiple jet state.

$$\theta_{k+1} = a_\theta \theta_k + g_\theta \vartheta_k$$  \hspace{1cm} (3.16)

Equation (3.16) is a stochastic model for the thrust. We assume $\vartheta_k$ is a white noise sequence with unit variance, and $\theta_0$ is Gaussian with zero mean and large covariance $P_{\theta_0}$. The parameters $a_\theta$ and $g_\theta$ determine the bandwidth and amplitude, respectively, of the system. The higher the bandwidth, then the larger the class of failures to be considered. Failures in the RCS with frequency content higher than the bandwidth of the Gauss-Markov model would be treated as additional noise in the system.

The above model is appended to the plant model of Equation (3.15) such that the jet thrust is now considered an estimator state rather than a command input. When the occasion calls for the design of filters for the Fault Detection and Isolation system, some of them will be based on the following augmented
dynamic system:

\[
\begin{bmatrix}
  x_{k+1} \\
  \theta_{k+1}
\end{bmatrix}
= \begin{bmatrix}
  A_k & T_k \\
  0 & a_\theta
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  \theta_k
\end{bmatrix}
+ \begin{bmatrix}
  B_k \\
  0
\end{bmatrix} u_k + \begin{bmatrix}
  G_k & 0 \\
  0 & g_\theta
\end{bmatrix}
\begin{bmatrix}
  d_k \\
  \vartheta_k
\end{bmatrix}
\]

\[
y_k = \begin{bmatrix}
  C_k & 0
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  \theta_k
\end{bmatrix}
+ \begin{bmatrix}
  E_k & 0
\end{bmatrix}
\begin{bmatrix}
  d_k \\
  \vartheta_k
\end{bmatrix}
\]

(3.17)

3.4 Model Limitations

For the purposes of this thesis, nonlinear effects as described below are left unmodeled.

- **Plume interaction.** Thrust levels can deviate significantly from their nominal output due to plume interaction effects with the aerodynamic flow field. During reentry, aft jets firing into the Orbiter’s slipstream affect the pressure field around the vehicle which may cause a net moment in an unanticipated direction.

- **Plume self-impingement.** This phenomenon occurs when a jet plume deflects off the Orbiter surfaces. The self-impingement moments caused by the rear pods firings in the downward direction may be significant. During reentry, however, side-firing jets used for yaw control are mostly used. These thrusters have relatively small self-impingement effects, so their inclusion does not add significant uncertainty to the fault detection and isolation problem.

- **Increased atmospheric pressure.** As the Orbiter descends through the atmosphere, the increased pressure reduces the effectiveness of the thrusters. Because of increased atmospheric pressure, there is less expansion of the combustion gases through the jet nozzles. The effect on the performance of the RCS thrusters has been empirically modeled as a decaying exponential. See [20] for details.
• Transient effects. The flight control software for the Reaction Control System commands the thrusters in a step-on/step-off fashion. Several factors in the actual hardware contribute to a delayed thrust time response. For example, it takes time for the valves to open and close and for the propellants to meet and combust. These operations take approximately 20 milliseconds after the firing command is issued. After a step-off command, it takes approximately 40 milliseconds for the thrust levels to die off to zero output.

<table>
<thead>
<tr>
<th>Mach Number</th>
<th>Angle of attack (deg)</th>
<th>Altitude (10^3 ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>12.5</td>
<td>77</td>
</tr>
<tr>
<td>3.0</td>
<td>16.0</td>
<td>90</td>
</tr>
<tr>
<td>4.0</td>
<td>20.5</td>
<td>105</td>
</tr>
<tr>
<td>5.0</td>
<td>25.0</td>
<td>120</td>
</tr>
<tr>
<td>5.7</td>
<td>29.0</td>
<td>130</td>
</tr>
<tr>
<td>7.5</td>
<td>35.0</td>
<td>150</td>
</tr>
<tr>
<td>8.8</td>
<td>38.0</td>
<td>158</td>
</tr>
<tr>
<td>10.0</td>
<td>40.0</td>
<td>170</td>
</tr>
</tbody>
</table>

Table 3.1: Summary of Linear Reentry Models

Appendix A presents the state-space matrices for eight linear models created for flight conditions during reentry ranging from Mach 10 to Mach 2. Table 3.1 summarizes the trim operating flight conditions for which the linear models were created.
Chapter 4

State-Space Filtering Methods

4.1 The Kalman Filter

Since its introduction in 1960, the Kalman filter has become the most widely used recursive state-space estimator for linear dynamic systems. Its application has been particularly pervasive in the areas of target tracking, aircraft and aerospace vehicle control, and process control. It has also found use in a wide variety of other fields such as image processing, geophysics, and environmental sciences. The Kalman filter may be thought of as the state-space equivalent to the frequency-domain Wiener filter, although it is more a generalization than just a simple extension. For example, the Kalman filter may be readily used in the multivariable case. Kalman filters may be designed as steady-state or time-varying estimators, as well as for continuous-time and discrete-time systems. In the latter case, which we consider throughout this thesis, the Kalman filter can be easily programmed for digital computer implementation.

By assuming an accurate representation of a system’s state-space dynamics and disturbance characteristics, the Kalman filter generates real-time (recursive), optimal estimates of the states and filtered outputs, based on noisy, past sensor measurements and knowledge of the applied controls. In particular, the state estimates are considered optimal because the Kalman filter attempts to minimize the expected value of the estimation error 2-norm at each instant over
some interval of time. In other words, the Kalman filter provides the linear minimum mean-square error estimate.

### 4.1.1 Derivation of the Kalman Filter Equations

We first consider a linear state-space system without control signal variation [12]:

\[
x_{k+1} = A_k x_k + G_k d_k \\
y_k = C_k x_k + E_k d_k \\
E\{d_k d_l^T\} = \delta(k - l) I
\]

where \(x_k\) is the state vector, \(y_k\) is the measurement vector, and \(d_k\) is a Gaussian white-noise disturbance vector of zero mean and unity covariance. The states are affected by process noise \((G_k d_k)\), while the measurements are corrupted by sensor noise \((E_k d_k)\). The matrices \(G_k\) and \(E_k\), therefore, define how disturbances enter the systems and may be appropriately adjusted to reflect any Gaussian disturbance characteristic.

As stated before, the criterion for finding the best estimate for the state of the system is a minimum mean-square error performance index. Since we are dealing with a stochastic system, we have

\[
\min_{\hat{x}_{(k|k)}} E\{\|e_k\|^2\} \equiv \min_{\hat{x}_{(k|k)}} E\{e_k^T e_k\}, \quad k \in [1, K] \\
\text{subject to} \quad \text{Equations (4.1) - (4.3)}
\]

with

\[
\hat{x}_{(k|k)} \equiv x_k - \hat{x}_{(k|k)} \quad \text{and} \quad e_k \equiv M_k \hat{x}_{(k|k)}
\]

and where we define \(\hat{x}_{(k|k)}\) is an estimate of \(x_k\) and \(e_k\) is the estimation error sequence weighted by some matrix \(M_k\). For the purpose of optimal full-state estimation, however, the weighting matrix is simply the identity, \(I\).

Equation (4.4) can be reformulated, using \(P_{(k|k)}\) to denote the covariance of
the estimation error, assuming that the estimator is unbiased.

\[
\min_{\hat{x}_{(k|k)}} E\{e_k^T e_k\} = \min_{\hat{x}_{(k|k)}} E\{\text{trace}\left(e_k e_k^T\right)\} \\
= \min_{\hat{x}_{(k|k)}} \text{trace}\left(E\{e_k e_k^T\}\right) \\
= \min_{\hat{x}_{(k|k)}} \text{trace}\{P_{(k|k)}\} \tag{4.6}
\]

The structure of the Kalman filter is illustrated in Figure 4-1. The solution to Equation (4.4) or equivalently, Equation (4.6), involves a recursive algorithm that relies on both the model of the system and incoming noisy measurements. Essentially, the Kalman filter evolves as a separate dynamical system that operates in parallel with the nominal plant dynamics. The Kalman filter takes as its input the output of the actual system, corrupted by sensor noise. The outputs from the Kalman filter are estimates of the plant states. These estimates have the minimum mean-square error possible when its representation of the true plant dynamics is accurate and the disturbances are Gaussian white-noise in nature.

Figure 4-1: Structure of the Kalman Filter

The first step in the recursive estimation algorithm is to use a priori information about the plant dynamics to predict the evolution of the state from one
sample time step to the next. The notation $\hat{x}_{(k|k-1)}$ will be used throughout this thesis to indicate the a priori estimate. Similarly, $P_{(k|k-1)}$ will refer to the covariance matrix of the a priori estimate. The update step allows the Kalman filter estimate to be corrected in the sense that the current measurement information is incorporated optimally to satisfy the minimum mean-square error performance criterion. In this case, the a posteriori estimate will be denoted by $\hat{x}_{(k|k)}$ since it includes information, i.e., the measurement, obtained at the current time step. Furthermore, we will use the notation $P_{(k|k)}$ to indicate the covariance of the a posteriori state estimate.

In the following equations, we make the simplifying assumption that the process noise and sensor noise are uncorrelated such that $G_k E_k^T = 0$. The steps may be summarized as follows:

**Prediction**

\[
\hat{x}_{(k|k-1)} = E\{A_{k-1} x_{k-1} + G_{k-1} d_{k-1}\} \\
= A_{k-1} \hat{x}_{(k-1|k-1)} \\
P_{(k|k-1)} = E\{\hat{x}_{(k|k-1)} \hat{x}_{(k|k-1)}^T\} \\
= E\{[A_{k-1} \hat{x}_{(k-1|k-1)} + G_{k-1} d_{k-1}] [A_{k-1} \hat{x}_{(k-1|k-1)} + G_{k-1} d_{k-1}]^T\} \\
= A_{k-1} P_{(k-1|k-1)} A_{k-1}^T + G_{k-1} G_{k-1}^T \tag{4.7}
\]

**Update**

\[
\hat{x}_{(k|k)} = \hat{x}_{(k|k-1)} + K_k \left[ y_k - C_k \hat{x}_{(k|k-1)} \right] \tag{4.9}
\]

\[
P_{(k|k)} = E\{\hat{x}_{(k|k)} \hat{x}_{(k|k)}^T\} \\
= E\{[I - K_k C_k] \hat{x}_{(k|k-1)} - K_k E_k d_k \} \left[ (I - K_k C_k) \hat{x}_{(k|k-1)} - K_k E_k d_k \right]^T \} \\
= (I - K_k C_k) P_{(k|k-1)} (I - K_k C_k)^T + K_k E_k E_k^T K_k^T \tag{4.10}
\]

Equation (4.8) shows how disturbance affects the uncertainty of the state.
estimate. The term $A_{k-1}P_{(k-1|k-1)}A_{k-1}^T$ is simply the propagation of the error covariance via the dynamics of the system. When the dynamics are stable, the error covariance decreases to zero with time. The presence of noise can only increase the uncertainty of the estimate as evidenced by the addition of the positive semi-definite term $G_{k-1}G_{k-1}^T$.

During the update stage, the Kalman filter incorporates the measurement information into its estimate through a corrective term as shown in Equation (4.9). The a posteriori estimate is the sum of the a priori estimate and some weighted innovations, $y_k - C_k \hat{x}_{(k|k-1)}$. The innovations sequence is precisely the difference between the actual output of the plant and the filter’s estimate of the output.

The only part of the Kalman filter that is not intuitively obvious is the calculation of the gain matrix, $K_k$. We notice that the a posteriori covariance is quadratic in $K_k$ and recall that the trace of the covariance matrix is equivalent to the squared estimation error needed to be minimized. By taking the partial derivative of $P_{(k|k)}$ with respect to $K_k$ and setting it equal to zero, the optimal value of the gain may be determined.

The partial derivative is

$$\frac{\partial \left[ \text{trace} \{ P_{(k|k)} \} \right]}{\partial K_k} = -2(I - K_kC_k) P_{(k|k-1)}C_k^T + 2K_kE_kE_k^T \quad (4.11)$$

Setting Equation (4.11) to zero and solving for $K_k$ yields the expression for the Kalman gain matrix

$$K_k^* = P_{(k|k-1)}C_k^T \left[ C_kP_{(k|k-1)}C_k^T + E_kE_k^T \right]^{-1} \quad (4.12)$$

Substituting for $K_k$ in Equation (4.10), the a posteriori error covariance can then be rewritten as

$$P_{(k|k)} = P_{(k|k-1)} - P_{(k|k-1)}C_k^T \left[ C_kP_{(k|k-1)}C_k^T + E_kE_k^T \right]^{-1} C_kP_{(k|k-1)}$$
\[ P^{-1}_{(k|k)} = P^{-1}_{(k|k-1)} + C_k^T \left( E_k E_k^T \right)^{-1} C_k \] (4.14)

In the last expression above, the inverse of the covariance matrix (a measure of uncertainty) can now be interpreted as a measure of information. By incorporating a measurement, Equation (4.14) shows that the information about the state is always increased from the a priori information.

Referring back to Equation (4.9), we see precisely how the Kalman filter appropriately incorporates measurement information. In particular, we notice that when the output of the true plant contains little measurement noise such that \( E_k E_k^T \) is small, and provided that \( C_k \) is in some sense invertible, i.e., \( C_k \) is square and nonsingular, or \( C_k \) has a left pseudo-inverse, \( C_k^+ = C_k^T (C_k C_k^T)^{-1} \), then we see that \( K_k \approx C_k^{-1} \), and the Kalman filter precisely inverts the measurement during the update stage. Over successive time steps, what the Kalman filter does is to use strictly the measurements and not its a priori model of the plant. On the other hand, when the measurement information is very poor, i.e., \( E_k E_k^T \) is large, then \( K_k \to 0 \), and the Kalman filter relies primarily on its internal representation of the system dynamics rather than on any measurements.

The two-step process of prediction and update may be combined to form a one-step recursive Kalman filter. In particular, we form the equations for the one-step forward predictor-to-predictor estimate.

\[
\hat{x}_{(k+1|k)} = A_k \hat{x}_{(k|k)} \\
= A_k \{ \hat{x}_{(k|k-1)} + K_k \left[ y_k - C_k \hat{x}_{(k|k-1)} \right] \} \\
= [A_k - A_k K_k C_k] \hat{x}_{(k|k-1)} + A_k K_k y_k \] (4.15)

The one-step estimator error and error covariance evolve according to

\[
\hat{x}_{(k+1|k)} = x_{k+1} - \hat{x}_{(k+1|k)} \\
= A_k x_k + G_k d_k - [A_k - A_k K_k C_k] \hat{x}_{(k|k-1)} - A_k K_k y_k
\]
\[ = A_k x_k + G_k d_k - [A_k - A_k K_k C_k] \hat{x}_{(k|k-1)} - A_k K_k C_k x_k - A_k K_k E_k d_k \]
\[ = (A_k - A_k K_k C_k) [x_k - \hat{x}_{(k|k-1)}] + (G_k - A_k K_k E_k) d_k \]
\[ = \tilde{A}_k \hat{x}_{(k|k-1)} + \tilde{G}_k d_k \quad (4.16) \]

\[ P_{(k+1|k)} = E\{ \hat{x}_{(k+1|k)}^T \hat{x}_{(k+1|k)} \} \]
\[ = E\{ [\tilde{A}_k \hat{x}_{(k|k-1)} + \tilde{G}_k d_k]^T [\tilde{A}_k \hat{x}_{(k|k-1)} + \tilde{G}_k d_k] \} \]
\[ = \tilde{A}_k P_{(k|k-1)} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T \quad (4.17) \]

where we have used the shorthand notations \( \tilde{A}_k = (A_k - A_k K_k C_k) \) and \( \tilde{G}_k = (G_k - A_k K_k C_k) \).

In the more general case, we place no restrictions on \( G_k E_k^T \) so that the process noise and sensor noise may be correlated. Furthermore, inclusion of command inputs is easily accomplished by appending the term \( B_k u_k \) to the a posteriori estimate. Since we assume that the inputs are precisely known, calculations for the error covariance and Kalman gain are not affected. The one-step recursion formulae in slightly modified form are given by the following:

\[ \hat{x}_{(k+1|k)} = \tilde{A}_k \hat{x}_{(k|k-1)} + B_k u_k + \tilde{K}_k^T y_k \quad (4.18) \]
\[ P_{(k+1|k)} = \tilde{A}_k P_{(k|k-1)} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T \quad (4.19) \]

where

\[ \tilde{A}_k = A_k - K_k^T C_k \quad (4.20a) \]
\[ \tilde{G}_k = G_k - K_k^T E_k \quad (4.20b) \]
\[ \tilde{K}_k' = [A_k P_{(k|k-1)} C_k^T + G_k E_k^T] \left[ C_k P_{(k|k-1)} C_k^T + E_k E_k^T \right]^{-1} \quad (4.20c) \]

All of the preceding equations concern the Kalman filter's behavior in the time-domain. In the frequency-domain, however, there is one other important interpretation for the Kalman filter we should consider. In steady-state, the weighted estimation error sequence \( e_k \) becomes a stationary random process with power spectral density \( S_{ee}(e^{j\omega}) \). In what follows, we denote \( G_{ee}(z) \) as the
transfer function from the disturbance to the weighted estimation error, and 
\( \sigma_i, i = 1, ..., N \) as its singular values. In the limit,

\[
\lim_{k \to \infty} E\{e_k^T e_k\} = \lim_{k \to \infty} E\{\text{trace} [e_k e_k^T]\} \\
= \text{trace} \left[ \lim_{k \to \infty} E\{e_k e_k^T\} \right] \\
= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \text{trace} \left[ S_{ee}(e^{j\omega}) \right] d\omega \\
= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \text{trace} \left[ G_{ed}(e^{j\omega}) G_{ed}^T(e^{-j\omega}) \right] d\omega \\
= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left[ \sum_{i=1}^{N} \sigma_i^2 \left[ G_{ed}(e^{j\omega}) \right] \right] d\omega \\
= \| G_{ed}(z) \|_{H_2}^2
\]

Therefore, the steady-state Kalman filter minimizes the \( H_2 \) norm of the transfer function from the disturbance to the weighted estimation error. The \( H_2 \) norm refers to the integral of the singular values of \( G_{ed} \) over all frequencies. In other words, the Kalman filter minimizes the average transmitted energy from the disturbance to the estimation error. It is for this reason that the Kalman filter is also known as the \( H_2 \) optimal filter.

### 4.1.2 Limitations in the Presence of Model Uncertainty

Based on Equations (4.18)-(4.20c), we see that the Kalman filter’s optimality is highly contingent upon the accuracy of its internal state-space representation of the plant dynamics. In particular, when the models are precisely known, the state and estimator error may be combined to form an augmented system. It is clear that there is no interaction between the states and estimation error. Regardless of whether the system is stable, i.e., \( (A_k, G_k) \) is reachable, \( (A_k, C_k) \) is detectable, the error dynamics will decay to zero.

\[
\begin{bmatrix}
    x_{k+1} \\
    \tilde{x}_{(k+1|k)}
\end{bmatrix} =
\begin{bmatrix}
    A_k & 0 \\
    0 & A_k - K_k C_k
\end{bmatrix}
\begin{bmatrix}
    x_k \\
    \tilde{x}_{(k|k-1)}
\end{bmatrix}
\]
\[ + \begin{bmatrix} B_k \\ 0 \end{bmatrix} u_k + \begin{bmatrix} G_k \\ G_k - K_k E_k \end{bmatrix} d_k \] \hfill (4.21)

However, if the true plant dynamics are described by the slightly perturbed system \( x_{k+1} = (A_k + \Delta A_k) x_k + B_k u_k + G_k d_k \), the plant states and the error states do not remain decoupled. The augmented dynamics, now including a perturbation in \( A \), are shown in Equation (4.22).

\[
\begin{bmatrix} x_{k+1} \\ \tilde{x}_{(k+1|k)} \end{bmatrix} = \begin{bmatrix} A_k + \Delta A_k & 0 \\ \Delta A_k & A_k - K_k C_k \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_{(k|k-1)} \end{bmatrix} + \begin{bmatrix} B_k \\ 0 \end{bmatrix} u_k + \begin{bmatrix} G_k \\ G_k - K_k E_k \end{bmatrix} d_k \] \hfill (4.22)

Even when \( \Delta A_k \) is small, the Kalman filter will not be optimal for the perturbed system. Of course, if the system were reachable and detectable, then the error decays nonetheless. However, if the plant is unstable, then the presence of any uncertainty in \( A \) will give rise to growing error. The error can only increase in magnitude as more matrices are perturbed (e.g. perturbations \( \Delta B_k, \Delta G_k, \Delta C_k, \Delta E_k \) exist in addition for \( \Delta A_k \)).

To illustrate how the Kalman filter’s performance degrades when there exists model mismatch, we look at the navigation equations (3.15) from Chapter 3. Specifically, we design a Kalman filter around the model for a particular trim operating condition (Mach Number = 7.5, \( \alpha = 35^\circ \), altitude = 150,000 feet). The Kalman filter will have a time-invariant plant model but a time-varying gain to provide good transient performance.

We then test it using noisy measurements from this nominal model as well as on corrupted output from a perturbed model (Mach Number = 8.8, \( \alpha = 38^\circ \), altitude = 158,000 feet). Since we are using this example for demonstration purposes only, we assume no failures in the attitude control system and make no effort to estimate the jet thrust levels. A commanded maneuver of 20\(^\circ\) change in bank angle is used in the following simulation. These commands
were generated using reentry flight control software based on closed-loop linear models as mentioned in [20]. The necessary elevons and rudder deflections and yaw axis jet firings are shown in Figure 4-2.

Figure 4-2: Inputs for 20° Bank Command

Figure 4-3 shows the time-domain behavior of the estimator errors, \( \hat{x}_{(k|k)}^i, \) \( i = 1, 3, 4, 6, \) i.e., errors in bank angle, sideslip angle, bank rate, and sideslip rate, respectively. We concern ourselves with only these errors since Equation (3.11) indicates that the elevon deflections affect only the longitudinal dynamics. There does exist some cross-coupling from the yaw jets to the lateral modes. However, the interactions between the two are relatively weak. A single RCS jet firing at nominal capacity (870 lb.) is roughly only comparable to one degree of elevon deflection. This explains why in Section 3.3 we augment a Gauss-Markov model for the entire jet thrust state to our estimator rather than just a representation of a jet failure. Nonetheless, aside from the yaw jet input, the longitudinal and lateral modes are decoupled from one another. We will use this fact in Chapter 5 to simplify some of our filter designs.

For the nominal plant, we see that the Kalman filter has all of its estimation errors near zero. On the other hand, for the perturbed plant, the Kalman filter relies too much on its internal representation of the dynamics and diverges from the true states. Almost immediately we can see how inaccurate plant knowledge
Figure 4-3: Estimation errors for the Kalman filter on a nominal and on a perturbed plant

can affect the filter’s performance. Note that the performance of the filters in the case of \( \hat{x}_{1(k)k} \) (bank angle) appears equal. However, Equation (3.11) shows that bank angle is relatively unobservable. Consequently, estimates of that state will be highly dependent on initial conditions.

<table>
<thead>
<tr>
<th>Squared Error in</th>
<th>Nominal KF</th>
<th>Perturbed KF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bank Angle</td>
<td>0.0564</td>
<td>0.0454</td>
</tr>
<tr>
<td>Sideslip Angle</td>
<td>0.6060</td>
<td>15.2111</td>
</tr>
<tr>
<td>Bank Rate</td>
<td>0.0902</td>
<td>5.5204</td>
</tr>
<tr>
<td>Sideslip Rate</td>
<td>0.0402</td>
<td>21.9326</td>
</tr>
</tbody>
</table>

Table 4.1: Kalman filter performance on a nominal plant and on a perturbed plant. Units are (deg)\(^2\) and (deg/sec)\(^2\).

The squared error for each of the longitudinal states is summarized in Table 4.1. The term “Nominal KF” refers to the Kalman filter designed around the operating condition of \((M = 7.5, \alpha = 35^\circ)\) and applied to noisy measurements from this model, while “Perturbed KF” refers to the same design filtering outputs from the system with \((M = 8.8, \alpha = 38^\circ)\). For the squared
angle errors, the units are \((\text{deg})^2\), while for the squared rate errors, the units are \((\text{deg/sec})^2\). We use the squared error comparison since it is intuitively similar to the variance of a random variable.

### 4.2 The \(H_\infty\) Minimax Filter

In the previous section, we derived the Kalman filter equations under the assumption that the disturbance vector \(d_k\) is Gaussian white-noise of zero mean and unity covariance. One implication of this assumption is that the estimator error sequences \(\hat{x}_{(k|k-1)}\) and \(\hat{x}_{(k|k)}\) are also white-noise with Gaussian distribution. Whiteness is critical to the propagation forward from \(\hat{x}_{(k|k-1)}\) to \(\hat{x}_{(k|k)}\).

If the noise were non-white or colored, then it is necessary to pass the disturbance through a whitening shaping filter and augmenting the plant dynamics. Such a procedure requires accurate knowledge of the noise characteristics which is often not available. As with modeling the plant dynamics, the representations for the noise may be incomplete or only approximate. Furthermore, incorporating additional dynamics into the Kalman filter may be undesirable as it increases computational complexity.

In the following sections, we will rely on a new approach to design estimators for discrete-time linear systems that are robust to plant and noise model uncertainties. This section, in particular, will concentrate on the latter problem of uncertain noise model. Rather than working in a stochastic setting, our approach tackles the filtering problem from a deterministic setting where we make no prior assumptions on the disturbance characteristics, save for a bound on its 2-norm. Essentially, a game-theoretic formulation is used in which the disturbance acts as an opponent to the estimator. Consequently, the performance criterion to minimize will be the transmitted energy from the disturbance to the estimation error. From our derivation will arise an estimator designed to minimize the error in the presence of worse-case noise. For simplicity, we assume zero initial conditions on the state and estimator. This turns out to be a special
case of the complete proof, including arbitrary initial conditions, that may be found in [14].

4.2.1 Derivation of Minimax Filter Equations

We present the approach to the minimax filter derivation as given in [14].

\[ d \rightarrow P \rightarrow y \rightarrow F \rightarrow \hat{x} \]

Figure 4-4: General input-Output representation of state estimation problem

An input-output representation of a nominal plant \( P \) and an estimator \( F \) are shown in Figure 4-4. We assume all inputs and outputs are real vectors. The disturbance input \( d \) includes both process and sensor noise.

The plant output is the measurement sequence \( y \), which is, in turn, fed into the filter \( F \), to produce a state estimate \( \hat{x} \). We consider a finite-horizon setting and define the following notation for \( k \in [0, N - 1] \).

\[
\begin{align*}
  d & \equiv [d_0, \ldots, d_{N-1}] \\
  y & \equiv [y_0, \ldots, y_{N-1}] 
\end{align*}
\]

The weighted state estimation error will be defined by

\[
\begin{align*}
  e & \equiv [e_1, \ldots, e_N] \\
  e_k & \equiv e_{(k|k-1)} = M_k \left[ x_k - \hat{x}_{(k|k-1)} \right] 
\end{align*}
\]  

(4.23)  

(4.24)

Note that the time interval for \( e \) is shifted since we are using the a priori estimate \( \hat{x}_{(k|k-1)} \) of \( x_k \) rather than the a posteriori estimate \( \hat{x}_{(k|k)} \) as in Equation (4.5).
The 2-norms for the sequences \( r \) and \( e \) are defined as:

\[
\|d\|_2 \equiv \left( \sum_{k=0}^{N-1} d_k^T d_k \right)^{1/2}
\]
\[
\|e\|_2 \equiv \left( \sum_{k=1}^{N} e_k^T e_k \right)^{1/2}
\]

The game-theoretic problem formulation assumes a linear state-space representation of the nominal plant as follows:

\[
x_{k+1} = A_k x_k + G_k d_k \quad (4.25)
\]
\[
y_k = C_k x_k + E_k d_k \quad (4.26)
\]
\[
\|d\|_2^2 \leq 1 \quad (4.27)
\]

The above system is essentially the same as Equations (4.1)-(4.3) for the Kalman filter, except that here we simply assume the disturbance is norm-bounded rather than white. Combining the first two equations with Equation (4.24), we may write the state-space representation in block form:

\[
\begin{bmatrix}
x_{k+1} \\
e_k \\
y_k
\end{bmatrix} =
\begin{bmatrix}
A_k & G_k & 0 \\
M_k & 0 & -M_k \\
C_k & E_k & 0
\end{bmatrix}
\begin{bmatrix}
x_k \\
d_k \\
\hat{x}_{(k|k-1)}
\end{bmatrix} \quad (4.28)
\]

We see how all the quantities of interest are related. The zero entries in the last column are obvious since the state estimate affects neither the plant state nor output. Similarly, the disturbance input does not enter into the weighted error.

While the above equations do not include perfectly known input sequences \( u_k \), they may be easily modified to include control signal variation using the following substitutions:

\[
r_k \leftarrow \begin{bmatrix} u_k \\ d_k \end{bmatrix} \quad (4.29)
\]
\begin{align*}
z_k & \leftarrow \begin{bmatrix} u_k \\ y_k \end{bmatrix} \quad (4.30) \\
B_k & \leftarrow \begin{bmatrix} B_k \\ G_k \end{bmatrix} \quad (4.31) \\
C_k & \leftarrow \begin{bmatrix} 0 \\ C_k \end{bmatrix} \quad (4.32) \\
D_k & \leftarrow \begin{bmatrix} I & 0 \\ D_k & E_k \end{bmatrix} \quad (4.33)
\end{align*}

The matrix $B_k$ describes how the inputs $u_k$ enter the system. The matrix $D_k$ allows for output feedthrough of the commands. The augmentations to the matrices $C_k$ and $D_k$ are of dimensions corresponding to $u_k$. Conceptually, the $u_k$ are considered as disturbances, but we measure them exactly. Therefore, they will make no contributions to the calculations for the error covariance or filter gain. For convenience, we derive the filter equations for no control input. The results will remain valid in the case of control signal variation given the substitutions of Equations (4.29)-(4.33) are made.

The game-theoretic estimator seeks to bound and minimize the induced norm of the operator from the input disturbances to the estimation error, for any possible disturbance. If $G$ is the mapping from $d$ to $e$, then our goal is to select a filter $F$ that achieves a bound on the induced 2-norm $\|G\|_{2}$ of $G$.

\[ \|G\|_{2}^{2} \equiv \sup_{d \neq 0} \frac{\|e\|_{2}^{2}}{\|d\|_{2}^{2}} < \gamma^{2} \quad (4.34) \]

In order to achieve the bound in Equation (4.34), we define a minimax or game-theoretic estimation problem that minimizes an objective function with respect to the state estimate $\hat{x}$ in the presence of the worst-case input $d$:

\begin{align*}
\min_{\hat{x}} \max_{d} J_1 \\
\text{subject to} & \quad \text{Equation (4.28)} \\
& \quad \|d\|_{2}^{2} \leq 1
\end{align*} \quad (4.35)
where

\[
J_1 \equiv \frac{1}{2} \| e \|^2_2
\]

\[
\equiv \frac{1}{2} \sum_{k=1}^{N} \left[ x_k - \hat{x}_{(k|k-1)} \right]^T M_k^T M_k \left[ x_k - \hat{x}_{(k|k-1)} \right]
\]

(4.36)

We choose the dynamics of the estimator to be of the same form as the one-step predictor-to-predictor Kalman filter.

\[
\hat{x}_{(k+1|k)} = (A_k - K_k C_k) \hat{x}_{(k|k-1)} + K_k y_k
\]

(4.37)

In this form, the estimator will be unbiased in the sense that \( x = \hat{x} \) in the absence of noise and initial error. The error dynamics \( \tilde{x}_{(k|k-1)} = x_k - \hat{x}_{(k|k-1)} \) can then be represented as

\[
\tilde{x}_{(k+1|k)} = (A_k - K_k C_k) \tilde{x}_{(k|k-1)} + (G_k - K_k E_k) d_k
\]

\[
= \tilde{A}_k \tilde{x}_{(k|k-1)} + \tilde{G}_k d_k
\]

(4.38)

The estimation problem may now be described by

\[
\min d \max \hat{x} \quad J_1
\]

subject to

\[
\hat{x}_{(k+1|k)} = \tilde{A}_k \tilde{x}_{(k|k-1)} + \tilde{G}_k d_k
\]

and \( \| d \|^2_2 \leq 1 \)

(4.39)

where \( K \equiv [ K_0, \ldots, K_{N-1} ] \). As the first step to solving this problem, we maximize with respect to \( d \), choosing the worst-case noise. The second step involves minimizing with respect to the gain \( K \). In [14], where a complete derivation is made including arbitrary initial conditions, it is shown that the worst-case disturbance and optimal choice of gain \( K \) satisfies a saddle point property. In other words, the stationary values \( K_k^*(d_k^*) \), \( k = 0, \ldots, N - 1 \) satisfy \( J_1(K_k^*, d_k^*) \leq J_1(K_k^*, d_k^*) \leq J_1(K_k, d_k^*) \). This implies that the solution would be
the same regardless of the order in which the optimization is carried.

If we let \( \lambda_0, \ldots, \lambda_{N-1} \) be the Lagrange multipliers associated with the error dynamics constraint in Equation (4.38) and \( \frac{\gamma^2}{2} \) be the multiplier associated with the disturbance norm constraint of Equation (4.27), then the augmented cost function is [14]:

\[
J_2 \equiv J_1 + \sum_{k=0}^{N-1} \left\{ \lambda^T_{k+1} \left[ \ddot{x}_{(k+1)k} - \tilde{A}_k \dot{x}_{(k)k-1} - \tilde{G}_k d_k \right] \right\} - \frac{\gamma^2}{2} \| d \|^2
\]

\[
= \frac{1}{2} \sum_{k=1}^{N} \dot{x}^T_{(k)k-1} M_k^T M_k \dot{x}_{(k)k-1} + \sum_{k=0}^{N-1} \lambda^T_{k+1} \left[ \ddot{x}_{(k+1)k} - \tilde{A}_k \dot{x}_{(k)k-1} - \tilde{G}_k d_k \right] - \frac{\gamma^2}{2} \sum_{k=0}^{N-1} d_k^T d_k
\]

\[
= \lambda^T_1 \left[ \ddot{x}_{(1)0} - \tilde{A}_0 \dot{x}_{(0)1} - \tilde{G}_0 d_0 \right] - \frac{\gamma^2}{2} \lambda^T_0 d_0
\]

\[
+ \sum_{k=1}^{N-1} \left\{ \frac{1}{2} \dot{x}^T_{(k)k-1} M_k^T M_k \dot{x}_{(k)k-1} + \lambda^T_{k+1} \left[ \ddot{x}_{(k+1)k} - \tilde{A}_k \dot{x}_{(k)k-1} - \tilde{G}_k d_k \right] - \frac{\gamma^2}{2} d_k^T d_k \right\}
\]

\[
+ \frac{1}{2} \dot{x}^T_{(N)N-1} M_N^T M_N \dot{x}_{(N)N-1}
\]

\[
= \lambda^T_1 \left[ -\tilde{A}_0 \ddot{x}_{(0)1} - \tilde{G}_0 d_0 \right] - \frac{\gamma^2}{2} \lambda^T_0 d_0
\]

\[
+ \sum_{k=1}^{N-1} \left\{ \lambda^T_k \dot{x}_{(k)k-1} \frac{1}{2} \dot{x}^T_{(k)k-1} M_k^T M_k \dot{x}_{(k)k-1} + \lambda^T_{k+1} \left[ -\tilde{A}_k \ddot{x}_{(k)k-1} - \tilde{G}_k d_k \right] - \frac{\gamma^2}{2} d_k^T d_k \right\}
\]

\[
+ \frac{1}{2} \dot{x}^T_{(N)N-1} M_N^T M_N \dot{x}_{(N)N-1} + \lambda^T_N \ddot{x}_{(N)N-1}
\]  

(4.40)

Note that \( \dot{x}_{(0)1} \) is zero since we assume zero initial conditions. Taking the variation for \( k = 0, \ldots, N-1 \), we get

\[
\delta J_2 = \sum_{k=1}^{N-1} \left[ \ddot{x}_{(k)k-1} M_k^T M_k - \lambda^T_{k+1} \tilde{A}_k + \lambda^T_k \right] \delta \dot{x}_{(k)k-1}
\]

\[
+ \sum_{k=0}^{N-1} \left[ -\gamma^2 d_k^T - \lambda^T_{k+1} \tilde{G}_k \right] \delta d_k
\]
\[ + \sum_{k=0}^{N-1} \left[ \tilde{x}_{(k+1|k)} - \tilde{A}_k \tilde{x}_{(k|k-1)} - \tilde{G}_k d_k \right]^T \delta \lambda_k \]  

(4.41)

Setting \( \delta J_2 = 0 \) gives (with \( M_0 = 0 \)),

\[
\begin{align*}
    d_k^* &= -\frac{1}{\gamma^2} \tilde{G}_k^T \lambda_{k+1} \\
    \lambda_k^* &= -M_k^T M_k \tilde{x}_{(k|k-1)} + \tilde{A}_k \lambda_{k+1} \\
    \tilde{x}_{(k+1|k)} &= \tilde{A}_k \tilde{x}_{(k|k-1)} + \tilde{G}_k d_k
    \end{align*}
\]  

(4.42a, 4.42b, 4.42c)

As expected when using Lagrange multipliers, Equation (4.42c) is simply a restatement of the constraint condition in Equation (4.38). Substituting for \( d_k^* \) into Equation (4.42c), we can write Equation (4.40) as a two-point boundary value problem:

\[
\begin{bmatrix}
    \tilde{x}_{(k+1|k)} \\
    -\lambda_k
\end{bmatrix} =
\begin{bmatrix}
    \tilde{A}_k & -\frac{1}{\gamma^2} \tilde{G}_k \tilde{G}_k^T \\
    M_k^T M_k & -\tilde{A}_k^T
\end{bmatrix}
\begin{bmatrix}
    \tilde{x}_{(k|k-1)} \\
    \lambda_{k+1}
\end{bmatrix}
\]  

(4.43a)

\[
\tilde{x}_{(0|-1)} = 0
\]  

(4.43b)

\[
\lambda_N = \text{free}
\]  

(4.43c)

The 2\( n \times 2n \) matrix on the right-hand side of Equation (4.43a) is the discrete-time Hamiltonian matrix, i.e., a matrix \( H \) such that \( J^{-1} HTJ = -H \), where

\[
J =
\begin{bmatrix}
    0 & I \\
    -I & 0
\end{bmatrix}
\]  

and

\[
J^{-1} = J^T = -J
\]

The equations involve a forward-moving recursion for \( \tilde{x}_{(k+1|k)} \) and a backward recursion for \( \lambda_k \). Note that if \( \tilde{A}_k \) is invertible for all \( k \), then we may transform Equation (4.43a) into a completely forward-moving recursion:

\[
\begin{bmatrix}
    \tilde{x}_{(k+1|k)} \\
    \lambda_{k+1}
\end{bmatrix} =
\begin{bmatrix}
    \left( \tilde{A}_k - \frac{1}{\gamma^2} \tilde{G}_k \tilde{G}_k^T M_k^T M_k \right) & -\frac{1}{\gamma^2} \tilde{G}_k \tilde{G}_k^T \tilde{A}_k^{-T} \\
    \tilde{A}_k^{-T} M_k^T M_k & \tilde{A}_k^{-T}
\end{bmatrix}
\begin{bmatrix}
    \tilde{x}_{(k|k-1)} \\
    \lambda_k
\end{bmatrix}
\]  

(4.44)
The associated Riccati equation may be derived by defining [14]:

\[ \ddot{x}_{(k|k-1)} = \frac{1}{\gamma^2} P_{(k|k-1)} \lambda_k \]
\[ = \frac{1}{\gamma^2} P_{(k|k-1)} \left[ M_k^T M_k \ddot{x}_{(k|k-1)} - \tilde{A}_k^T \lambda_{k+1} \right] \]
\[ = -\frac{1}{\gamma^2} \left[ I - \frac{1}{\gamma^2} P_{(k|k-1)} M_k^T M_k \right]^{-1} P_{(k|k-1)} \tilde{A}_k^T \lambda_{k+1} \]
\[ = -\frac{1}{\gamma^2} \left[ P_{(k|k-1)}^{-1} - \frac{1}{\gamma^2} M_k^T M_k \right]^{-1} \tilde{A}_k^T \lambda_{k+1} \tag{4.45} \]

Substituting for \( \ddot{x}_{(k|k-1)} \) from Equation (4.45) into the top row of Equation (4.43a), we get

\[ \ddot{x}_{(k+1|k)} = -\frac{1}{\gamma^2} \left[ \tilde{A}_k \left( P_{(k|k-1)}^{-1} - \frac{1}{\gamma^2} M_k^T M_k \right)^{-1} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T \right] \lambda_{k+1} \tag{4.46} \]

Equating Equation (4.45) at time \((k+1)\) with Equation (4.46),

\[ -\frac{1}{\gamma^2} P_{(k+1|k)} \lambda_{k+1} = -\frac{1}{\gamma^2} \left[ \tilde{A}_k \left( P_{(k|k-1)}^{-1} - \frac{1}{\gamma^2} M_k^T M_k \right)^{-1} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T \right] \lambda_{k+1} \tag{4.47} \]

From Equations (4.43a)-(4.43c), when we assume \( \tilde{A}_k \) is invertible, and since \( \lambda_N \) is free, \( \lambda_k \) may essentially take on any value. Consequently, Equation (4.47) must be valid for any arbitrary value of \( \lambda_{k+1} \). Therefore, we have the following:

\[ P_{(k+1|k)} = \tilde{A}_k H_k^{-1} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T \tag{4.48} \]
\[ H_k = P_{(k|k-1)}^{-1} - \frac{1}{\gamma^2} M_k^T M_k \tag{4.49} \]

The derivation in [14] gives the conditions for which \( P_{(k|k-1)} \) must be positive definite. Since it is assumed that \( \tilde{A}_k \) is invertible, from Equation (4.48), we find that this condition is satisfied when \( H_k \) is positive definite also. As a result, the free parameter \( \gamma \) in Equation (4.49) is bounded below by the requirement that \( H_k > 0 \) for all \( k \). Essentially, it is the variation of the parameter \( \gamma \) that makes
the minimax filter different from the Kalman filter. When there does exist a solution \( H_k \), there exists a filter described by Equation (4.37) that guarantees the bound of Equation (4.34).

The optimal gain \( K_k \) may now be found by substituting the worst-case noise \( d_k \) from Equation (4.42a) and \( \tilde{x}_{(k|k-1)} \) from Equation (4.45) into the cost function \( J \) of Equation (4.36) and minimizing with respect to \( K_k \) [14].

\[
J_3 = \max_d J_1
= \sum_{k=1}^{N} \tilde{x}_{(k|k-1)}^T M_k^T M_k \tilde{x}_{(k|k-1)}
= \sum_{k=0}^{N-1} \tilde{x}_{(k+1|k)}^T M_{k+1}^T M_{k+1} \tilde{x}_{(k+1|k)}
\] (4.50)

Then,

\[
J_3 = \frac{1}{2} \sum_{k=0}^{N-1} \left[ \tilde{x}_{(k|k-1)}^T \tilde{A}_k^T + d_k^T \tilde{G}_k^T \right] M_{k+1}^T M_{k+1} \left[ \tilde{A}_k \tilde{x}_{(k|k-1)} + \tilde{G}_k d_k \right]
= \frac{1}{2} \sum_{k=0}^{N-1} \left[ \tilde{x}_{(k|k-1)}^T \tilde{A}_k^T M_{k+1}^T M_{k+1} \tilde{A}_k \tilde{x}_{(k|k-1)} + d_k^T \tilde{G}_k^T M_{k+1}^T M_{k+1} \tilde{G}_k d_k \right]
= \frac{1}{2} \sum_{k=0}^{N-1} \left\{ \left[ -\frac{1}{\gamma^2} H_k^{-1} \tilde{A}_k \lambda_{k+1} \right]^T \begin{bmatrix} -\frac{1}{2} H_k^{-1} \tilde{A}_k^T \lambda_{k+1} \\ \frac{1}{2} \tilde{G}_k^T \tilde{A}_k \lambda_{k+1} \end{bmatrix} \right\} + \left[ \left[ -\frac{1}{\gamma^2} \tilde{G}_k^T \lambda_{k+1} \right]^T \begin{bmatrix} \tilde{A}_k^T M_{k+1}^T M_{k+1} \tilde{A}_k \lambda_{k+1} \end{bmatrix} \right]
= \frac{1}{2\gamma^4} \sum_{k=0}^{N-1} \left[ \lambda_{k+1} A_k H_k^{-1} \tilde{A}_k^T M_{k+1}^T M_{k+1} \tilde{A}_k H_k^{-1} \lambda_{k+1} \right.
+ \left. \lambda_{k+1} \tilde{G}_k \tilde{G}_k^T M_{k+1}^T M_{k+1} \tilde{G}_k \tilde{G}_k^T \lambda_{k+1} \right]
= \frac{1}{2\gamma^4} \sum_{k=0}^{N-1} \lambda_{k+1} \left( \tilde{A}_k H_k^{-1} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T \right)^T M_{k+1}^T M_{k+1} \left( \tilde{A}_k H_k^{-1} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T \right) \lambda_{k+1}
= \frac{1}{2\gamma^4} \sum_{k=0}^{N-1} \text{trace}\left\{ \left( \lambda_{k+1} \lambda_{k+1} \right) \left( \tilde{A}_k H_k^{-1} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T \right)^T M_{k+1}^T M_{k+1} \right\} (4.51)

Only \( \tilde{A}_k \) and \( \tilde{G}_k \) are functions of \( K_k \). When we differentiate, the result is the
following:

\[
\frac{\partial J_3}{\partial K_k} = \frac{1}{\gamma^4} \sum_{k=0}^{N-1} \text{trace}\left\{ \left( \lambda_{k+1} \lambda_{k+1}^T \right) \left( \tilde{A}_k H_k^{-1} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T \right)^T M_{k+1}^T M_{k+1} \right\}
\]

\[
= \frac{1}{\gamma^4} \sum_{k=0}^{N-1} \text{trace}\left\{ \left( \lambda_{k+1} \lambda_{k+1}^T \right) \left( \tilde{A}_k H_k^{-1} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T \right)^T M_{k+1}^T M_{k+1} \right\}
\]

\[
= \frac{\partial}{\partial K_k} \left[ (A_k - K_k C_k) H_k^{-1} (A_k - K_k C_k)^T + (G_k - K_k E_k) (G_k - K_k E_k)^T \right]\}
\]

(4.52)

When we set \( \partial J / \partial K_k = 0 \), the partial derivative vanishes when

\[2(A_k - K_k C_k) H_k^{-1} C_k^T + 2(G_k - K_k E_k) E_k^T = 0\]

This yields the optimum value for \( K_k \):

\[K_k^* = \left[ A_k H_k^{-1} C_k + G_k E_k^T \right] \left[ C_k H_k^{-1} C_k^T + E_k E_k^T \right]^{-1}\]

(4.53)

The one-step predictor is therefore given by Equation (4.37) with gain matrix \( K_k \) given by Equation (4.53) which comes from an discrete-time algebraic Riccati equation satisfying Equations (4.48) and (4.49).

### 4.2.2 Estimator Properties

For convenience, we repeat the equations for the minimax filter dynamics:

\[
\hat{x}_{(k+1|k)} = \tilde{A}_k \hat{x}_{(k|k-1)} + K_k y_k
\]

\[
P_{(k+1|k)} = \tilde{A}_k H_k^{-1} \tilde{A}_k^T + \tilde{G}_k \tilde{G}_k^T
\]

where

\[
\tilde{A}_k = A_k - K_k C_k
\]
\[ \tilde{G}_k = G_k - K_k E_k \]
\[ H_k = P_{(k|k-1)}^{-1} - \frac{1}{\gamma^2} M_k^T M_k \]
\[ K_k = \left[ A_k H_k^{-1} C_k + G_k E_k^T \right] \left[ C_k H_k^{-1} C_k^T + E_k E_k^T \right]^{-1} \]

Comparing the expression here for \( P_{(k+1|k)} \) with that for the Kalman filter in Equation (4.19), we see that they are similar. In the minimax filter, \( H_k^{-1} \) replaces \( P_{(k|k-1)} \) at each time step. Strictly speaking then, the \( P_k \) used for the minimax filter are not covariances of the estimation error. Rather, the \( P_k \) are just symmetric positive definite matrices that satisfy a modified filter algebraic Riccati equation. They are used only in calculating the filter gain matrix \( K_k \). Recall that in the stochastic setting, the inverse of the covariance matrix may be interpreted as a measure of information about the states.

Here, we interpret that the minimax estimator is reducing the amount of a priori information about the system at each time step. In particular, during the intermediate calculation for \( H_k \), a certain amount of information proportional to \( \frac{1}{\gamma^2} \) is removed from \( P_{(k|k-1)} \) before updating to \( P_{(k+1|k)} \). As \( \gamma \to \infty \), \( H_k^{-1} \to P_{(k|k-1)} \), and we recover the Kalman filter. On the other hand, there is a lower bound \( \gamma_{min} \) on \( \gamma \) such that the matrix \( H_k \) is positive definite for all \( k \). This represents the minimum achievable bound on \( ||G||_{L2} \) in Equation (4.34).

In terms of the minimum mean-square error criterion of the Kalman filter, the minimax filter presented in the previous section is suboptimal. However, it does provide the best estimate in the presence of the worst-case error given by Equation (4.42a). Since \( \gamma \) is a free design parameter, we may use it to trade off least mean-square error nominal performance with worst-case noise estimation error performance. It is clear that, in practice, we should always choose \( \gamma_{min} < \gamma < \infty \). While using \( \gamma = \gamma_{min} \) may provide the tightest achievable bound on \( ||G||_{L2} \), such a design would be too conservative since the worst-case disturbance does not occur in nature. Choosing \( \gamma = \infty \), i.e., designing a Kalman filter, implies we have a perfect model. This is never the case as there are always
modeling errors such as parametric uncertainties, nonlinearities, and unmodeled high-frequency dynamics.

Furthermore, the weighting matrix $M_k$ may also be used to achieve different filter designs. Unlike the Kalman filter which gives the best estimate for all of the states, $M_k$ may be chosen to emphasize more heavily different components of the system.

One advantage of the Kalman filter as presented in Equations (4.7)-(4.10) is the distinct prediction and update steps. This form is useful in multi-rate systems where the sampling rate for the plant propagation is different than the rate at which measurements are taken. In what follows, we present without proof equations for prediction and update steps for the minimax filter that are similar to those of the two-step Kalman filter. For simplicity, we assume that the process and measurement disturbances are uncorrelated so that $G_k E_k^T = 0$.

**Prediction**

\[
\hat{x}_{(k|k-1)} = A_{k-1} \hat{x}_{(k-1|k-1)} \tag{4.54}
\]

\[
P_{(k|k-1)} = A_{k-1} P_{(k-1|k-1)} A_{k-1}^T + G_{k-1} G_{k-1}^T \tag{4.55}
\]

**Update**

\[
\hat{x}_{(k|k)} = \hat{x}_{(k|k-1)} + K_k^u \left[ y_k - C_k \hat{x}_{(k|k-1)} \right] \tag{4.56}
\]

\[
P_{(k|k)} = (I - K_k C_k) H_k^{-1} \tag{4.57}
\]

\[
P_{(k|k)}^{-1} = H_k + C_k^T \left( E_k E_k^T \right)^{-1} C_k \tag{4.58}
\]

where

\[
H_k = P_{(k|k-1)}^{-1} - \frac{1}{\gamma^2} M_k^T M_k \tag{4.59}
\]

\[
K_k^u = H_k^{-1} C_k^T \left( C_k H_k^{-1} C_k^T + E_k E_k^T \right)^{-1} \tag{4.60}
\]
These forms of the minimax filter equations are based on the intuitive assumption that a prediction step propagates the state estimate based solely on the a priori model dynamics, while the update step provides a correction to the estimator via the incoming measurement information. Unlike the Kalman filter, however, we use $H_k$ to calculate the filter gain rather than the matrix $P_{(k|k-1)}$. The intermediate calculation for $H_k$ is roughly equivalent to an information removal step. Removing information is justified by the fact that we are using an uncertain noise model. The disturbance $d_k$ is no longer assumed to be white but is assumed only to be of bounded norm. Consequently, the filter necessarily must weight incoming measurement more heavily in order not to be overconfident of its internal representation of the plant. Since the measurements are weighted by $K_k$, the gain is increased by an increase in $H_k^{-1}$ over $P_{(k|k-1)}$ to achieve this end.

As in Section 4.1.1, let us look at the properties of the steady-state minimax filter before proceeding with a numerical example. Like the Kalman filter, the game-theoretic minimax filter may be extended to the infinite-horizon setting for linear time-invariant plants. When we switch to this formulation, sequences must be square summable over an infinite horizon. As a result, the finite-horizon 2-norm defined previously must be replaced by an equivalent quantity over an infinite horizon.

**Proposition 4.1** Let a stable linear system have transfer matrix $G_{ed}(z)$, and let $G$ denote the linear mapping from the square summable input sequence to the square summable output sequence. Then, the induced 2-norm of $G$ coincides with the $H_\infty$ norm of $G_{ed}(z)$.

$$
\sup_{d \neq 0} \frac{\|e\|_2}{\|d\|_2} = \|G\|_2 \equiv \|G_{ed}(z)\|_{H_\infty} \equiv \sup_{\omega} \sigma_{\text{max}} \left[ G_{ed}(e^{j\omega}) \right] \quad (4.61)
$$

**Proof [3]**: Let us denote by $d$ the input sequence and $e = Gd$ its related output
sequence. By Parsevaal’s identity, we have the following:

\[ \sum_{k=1}^{\infty} e_k e_k^T = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |S_{ee}(e^{j\omega})|^2 d\omega \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |G_{ed}(e^{j\omega})D(e^{j\omega})|^2 d\omega \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left[ \sum_{i=1}^{N} \sigma_i^2 \left( G_{ed}(e^{j\omega}) \right) \right] |D(e^{j\omega})|^2 d\omega \]

\[ \leq \frac{1}{2\pi} \int_{-\pi}^{+\pi} \sigma_{max}^2 \left[ G_{ed}(e^{j\omega}) \right] |D(e^{j\omega})|^2 d\omega \]

\[ \leq \|G_{ed}(z)\|^2_{H_\infty} \frac{1}{2\pi} \int_{-\pi}^{+\pi} |D(e^{j\omega})|^2 d\omega \]

\[ = \|G_{ed}(z)\|^2_{H_\infty} \sum_{k=0}^{\infty} d_k d_k^T \]

The final equality arises again from Parsevaal’s identity. We see that
\[ \|G\|_{i2} \leq \|G_{ed}(z)\|_{H_\infty}. \]

Conversely, let us suppose that \( \gamma < \|G_{ed}\|_{H_\infty} \). This implies that there exists some frequency \( \Omega \) for which \( \sigma_{max} \left[ G_{ed}(e^{j\Omega}) \right] > \gamma \). By continuity, there must also exist some \( \eta > 0 \) such that \( \sigma_{max} \left[ G_{ed}(e^{j\omega}) \right] > \gamma \) for all \( \omega \) in the interval \([-\Omega - \eta, -\Omega + \eta]\) as well as in the interval \([\Omega - \eta, \Omega + \eta]\). If we were to choose a disturbance to be zero outside this frequency range such that for each frequency inside the range, it coincides with the eigenvector corresponding to the largest eigenvalue of \( G_{ed}^T(e^{-j\omega})G_{ed}(e^{j\omega}) \) in these frequencies, then the corresponding output \( e \) is given by the following:

\[ \|e\|^2_2 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |S_{ee}(j\omega)|^2 d\omega \]

\[ = \frac{1}{2\pi} \left[ \int_{-\Omega-\eta}^{-\Omega+\eta} |S_{ee}(e^{j\omega})|^2 d\omega + \int_{\Omega-\eta}^{\Omega+\eta} |S_{ee}(e^{j\omega})|^2 d\omega \right] \]

\[ \geq \frac{1}{2\pi} \left[ \int_{-\Omega-\eta}^{-\Omega+\eta} \gamma^2 |D(e^{j\omega})|^2 d\omega + \int_{\Omega-\eta}^{\Omega+\eta} \gamma^2 |D(e^{j\omega})|^2 d\omega \right] \]

\[ = \gamma^2 \|d\|^2_2 \]

Hence \( \|G\|_{i2} \geq \|G_{ed}(z)\|_{H_\infty} \) \( \square \)

Proposition 4.1 shows that in steady-state, the game-theoretic filter problem
formulation reduces to bounding the $H_\infty$ norm of the transfer function from the disturbance to the weighted estimation error. Recalling that the steady-state Kalman filter minimizes the $H_2$ norm of that same transfer function, we see that $\gamma$ may be treated as a frequency-domain design parameter trading off $H_2$ with $H_\infty$ performance. Therefore, a filter using the lower achievable bound $\gamma_{\text{min}}$ is known as a pure $H_\infty$ filter. The filter is then optimized to minimize the weighted estimation error in the presence of sinusoidal noise at the worst-case frequency.

4.2.3 Experiments with the Minimax Filter

We revisit the navigation problem investigated in Section 4.1.2 and compare the performance of the $H_\infty$ minimax filter with that of the Kalman filter. As mentioned previously, performance improvements in the off-nominal case will be made at the expense of nominal performance. For our navigation problem, however, it is found that the improvements in performance gained by using an $H_\infty$ minimax filter for the perturbed case are marginal in comparison to using a Kalman filter. We investigate this point further by implementing more traditional techniques to “robustify” the Kalman filter such as the overdesigned Kalman filter [10] and the exponentially-forgetting Kalman filter. In all cases, none of the filters performed well for the off-nominal plant. The most deleterious source of uncertainty comes from an inaccurate plant model rather than an inaccurate noise model. Consequently, an alternative approach using a filter robust to these plant model uncertainties is necessary to achieve the requisite performance.

As before, the filters designs use time-invariant plants but employ a time-varying gain for improved transient response. The terms “Nominal KF” and “Nominal $H_\infty$” refer to Kalman filters and minimax filters designed around the nominal operating condition ($M = 7.5, \alpha = 35^\circ$) and applied to noisy measurements from this system. The terms “Perturbed KF” and “Perturbed $H_\infty$” have similar denotations, referring to the same filter designs applied to
the off-nominal system \((M = 8.8, \alpha = 38^\circ)\). The results of three types of simulation are used to gauge the effectiveness of the two filters when different noise models are assumed.

The first simulation serves as a standard of comparison. In this experiment, white-noise disturbances corrupt the state evolution and measurements. The Kalman filter should have the smaller estimation errors since it is the optimal estimator in the presence of Gaussian noise. By comparing these results, we can see how much is sacrificed in trading off mean-square error performance with worst-case noise performance.

A second experiment assumes that the disturbances are the output of some shaping filters. In particular, we start with several channels of Gaussian white-noise and pass them through first-order high-pass and low-pass filters. Since process noise disturbances are typically concentrated at the low end of the frequency spectrum, white noise is passed through a low-pass filter with a break frequency of 1.0 rad/sec. Sensor noise often occurs at higher frequencies; therefore, we send white noise through a high-pass filter with a break frequency of \(10^4\) rad/sec.

As before, we concern ourselves only with the estimation errors in bank angle, sideslip angle, bank rate, and sideslip rate. It is found that the lowest possible value for \(\gamma\) is \(\gamma_{\text{min}} = 99.321\). We used this value in simulating the minimax filter. Sample statistics for these two experimental runs are summarized in Tables 4.2 and 4.3. For errors in the attitude angles, the units are \((\deg)^2\), while for the errors in attitude rates, the units are \((\deg/sec)^2\).

<table>
<thead>
<tr>
<th>Squared Error</th>
<th>Nominal KF</th>
<th>Nominal (H_\infty)</th>
<th>Perturbed KF</th>
<th>Perturbed (H_\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bank Angle</td>
<td>0.0563548</td>
<td>0.0791384</td>
<td>0.0454291</td>
<td>1.7542575</td>
</tr>
<tr>
<td>Sideslip Angle</td>
<td>0.6059740</td>
<td>0.6059902</td>
<td>15.2111412</td>
<td>15.2111419</td>
</tr>
<tr>
<td>Bank Rate</td>
<td>0.0902142</td>
<td>0.0902142</td>
<td>5.5204113</td>
<td>5.5204113</td>
</tr>
<tr>
<td>Sideslip Rate</td>
<td>0.0401727</td>
<td>0.0401727</td>
<td>21.9326177</td>
<td>21.9326177</td>
</tr>
</tbody>
</table>

Table 4.2: Kalman filter and minimax filter performance for white-noise disturbance. Units are \((\deg)^2\) and \((\deg/sec)^2\).
<table>
<thead>
<tr>
<th>Squared Error</th>
<th>Nominal KF</th>
<th>Nominal $H_\infty$</th>
<th>Perturbed KF</th>
<th>Perturbed $H_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bank Angle</td>
<td>0.0018915</td>
<td>0.1283327</td>
<td>0.0208538</td>
<td>1.6990361</td>
</tr>
<tr>
<td>Sideslip Angle</td>
<td>0.9287245</td>
<td>0.9287427</td>
<td>15.2236719</td>
<td>15.2236730</td>
</tr>
<tr>
<td>Bank Rate</td>
<td>0.0351882</td>
<td>0.0351883</td>
<td>5.5121423</td>
<td>5.5121423</td>
</tr>
<tr>
<td>Sideslip Rate</td>
<td>0.0116235</td>
<td>0.0116235</td>
<td>21.9303987</td>
<td>21.9303986</td>
</tr>
</tbody>
</table>

Table 4.3: Kalman filter and minimax filter performance for shaped disturbance. Units are (deg)$^2$ and (deg/sec)$^2$.

For the final simulation, we use the worst-case disturbance. Substituting Equation (4.45) into Equation (4.42a), the worst-case noise may related to the estimation error as follows:

$$d_k = -\frac{1}{\gamma^2} \tilde{G}_k^T \lambda_{k+1}$$

$$= \tilde{G}_k^T P_{(k+1|k)}^{-1} \tilde{z}_{(k+1|k)}$$  \hspace{1cm} (4.62)

The worst-case noise actually depends on future information. The Riccati matrix $P_k$ evolves independently of the estimation error, so that piece of information is available, as well as the filter gain $K_k$. In our simulation, we use the two-step prediction and update forms of the filter equations, so that

$$P_{(k+1|k)} = A_k P_{(k|k)} A_k^T + G_k \gamma_k^2$$

The future estimation error, on the other hand, will depend on the current disturbance via incorporation of the current measurement since $\tilde{x}_{(k+1|k)} = A_k \tilde{x}_{(k|k)} + G_k d_k$. Therefore, the following approximation is used [11].

$$d_k \approx (G_k - K_k E_k)^T \left[ A_k P_{(k|k)} A_k^T + G_k G_k^T \right]^{-1} (A_k \tilde{x}_{(k|k)} + G_k d_{k-1})$$ \hspace{1cm} (4.63)

Note that since $P_{(k|k)}^{-1} = \left( P_{(k|k-1)}^{-1} - \frac{1}{\gamma^2} M_k^T M_k \right) + C_k (E_k E_k^T)^{-1} C_k^T$, the worst-case disturbances for the minimax filter and the Kalman filter will differ. Nevertheless, a comparison between the two filters when subjected to a disturbance in the worst-case direction shows that the minimax filter offers some improvement in performance. Results are summarized in Table 4.4. As before, the units on
the errors in attitude angles are \((\text{deg})^2\), while the units for the errors in attitude rates are \((\text{deg/sec})^2\). Note that the actual worst-case noise would have induced even larger errors for the estimators. In the case of the \(H_\infty\) minimax filter, the worst-case disturbance is shown in Figure 4-5.

<table>
<thead>
<tr>
<th>Squared Error</th>
<th>Nominal (KF)</th>
<th>Nominal (H_\infty)</th>
<th>Perturbed (KF)</th>
<th>Perturbed (H_\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bank Angle</td>
<td>2.1379886</td>
<td>2.1288431</td>
<td>2.1230504</td>
<td>3.895099</td>
</tr>
<tr>
<td>Sideslip Angle</td>
<td>0.2102403</td>
<td>0.2102325</td>
<td>15.1642168</td>
<td>15.164217</td>
</tr>
<tr>
<td>Bank Rate</td>
<td>0.5072305</td>
<td>0.5072300</td>
<td>5.5623050</td>
<td>5.5623071</td>
</tr>
<tr>
<td>Sideslip Rate</td>
<td>0.1754816</td>
<td>0.1754773</td>
<td>21.9005954</td>
<td>21.9005957</td>
</tr>
</tbody>
</table>

Table 4.4: Kalman filter and minimax filter performance in the presence of worst-case disturbance. Units are \((\text{deg})^2\) and \((\text{deg/sec})^2\).

Figure 4-5: Simulated worst-case noise input

Note that there do exist other methods for increasing the robustness of the Kalman filter in the presence of noise uncertainty. What essentially happens in a divergent filter, i.e., a filter whose mean estimation error is biased, is that over time, its gain becomes too small. It is in this case that the filter is often referred to as becoming too confident of its internal model of the system. When the filter
gain approaches zero, that turns the filter off from new data and forces it to run in prediction mode. Only very weak corrections are made between prediction and observation. What needs to be done, then, is to raise the filter’s sensitivity to incoming measurements. Increasing the Kalman gain may be accomplished by increasing the magnitude of the process noise covariance in the filter design.

We repeat discrete-time Kalman filter prediction and update equations for the estimation error covariance here for convenience:

\[
P_{(k+1|k)} = A_k P_{(k|k)} A_k^T + G_k G_k^T
\]

\[
P_{(k+1|k+1)}^{-1} = P_{(k+1|k)}^{-1} + C_{k+1}^T (E_{k+1}^T E_{k+1})^{-1} C_{k+1}
\]

\[
K_{k+1} = P_{(k+1|k+1)} C_{k+1}^T (E_{k+1}^T E_{k+1})^{-1}
\]

When certain states are poorly or inadequately modeled, increasing the process noise covariance by changing elements of \(G_k\) tells the filter that there are uncertainties. Clearly, overdesigning \(G_k\) will lead to an increase in \(K_{k+1}\). The Kalman filter will then incorporate a greater correction over its internally predicted estimate by paying more attention to the incoming measurements. The estimate, unfortunately, will no longer be optimal in the mean-squared error sense. Since the design accounts for more process than there actually exists in the true plant, the filter estimate will, in fact, be quite noisy. Such a degradation in performance is the trade off that must be made for increased robustness.

A greater understanding may be had by an analysis in the frequency domain. We consider the discrete-time transfer function from the measurements to the state estimates in steady-state. Taking the z-transform of the one-step Kalman filter equation \(\hat{x}_{(k+1|k)} = (A - AK_\infty C) \hat{x}_{(k|k-1)} + AK_\infty y_k\), we obtain the following:

\[
\frac{\hat{X}(z)}{Y(z)} = \frac{AK_\infty}{zI - (A - AK_\infty C)}
\]

This may be recognized as a kind of multi-dimensional low-pass filter. The poles
of the transfer function may be found by setting the denominator equal to zero.

\[
\begin{align*}
  z &= (A - AK_\infty C) \\
  &= A \left[ I - P_\infty C^T (C P_\infty C^T + EE^T)^{-1} C \right] \\
  &= A \left\{ I - (AP_\infty A^T + GG^T) C^T \left[ C(AP_\infty A^T + GG^T) C^T + EE^T \right]^{-1} C \right\}
\end{align*}
\]

(4.64)

For simplicity, let us consider only the scalar case. There are two extreme
situations to consider in the analysis. When there is little process noise, such
that \( G \to 0 \), then \( P_\infty \to 0 \), and \( z \to A \). The steady-state Kalman filter behaves
asymptotically like a simple low-pass filter. As mentioned earlier, measurement
noise usually occurs at high frequencies. With this in mind, we see that the
Kalman filter relies heavily on its internal model and filters everything out. On
the other hand, as \( G \to \infty \), so that the process noise approaches infinity, the pole
in Equation (4.64) goes to zero. As a result, we may borrow terminology from
steady-state frequency-domain analytic techniques and say that the transfer
function from the measurements to the state estimate acquires an increasingly
infinite bandwidth. Rather than attenuate input signals with frequency content
above a certain cutoff frequency, the filter allows noisy measurements across all
frequencies. In other words, the Kalman filter completely disregards its a priori
model and merely inverts the noisy measurements.

Simulation results for various overdesigned Kalman filters are presented in
Table 4.5. Squared errors in bank and sideslip angle are expressed in units of
(deg)^2, while squared errors in bank and sideslip rate are expressed in units
of (deg/sec)^2. All the filter designs are based on the nominal model with
\( (M = 7.5, \alpha = 35^\circ) \). For simplicity, the process noise input matrix for the
filter design is taken as a scalar multiple of process noise matrix for the nominal
system, such that \( G_{k}^{des} = aG_{k}^{nom} \). Note that as the parameter \( a \) increases, the
general trend is for the estimation errors for the nominal case increase, while
those for the perturbed case decrease. The filter is becoming more robust to
plant model disturbances only in the sense that it relying more heavily on the incoming measurement and ignoring its internal model. To generate the state estimates, the filter is merely inverting the measurements with only moderate filtering of the sensor noise.

<table>
<thead>
<tr>
<th>$a$</th>
<th>Squared Error</th>
<th>Bank Angle</th>
<th>Sideslip Angle</th>
<th>Bank Rate</th>
<th>Sideslip Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Nominal KF</td>
<td>0.05635</td>
<td>0.60597</td>
<td>0.09021</td>
<td>0.04017</td>
</tr>
<tr>
<td></td>
<td>Perturbed KF</td>
<td>0.04543</td>
<td>15.21114</td>
<td>5.52041</td>
<td>21.93262</td>
</tr>
<tr>
<td>10</td>
<td>Nominal KF</td>
<td>0.05579</td>
<td>0.61193</td>
<td>0.18646</td>
<td>0.04417</td>
</tr>
<tr>
<td></td>
<td>Perturbed KF</td>
<td>0.05400</td>
<td>6.39066</td>
<td>0.26702</td>
<td>8.52069</td>
</tr>
<tr>
<td>20</td>
<td>Nominal KF</td>
<td>0.05554</td>
<td>0.59452</td>
<td>0.25173</td>
<td>0.04702</td>
</tr>
<tr>
<td></td>
<td>Perturbed KF</td>
<td>0.05520</td>
<td>4.82442</td>
<td>0.25848</td>
<td>4.77962</td>
</tr>
<tr>
<td>100</td>
<td>Nominal KF</td>
<td>0.05545</td>
<td>0.51545</td>
<td>0.37061</td>
<td>0.07894</td>
</tr>
<tr>
<td></td>
<td>Perturbed KF</td>
<td>0.05539</td>
<td>3.97611</td>
<td>0.37065</td>
<td>0.99824</td>
</tr>
</tbody>
</table>

Table 4.5: Overdesigned Kalman filter performance for white-noise disturbance. Units are (deg)$^2$ and (deg/sec)$^2$.

An alternative method works by shortening the Kalman filter’s memory and involves exponential age-weighting of the data. In this approach, old data are given exponentially decreasing weight so that the estimate relies on relatively recent measurements. This keeps the filter sensitive to incoming observations. Note that there are other methods of implementing a finite memory recursive filter but are usually more computationally complex than the standard recursive Kalman filter. At each time step $k$, we resolve the estimation problem for $x_k$, assuming that the process noise and sensor noise intensities are exponentially increased at previous time steps. In other words, we replace $G_lG_l^T$ for $l \leq k - 1$, and $E_lE_l^T$ for $l \leq k$ by the following:

$$
\alpha^{(k-l-1)} G_lG_l^T \leftarrow G_lG_l^T
$$

$$
\alpha^{(k-l)} E_lE_l^T \leftarrow E_lE_l^T
$$

where we must specify $\alpha > 1$. While these substitutions appear to change radically the state estimation problem, the solution may be reduced to a simple
recursive estimator nearly identical to the two-step Kalman filter:

\[
\hat{x}^{ef}_{(k|k-1)} = A_k \hat{x}^{ef}_{(k-1|k-1)} \\
\hat{x}^{ef}_{(k|k)} = \hat{x}^{ef}_{(k|k-1)} + K_k \left[ y_k - C_k \hat{x}^{ef}_{(k|k-1)} \right] \tag{4.68}
\]

where

\[
P^{ef}_{(k|k-1)} = \alpha A_{k-1} P^{ef}_{(k-1|k-1)} A_{k-1}^T + G_{k-1} G_{k-1}^T \tag{4.69}
\]

\[
K^e_k = P^{ef}_{(k|k-1)} C_k^T \left[ C_k P^{ef}_{(k|k-1)} C_k^T + E_k E_k^T \right]^{-1} \tag{4.70}
\]

\[
P^{ef}_{(k|k)} = P^{ef}_{(k|k-1)} - P^{ef}_{(k|k-1)} C_k^T \left[ C_k P^{ef}_{(k|k-1)} C_k^T + E_k E_k^T \right]^{-1} P^{ef}_{(k|k-1)} \tag{4.71}
\]

The superscript "ef" denotes exponentially-forgetting. The only difference between Equations (4.67)-(4.71) and those of the two-step Kalman filter is the presence of the term \( \alpha \) in Equation (4.69). As with the \( H_{\infty} \) minimax filter and the overdesigned Kalman filter, this modification increases the gain matrix so that the filter remains sensitive to incoming measurements. Strictly speaking, of course, the Riccati matrix \( P_k \) is no longer the error estimation covariance. It may still be interpreted, nonetheless, as a measure of uncertainty (or, if using \( P_k^{-1} \), a measure of information) about the state.

Simulation results for Kalman filters with different degrees of exponential forgetting are summarized in Table 4.6. The squared errors in attitude angles are in \((\text{deg})^2\), while the squared errors in the attitude rates are in \((\text{deg/sec})^2\). As with the overdesigned filters, all designs are based on the nominal model with \((M = 7.5, \alpha = 35^\circ)\). Here, estimation errors in both the nominal case and the perturbed case decrease as the \( \alpha \) increases and the sliding observation window becomes more narrow. Note that in our case, the degree to which exponential age-weighting may be used is limited by numerical issues.

All this aside, the results presented in Tables 4.2, 4.3, and 4.4 are somewhat mixed when we consider the nominal case. Obviously, the Kalman filter is the better filter when the disturbances are white. However, the differences
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Squared Error</th>
<th>Bank Angle</th>
<th>Sideslip Angle</th>
<th>Bank Rate</th>
<th>Sideslip Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Nominal KF</td>
<td>0.05635</td>
<td>0.60597</td>
<td>0.09021</td>
<td>0.04017</td>
</tr>
<tr>
<td></td>
<td>Perturbed KF</td>
<td>0.04543</td>
<td>15.21114</td>
<td>5.52041</td>
<td>21.93262</td>
</tr>
<tr>
<td>1.001</td>
<td>Nominal KF</td>
<td>0.05520</td>
<td>0.60593</td>
<td>0.09011</td>
<td>0.03860</td>
</tr>
<tr>
<td></td>
<td>Perturbed KF</td>
<td>0.15280</td>
<td>11.79518</td>
<td>4.20467</td>
<td>16.99186</td>
</tr>
<tr>
<td>1.002</td>
<td>Nominal KF</td>
<td>0.05410</td>
<td>0.60591</td>
<td>0.09000</td>
<td>0.03827</td>
</tr>
<tr>
<td></td>
<td>Perturbed KF</td>
<td>0.28014</td>
<td>9.44996</td>
<td>3.28368</td>
<td>12.98577</td>
</tr>
<tr>
<td>1.005</td>
<td>Nominal KF</td>
<td>0.05107</td>
<td>0.60589</td>
<td>0.08969</td>
<td>0.04029</td>
</tr>
<tr>
<td></td>
<td>Perturbed KF</td>
<td>0.66903</td>
<td>6.85918</td>
<td>2.20203</td>
<td>6.94640</td>
</tr>
</tbody>
</table>

Table 4.6: Exponentially-forgetting Kalman filter performance for white-noise disturbance. Units are (deg)$^2$ and (deg/sec)$^2$.

between it and the $H_\infty$ minimax filter are marginal for our particular navigation problem. When the process and sensor noise are outputs of low-pass and high-pass shaping filters, respectively, the two perform almost identically. (We should ignore errors in the bank angle state since it is relatively unobservable). Even in the presence of the worst-case noise, any performance gains made by the $H_\infty$ filter are very small. There is no instance when either of the filters perform well on the perturbed system. The $H_\infty$ minimax filter, designed to be robust in the presence of noise model uncertainty, is insufficient in providing good state estimates for our particular navigation problem. In all proceeding simulations, it is sufficient to use Gaussian white-noise when comparing filter performance since, for our system, plant model uncertainty is the dominant factor affecting performance. In the following section, we derive an estimator that is robust to both uncertain plant dynamics and noise model.

4.3 The Robust Game-Theoretic Filter

In this section, we generalize the state estimation problem to include not only an uncertain noise model but also uncertainties in the model dynamics themselves. Robustness to noise model is an important contribution to state estimation theory; however, in many cases, the most deleterious uncertainty source comes from
inaccurate plant modeling. As mentioned previously, there will always exist to some degree uncertainty about the plant dynamics. A linearized model derived from a nonlinear system, for example, will break down when the state moves beyond the equilibrium operating condition. In fact, all physical systems contain nonlinearities which must be ignored when using a linear model. Furthermore, high-frequency dynamics are often discarded to limit the size of plant model and make the system computationally tractable.

Our goal now is to minimize the estimation error for an entire family of disturbances and plants. Rather than approach the problem from a stochastic setting like in the derivation of the Kalman filter, we again work in a determinist setting, assuming no a priori information about the perturbation structure or disturbance characteristics. The only assumptions we make are bounds on the uncertainty 2-norm and on the disturbance 2-norm. As in the derivation of the minimax filter, we assume zero initial conditions on the plant and estimator. A complete derivation, including arbitrary initial conditions may be found in [14].

4.3.1 Derivation of the Robust Minimax Filter Equations

\[ \Delta \]

\[ P \quad \eta \quad d \]

\[ F \quad \epsilon \quad y \]

\[ \hat{x} \]

Figure 4-6: General representation of robust estimation problem

Figure 4-6 shows a general input-output representation of a nominal plant $P$ with modeling uncertainties $\Delta$ and an estimator $F$. Figure 4-6 differs from Figure 4-4 only in the inclusion of the plant model uncertainty $\Delta$. Note that the actual plant should not be interpreted as a parallel combination of the nominal plant $P$ and another system with transfer function matrix $\Delta$. Nonetheless, $d$, $y$, $\hat{x}$, and $e$ retain the same definitions as before in Section 4.2.1. The signals
\( \epsilon \) and \( \eta \) reflect the interactions between the nominal plant and the uncertainty and are represented as the following sequences:

\[
\begin{align*}
\epsilon & \equiv [\epsilon_0, \ldots, \epsilon_{N-1}] \\
\eta & \equiv [\eta_0, \ldots, \eta_{N-1}]
\end{align*}
\]

The finite-horizon 2-norms for these sequences are defined as:

\[
\|\epsilon\|_2 \equiv \left( \sum_{k=0}^{N-1} \epsilon_k^T \epsilon_k \right)^{1/2}
\]

\[
\|\eta\|_2 \equiv \left( \sum_{k=0}^{N-1} \eta_k^T \eta_k \right)^{1/2}
\]

The open-loop may be defined by the following set of linear state-space equations:

\[
\begin{align*}
x_{k+1} & = A_k x_k + Q_k \eta_k + G_k d_k \\
\epsilon_k & = S_k x_k + L_k d_k \\
e_k & = M_k \left[ x_k - \hat{x}_{(k|k-1)} \right] \\
y_k & = C_k x_k + R_k \eta_k + E_k d_k
\end{align*}
\]

(4.72a)  (4.72b)  (4.72c)  (4.72d)

The matrices \( S_k \) and \( L_k \) represent how the state and disturbance interact with the perturbation \( \Delta \) and come from a singular value decomposition of the uncertainties in \( A_k, G_k, C_k, \) and \( E_k \). Elaboration of exactly how the uncertainty matrices arise is given in Appendix C. The state-space equations may be combined and written in block form as follows:

\[
\begin{bmatrix}
  x_{k+1} \\
  \epsilon_k \\
e_k \\
y_k
\end{bmatrix} =
\begin{bmatrix}
  A_k & Q_k & 0 \\
  S_k & T_k & 0 \\
  M_k & 0 & -M_k \\
  C_k & \mathcal{E}_k & 0
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  \xi_k \\
\hat{x}_{(k|k-1)}
\end{bmatrix}
\]

(4.73)
Note that in this representation, we have made the following substitutions into the nominal model:

\[
\begin{align*}
\xi_k & \leftarrow \begin{bmatrix} \eta_k \\ d_k \end{bmatrix} \\
Q_k & \leftarrow \begin{bmatrix} Q_k & G_k \end{bmatrix} \\
T_k & \leftarrow \begin{bmatrix} 0 & L_k \end{bmatrix} \\
E_k & \leftarrow \begin{bmatrix} R_k & E_k \end{bmatrix}
\end{align*}
\] (4.74) (4.75) (4.76) (4.77)

The sequence \( \xi \) represents a generalized disturbance vector consisting of the output \( \eta \) of the perturbation \( \Delta \) and the original noise disturbance \( d \). The block form equations may be modified further to include control signal variation via the substitutions given previously in Equations (4.29)-(4.33). Moreover, neglected dynamics, such as gyroscope biases, may be incorporated back into the estimator model. To do this, one would naturally have to add some additional states. The zero entries in the last column are due to the fact that the state estimate does not affect the plant dynamics. In a similar fashion, the generalized disturbance input does not enter into the definition of the weighted error. Note that restricting \( L_k = 0 \) (and consequently, \( T_k = 0 \)), allows only parametric uncertainties in the matrices \( A_k \) and \( C_k \) to be handled.

As with the nominal \( H_\infty \) filter, the derivation presented in this section will be concerned with the a priori estimator. Similarly, the filter objective is to bound and minimize the induced norm of the operator from the input disturbances to the estimation error. For the robust \( H_\infty \) filter, however, there is the additional requirement of achieving this induced norm bound for all possible perturbations \( \Delta \). Mathematically, these objectives translate into the following performance criterion:

\[
\| \mathcal{G} \|_2^2 \equiv \sup_{d \neq 0} \frac{\| e \|_2^2}{\| d \|_2^2} < \gamma^2
\]
\[ \forall \Delta \exists \| \Delta \|_{i2} \equiv \sup_{\varepsilon \neq 0} \frac{\| \eta \|_2}{\| \varepsilon \|_2} < \frac{1}{\gamma^2} \quad (4.78) \]

To achieve the performance objective of Equation (4.78), a new objective function is formed that treats the perturbation output \( \eta \) as an additional exogenous disturbance input to the nominal plant \( P \) and treats the perturbation input \( \varepsilon \) as an additional estimation error term. The new criterion may be expressed as follows:

\[ J_1 < \gamma^2 \quad \forall \Delta \exists \| \Delta \|_{i2}^2 < \frac{1}{\gamma^2} \quad (4.79) \]

where

\[ J_1 \equiv \sup_{(d, \eta) \neq 0} \frac{\| \varepsilon \|_2^2 + \| \eta \|_2^2}{\| d \|_2^2 + \| \eta \|_2^2} \quad (4.80) \]

Equation (4.79) is a restatement of the small gain theorem. As explained in Appendix D, robust stability for interconnected feedback systems is guaranteed when the loop gains are less than unity. Because we consider further the interactions between the nominal plant and the perturbation, Equation (4.79) becomes a sufficient condition for robust performance. It is now rather simple to show that if this condition is satisfied, then the original performance condition of Equation (4.78) is also satisfied for the entire class of uncertainties \( \Delta \) with bounded induced norm. This is shown in the Proposition 4.2. Note that a complete derivation that includes arbitrary initial conditions is presented in [14].

**Proposition 4.2** If \( J_1 < \gamma^2 \), and if \( \| \Delta \|_2^2 < \frac{1}{\gamma^2} \), then \( \| \mathcal{G} \|_{i2}^2 < \gamma^2 \).

**Proof** [14]: \( J_1 < \gamma^2 \) implies that

\[ \| \varepsilon \|_2^2 + \| \eta \|_2^2 < \gamma^2 \left( \| d \|_2^2 + \| \eta \|_2^2 \right) \]

However, from the bound on the perturbation \( \Delta \),

\[ \| \eta \|_2^2 < \frac{1}{\gamma^2} \| \varepsilon \|_2^2 \]
Combining these equations gives the following:

\[ \|e\|_2^2 < \gamma^2 \|d\|_2^2 \]

Noting the definition from Equation (4.78), we have shown \( \|\mathcal{G}\|_2^2 < \gamma^2 \).

Proposition 4.2 assumes a rather general class of perturbations and does not take advantage of any of the structure to the uncertainty. Because of this, the robust estimator is rather conservative. A less conservative estimator would take the structure of the uncertainty into account, such as the structured singular value, or \( \mu \), estimator [8]. However, \( \mu \) is not a norm and may not be easily computed. Furthermore, \( \mu \) estimators are restricted to steady-state rather than time-varying implementations.

Another source of conservatism is that Proposition 4.2 is robust to even complex perturbations, although most uncertainties encountered in practice are real. For example, the method described in Appendix C considers only real perturbations. Therefore, the robust estimator is robust to model uncertainties that do not actually occur.

The bound on the objective function as defined in Equation (4.79) may be solved by defining a game-theoretic estimation problem similar to Equation (4.36) that was formulated in Section 4.2.1. Once again, we attempt to minimize the performance criterion with respect to the state estimate \( \hat{x} \) while in the presence of the worst-case disturbance, which in this case is now \( \xi \):

\[
\min_{\hat{x}} \max_{\xi} \quad \overline{J}_2
\]

subject to

Equation (4.73)

where

\[
\overline{J}_2 \equiv \frac{1}{2} \|e\|_2^2 + \frac{1}{2} \|e\|_2^2 - \frac{\gamma^2}{2} \|\xi\|_2^2
\]

The solution to the robust estimation problem is derived in two stages.
Each stage requires the solution to a discrete-time algebraic Riccati equation. In the first stage, we bound the terms in the performance criterion introduced for robustness to plant model uncertainty but not affected by the estimate, \( \hat{x} \). A change of variables is used to eliminate the sequence \( \epsilon \). Consequently, the problem is reduced to a simpler one. In particular, the second stage involves solving essentially the same problem formulated for the \( H_\infty \) minimax filter regarding the optimal estimate in the presence of the worst-case noise. Once again, we note that for simplicity, we assume zero initial conditions on the state and estimator. This is a special case of the complete proof, including arbitrary initial conditions, that may be found in [14].

The first Riccati equation arises by completing the square as shown in the following theorem [14]:

**Theorem 4.1** The discrete-time algebraic Riccati equation

\[
X_k \equiv A_k^T X_{k+1} A_k + S_k^T S_k + \frac{1}{\gamma^2} F_k Z_k^{-1} F_k^T \\
X_N = 0
\]  

(4.83)

where

\[
F_k \equiv S_k^T T_k + A_k^T X_{k+1} Q_k
\]  

(4.84)

\[
Z_k \equiv I - \frac{1}{\gamma^2} \left( T_k^T T_k + Q_k^T X_{k+1} Q_k \right)
\]  

(4.85)

has a solution such that \( X_k \geq 0, Z_k \geq 0, \forall k \in [0, \ldots, N - 1] \), if and only if

\[
\xi_k^* \equiv \frac{1}{\gamma^2} Z_k^{1/2} F_k^T x_k
\]  

(4.86)

\[
\bar{\xi}_k \equiv Z_k^{1/2} \xi_k - \xi_k^*
\]  

(4.87)

and result in

\[
\frac{1}{2} (\|\epsilon\|_2^2 - \gamma^2 \|\xi\|_2^2) = -\frac{\gamma^2}{2} \|\bar{\xi}\|_2^2
\]  

(4.88)
Proof: Sufficiency[14]. By adding the identically zero term,

\[ \sum_{k=1}^{N-1} \frac{1}{2} \left( x_{k+1}^T X_{k+1} x_{k+1} - x_k^T X_k x_k \right) \]

to

\[ \frac{1}{2} \| \epsilon \|_2^2 - \frac{\gamma^2}{2} \| \xi \|_2^2 \]

and substituting for \( x_{k+1} \) from Equation (4.73) as well as for \( X_k \) from Equation (4.83), yields, after some rearrangement, the following:

\[ \frac{1}{2} \| \epsilon \|_2^2 - \frac{\gamma^2}{2} \| \xi \|_2^2 = -\frac{\gamma^2}{2} \sum_{k=0}^{N-1} \left( \xi_k^T Z_k \xi_k - \frac{2}{\gamma^2} x_k^T F_k \xi_k + \frac{1}{\gamma^4} x_k^T F_k Z_k^{-1} F_k^T x_k \right) \]

\[ = -\frac{\gamma^2}{2} \sum_{k=0}^{N-1} \| Z_k^{1/2} \xi_k - \frac{1}{\gamma^2} Z_k^{-1/2} F_k^T x_k \|_2^2 \]

\[ = -\frac{\gamma^2}{2} \| \xi \|_2^2 \]

Necessity[14]: Assume \( k = k^* \) is the largest time step such that \( Z_k \) has a nonpositive eigenvalue. Adding the identically zero term

\[ \sum_{k=k^*}^{N-1} \frac{1}{2} \left( x_{k+1}^T X_{k+1} x_{k+1} - x_k^T X_k x_k \right) + \frac{1}{2} x_{k^*}^T X_{k^*} x_{k^*} \]

to \( \frac{1}{2} \| \epsilon \|_2^2 - \frac{\gamma^2}{2} \| \xi \|_2^2 \) yields, after some algebraic manipulation,

\[ \frac{1}{2} \| \epsilon \|_2^2 - \frac{\gamma^2}{2} \| \xi \|_2^2 = \frac{1}{2} \sum_{k=0}^{k^*-1} \xi_k^T \xi_k + \frac{1}{2} x_{k^*}^T X_{k^*} x_{k^*} \]

\[ -\frac{\gamma^2}{2} \sum_{k=k^*}^{N-1} \left( Z_k^{1/2} \xi_k - \xi_k^* \right) \]

Over the interval \( k = 0, \ldots, k^*-1, \) we may choose \( \xi_k = 0, \) giving \( x_k = 0 \) for \( k = 0, \ldots, k^* \). Note that \( x_0 = 0 \) by assumption. The terms in the summations for \( \epsilon_k^T \epsilon_k \) and for \( \xi_k^T \xi_k \), as well as the term \( \frac{1}{2} x_{k^*}^T X_{k^*} x_{k^*} \) now all vanish. The terms
in the final summation become

\[-\frac{\gamma^2}{2} \left[ \xi_k \cdot Z_k \cdot \xi_k + \sum_{k=k^*}^{N-1} \left( Z_k^{1/2} \xi_k - \xi_k^* \right) \right] \]

Since, at the time step \( k = k^* \), \( Z_k^* \) has a nonpositive eigenvalue, it is possible to choose a \( \xi_k^* \neq 0 \) so that the term \( \xi_k^* \cdot Z_k^* \cdot \xi_k^* \) is nonpositive. To cancel the terms in the summation, we choose \( \xi_k = Z_k^{-1/2} \xi_k^* \). Consequently, \( \frac{1}{2}\|e\|_2^2 - \frac{\gamma^2}{2}\|\xi\|_2^2 \geq 0 \), leading to a contradiction. \( \square \)

During the second stage of the solution, we find a state \( \varepsilon \) estimate \( \tilde{x} \) such that the objective function \( \overline{J}_2 < 0 \). Theorem 4.1 helps by simplifying the problem through a change of variables. First, we must rewrite the state and observation equations of Equation (4.73) in terms of \( \tilde{\xi}_k \). Substituting from Equations (4.86) and (4.87), \( \xi_k = \frac{1}{\gamma^2} Z_k^{-1} F_k^T x_k + Z_k^{-1/2} \tilde{\xi}_k \) into these equations yields the following:

\[
\begin{align*}
x_{k+1} &= \bar{A}_k x_k + \bar{C}_k \tilde{\xi}_k \\
y_k &= \bar{C}_k x_k + \bar{E}_k \tilde{\xi}_k
\end{align*}
\]

where

\[
\begin{align*}
\bar{A}_k &= A_k + \frac{1}{\gamma^2} Q_k Z_k^{-1} F_k^T \\
\bar{C}_k &= Q_k Z_k^{-1/2} \\
\bar{C}_k &= C_k + \frac{1}{\gamma^2} E_k Z_k^{-1} F_k^T \\
\bar{E}_k &= E_k Z_k^{1/2}
\end{align*}
\]

We can then redefine our optimization problem for \( \overline{J}_2 \) as follows:

\[
\min_{\tilde{x}} \max_{\tilde{\xi}} \overline{J}_2 \equiv \min_{\tilde{x}} \max_{\tilde{\xi}} \left( \frac{1}{2}\|e\|_2^2 - \frac{\gamma^2}{2}\|\tilde{\xi}\|_2^2 \right)
\]

subject to Equations (4.89a) and (4.89b)
Since \( d \) is bounded by assumption and \( \eta \) is bounded as a consequence of the small gain theorem, \( \xi \) is bounded and so is \( \hat{\xi} \). Therefore, the criterion of Equation (4.91) is equivalent to the one considered in the derivation of the \( H_\infty \) minimax filter. The one-step estimator equations are in the standard form:

\[
\hat{x}_{(k+1|k)} = (A_k - \bar{K}_kC_k)\hat{x}_{(k|k-1)} + \bar{K}_ky_k
\] (4.92)

with filter gain given by

\[
\bar{K}_k = \left[ A_k\bar{H}_k^{-1}C_k^T + C_k\bar{E}_k^T \right] \left[ C_k\bar{H}_k^{-1}C_k^T + \bar{E}_k\bar{E}_k^T \right]^{-1}
\] (4.93)

\[
\bar{H}_k = \bar{P}^{-1}_{(k|k-1)} - \frac{1}{\gamma^2} M_k^T M_k
\] (4.94)

The Riccati matrix is a positive definite solution to the following:

\[
\bar{P}_{(k+1|k)} = (A_k - \bar{K}_kC_k)\bar{H}_k^{-1}(A_k - \bar{K}_kC_k)^T + (C_k - \bar{K}_k\bar{E}_k)(C_k - \bar{K}_k\bar{E}_k)^T, \quad k = 0, \ldots, N-1
\] (4.95)

As with the nominal \( H_\infty \) filter, \( \bar{P}_{(k|k-1)} \) must be positive definite. When there exist positive definite solutions \( \bar{H}_k \), this condition is met. Of course, the free parameter \( \gamma \) in Equation (4.94) is lower bounded such that \( \bar{H}_k > 0 \) for all \( k \). Finally, when there are also positive definite solutions \( X_k \) and \( Z_k \) in Equations (4.83) and (4.85), respectively, then there exists an estimator described by Equation (4.92) that achieves the norm bound on \( \mathcal{G} \) in Equation (4.78).

### 4.3.2 Summary of Robust \( H_\infty \) Estimator Equations

For convenience, the one-step predictor-to-predictor estimator equations are collected below:

\[
\hat{x}_{(k+1|k)} = (A_k - \bar{K}_kC_k)\hat{x}_{(k|k-1)} + \bar{K}_ky_k
\]

\[
\bar{K}_k = \left[ A_k\bar{H}_k^{-1}C_k^T + C_k\bar{E}_k^T \right] \left[ C_k\bar{H}_k^{-1}C_k^T + \bar{E}_k\bar{E}_k^T \right]^{-1}
\]
where

\[
\bar{A}_k = A_k + \frac{1}{\gamma^2} Q_k Z_k^{-1} F_k^T 
\]
(4.96a)

\[
\bar{G}_k = Q_k Z_k^{-1/2} 
\]
(4.96b)

\[
\bar{C}_k = C_k + \frac{1}{\gamma^2} E_k Z_k^{-1} F_k^T 
\]
(4.96c)

\[
\bar{E}_k = E_k Z_k^{1/2} 
\]
(4.96d)

\[
F_k = S_k^T T_k + A_k^T X_{k+1} Q_k 
\]
(4.96e)

\[
\bar{H}_k = \bar{P}_{(k|k-1)}^{-1} - \frac{1}{\gamma^2} M_k^T M_k 
\]
(4.96f)

\[
Z_k = I - \frac{1}{\gamma^2} (T_k^T T_k + Q_k^T X_{k+1} Q_k) 
\]
(4.96g)

and the matrices \( X_k \) and \( \bar{P}_k \) are positive definite solutions, respectively, to the following discrete-time Riccati equations:

\[
X_k = A_k^T X_{k+1} A_k + S_k^T S_k + \frac{1}{\gamma^2} F_k Z_k^{-1} F_k^T 
\]
(4.97a)

\[
X_N = 0 
\]

\[
\bar{P}_{(k+1|k)} = (\bar{A}_k - \bar{K}_k \bar{C}_k) \bar{H}_k^{-1} (\bar{A}_k - \bar{K}_k \bar{C}_k)^T 
\]

\[
+ (\bar{G}_k - \bar{K}_k \bar{E}_k) (\bar{G}_k - \bar{K}_k \bar{E}_k)^T 
\]
(4.97b)

In the above equations, it is also assumed that \( Z_k \) and \( H_k \) are positive definite matrices.

The solution to the robust state estimation problem is an extension of both the \( H_\infty \) optimal estimator and the Kalman filter for nominal systems. When there exist no model perturbations, then \( Q_k = 0, R_k = 0, S_k = 0, \) and \( T_k = 0 \) in Equations (4.72a), (4.72b), and (4.72d) so that the Riccati Equation (4.83) is superfluous. The estimator is reduced to solving one Riccati equation based on the nominal plant dynamics only. The modified Riccati Equation (4.95) and filter gain from Equation (4.93) are then the same as those of the nominal \( H_\infty \) minimax filter. These equations are, of course, simply the Kalman filter equations except with the term \( H_k = (P_{(k|k-1)}^{-1} - \frac{1}{\gamma^2} M_k^T M_k) \) replacing the a
priori information matrix $P_{(k|k-1)}^{-1}$. Now, however, the parameter $\gamma$ does not just trade off nominal performance with worse-case noise estimation performance. As $\gamma \to \infty$, one recovers the Kalman filter as before. As $\gamma \to \gamma_{\text{min}}$, on the other hand, one becomes more robust to both disturbance and plant modeling error. The smaller $\gamma$ becomes, the more conservative one is.

As with the nominal $H_\infty$ filter equations, the robust minimax filter equations may be written in prediction and update forms. These equations are precisely the same as those given in Equations (4.54)-(4.60) except with $(A_k, Q_k, C_k, E_k)$ replaced with $(\tilde{A}_k, \tilde{G}_k, \tilde{C}_k, \tilde{E}_k)$, respectively, as the filter model. The Riccati matrices $(P_{(k+1|k)}, P_{(k|k)}, H_k)$ and filter gain matrix $K_k^w$ would then be suitably altered.

4.3.3 Implementation

Examining the performance objective of Equation (4.78), there appears to be a contradiction in that allowing a larger uncertainty $\Delta$ results in a tighter bound, i.e., better performance. We realize upon further reflection, however, that this is not the case. As mentioned in Section 4.1.1, the Kalman filter minimizes the $H_2$ norm of the transfer function from the disturbance to the weighted estimation error. The $H_2$ norm is calculated by taking the integral of all the singular values of $G_{cd}$ over all frequencies and is a measure of transmitted energy. In Section 4.2.2, it was found that the game-theoretic formulation for uncertain noise model minimizes the $H_\infty$ norm. In other words, game-theoretic filters work by bounding only the maximum amplification from disturbance to error without regard to the amount of energy transmitted from disturbance to estimation error. Of course, in cases where the bound $\gamma$ is large, the $H_2$ and $H_\infty$ solutions will be equivalent. The obvious example is allowing $\gamma \to \infty$ in order to recover the Kalman filter. Nevertheless, understanding the differences between the $H_2$ and $H_\infty$ optimizations is the key to unravelling the apparent paradox.

A robust $H_\infty$ filter is able to bound the maximum amplification from dis-
turbances to estimation errors in the presence of plant model perturbations $\Delta$ but does nothing to minimize the transmitted energy. In fact, because the $H_\infty$ bound extends over all frequencies, the minimum achievable bound for Equation (4.78) will actually allow for infinite energy transmitted from disturbance to error. Further discussion of the properties of robust $H_\infty$ filter in steady-state may be found in [14],[11], and [10].

Before applying the robust minimax filter to the navigation problem example of Section 4.1.2, let us recall that the robust estimation problem is solved in two stages. The first stage bounds terms in the performance objective related to robustness to plant model uncertainty not affected by the estimate. During the second stage, an estimator is found to bound and minimize the transmitted energy from the augmented disturbance to the augmented estimation error. Although Equations (4.96a)-(4.97b) share the same parameter $\gamma$, there is no strict requirement that the same parameter be used in actually implementing the bounds for the first and second stages. As mentioned previously, the performance criterion of Equation (4.78) is a conservative bound. Therefore, for actual implementation, we depart from the theory and use two different parameters in an attempt to improve performance. Indeed, one may imagine the minimax estimation problems of Equation (4.82) and Equation (4.91) to be rewritten, respectively, as follows:

\[
\min_{\hat{x}} \max_{\xi} \tilde{J}_2 = \min_{\hat{x}} \max_{\xi} \left( \frac{1}{2} \|e\|^2_2 + \frac{1}{2} \|\epsilon\|^2_2 - \frac{\kappa^2}{2} \|\xi\|^2_2 \right) \tag{4.98}
\]

\[
\min_{\hat{x}} \max_{\xi} \tilde{J}_2 = \min_{\hat{x}} \max_{\xi} \left( \frac{1}{2} \|e\|^2_2 - \frac{\gamma^2}{2} \|\xi\|^2_2 \right) \tag{4.99}
\]

where we would use $\kappa$ to solve the first Riccati equation and use $\gamma$ to solve the second Riccati equation. In other words, $\kappa$ controls robustness to plant model uncertainty, while $\gamma$ controls robustness to noise model uncertainty. We require $\kappa \leq \gamma$ in order to satisfy the robust stability conditions of the small gain theorem. Formally, however, the theory does not guarantee that the bound of Equation (4.78) will be met.
To summarize, the robust filter design process may be broken down into the following steps. Given a state-space representation of the dynamics in the form of Equation 4.77:

- Select a value for the parameter $\kappa$,
- Iterate and solve the first Riccati equation (4.83),
- Check the conditions $X_k \geq 0$ and $Z_k > 0$,
- Select an estimation error weighting matrix $M_k$,
- Select a value for the parameter $\gamma$,
- Iterate and solve the second Riccati equation (4.95),
- Check the condition $\overline{H}_k > 0$.

Note that determining how much uncertainty to include in the nominal system matrices to determine $Q_k$ and $L_k$ is a question of engineering judgment. Large perturbations incorporated into the system make for conservative designs. The conservatism arising from using large perturbations may be mitigated, of course, by increasing the design parameters $\kappa$ and $\gamma$.

In what follows, our robust filter designs incorporate a relatively small amount of uncertainty, but we always choose the minimum $\kappa$. This is equivalent to including a larger uncertainty but choosing $\kappa > \kappa_{\text{min}}$. We are essentially designing for the the worst-case effect of a small uncertainty in the plant model, and that is equivalent to designing for the average effect of a larger uncertainty. In either case, the performance is the same.

Recall from the application of the minimax filter in the presence of the worst-case disturbance (Section 4.2.3), that the improvement in filter performance for our navigation problem over the Kalman filter is marginal. This is because the dominant factor affecting the estimation error is uncertainty in the plant model rather than any uncertainty in the noise model. Therefore, what we assume for
all subsequent simulations and designs is that the disturbances are white. In solving the second modified Riccati equation, it is then sufficient to set $\gamma = \infty$. What this means is that we are actually designing a Kalman filter based on the fictitious plant $(\tilde{A}_k, \tilde{G}_k, \tilde{C}_k, \tilde{E}_k)$. Note that when we set $\gamma = \infty$, the estimation error weighting matrix $M_k$ does not affect the filter design. Consequently, the robust minimax filter seeks the best robust estimate for all the states of the system.

<table>
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<td>Nominal Plant</td>
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<tr>
<td>$\Delta A$</td>
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Table 4.7: Definitions for Robust Filter Design

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<th>Robust Filter 2</th>
<th>Robust Filter 3</th>
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<tbody>
<tr>
<td>$dA$</td>
<td>$\Delta A$</td>
<td>$0.1 \times \Delta A$</td>
<td>$0.01 \times \Delta A$</td>
<td>$0.001 \times \Delta A$</td>
</tr>
<tr>
<td>$dB$</td>
<td>$\Delta B$</td>
<td>$0.5 \times \Delta B$</td>
<td>$0.05 \times \Delta B$</td>
<td>$0.005 \times \Delta B$</td>
</tr>
<tr>
<td>$dG$</td>
<td>$\Delta G$</td>
<td>$0.5 \times \Delta G$</td>
<td>$0.05 \times \Delta G$</td>
<td>$0.005 \times \Delta G$</td>
</tr>
<tr>
<td>$dC$</td>
<td>$10^{-4} \times E_{\text{nom}}$</td>
<td>$10^{-4} \times C_{\text{nom}}$</td>
<td>$10^{-4} \times C_{\text{nom}}$</td>
<td>$10^{-4} \times C_{\text{nom}}$</td>
</tr>
<tr>
<td>$dE$</td>
<td>$0.005 \times E_{\text{nom}}$</td>
<td>$0.005 \times E_{\text{nom}}$</td>
<td>$0.005 \times E_{\text{nom}}$</td>
<td>$0.005 \times E_{\text{nom}}$</td>
</tr>
<tr>
<td>$\kappa_{\text{min}}$</td>
<td>0.381</td>
<td>0.210</td>
<td>0.083</td>
<td>0.057</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Table 4.8: Robust filter design parameters for navigation problem example

The design parameters for the robust filter are selected through trial and error to obtain good time-domain results. As mentioned before, the robust filter's performance may be rather conservative since it is robust to a large class of model perturbations. With this in mind, we actually vary the amount of uncertainty during the design process in hopes of finding a good compromise between nominal and robust performance. In other words, we define the uncertainties using $(dA, dB, dC, dE)$ rather than using strictly $(\Delta A, \Delta B, \Delta C, \Delta E)$. 

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Table 4.8 indicates explicitly how we modify the amount of uncertainty in the system. Note that we are now including control signal variation so that it is necessary to perform the additional substitutions given in Equations (4.29)-(4.33). As a result, we have \( \Delta B \equiv [\Delta B \ \Delta G] \) in performing the singular value decomposition for \( N \), as given in Appendix C. The parameter \( \kappa_{\text{min}} \) is found by solving Equation (4.83) in steady-state. Since we are applying the robust filter equations to time-invariant plant models, then it is sufficient to have the filter design model be time-invariant as well. That is, the filter plant models are described by Equations (4.90a)-(4.90d) but with all subscripts \( k \) dropped. Because we now assume white-noise process and sensor disturbances, we set the parameter \( \gamma \) equal to infinity.

Estimation error results for a number of robust filter designs are summarized in Table 4.9. Their design parameters are summarized in Table 4.8. “Nominal plant” refers to a robust filter design applied to the nominal system with \( (\theta = 7.5, \alpha = 35^\circ) \). “Perturbed plant” refers to a robust filter design applied to the off-nominal system with \( (\theta = 8.8, \alpha = 38^\circ) \). Robust filter 3 appears to give the best results from among the designs. Its estimation errors are plotted in Figure 4-7. Note, however, that the robust designs of Table 4.8 have \( \kappa = \kappa_{\text{min}} \). As mentioned previously, the smaller \( \kappa \) becomes, the more conservative the filter design since nominal performance may be greatly sacrificed to insure robust performance. It may be possible to increase \( \kappa \) above \( \kappa_{\text{min}} \) and obtain less conservative results. While robustness to plant model uncertainties may be of great benefit, however, one may be able only to tolerate a certain level of degradation in nominal performance.

For a final comparison, we repeat the sample statistics for the Kalman filter (KF), an overdesigned Kalman filter (OKF) with \( a = 100 \), an exponentially-forgetting Kalman filter (EF) with \( \alpha = 1.005 \), and Robust filter 3 (ROB) in Table 4.10. The robust filter’s performance in the nominal case compares favorably with the Kalman filter designs. Furthermore, among the filters, only the robust filter manages to give acceptable estimates for the perturbed plant.
<table>
<thead>
<tr>
<th>Filter</th>
<th>Squared Error</th>
<th>Bank Angle</th>
<th>Sideslip Angle</th>
<th>Bank Rate</th>
<th>Sideslip Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Nominal plant</td>
<td>0.0446</td>
<td>0.6899</td>
<td>0.2549</td>
<td>0.3643</td>
</tr>
<tr>
<td></td>
<td>Perturbed plant</td>
<td>0.0462</td>
<td>1.0597</td>
<td>0.2548</td>
<td>0.3642</td>
</tr>
<tr>
<td>2</td>
<td>Nominal plant</td>
<td>0.0420</td>
<td>0.6709</td>
<td>0.2217</td>
<td>0.3294</td>
</tr>
<tr>
<td></td>
<td>Perturbed plant</td>
<td>0.0442</td>
<td>1.0453</td>
<td>0.2220</td>
<td>0.3294</td>
</tr>
<tr>
<td>3*</td>
<td>Nominal plant</td>
<td>0.0232</td>
<td>0.6206</td>
<td>0.1370</td>
<td>0.2190</td>
</tr>
<tr>
<td></td>
<td>Perturbed plant</td>
<td>0.0276</td>
<td>0.9664</td>
<td>0.1543</td>
<td>0.2359</td>
</tr>
<tr>
<td>4</td>
<td>Nominal plant</td>
<td>0.0245</td>
<td>0.6112</td>
<td>0.0965</td>
<td>0.1277</td>
</tr>
<tr>
<td></td>
<td>Perturbed plant</td>
<td>0.0187</td>
<td>0.9011</td>
<td>0.2279</td>
<td>0.3799</td>
</tr>
</tbody>
</table>

Table 4.9: Performance results for different robust filter designs. Units are $(\text{deg})^2$ and $(\text{deg/sec})^2$.

Figure 4-7: Estimation errors for Robust filter 3 on a nominal and on a perturbed plant.
<table>
<thead>
<tr>
<th></th>
<th>Squared Error</th>
<th>Bank Angle</th>
<th>Sideslip Angle</th>
<th>Bank Rate</th>
<th>Sideslip Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>KF</td>
<td>Nominal Plant</td>
<td>0.05635</td>
<td>0.60597</td>
<td>0.09021</td>
<td>0.04017</td>
</tr>
<tr>
<td></td>
<td>Perturbed Plant</td>
<td>0.04543</td>
<td>15.21114</td>
<td>5.52041</td>
<td>21.93262</td>
</tr>
<tr>
<td>OKF</td>
<td>Nominal Plant</td>
<td>0.05545</td>
<td>0.51545</td>
<td>0.37061</td>
<td>0.07894</td>
</tr>
<tr>
<td></td>
<td>Perturbed Plant</td>
<td>0.05539</td>
<td>3.97611</td>
<td>0.37065</td>
<td>0.99824</td>
</tr>
<tr>
<td>EF</td>
<td>Nominal Plant</td>
<td>0.05107</td>
<td>0.60589</td>
<td>0.08969</td>
<td>0.04029</td>
</tr>
<tr>
<td></td>
<td>Perturbed Plant</td>
<td>0.66903</td>
<td>6.85918</td>
<td>2.20203</td>
<td>6.94640</td>
</tr>
<tr>
<td>ROB</td>
<td>Nominal Plant</td>
<td>0.02323</td>
<td>0.62057</td>
<td>0.13702</td>
<td>0.21904</td>
</tr>
<tr>
<td></td>
<td>Perturbed Plant</td>
<td>0.02756</td>
<td>0.96638</td>
<td>0.15426</td>
<td>0.23586</td>
</tr>
</tbody>
</table>

Table 4.10: Comparison of performance among the Kalman filter, two robustified Kalman filters, and the Robust filter. Units are (deg)$^2$ and (deg/sec)$^2$.

In the next chapter we design a robust fault detection and isolation architecture for the space shuttle Orbiter’s attitude control system. As mentioned in Chapter 3, a robust FDI system would distinguish between failures in the control aerosurfaces and failures in the Reaction Control System. This is in contrast to the navigation problem given in this chapter where we are concerned with robust estimates of the attitude states. While still important, we will now consider these estimates in terms of the innovations sequence to detect failures in the elevons. An explicit model of the jet dynamics, now considered as a bias, will be augmented to the state-space model in order to provide estimates of the RCS thrust levels directly.
Chapter 5

Robust Fault Detection and Isolation

In the last section of the previous chapter, we applied the robust filter equations to our navigation problem and found the robust estimates of the attitude states in the presence of plant model uncertainty. In this chapter, we wish to apply the same filter equations to the problem of failure detection and isolation in the space shuttle Orbiter’s attitude control system. In particular, we investigate the use of the robust filter as a residual generator to ascertain the presence of failures.

Kalman filters are typically used because their residuals are white and zero-mean. Shifts in the innovations sequence of the Kalman filter are straightforward to identify. Hence, a threshold for fault detection may be set for the magnitude of the residual. Above this threshold, a failure is indicated. However, as we will see in Section 5.1, when model uncertainties exist, terms associated with the perturbations in the system matrices appear in the residuals. As a result, the residuals are biased rather than zero-mean and may indicate a false alarm. Robust filters, on the other hand, may be used since they are insensitive to model uncertainty.

As we shall see in Section 5.2, the robust filter still remains sensitive to abrupt changes or inconsistencies in the inputs. Unlike the Kalman filter, there-
fore, an acceptable detection threshold may be set that is not too high as to increase the chances of missed detections and is not too low so as to increase the likelihood of false alarm. One drawback to this method, however, is that it does not perform proper isolation. To this end, we develop a robust jet thrust estimator to run in parallel with the robust residual generator. This estimator is presented in Section 5.3. With these two estimator designs operating in parallel to provide proper fault isolation, we discuss the robust FDI architecture in Section 5.4.

5.1 Kalman Filters for FDI

As mentioned in the introduction to this thesis, many engineering systems rely on fault detection and isolation algorithms to detect anomalies in their system subcomponents. Analytic observers are implemented when sensor hardware redundancy is infeasible or undesirable. These estimators use both the output measurements from the system as well as a priori knowledge of the dynamics to provide the necessary information.

To this end, we consider sudden changes in one or more of the parameters of the system as additive failures, meaning that there are recognizable changes in the mean of the observations. In other words, we may look simply for shifts in the innovations sequence or residuals of the measurements. Non-additive failures such as those resulting in changes to the spectral content of the output will not be considered.

When a dynamic system is accurately modeled, the Kalman filter serves as the optimal state estimator in the presence of Gaussian white-noise disturbances. Two important characteristics of the Kalman filter are that its residuals are white and zero-mean. These properties, therefore, also make the Kalman filter especially attractive for use in fault detection since it will be relatively straightforward to notice shifts in its innovations sequence. We begin by formulating the following hypotheses for failure detection.
When the system operates with no failures \((H_0)\) and when the system undergoes an actuator failure \((H_1)\), we consider the following:

\[
\begin{align*}
H_0 & : \quad x_{k+1} = A_k x_k + B_k u_k + G_k d_k \\
y_k & = C_k x_k + E_k d_k \\
H_1 & : \quad x_{k+1} = A_k x_k + B_k u_k + G_k d_k + W_k s_{k-\tau^*} \nu \\
y_k & = C_k x_k + E_k d_k
\end{align*}
\] (5.1a)

where \(x_k\), \(y_k\), \(u_k\), and \(d_k\) take on their usual definitions. The matrix \(W_k\) describes how the failure is injected into the system. Note that for our system in Equation (3.15), we are considering failures in the control aerosurfaces and failures in the Reaction Control System. Therefore, \(W_k\) is equal to the column vector of \(B_k\) representing an elevon or rudder in the plant dynamics, or it is equal to \(T_k\), respectively. The parameter \(\tau^*\) is the unknown time at which the failure occurs, \(s_k\) is the unit step function, and the quantity \(\nu\) is the generally unknown magnitude of the step failure.

When we consider the innovations sequence \(\varrho_k\), which is white:

\[
\varrho_k = y_k - C_k \hat{x}_{(k|k-1)}
\]

and \(\hat{x}_{(k|k-1)} \equiv E\{x_k|y_0, \ldots, y_{(k-1)}\}\) is given by the Kalman filter, then the aforementioned hypotheses of Equations (5.1a)-(5.1b) may be expressed as the following:

\[
\begin{align*}
H_0 & : \quad \varrho_k = \varrho_k^o \\
H_1 & : \quad \varrho_k = \varrho_k^o + V_k(\tau^*) \nu
\end{align*}
\] (5.2a)

where \(V_k(\tau^*) \nu\) is the failure signature, which may also be computed recursively using a Kalman filter as is shown in [14] when considering the Generalized Likelihood Ratio Test (GLRT). What Equations (5.2a)-(5.2b) tell us, however,
is that when a failure is present at time $\tau^*$, the output residuals will become biased by an amount $V_k(\tau^*) \nu$. Therefore, it is sufficient to set a threshold value for the residual above which a failure is declared.

The main weakness of using a Kalman filter for fault detection, of course, is the assumption of an accurately modeled plant. As was shown previously in Section 4.1.2, the performance of the Kalman filter as a state estimator degrades severely in the presence of plant model uncertainty. Even when the system is only slightly perturbed, we also realize that the Kalman filter for failure detection and isolation may inaccurately declare a failure when none is present.

Calculating the innovations based on the augmented plant dynamics given in Equation (4.22), we have

$$
\varrho_{k+1} = y_{k+1} - C_{k+1} \hat{x}_{(k+1|k)} \\
= C_{k+1} x_{k+1} - C_{k+1} \left[ (A_k - K_k C_k) \hat{x}_{(k|k-1)} + B_k u_k + K_k y_k \right] \\
= C_{k+1} \left[ (A_k + \triangle A_k) x_k + B_k u_k + G_k d_k \right] \\
- C_{k+1} \left[ (A_k - K_k C_k) \hat{x}_{(k|k-1)} + B_k u_k + K_k (C_k x_k + E_k d_k) \right] \\
= C_{k+1} (A_k - K_k C_k) \hat{x}_{(k|k-1)} + C_{k+1} (G_k - K_k E_k) d_k \\
+ C_{k+1} \triangle A_k x_k \\
= \varrho^*_k + C_{k+1} \triangle A_k x_k \\
(5.3)
$$

As is apparent from the last line of Equation (5.3), the term $C_{k+1} \triangle A_k x_k$ caused by the perturbation in $A_k$ will cause the residual to be biased and may easily be mistaken for a failure. The bias can only increase in magnitude in the presence of additional perturbations $\triangle B_k, \triangle G_k, \triangle C_k,$ and $\triangle E_k$.

To illustrate this point with a concrete example, we plot the residuals for the Kalman filter designed and tested in Section 4.1.2. A commanded maneuver of $20^\circ$ change in bank angle is used with no failures in either the elevons or RCS thrusters. Because of the decoupling between the longitudinal and lateral
models, we know a priori that the residual in the second measurement for pitch rate will be white and exclude it from our presentation. Immediately we can see that the perturbed residuals are not white. It would be difficult to set an appropriate threshold level for detecting failures because of model mismatch. A threshold set too low would give rise to excessive numbers of false alarm, while a threshold set too high would increase the probability of missed detections. The Kalman filter cannot distinguish between plant model uncertainty and actual failures in the attitude control system.

The frequency content for the residuals is also plotted in Figure 5-2. These results are for verification only and are obtained from post-processing the data rather than done on-line. As can be seen, the residuals for the Kalman filter on the nominal plant has a white power spectral density, i.e., equal intensity across all frequencies. The residuals for the mismatched Kalman filter on the perturbed plant are clearly not white.
Figure 5-2: Power spectral densities for the Kalman filter residuals in the absence of failures for the nominal plant (above) and for the perturbed plant (below)

5.2 Robust Filters for FDI

The basic idea behind using a Kalman filter for fault detection is to look for discrepancies between the plant’s output relative to the filter’s estimate of that output. As demonstrated in the previous section, the limitation of the Kalman filter and other model-based estimators is their dependence on an accurate representation of the system dynamics. Plant models used in practice are often incomplete; therefore, such fault detection and isolation schemes will have problems distinguishing between true failures and false alarms induced by plant model inaccuracies.

A robust FDI system may be designed to overcome this problem. Accurately distinguishing between model uncertainties and failures would allow for a reduction in excessive false alarms and missed detections. In this section we wish to investigate the use of the robust filter for fault detection and isolation in the space shuttle Orbiter’s attitude control system. We employ the same methodology as in applying the Kalman filter, i.e., checking the filter residual.
Strictly speaking, however, this approach succeeds for the Kalman filter because its residual is white. Any analytic examination of the residual for the robust filter will show that this is not the case since the estimator design model is based on the fictitious system \((\tilde{A}_k, \tilde{C}_k, \tilde{E}_k)\). Nevertheless, we compute the robust filter's residuals in the “failed” and “no-failed” cases to see if an empirically determined threshold may be set for failure detection.

5.2.1 No Failures Case

In Section 4.3.3, we designed a number of robust filters to get good state estimation results. The parameters for the final design we choose as our residual generator is summarized in Table 5.1.

<table>
<thead>
<tr>
<th></th>
<th>Robust Filter Design Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal Plant</td>
<td>(M = 7.5, \alpha = 35^\circ)</td>
</tr>
<tr>
<td>Perturbed Plant</td>
<td>(M = 8.8, \alpha = 38^\circ)</td>
</tr>
<tr>
<td>(dA)</td>
<td>(0.01 \times</td>
</tr>
<tr>
<td>(dB)</td>
<td>(0.05 \times</td>
</tr>
<tr>
<td>(dG)</td>
<td>(0.05 \times</td>
</tr>
<tr>
<td>(dC)</td>
<td>(10^{-4} \times C_{\text{nom}})</td>
</tr>
<tr>
<td>(dE)</td>
<td>(0.005 \times E_{\text{nom}})</td>
</tr>
<tr>
<td>(\kappa_{\text{min}})</td>
<td>0.083</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>(\infty)</td>
</tr>
</tbody>
</table>

Table 5.1: Robust filter design for residual generation in fault detection

We simulate the robust filter using the same 20° change in bank angle with no failures as done in Section 5.1. The residuals generated from this experiment will serve as a benchmark for comparison in the cases of elevon or RCS failure. As stated previously, the strength of using a Kalman filter for fault detection is that its residuals are white. Post-processing the residuals for the robust filter, we plot their frequency content in Figure 5-3 and verify that its residuals are not white. Note that only the first and third measurement residuals are shown since the longitudinal and lateral modes are decoupled.
Figure 5-3: Power spectral densities for the robust filter residuals in the absence of failures for the nominal plant (above) and for the perturbed plant (below)

However, Figure 5-4 showing the behavior of the robust filter residuals in the time-domain indicates that a tolerable detection threshold for failures may still be set. This is because the filter is robust to plant model uncertainties. Consequently, the presence of actual failures may still be detectable through shifts in the output residuals.

5.2.2 Elevon Failure Case

Now we consider the case where there is a failure in the elevon control surfaces. We do not consider failures in the rudder since it is relatively ineffective for yaw control throughout the two operating conditions used ($M = 7.5$, $\alpha = 35^\circ$ and $M = 8.8$, $\alpha = 38^\circ$) for our examples. At these relatively high angles of attack, the Orbiter's body and wings shadow the rudder, decreasing its utility.

We mention in Section 2.2.1 that the elevons operate in such a manner as to recreate the effects of conventional elevator and aileron deflections. The elevon acting as an elevator controls pitching moment, while acting as an aileron effects roll torque. These two modes of motion are independent, however, so that there
Figure 5-4: Output residuals for the robust filter in the absence of failures for the nominal plant (above) and for the perturbed plant (below). Units are (deg) and (deg/sec).

may be a change in bank angle without affecting angle of attack. This may easily be seen from examining Equations (2.1) and (2.2). When the port and starboard elevons deflect an equal amount in the same direction, there is a net elevator deflection but zero aileron deflection. Similarly, when the port and starboard elevons deflect an equal amount but in opposite direction, there is zero elevator deflection but a net aileron deflection. Therefore, for the 20° change in bank angle considered in our examples, we need only be concerned with failures in the bank input command.

Figure 5-5 shows the commanded and actual elevon (aileron) deflections when there is a failure injected at $t = t^* = 7.69$ seconds and lasting for the remainder of the simulation. The output residuals are shown in Figure 5-6. For simplicity, we assume the elevon fails by remaining stuck at a particular angle. The commanded and actual thrust inputs remain the same since we also assume no simultaneous or overlapping failures occur. Based on these results, a detection threshold may be set at $V_e^* = 0.018$, for example. When the magnitude
Figure 5-5: Commanded and actual (failed) elevon deflections for a 20° change in bank angle

Figure 5-6: Output residuals for the robust filter in the presence of an elevon failure for the nominal plant (above) and for the perturbed plant (below). Units are (deg) and (deg/sec).
of one of the residuals is below this threshold, no failure is detected. Conversely, equal to or above this threshold, a failure is signalled. In other words,

\[ |e_k^i| \geq \nu_e^* \quad \text{Failure} \]
\[ |e_k^i| < \nu_e^* \quad \text{No Failure} \]

A more conservative or less conservative threshold may be selected. A more conservative threshold has the advantages of fewer false alarms but has the disadvantage of increased time between the onset of failure and its detection. A less conservative threshold decreases the detection lag but may lead to an increased rate of false alarms.

### 5.2.3 RCS Failure Case

Analogous results may be obtained in the case when there is a thruster failure. As in the previous section, we assume no overlapping failures, i.e., when a jet fails to fire, there is no simultaneous failure in the elevon. Figure 5-7 shows the commanded and actual thruster firings used for our simulation. At \( t = 4.925 \) seconds, one of two yaw jets fails to fire and remains failed off for 1.395 seconds. The output residuals are shown in Figure 5-8. Unlike the elevon failure case, we do not consider a jet failed for the remainder of the simulation run. The Redundancy Management (RM) system described in Section 2.3.3 already provides some measure of automatic fault detection for the RCS thrusters.

Figure 5-8 shows little indication of a shift in the mean of its residual. Improved results may be obtained by smoothing or averaging the residual over a moving window. The smoothed residuals are plotted in Figure 5-9. At each time \( k \), we consider an observation window of five time steps, i.e., the current residual is averaged with the residuals from the previous four time steps. It is this average that is then used to compare with the threshold value for fault detection. Based on Figure 5-9, we may set a threshold of \( \nu_\phi^* = 0.003 \), for example. While this has the advantage of eliminating noise in the residual,
Figure 5-7: Commanded and actual (failed) thruster firings for a 20° change in bank angle.

Figure 5-8: Output residuals for the robust filter in the presence of an RCS thruster failure for the nominal plant (above) and for the perturbed plant (below). Units are (deg) and (deg/sec).
it has the disadvantage of increased detection lag time because of the delay required by using the previous residuals.

![Graphs showing data](image)

Figure 5-9: Smoothed output residuals for the robust filter in the presence of an RCS thruster failure for the nominal plant (above) and for the perturbed plant (below). Units are (deg) and (deg/sec).

Although we may select appropriate thresholds for fault detection, there still remains the question of fault isolation. For example, if at time step $k^*$, we have $|\hat{\theta}_k^i| = 0.02$. We have

$$|\hat{\theta}_k^i| > \nu_e^* = 0.018$$

$$|\hat{\theta}_k^i| > \nu_\theta^* = 0.003$$

The magnitude of the residual is greater than both thresholds, so we have detected a failure but cannot resolve whether it is a failure in the elevon or a failure in the jet. An additional criterion is needed to guarantee proper fault isolation.
5.3 Jet Thrust Estimator

In Section 3.3, we formulated a stochastic model of the multiple jet thrust state. By augmenting this to our system as an additional state for the filter design, we are now attempting to estimate the jet thrust. The augmented dynamics of Equation (3.17) are used to design the jet thrust estimator.

\[
\begin{bmatrix}
    x_{k+1} \\
    \theta_{k+1}
\end{bmatrix} = \begin{bmatrix}
    A_k & T_k \\
    0 & a_\theta
\end{bmatrix} \begin{bmatrix}
    x_k \\
    \theta_k
\end{bmatrix} + \begin{bmatrix}
    B_k \\
    0
\end{bmatrix} u_k + \begin{bmatrix}
    G_k & 0 \\
    0 & g_\theta
\end{bmatrix} \begin{bmatrix}
    d_k
\end{bmatrix} \\

y_k = \begin{bmatrix}
    C_k & 0
\end{bmatrix} \begin{bmatrix}
    x_k \\
    \theta_k
\end{bmatrix} + \begin{bmatrix}
    E_k & 0
\end{bmatrix} \begin{bmatrix}
    d_k
\end{bmatrix} \tag{5.4}
\]

This system defines a robust estimation problem as in Section 4.3. The solution to this problem is also considered in [2]. As in the the examples presented in this thesis, we look at time-invariant plant models of the space shuttle Orbiter's dynamics at different trim operating conditions to define the model uncertainty $\Delta$. To apply the recursive estimator equations, we simply define:

\[
\dot{\mathbf{A}} \equiv \begin{bmatrix}
    A & T \\
    0 & a_\theta
\end{bmatrix} \tag{5.5}
\]

\[
\dot{\mathbf{B}} \equiv \begin{bmatrix}
    B \\
    0
\end{bmatrix} \tag{5.6}
\]

\[
\dot{\mathbf{G}} \equiv \begin{bmatrix}
    G & 0 \\
    0 & g_\theta
\end{bmatrix} \tag{5.7}
\]

\[
\dot{\mathbf{C}} \equiv \begin{bmatrix}
    C & 0
\end{bmatrix} \tag{5.8}
\]

\[
\dot{\mathbf{E}} \equiv \begin{bmatrix}
    E & 0
\end{bmatrix} \tag{5.9}
\]

Similar expressions may be formed for the augmented state $\dot{x}_k$, the augmented output $\dot{y}_k$, the augmented input $\dot{u}_k$, and the augmented disturbance $\dot{d}_k$.

Recalling that the longitudinal modes for bank and sideslip and the lateral
modes for angle of attack are decoupled, we simplify the design process by employing a reduced-order filter. For the filter design model, we consider only the longitudinal modes so that the order of the system is reduced from seven states (six attitude states and the multiple jet state) to only five states. An estimator using all seven states would be a relatively straightforward extension of the work done in [2].

Initial results in the time-domain, however, indicated that the plant model uncertainties were too large to obtain acceptable results. In other words, it was necessary during the design phase to decrease the amount of uncertainty in the system. Rather than decrease the uncertainty in the system matrices directly, as in Section 4.3.3, we altered how the plant interacts with the uncertainties. Design weights are placed on the columns of the matrices \( \tilde{Q} \) and \( \tilde{R} \) found from the singular value decomposition of the matrix \( \tilde{N} \) defined in Appendix C:

\[
\tilde{N} = \begin{bmatrix}
\Delta \tilde{A} & \Delta \tilde{B} \\
d\tilde{C} & d\tilde{E}
\end{bmatrix} = \begin{bmatrix}
\tilde{Q} \\
\tilde{R}
\end{bmatrix} \begin{bmatrix}
\tilde{S} \\
\tilde{L}
\end{bmatrix}
\] (5.10)

As in Equations (4.72a) and (4.72d), recall that

\[
\begin{align*}
\tilde{x}_{k+1} &= \tilde{A} \tilde{x}_k + \tilde{Q} \eta_k + \tilde{G} \tilde{d}_k \\
\tilde{y}_k &= \tilde{C} \tilde{x}_k + \tilde{R} \eta_k + \tilde{E} \tilde{d}_k
\end{align*}
\]

After performing this singular value decomposition, it is found that \( \tilde{Q} \in \mathcal{R}^{5 \times 6} \) and that \( \tilde{R} \in \mathcal{R}^{2 \times 6} \). The design weights are placed on the columns of these matrices in order to control how the plant interacts with the perturbation. In solving the discrete-time algebraic Riccati Equation (4.83), we have the following:
\[
Q = \varphi_1 \begin{bmatrix} \dot{\varphi}_{11} \\ \vdots \\ \dot{\varphi}_{51} \end{bmatrix} + \varphi_2 \begin{bmatrix} \dot{\varphi}_{12} \\ \vdots \end{bmatrix} + \ldots + \varphi_6 \begin{bmatrix} \dot{\varphi}_{16} \\ \vdots \end{bmatrix} \tag{5.11}
\]
\[
R = \psi_1 \begin{bmatrix} \dot{\psi}_{11} \\ \dot{\psi}_{21} \end{bmatrix} + \psi_2 \begin{bmatrix} \dot{\psi}_{12} \\ \dot{\psi}_{22} \end{bmatrix} + \ldots + \psi_6 \begin{bmatrix} \dot{\psi}_{16} \\ \dot{\psi}_{26} \end{bmatrix} \tag{5.12}
\]

where \( \varphi_1, \varphi_2, \ldots, \varphi_6 \) and \( \psi_1, \psi_2, \ldots, \psi_6 \) are positive, scalar design weights. Furthermore, it is found that overdesigning the process noise input matrix improves performance. The process noise input matrix for the filter design is

\[
G = \begin{bmatrix} a_1 G_1 & a_2 G_2 & a_3 G_3 \end{bmatrix} \tag{5.13}
\]

where \( a_1, a_2, \) and \( a_3 \) are additional design weights placed on the columns \( G_1, G_2, \) and \( G_3, \) respectively, of the nominal \( G \) matrix.

Table 5.2 summarizes all the parameters and design weights used for the robust jet thrust estimator.

<table>
<thead>
<tr>
<th></th>
<th>Robust Filter Design Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal Plant</td>
<td>( M = 7.5, \alpha = 35^\circ )</td>
</tr>
<tr>
<td>Perturbed Plant</td>
<td>( M = 8.8, \alpha = 38^\circ )</td>
</tr>
<tr>
<td>( \Delta \bar{A} )</td>
<td>( \bar{A}<em>{\text{nom}} - \bar{A}</em>{\text{pert}} )</td>
</tr>
<tr>
<td>( \Delta \bar{B} )</td>
<td>( \bar{B}<em>{\text{nom}} - \bar{B}</em>{\text{pert}} )</td>
</tr>
<tr>
<td>( \Delta \bar{G} )</td>
<td>( \bar{G}<em>{\text{nom}} - \bar{G}</em>{\text{pert}} )</td>
</tr>
<tr>
<td>( d\bar{C} )</td>
<td>( 10^{-4} \times \bar{C}_{\text{nom}} )</td>
</tr>
<tr>
<td>( d\bar{E} )</td>
<td>( 0.005 \times \bar{E}_{\text{nom}} )</td>
</tr>
<tr>
<td>[ ( \varphi_1 \varphi_2 \varphi_3 \varphi_4 \varphi_5 \varphi_6 ) ]</td>
<td>[0.1 0.1 10 1.0 1.0 1.0]</td>
</tr>
<tr>
<td>[ ( \psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_6 ) ]</td>
<td>[10 1.0 3.0 0.2 1.0 1.0]</td>
</tr>
<tr>
<td>[ ( a_1 a_2 a_3 ) ]</td>
<td>[7.5 10 7.5]</td>
</tr>
<tr>
<td>( \kappa_{\text{min}} )</td>
<td>0.0440</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

Table 5.2: Robust jet thrust estimator design parameters and weights
A typical albeit fictitious RCS firing pattern is used to test the effectiveness of the robust jet estimator in estimating the thrust level and in detecting failures. This pattern is plotted in Figure 5-10.

![Thrust vs Time Graph](image)

**Figure 5-10:** Typical thruster firing pattern for RCS simulation

In the theory of fault detection and isolation, Gauss-Markov models are very effective in representing unknown failure models. However, they are only approximations and possess limitations in their implementation. For example, the jet command given in Figure 5-10 is mostly comprised of step changes in the thruster magnitude. No linear system, such as the simple, scalar, high-bandwidth Gauss-Markov model used to represent the multiple jet state, can represent a step input. To overcome this difficulty, we implement the robust jet thrust estimator as a transient filter in the sense that whenever a new number of thrusters fire, the covariance of the jet state is reset to a large value. By so doing, the estimator is able to track abrupt changes in the jet state by heavily weighting the incoming measurements.

In fact, to keep the filter sensitive to any and all abrupt change in the condition of the thruster output, i.e., failures in the jets, the transient robust filters are also simulated where the jet state covariance is arbitrarily reset every 40 time steps or 0.2 seconds. This is equivalent to implementing the filters as finite memory filters where all date if ignored beyond a certain point of time.
in the past. Although we implement this on a somewhat ad hoc basis, we nevertheless get acceptable performance.

Simulation results comparing the Kalman filter to the robust jet estimator are summarized in Table 5.3. As in previous comparisons, the Kalman filter is designed around the nominal operating condition with \( M = 7.5, \alpha = 35^\circ \). Our performance criterion is the sum of squared error in the jet estimate, i.e.,

\[
\|\theta - \hat{\theta}\|_2^2 = \sum_{k=1}^{N} (\theta_k - \hat{\theta}_k)^2,
\]

since that is similar to the variance of a random variable. The term "Steady-state filter" refers to a robust jet estimator whose covariance is not arbitrarily reset. Strictly speaking, it is still a transient filter since the jet state covariance is still reset when a new number of jets is commanded. The term "Transient filter" refers to the jet estimator whose covariance is continually reset in order to keep it sensitive to abrupt failures.

The cases considered are one in which the thrusters fire as commanded and one in which there is a jet failure. For both cases, elevons deflections are commanded, but we do not assume any failure in the aerosurfaces. The term "Commanded Jet" refers to this case without failures. For the latter case, one of four jets commanded fails to fire at \( t = 0.35 \) seconds and remains off until \( t = 0.74 \) seconds at which time it actually does fire. The term "Failed Jet" refers to this situation with an anomaly in the thruster pattern. Finally, the term "Nominal Plant" refers to the estimators tested on noisy measurements from the nominal system, while "Perturbed Plant" refers to the same designs filtering outputs from the perturbed system with \( M = 8.8, \alpha = 38^\circ \).

For all the cases shown in Table 5.3, regardless of the presence of failure and regardless of whether a steady-state of a transient filter is used, the Kalman filter estimates are severely degraded in the presence of plant perturbation. The robust jet thrust estimator, on the other hand, performs similarly in both the nominal and perturbed cases. While it does sacrifice some nominal performance, when compared to the Kalman filter, it clearly makes up for that sacrifice in robust performance. Moreover, the transient robust filter outperforms the transient Kalman filter, even for the nominal plant. This may be explained in that
<table>
<thead>
<tr>
<th></th>
<th>Kalman filter</th>
<th>Robust filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commanded Jet</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Steady-state filter)</td>
<td>Nominal Plant</td>
<td>1.9282</td>
</tr>
<tr>
<td></td>
<td>Perturbed Plant</td>
<td>22.990</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.3329</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.8438</td>
</tr>
<tr>
<td>Commanded Jet</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Transient filter)</td>
<td>Nominal Plant</td>
<td>5.0161</td>
</tr>
<tr>
<td></td>
<td>Perturbed Plant</td>
<td>23.718</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.8658</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5.0349</td>
</tr>
<tr>
<td>Failed Jet</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Steady-state filter)</td>
<td>Nominal Plant</td>
<td>6.4095</td>
</tr>
<tr>
<td></td>
<td>Perturbed Plant</td>
<td>23.655</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5.5288</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7.3224</td>
</tr>
<tr>
<td>Failed Jet</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Transient filter)</td>
<td>Nominal Plant</td>
<td>6.8391</td>
</tr>
<tr>
<td></td>
<td>Perturbed Plant</td>
<td>21.799</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.4167</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6.7050</td>
</tr>
</tbody>
</table>

Table 5.3: Sum of squared error ($10^7$ lb$^2$) in jet estimate

the robust filter has less transient instability than the Kalman filter when the covariance is reset. When a failure does not occur, obviously, the steady-state filter will give better performance.

Before leaving this section, we plot the performance of the transient robust jet thrust estimator during the simulation that includes an RCS failure. The first two seconds of the simulation are shown in Figure 5-11. The transients in the plot are due to the filter’s restarting. Overall, these results demonstrate clearly that while the transient filter is insensitive to model perturbation, it is still highly sensitive to unexpected jet malfunction.

### 5.4 Robust FDI Architecture

We now have all the components necessary for a complete Fault Detection and Isolation (FDI) architecture. By using the two robust filters in parallel, we are able to detect failures in either the control aerosurfaces or in the Reaction Control System and perform failure isolation. Figure 5-12 shows the filter architecture used for attitude control system FDI.

In the architecture of Figure 5-12, robust filter, $F_1$, is designed to detect failures by checking whether a detection threshold has been exceeded by the residual. The second robust filter, $F_2$, is designed to provide accurate thrust
Figure 5-11: Jet failure detection using a transient robust estimator

Figure 5-12: Robust Filter Architecture
estimates. Note that an alternative design where an elevon failure state is introduced using an additional Gauss-Markov model is also possible [14]. Other techniques are possible, as explained in [7].

In general, one of the strengths of robust filter design is that there exist various ways of representing the uncertainty. Different filter designs may also be developed by varying estimation error weighting matrices should the complete robust \( H_\infty \) filter equations be used. Flexibility of design allows each of the two filters to be tuned to a different set of states.

The detection logic in Figure 5-12 works as follows. Recall from Section 5.2 that two detection thresholds may be set. A minimum threshold may be set to detect failures in the jet input, while a larger threshold may be set for elevon failures. When the lesser of the detection thresholds is surpassed, robust filter \( F_2 \) is checked for verification. When the greater detection threshold is exceeded by the residual of robust filter \( F_1 \), the thrust estimate of robust filter \( F_2 \) is checked to see if an RCS thruster has failed. If robust filter \( F_2 \) does not indicate a failure in the jet, then it must be an elevon that fails. Essentially, robust filter \( F_2 \) provides an independent check of the faults detected by robust filter \( F_1 \). Since we do not consider simultaneous failures in this thesis, using these two filters in parallel allows for proper isolation of both kinds of failures in the space shuttle Orbiter's attitude control system. As represented above, the two filters operate independently; however, they may be designed to interact with one another.
Chapter 6

Conclusions

This thesis applies robust $H_\infty$ estimation to the problem of failure detection and isolation for reentry vehicle attitude control systems. The objectives are to detect a failure rapidly and to determine accurately whether the fault occurred in the control aerosurfaces or in the Reaction Control System. Correct isolation is essential to the second objective. The problem is particularly difficult during reentry into the atmosphere since there will be rapid variations in Mach number and large uncertainties in the vehicle aerodynamic properties.

While searching for a solution, we also demonstrate that robust filters may be used for aircraft attitude determination. Specifically, robust estimates of the attitude angles and attitude rates during a change in bank angle maneuver is found. A Kalman filter designed around the nominal operating condition is shown to be highly sensitive to plant model perturbations. A robust filter designed with a priori knowledge of the perturbations in the state-space matrices is capable of handling such perturbations much better.

6.1 Summary of thesis content

In Chapter 2 the attitude control effectors for the space shuttle Orbiter are described. The control aerosurfaces include the elevons, rudder, speedbrake, and bodyflap. The Reaction Control System augments the aerosurfaces throughout
reentry and are the sole attitude effectors during on-orbit operation.

Chapter 3 develops the rotational dynamics state-space models used throughout this thesis. Disturbance models for process and sensor noise are developed, and a Gauss-Markov model for the thrust is also presented. Unmodeled nonlinear effects are discussed.

We examine various filtering methods in Chapter 4. The discrete-time Kalman filter equations are presented, and we demonstrate its sensitivity to model uncertainty in the context of the attitude determination problem. An $H_\infty$ or game-theoretic minimax filter is evaluated for its effectiveness in providing robust estimates in the presence of uncertain noise model. A design parameter $\gamma$ is found to trade off $H_2$ and $H_\infty$ performance. For our navigation problem only, however, the most deleterious source of uncertainty comes from inaccurate plant modeling rather than worst-case noise. In other words, a $H_\infty$ minimax filter alone is insufficient in estimating the attitude states in the presence of plant model uncertainty. Other methods for robustifying the Kalman filter, such as overdesigning the process noise and exponential age-weighting of past observations, are also examined.

An extension of the Kalman filter and the $H_\infty$ minimax filter, i.e., the robust $H_\infty$ minimax filter, is presented to minimize the estimation error for an entire class of disturbances and plants. For this estimator, the performance criterion bounding the transmitted energy from the augmented disturbance to the augmented estimation error defines a small gain condition. According to the theory, solution involves solving two algebraic Riccati equations that are dependent on the parameter $\gamma$. However, the criterion of Equation (4.78) is a conservative one. During implementation, we solve the first and second Riccati equations independently with different parameters, $\kappa$ and $\gamma$, respectively. It is found that the parameter $\kappa$ may be used to control robustness to plant model uncertainty, while the parameter $\gamma$ may be used to add robustness to noise model uncertainty. The only requirement we make is that $\kappa \leq \gamma$ in order to satisfy the robust stability conditions of the small gain theorem. The robust
filter's insensitivity to model perturbations is demonstrated while applying it to the navigation problem.

In Chapter 5, a methodology for failure detection is discussed. The effectiveness of using the Kalman filter as a residual generator is limited by its assumption of an accurately modeled system. Instead, a robust filter is used although its innovations are not white in frequency content. Nevertheless, a robust filter may be used effectively to set a minimum fault detection threshold that does not give rise to excessive numbers of false alarms or missed detections. To deal with the issue of fault isolation, the so-called robust jet thrust estimator is developed that provides a direct estimate of the RCS output. Used in parallel with a filter designed to provide a robust residual, this filter defines a robust FDI architecture. The results obtained show that a robust filter may be insensitive to model perturbations while remaining sensitive to failures.

6.2 Suggestions for Future Work

The results obtained in this thesis suggest several immediate avenues of exploration:

1. *Lateral Dynamics.* The most obvious and direct extension of the work in this thesis would be to include a change in angle of attack maneuver. The lateral modes would then be excited, and we could then see whether the robust filters designed in Section 4.3.3 provide for robustness to plant model uncertainty in the lateral dynamics. Moreover, we could design a full-order estimator rather than a reduced-order filter as done in Section 5.3. As remarked, including the lateral dynamics in the filter design would be a relatively straightforward extension.

2. *Unmodeled Nonlinear Effects.* As mentioned in Chapter 3, certain nonlinear effects such as plume interaction, plume self-impingement,
and the effects of increased atmospheric pressure are left unmodeled. Plume interaction is particularly important since interaction with the aerodynamic flow field presents a problem in estimating RCS thruster levels. The effects of plume flow field interaction modeling errors may be difficult to separate from actual failures in the RCS.

3. **Gain Scheduling of Robust Filters.** Estimation over the entire reentry trajectory would require either a nonlinear filter or a system of gain scheduled linear filters. As mentioned in Chapter 1, rapid gain scheduling of Kalman filters may not be possible due to the large data storage requirements entailed by the necessarily large number of filter designs. With robust filters, fewer designs are needed; however, there still needs to be developed some blending logic to incorporate all the measurements from the various robust filters running in parallel.

4. **On-board Flight Experiment.** An on-board flight experiment of robust filters for either attitude determination or failure detection and isolation would be desirable. The only requirement would be to obtain all the sensor information from the flight control software. However, the current flight control software aboard the space shuttle Orbiter operates at 12.5 Hz. The designs developed in this thesis are based on state-spaced models discretized with a sampling period of $T = 0.005$ seconds. New designs would have to be developed using a slower time step. Alternatively, a self-contained measurement package operating at a higher sampling rate could be placed aboard the Orbiter in a Get-Away-Special container (GAS can).

Furthermore, future directions for research include the following:

1. **Propulsion Model.** The work done in this thesis lumped the effect of the RCS jets into a single multiple jet firing command. Research into
applying a detailed Reaction Control System jet propulsion model would allow for isolation of precisely which jet failed.

2. **Robust Navigation Filter.** Based on the results of Section 4.3.3, it is possible to design a robust navigation filter for attitude states. Such a filter design would have broad application to aircraft and spacecraft dynamics. In these applications, the greatest amount of uncertainty comes from plant modeling errors. For example, phenomena such as fuel slosh cause uncertainties in the moments and products of inertia of an aircraft. Neglected dynamics include those associated with structural flexibility and aeroelastic effects. A robust filter would be insensitive to these uncertainties in the plant model and provide better estimates than a nominal Kalman filter. Furthermore, in satellite tracking problems, a robust filter may be able to compensate for GPS multipath errors because of the filter's increased disturbance rejection properties.

3. **Operation in a Controller.** It would be interesting to examine the effect of using a robust filter in a closed-loop controller. Modern control methodologies, such as Linear Quadratic Gaussian (LQG) and $H_\infty$ control, require the solution of two Riccati equations. One equation determines the estimator gain, while the other calculates the controller gain. The full robust $H_\infty$ minimax filter equations already involve solutions to two Riccati equations. The complexity and efficiency of a controller involving three Riccati equations, i.e., two for the estimator and one for the control gain, are yet to be determined.

4. **Structured Uncertainty.** As mentioned in Chapter 4, the robust filter equations do not take the structure of the uncertainty into account. The derivation assumes a rather general class of perturbations, causing the robust estimator to be overly conservative. Estimators using the structured singular value $\mu$ are available only for
steady-state. Research to develop analogous filters for finite time-horizon and time-varying systems would be useful.
Appendix A

Orbiter Linearized Models

The models considered throughout this thesis are linearizations of the longitudinal and lateral dynamics of the space shuttle Orbiter at various Mach Numbers, angles of attack, and constant altitudes [29]. Here, they are given in continuous-time. We later discretize them using a sampling period of $T = 0.005$ seconds. The six states are as follows: bank angle ($\text{deg}$), angle of attack ($\text{deg}$), sideslip angle ($\text{deg}$), bank rate ($\text{deg/sec}$), time rate of change angle of attack ($\text{deg/sec}$), and sideslip rate ($\text{deg/sec}$). The inputs are aileron deflection ($\text{deg}$), elevator deflection ($\text{deg}$), rudder deflection ($\text{deg}$), and the multiple jet command ($\text{lb}$).

\begin{align*}
\text{Mach Number} & = 10.0 \\
\text{Angle of attack} & = 40.0 \text{ degrees} \\
\text{Altitude} & = 170 \text{ kft}
\end{align*}

\begin{align*}
A &=
\begin{bmatrix}
0 & 0 & 0 & 1.0000e+00 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000e+00 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0000e+00 \\
0 & 0 & 3.4432e+00 & -9.5301e-03 & 0 & 0 \\
0 & -3.5053e-01 & 0 & 0 & -1.4820e-02 & 0 \\
0 & 0 & -2.2451e+00 & 2.2090e-03 & 0 & -1.0579e-02
\end{bmatrix} \\
B &=
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1.4347e+00 & 0 & 0 & -1.2211e-04 & 0 & 0 \\
0 & -5.4081e-01 & 0 & -8.9965e-06 & 0 & 0 \\
1.3318e+00 & 0 & 0 & -5.2927e-04 & 0 & 0
\end{bmatrix}
\end{align*}

\begin{align*}
\text{Mach Number} & = 8.8 \\
\text{Angle of attack} & = 38.0 \text{ degrees}
\end{align*}
Altitude = 158 kft

\[
A =
\begin{bmatrix}
0 & 0 & 0 & 1.0000e+00 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000e+00 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0000e+00 \\
0 & 0 & -3.5397e+00 & -1.0727e-02 & 0 & 0 \\
0 & -3.2230e-01 & 0 & 0 & -1.6611e-02 & 0 \\
0 & 0 & -2.1635e+00 & 2.6479e-03 & 0 & -1.3360e-02
\end{bmatrix}
\]

\[
B =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1.2312e+00 & 0 & 0 & -1.4051e-04 \\
-4.7271e-01 & 0 & -8.9965e-06 \\
1.1430e+00 & 0 & 2.0410e-05 & -5.2468e-04
\end{bmatrix}
\]

Mach Number = 7.5
Angle of attack = 35.0 degrees
Altitude = 150 kft

\[
A =
\begin{bmatrix}
0 & 0 & 0 & 1.0000e+00 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000e+00 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0000e+00 \\
0 & 0 & -3.4236e+00 & -1.2405e-02 & 0 & 0 \\
-3.1866e-01 & 0 & 0 & -1.7474e-02 & 0 & 0 \\
0 & -1.7297e+00 & 2.2207e-03 & 0 & -1.4775e-02
\end{bmatrix}
\]

\[
B =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1.3042e+00 & 0 & 2.9345e-02 & -1.6778e-04 \\
-3.6418e-01 & 0 & -8.9965e-06 \\
8.6483e-01 & 0 & 2.1621e-02 & -5.1661e-04
\end{bmatrix}
\]

Mach Number = 5.7
Angle of attack = 29.0 degrees
Altitude = 130 kft

\[
A =
\begin{bmatrix}
0 & 0 & 0 & 1.0000e+00 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000e+00 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1.0000e+00 \\
0 & 0 & -3.6779e+00 & -1.9158e-02 & 0 & 0 \\
0 & -2.7314e-01 & 0 & 0 & -2.5308e-02 & 0 \\
0 & 0 & -1.4529e+00 & 1.4077e-03 & 0 & -2.1612e-02 \\
\end{bmatrix}
\]

\[
B =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1.5651e+00 & 0 & 7.8254e-03 & 2.2086e-04 & 0 \\
0 & 3.6874e-01 & 0 & -8.9965e-06 & 0 \\
8.8213e-01 & 0 & 2.5945e-02 & -4.9624e-04 & 0 \\
\end{bmatrix}
\]

**Mach Number = 5.0**

**Angle of attack = 25.0 degrees**

**Altitude = 120 kft**

\[
\begin{bmatrix}
0 & 0 & 0 & 1.0000e+00 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000e+00 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0000e+00 \\
0 & 0 & -3.6936e+00 & -2.2873e-02 & 0 & 0 \\
0 & -2.4173e-01 & 0 & 0 & -2.8993e-02 & 0 \\
0 & 0 & -1.1021e+00 & 7.1514e-04 & 0 & -2.8566e-02 \\
\end{bmatrix}
\]

\[
B =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1.7083e+00 & 0 & 2.3085e-01 & -2.5494e-04 & 0 \\
0 & 3.8676e-01 & 0 & -8.9965e-06 & 0 \\
9.1845e-01 & 0 & 7.9599e-02 & -4.7963e-04 & 0 \\
\end{bmatrix}
\]

**Mach Number = 4.0**

**Angle of attack = 20.5 degrees**

**Altitude = 105 kft**

\[
\begin{bmatrix}
0 & 0 & 0 & 1.0000e+00 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000e+00 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0000e+00 \\
0 & 0 & -3.1302e+00 & -3.1032e-02 & 0 & 0 \\
0 & -1.8209e-01 & 0 & 0 & -3.8980e-02 & 0 \\
0 & 0 & -1.1762e+00 & -6.2403e-04 & 0 & -3.9287e-02 \\
\end{bmatrix}
\]
\[ B = \]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1.8259e+00 & 0 & 4.6952e-01 & -1.6778e-04 \\
0 & -4.0060e-01 & 0 & -8.9965e-06 \\
6.9187e-01 & 0 & 2.5599e-01 & -5.1661e-04 \\
\end{bmatrix}
\]

Mach Number = 3.0  
Angle of attack = 16.0 degrees  
Altitude = 90 kft

\[ A = \]

\[
\begin{bmatrix}
0 & 0 & 0 & 1.0000e+00 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000e+00 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0000e+00 \\
0 & 0 & -3.3649e+00 & -5.2238e-02 & 0 & 0 \\
0 & -1.9575e-01 & 0 & 0 & -4.3764e-02 & 0 \\
0 & 0 & -7.4376e-01 & -8.9438e-05 & 0 & -6.2122e-02 \\
\end{bmatrix}
\]

\[ B = \]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1.9629e+00 & 0 & 8.4123e-01 & -1.6778e-04 \\
0 & -3.9150e-01 & 0 & -8.9965e-06 \\
6.6938e-01 & 0 & 3.7188e-01 & -5.1661e-04 \\
\end{bmatrix}
\]

Mach Number = 2.0  
Angle of attack = 12.5 degrees  
Altitude = 77 kft

\[ A = \]

\[
\begin{bmatrix}
0 & 0 & 0 & 1.0000e+00 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000e+00 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0000e+00 \\
0 & 0 & -4.1735e+00 & -9.4747e-02 & 0 & 0 \\
0 & -5.8270e-01 & 0 & 0 & -7.1495e-02 & 0 \\
0 & 0 & -1.3837e+00 & 4.2591e-03 & 0 & -1.0431e-01 \\
\end{bmatrix}
\]

\[ B = \]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2.6085e+00 & 0 & 1.0434e+00 & -1.6778e-04 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & -8.9965e-06 \\
-6.1911e-01 & 4.8431e-01 \\
4.8431e-01 & 0.8192 \\
4.8431e-01 & 0.8192 \\
\end{bmatrix}
\]

For all of the linearized models, the output matrix is given by

\[
C = 
\begin{bmatrix}
0 & 0 & 0 & 0.8192 & 0 & 0.5736 \\
0 & 0 & 0 & 0 & 1.0000 & 0 \\
0 & 0 & 0 & 0.5736 & 0 & -0.8192 \\
\end{bmatrix}
\]

In Chapter 4, a number of filters are designed and simulated, using the discretized versions of the Orbiter rotational dynamic models. For the results of Section 4.1.2, the Kalman filter is designed around the model with Mach Number \( M = 7.5 \), \( \alpha = 35^\circ \), and altitude = 150,000 feet. Similarly, the nominal \( H_\infty \) minimax filter of Section 4.2.3 is also designed around this linearized plant.

**Nominal plant, \( M = 7.5 \)**

**Angle of attack = 35 degrees**

**Altitude = 150 kft**

\[
A_{\text{nom}} = 
\begin{bmatrix}
1.0000e+00 & 0 & -4.2794e-05 & 4.9998e-03 & 0 & -7.1323e-08 \\
0 & 1.0000e+00 & 0 & 0 & 4.9998e-03 & 0 \\
0 & 0 & 9.9998e-01 & 2.7757e-08 & 0 & 4.9998e-03 \\
0 & 0 & -1.7117e-02 & 9.9994e-01 & 0 & -4.2793e-05 \\
0 & -1.5932e-03 & 0 & 0 & 9.9991e-01 & 0 \\
0 & 0 & -8.6480e-03 & 1.1103e-05 & 0 & 9.9990e-01 \\
\end{bmatrix}
\]

\[
B_{\text{nom}} = 
\begin{bmatrix}
1.6302e-05 & 0 & 3.6681e-07 & -2.0971e-09 \\
0 & -4.5522e-06 & 0 & -1.1245e-10 \\
1.0810e-05 & 0 & 2.7025e-07 & -6.4574e-09 \\
6.5209e-03 & 0 & 1.4672e-04 & -8.3882e-07 \\
0 & -1.8208e-03 & 0 & -4.4980e-08 \\
4.3240e-03 & 0 & 1.0810e-04 & -2.5829e-06 \\
\end{bmatrix}
\]

\[
G_{\text{nom}} = 
\begin{bmatrix}
8.1513e-07 & 0 & -9.6379e-14 & -2.0971e-10 & 0 & 0 & 0 \\
0 & -2.2761e-07 & 0 & -1.1245e-11 & 0 & 0 & 0 \\
3.0169e-12 & 0 & 1.3513e-08 & -6.4574e-10 & 0 & 0 & 0 \\
3.2605e-04 & 0 & -7.7103e-11 & -8.3882e-08 & 0 & 0 & 0 \\
0 & -9.1042e-05 & 0 & -4.4980e-09 & 0 & 0 & 0 \\
1.8101e-09 & 0 & 5.4050e-06 & -2.5829e-07 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
C_{\text{nom}} = 
\begin{bmatrix}
0 & 0 & 0 & 8.1915e-01 & 0 & 5.7358e-01 \\
0 & 0 & 0 & 0 & 1.0000e+00 & 0 \\
\end{bmatrix}
\]
To define parametric uncertainties for the robust filters designed in Section 4.3.3, we choose the nominal plant as the model with Mach Number = 7.5, $\alpha = 35^\circ$, and altitude = 150,000 feet. The perturbed model is chosen with Mach Number = 8.8, $\alpha = 38^\circ$, and altitude = 158,000 feet. All of the model parameters and filter matrices are given in the following.
Robust filter 1
kappa_min = 0.381
gamma = Inf

d delta =
\begin{array}{cccc}
0 & 0 & 1.4510e-06 & 2.0977e-08 \\
0 & 4.5527e-08 & 0 & 0 \\
0 & 0 & 5.4224e-06 & 5.3408e-09 \\
0 & 0 & 5.8040e-04 & 8.3904e-06 \\
0 & 1.0212e-05 & 0 & 0 \\
0 & 0 & 2.1689e-03 & 2.1363e-06 \\
\end{array}
0 2.4184e-09 0 1.0714e-08 0 8.6530e-09 0 1.4511e-06 0 4.2702e-06 0 1.6536e-06

d beta =
\begin{array}{cccc}
9.1293e-07 & 0 & 3.6681e-07 & 3.4083e-10 \\
0 & 1.3566e-06 & 0 & 1.6093e-16 \\
3.4766e-06 & 0 & 2.7000e-07 & 1.0092e-10 \\
3.6517e-04 & 0 & 1.4672e-04 & 1.3633e-07 \\
0 & 5.4262e-04 & 0 & 9.6385e-14 \\
1.3906e-03 & 0 & 1.0800e-04 & 4.0366e-08 \\
\end{array}

d gamma =
\begin{array}{cccc}
4.5645e-08 & 0 & 9.6285e-14 & 3.4083e-11 \\
0 & 6.7826e-08 & 0 & 1.6093e-17 \\
3.7902e-13 & 0 & 1.3500e-08 & 1.0092e-11 \\
1.8257e-05 & 0 & 7.7027e-11 & 1.3633e-08 \\
0 & 2.7131e-05 & 0 & 9.6385e-15 \\
2.2741e-10 & 0 & 5.3999e-06 & 4.0366e-09 \\
\end{array}
0 0 0 0

d delta =
\begin{array}{cccc}
0 & 0 & 0 & 8.1915e-05 \\
0 & 0 & 0 & 1.0000e-04 \\
0 & 0 & 0 & 5.7358e-05 \\
\end{array}
0 0 0 -8.1915e-05

d gamma =
\begin{array}{cccc}
0 & 0 & 0 & 3.5355e-05 \\
0 & 0 & 0 & 3.5355e-05 \\
0 & 0 & 0 & 3.5355e-05 \\
\end{array}
0 0 0 0 0 3.5355e-05

Abar =
\begin{array}{cccccccccccc}
1.0000e+00 & -1.8693e-20 & -4.2639e-05 & 4.9999e-03 & 1.2333e-18 & -4.8162e-08 \\
0 & 1.0000e+00 & -4.4130e-17 & -2.4312e-18 & 4.9998e-03 & 3.5479e-18 \\
0 & -5.7559e-20 & 9.9998e-01 & 4.9359e-08 & 3.7992e-18 & 4.9999e-03 \\
0 & -7.4753e-18 & -1.7055e-02 & 9.9994e-01 & 4.9316e-16 & -3.3527e-05 \\
0 & -1.5936e-03 & -1.7652e-14 & -9.7161e-16 & 9.9992e-01 & 1.4190e-15 \\
\end{array}
| 0 | -2.3031e-17 | -8.5826e-03 | 1.9744e-05 | 1.5202e-15 | 9.9994e-01 |

\( B_{\text{bar}} = \)

**Columns 1 through 6**

| -1.5591e-19 | -5.8290e-05 | 1.5632e-17 | 3.2145e-19 | -3.5186e-08 | -5.8493e-18 |
| 1.3404e-02 | -9.4250e-16 | -9.9274e-03 | -6.2942e-08 | 1.3039e-18 | 3.0369e-03 |
| 1.0034e-16 | -2.3316e-02 | 4.3014e-16 | -2.3862e-18 | -1.5470e-05 | -1.2685e-16 |

**Columns 7 through 12**

| -2.5663e-08 | 1.3200e-17 | 5.6609e-09 | 6.5384e-03 | -4.8007e-18 | 1.4835e-04 |
| 9.4804e-08 | 3.0761e-18 | 1.4799e-09 | 4.3357e-03 | 1.4739e-18 | 1.0918e-04 |

**Columns 13 through 18**

| -2.0952e-09 | 8.1523e-07 | -1.4149e-21 | 1.6181e-10 | -2.0952e-10 | 5.8953e-13 |
| -6.4563e-09 | 7.2502e-11 | 1.7563e-21 | 1.3620e-08 | -6.4563e-10 | 5.8710e-13 |
| -4.4980e-08 | -2.9229e-15 | -9.1213e-05 | 2.0645e-16 | -4.4980e-09 | 1.6551e-16 |
| -2.5825e-06 | 2.9604e-08 | 8.0515e-20 | 5.4481e-06 | -2.5825e-07 | 2.3484e-10 |

**Columns 19 through 20**

| -2.2161e-23 | 8.9337e-13 |
| 3.4475e-12 | 2.0901e-19 |
| -2.6221e-23 | 6.9526e-13 |
| 7.3011e-22 | 3.5734e-10 |
| 1.3790e-09 | 8.3532e-17 |
| -5.4789e-21 | 2.7810e-10 |

\( C_{\text{bar}} = \)

| 0 | 5.0582e-23 | -1.3891e-08 | 8.1915e-01 | 2.8893e-20 | 5.7358e-01 |
| 0 | -4.8747e-10 | -2.6039e-21 | -7.0821e-22 | 1.0000e+00 | 2.5424e-22 |
| 0 | -5.1368e-22 | -8.3754e-09 | 5.7358e-01 | 1.3941e-20 | -8.1915e-01 |

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\[ \bar{E} = \]

Columns 1 through 6

\[
\begin{array}{cccccc}
3.3431e-06 & -2.8554e-18 & -2.9370e-03 & -4.7806e-03 & 1.7583e-16 & -8.6338e-03 \\
5.2007e-21 & -3.5024e-05 & 7.6816e-17 & -7.6822e-17 & 1.0299e-02 & 2.2822e-16 \\
5.9185e-07 & 2.8338e-18 & -1.5394e-03 & 9.1220e-03 & 1.8667e-16 & -4.5247e-03 \\
\end{array}
\]

Columns 7 through 12

\[
\begin{array}{cccccc}
9.4194e-10 & -5.1109e-18 & 4.2172e-15 & 1.1023e-09 & -2.6425e-18 & -2.4426e-08 \\
1.4295e-18 & 1.2200e-12 & 3.5836e-19 & 4.1786e-20 & -5.3496e-09 & -5.4272e-19 \\
-1.3449e-09 & -1.3879e-18 & 4.4731e-15 & 8.2822e-10 & 2.7114e-19 & -1.2777e-08 \\
\end{array}
\]

Columns 13 through 18

\[
\begin{array}{cccccc}
-1.3617e-11 & -1.9771e-09 & -1.2397e-18 & 1.5679e-10 & -1.3617e-12 & 1.6012e-10 \\
\end{array}
\]

Columns 19 through 20

\[
\begin{array}{cccc}
1.5084e-19 & 1.6012e-10 \\
7.0714e-03 & 1.2220e-18 \\
6.9195e-19 & 7.0714e-03 \\
\end{array}
\]

Robust filter 2

kappa_min = 0.210

\[ \gamma = \infty \]

\[ dA = \]

\[
\begin{array}{cccccc}
0 & 0 & 1.4510e-07 & 2.0977e-09 & 0 & 2.4184e-10 \\
0 & 4.5527e-09 & 0 & 0 & 1.0714e-09 & 0 \\
0 & 0 & 5.4224e-07 & 5.3408e-10 & 0 & 8.6530e-10 \\
0 & 0 & 5.8040e-05 & 8.3904e-07 & 0 & 1.4511e-07 \\
0 & 1.8212e-06 & 0 & 0 & 4.2702e-07 & 0 \\
0 & 0 & 2.1689e-04 & 2.1363e-07 & 0 & 1.6536e-07 \\
\end{array}
\]

\[ dB = \]

\[
\begin{array}{cccc}
4.5646e-07 & 0 & 1.8340e-07 & 1.7041e-10 \\
0 & 6.7828e-07 & 0 & 8.0464e-17 \\
1.7383e-06 & 0 & 1.3500e-07 & 5.0458e-11 \\
1.8259e-04 & 0 & 7.3360e-05 & 6.8165e-08 \\
0 & 2.7131e-04 & 0 & 4.8193e-14 \\
6.9530e-04 & 0 & 5.3999e-05 & 2.0183e-08 \\
\end{array}
\]
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**Columns 1 through 6**

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**Columns 7 through 12**

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**Columns 13 through 18**

-2.0912e-09  8.1545e-07  2.4413e-21  4.7271e-10  -2.0912e-10  -1.5400e-12
-1.1245e-10  -1.2068e-17  -2.2831e-07  7.9143e-19  -1.1245e-11  1.8476e-19
-4.4981e-08  -4.8345e-16  -9.1324e-05  3.1659e-16  -4.4981e-09  7.3890e-17
-2.5814e-06  8.5804e-08  -9.6443e-20  5.5306e-06  -2.5814e-07  -3.7356e-10

**Columns 19 through 20**

8.3168e-23  2.9431e-13
1.9568e-12  9.6551e-20
-3.6514e-23  2.4101e-13
3.7392e-20  1.1772e-10
7.8270e-10  3.8536e-17
-1.9522e-20  9.6402e-11

Cbar =

| 0  | -7.5695e-22 | -3.9064e-09 | 8.1916e-01 | 4.6131e-20 | 5.7358e-01 |
| 0  | -1.2932e-10 | -5.0523e-21 | -1.2907e-22 | 1.0000e+00 | 7.1273e-22 |
| 0  | -5.1319e-22 | -1.6824e-09 | 5.7358e-01 | 2.6805e-20 | -8.1916e-01 |

Ebar =

**Columns 1 through 6**

2.2641e-06  -5.6715e-18  9.1124e-03  -4.7985e-03  -1.1732e-16  -6.4482e-05
-1.2578e-21  -1.1253e-05  -2.6526e-16  -2.4112e-16  -1.0299e-02  1.0541e-18

**Columns 7 through 12**

2.9800e-11  -1.6253e-18  3.6732e-16  -2.8191e-10  1.8474e-18  -1.0957e-08
1.5797e-19  3.8688e-14  -1.7432e-18  1.7363e-20  -3.0366e-09  -9.2312e-20
-4.2525e-11  -1.4648e-18  3.8356e-16  3.2658e-10  1.1265e-18  -5.7007e-09

**Columns 13 through 18**

-1.1627e-11  -1.6633e-09  -1.2911e-18  1.2965e-10  -1.1627e-12  7.0720e-03
-1.1391e-16  8.7496e-19  -1.5183e-10  -1.1656e-18  -1.1505e-17  -1.2395e-18
Columns 19 through 20

4.4592e-19  1.2259e-11
7.0720e-03  -2.1741e-18
-8.6827e-19  7.0720e-03

Robust filter 3
kappa_min = 0.083
gamma = Inf

\[ a = \]
\[
\begin{pmatrix}
0 & 0 & 1.4510e-08 & 2.0977e-10 & 0 & 2.4184e-11 \\
0 & 4.5527e-10 & 0 & 0 & 1.0714e-10 & 0 \\
0 & 0 & 5.4224e-08 & 5.3408e-11 & 0 & 8.6530e-11 \\
0 & 0 & 5.8040e-06 & 8.3904e-08 & 0 & 1.4511e-08 \\
0 & 1.8212e-07 & 0 & 0 & 4.2702e-08 & 0 \\
0 & 0 & 2.1689e-05 & 2.1363e-08 & 0 & 1.6536e-08
\end{pmatrix}
\]

\[ b = \]
\[
\begin{pmatrix}
4.5646e-08 & 0 & 1.8340e-08 & 1.7041e-11 \\
0 & 6.7828e-08 & 0 & 8.0464e-18 \\
1.7383e-07 & 0 & 1.3500e-08 & 5.0458e-12 \\
1.8259e-05 & 0 & 7.3360e-06 & 6.8165e-09 \\
0 & 2.7131e-05 & 0 & 4.8193e-15 \\
6.9530e-05 & 0 & 5.3999e-06 & 2.0183e-09
\end{pmatrix}
\]

\[ c = \]
\[
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0 & 3.3914e-09 & 0 & 8.0464e-19 & 0 & 0 & 0 \\
1.8951e-14 & 0 & 6.7499e-10 & 5.0458e-13 & 0 & 0 & 0 \\
9.1287e-07 & 0 & 3.8514e-12 & 6.8165e-10 & 0 & 0 & 0 \\
0 & 1.3565e-06 & 0 & 4.8193e-16 & 0 & 0 & 0 \\
1.1371e-11 & 0 & 2.6999e-07 & 2.0183e-10 & 0 & 0 & 0
\end{pmatrix}
\]

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0 & 0 & 0 & 0 & 1.0000e-04 & 0 \\
0 & 0 & 0 & 5.7358e-05 & 0 & -8.1915e-05
\end{pmatrix}
\]

\[ Abar = \]
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Bbar =

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-1.7321e-08 & 6.1292e-18 & 8.2493e-09 & 2.0989e-05 & 2.5677e-19 & -1.5685e-06 \\
-8.9278e-06 & -5.7670e-16 & -4.9949e-07 & 2.2728e-03 & -2.3374e-16 & 2.3176e-03 \\
7.6172e-17 & -4.1836e-06 & 3.4332e-15 & 1.8366e-16 & -5.2120e-03 & -9.9761e-17 \\
-6.5014e-06 & 1.2728e-15 & 1.7009e-06 & 8.3955e-03 & -1.2572e-17 & -6.2739e-04 \\
\end{tabular}
\end{verbatim}

Columns 7 through 12

\begin{verbatim}
\begin{tabular}{cccccc}
2.7303e-17 & -3.6051e-07 & 2.2987e-14 & -2.4450e-17 & -4.5612e-06 & -5.2015e-17 \\
9.5451e-09 & 1.7080e-17 & 2.0258e-10 & 4.3458e-03 & -2.8608e-17 & 1.1013e-04 \\
\end{tabular}
\end{verbatim}

Columns 13 through 18

\begin{verbatim}
\begin{tabular}{cccccc}
-6.4550e-09 & 1.3967e-10 & -4.9807e-21 & 1.3713e-08 & -6.4550e-10 & -5.2858e-12 \\
-2.5820e-06 & 5.6496e-08 & -1.7203e-18 & 5.4851e-06 & -2.5820e-07 & -2.1143e-09 \\
\end{tabular}
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Columns 19 through 20

\begin{verbatim}
\begin{tabular}{cccccc}
-8.3968e-21 & -3.6312e-10 & 1.4197e-09 & 3.6646e-17 & 1.0432e-20 & -2.2361e-10 \\
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\end{verbatim}

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**Robust filter 4**

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gamma = Inf

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\[ dG = \]
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0 & 3.3914e-10 & 0 & 8.0464e-20 & 0 & 0 \\
1.8951e-15 & 0 & 6.7499e-11 & 5.0458e-14 & 0 & 0 \\
9.1287e-08 & 0 & 3.8514e-13 & 6.8165e-11 & 0 & 0 \\
0 & 1.3565e-07 & 0 & 4.8193e-17 & 0 & 0 \\
1.1371e-12 & 0 & 2.6999e-08 & 2.0183e-11 & 0 & 0
\end{array}
\]

\[ dC = \]
\[
\begin{array}{cccccc}
0 & 0 & 0 & 8.1915e-05 & 0 & 5.7358e-05 \\
0 & 0 & 0 & 0 & 1.0000e-04 & 0 \\
0 & 0 & 0 & 5.7358e-05 & 0 & -8.1915e-05
\end{array}
\]

\[ dE = \]
\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 3.5355e-05 & 0 \\
0 & 0 & 0 & 0 & 0 & 3.5355e-05 \\
0 & 0 & 0 & 0 & 0 & 3.5355e-05
\end{array}
\]

\[ Abar = \]
\[
\begin{array}{cccccc}
1.0000e+00 & -1.0183e-18 & -4.2800e-05 & 4.9999e-03 & 5.8065e-17 & -2.6193e-08 \\
0 & 1.0000e+00 & -3.1132e-17 & 5.7052e-17 & 4.9998e-03 & 1.7760e-16 \\
0 & -3.1355e-18 & 9.9998e-01 & 7.3395e-08 & 1.7888e-16 & 4.9998e-03 \\
0 & -4.0729e-16 & -1.7120e-02 & 9.9996e-01 & 2.3233e-14 & -2.4741e-05 \\
0 & -1.5933e-03 & -1.2452e-14 & 2.2820e-14 & 9.9991e-01 & 7.1040e-14 \\
\end{array}
\]

\[ Bbar = \]

Columns 1 through 6

\[-1.9256e-09 & -6.5891e-17 & -1.8521e-10 & -1.7968e-06 & -2.0655e-19 & -1.8323e-06 \\
2.9172e-15 & -3.9117e-07 & 1.0886e-13 & -1.1293e-16 & -1.6482e-03 & 3.0054e-17 \\
\]

Columns 7 through 12
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**Ebar**

**Columns 1 through 6**

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In Section 5.3, a robust estimator is designed to estimate the multiple jet thrust state. Its design matrices are given as follows:

Robust jet thrust estimator

\[
\begin{align*}
\kappa_{\text{min}} &= 0.0440 \\
\gamma &= \infty
\end{align*}
\]

\[
\begin{align*}
A & = \\
&= \begin{bmatrix}
1.0000e+00 & -6.0657e-05 & 4.9998e-03 & -2.2796e-08 & -4.7956e-09 \\
0 & 9.9997e-01 & -2.3404e-08 & 4.9998e-03 & -8.2335e-09 \\
0 & -2.4263e-02 & 9.9991e-01 & -2.3384e-05 & -1.9182e-06 \\
0 & -1.3380e-02 & -9.3613e-06 & 9.9992e-01 & -3.2934e-06 \\
0 & 4.0822e-13 & 8.5864e-16 & 1.2577e-15 & 1.0000e+00
\end{bmatrix}
\end{align*}
\]

\[
B = \\
= \begin{bmatrix}
1.5512e-06 & -6.2328e-06 & 1.0876e-06 & -7.6027e-08 & 1.0208e-05 & -2.1374e-06 \\
1.2645e-05 & 7.4668e-07 & 4.0496e-06 & -7.4888e-08 & 3.8146e-05 & 5.7198e-07 \\
5.0581e-03 & 3.0584e-04 & -3.2495e-06 & 2.2444e-06 & -9.5363e-08 & -1.4299e-09 \\
\end{bmatrix}
\]

Columns 1 through 6

\[
\begin{align*}
&= \begin{bmatrix}
1.5512e-06 & -6.2328e-06 & 1.0876e-06 & -7.6027e-08 & 1.0208e-05 & -2.1374e-06 \\
1.2645e-05 & 7.4668e-07 & 4.0496e-06 & -7.4888e-08 & 3.8146e-05 & 5.7198e-07 \\
5.0581e-03 & 3.0584e-04 & -3.2495e-06 & 2.2444e-06 & -9.5363e-08 & -1.4299e-09 \\
\end{bmatrix}
\end{align*}
\]

Columns 7 through 12

\[
\begin{align*}
&= \begin{bmatrix}
7.5507e-08 & -5.6795e-21 & 9.9595e-07 & 1.6826e-05 & 1.6081e-21 & 5.2684e-07 \\
2.4913e-03 & -2.3231e-19 & -1.0595e-05 & 1.0167e-02 & -3.0734e-19 & 3.0419e-04 \\
\end{bmatrix}
\end{align*}
\]

Columns 13 through 14

\[
\begin{align*}
&= \begin{bmatrix}
7.5507e-08 & -5.6795e-21 & 9.9595e-07 & 1.6826e-05 & 1.6081e-21 & 5.2684e-07 \\
2.4913e-03 & -2.3231e-19 & -1.0595e-05 & 1.0167e-02 & -3.0734e-19 & 3.0419e-04 \\
\end{bmatrix}
\end{align*}
\]

143
\begin{verbatim}
1.5691e-09  1.8269e-09
1.0438e-09  1.2007e-09
6.3564e-07  7.3073e-07
4.1753e-07  4.8028e-07
-1.7839e-17 -2.2064e-17

Cbar =
0  -3.1357e-07  8.1925e-01  5.7365e-01  -3.3339e-10
0  -2.3829e-07  5.7365e-01  -8.1925e-01  -3.5517e-10

Ear =

Columns 1 through 6

-2.5840e-05  3.6800e-05  -2.2308e-02  -1.4250e-03  9.3915e-10  -4.6235e-14
-4.5597e-06  3.8781e-05  2.1376e-02  -1.4872e-03  -1.3424e-09  3.4133e-14

Columns 7 through 12

1.7750e-08  3.0417e-19  -1.2619e-09  6.7993e-07  2.3282e-18  1.1745e-07
1.9033e-08  3.8939e-21  -1.5609e-09  7.8248e-07  -2.6817e-18  1.2178e-07

Columns 13 through 14

7.0927e-03  -3.2706e-10
-3.2706e-10  7.0927e-03
\end{verbatim}
Appendix B

Reentry Model Parameters

The forces and moments acting on the space shuttle Orbiter, like any other aircraft, are defined in terms of dimensionless aerodynamic coefficients [24]. We have the following:

\[ \text{drag, } D = \bar{q}SC_D \]  
(B.1)

\[ \text{lift, } L = \bar{q}SC_L \]  
(B.2)

\[ \text{sideforce, } Y = \bar{q}SC_Y \]  
(B.3)

\[ \text{rolling moment, } L = \bar{q}ScC_l \]  
(B.4)

\[ \text{pitching moment, } M = \bar{q}ScC_M \]  
(B.5)

\[ \text{yawing moment, } N = \bar{q}ScC_N \]  
(B.6)

where we had previously defined in Chapter 3, the following:

\[ \bar{q} = \text{dynamic pressure}(lb/ft^2) \]

\[ S = \text{wing reference area (ft}^2) \]

\[ c = \text{reference chord (ft)} \]

\[ b = \text{wingspan (ft)} \]
The dimensionless stability and control derivatives relevant to this thesis are divided between the longitudinal and lateral dynamics.

**Longitudinal dimensionless derivatives**

\[
C_{D_{e\alpha}} = \frac{\partial C_R}{\partial e_l} \quad C_{M_{\alpha}} = \frac{\partial C_M}{\partial \alpha} \\
C_{L_{\alpha}} = \frac{\partial C_L}{\partial \alpha} \quad C_{M_{\eta}} = \frac{2V_T}{c} \cdot \frac{\partial C_M}{\partial q} \\
C_{L_{s e}} = \frac{\partial C_L}{\partial e_l} \quad C_{M_{s e}} = \frac{\partial C_M}{\partial e_l}
\]

**Lateral dimensionless derivatives**

\[
C_{Y_{\beta}} = \frac{\partial C_Y}{\partial \beta} \quad C_{l_{\beta}} = \frac{\partial C_l}{\partial \beta} \\
C_{n_{\beta}} = \frac{\partial C_n}{\partial \beta} \quad C_{l_{\phi}} = \frac{\partial C_l}{\partial \phi} \\
C_{n_{s a}} = \frac{\partial C_n}{\partial a_i} \
C_{l_{s a}} = \frac{\partial C_l}{\partial a_i} \\
C_{n_{s r}} = \frac{\partial C_n}{\partial r} \frac{\partial r}{\partial r} \quad C_{l_{s r}} = \frac{\partial C_l}{\partial r} \frac{\partial r}{\partial r}
\]

<table>
<thead>
<tr>
<th>m (slugs)</th>
<th>I_{xx} (sl·ft²)</th>
<th>I_{yy} (sl·ft²)</th>
<th>I_{zz} (sl·ft²)</th>
<th>I_{xz} (sl·ft²)</th>
<th>S (ft²)</th>
<th>c (ft)</th>
<th>b (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5640</td>
<td>0.805e6</td>
<td>5.85e6</td>
<td>6.07e6</td>
<td>0.14e6</td>
<td>2690</td>
<td>39.6</td>
<td>78.06</td>
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</table>

Table B.1: Vehicle Constants
<table>
<thead>
<tr>
<th>Mach Number</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
<th>5.0</th>
<th>5.7</th>
<th>7.4</th>
<th>8.8</th>
<th>10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Altitude (ft)</td>
<td>70000</td>
<td>90000</td>
<td>105000</td>
<td>120000</td>
<td>130000</td>
<td>150000</td>
<td>158000</td>
<td>170000</td>
</tr>
<tr>
<td>$\alpha$ (deg)</td>
<td>12.5</td>
<td>16</td>
<td>20.5</td>
<td>25</td>
<td>29</td>
<td>35</td>
<td>38</td>
<td>40</td>
</tr>
<tr>
<td>$V_T$ (ft/sec)</td>
<td>1950</td>
<td>3000</td>
<td>4000</td>
<td>5100</td>
<td>5900</td>
<td>8300</td>
<td>9500</td>
<td>109000</td>
</tr>
<tr>
<td>$q$ (lb/ft$^2$)</td>
<td>200</td>
<td>215</td>
<td>200</td>
<td>177</td>
<td>150</td>
<td>125</td>
<td>118</td>
<td>110</td>
</tr>
<tr>
<td>$\phi_1/\phi_{\infty}$</td>
<td>7.72e-4</td>
<td>7.18e-4</td>
<td>7.72e-4</td>
<td>8.72e-4</td>
<td>1.05e-3</td>
<td>1.23e-4</td>
<td>1.31e-3</td>
<td>1.47e-3</td>
</tr>
<tr>
<td>$C_{M_{\alpha}}$</td>
<td>-0.17</td>
<td>-0.1</td>
<td>-0.11</td>
<td>-0.12</td>
<td>-0.135</td>
<td>-0.16</td>
<td>-0.22</td>
<td>-0.27</td>
</tr>
<tr>
<td>$C_{M_\delta}$</td>
<td>-0.16</td>
<td>-0.05</td>
<td>-0.05</td>
<td>-0.075</td>
<td>-0.1</td>
<td>-0.14</td>
<td>-0.15</td>
<td>-0.175</td>
</tr>
<tr>
<td>$C_D$</td>
<td>0.23</td>
<td>0.22</td>
<td>0.3</td>
<td>0.35</td>
<td>0.48</td>
<td>0.65</td>
<td>0.8</td>
<td>0.84</td>
</tr>
<tr>
<td>$C_{L_{\alpha}}$</td>
<td>1.8</td>
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<td>1.63</td>
<td>1.6</td>
<td>1.6</td>
<td>1.4</td>
<td>1.0</td>
<td>1.19</td>
</tr>
<tr>
<td>$C_{L_{\delta}}$</td>
<td>-1.1</td>
<td>-0.9</td>
<td>-1.1</td>
<td>-1.1</td>
<td>-1.2</td>
<td>-1.3</td>
<td>-1.5</td>
<td>-1.6</td>
</tr>
<tr>
<td>$C_L$</td>
<td>0.4</td>
<td>0.38</td>
<td>0.5</td>
<td>0.57</td>
<td>0.7</td>
<td>0.8</td>
<td>0.87</td>
<td>0.86</td>
</tr>
<tr>
<td>$C_{D_{1+\alpha}}$</td>
<td>0.04</td>
<td>0.04</td>
<td>0.045</td>
<td>0.064</td>
<td>0.08</td>
<td>0.11</td>
<td>0.14</td>
<td>0.146</td>
</tr>
<tr>
<td>$C_{L_{1+\alpha}}$</td>
<td>0.2</td>
<td>0.16</td>
<td>0.135</td>
<td>0.12</td>
<td>0.13</td>
<td>0.15</td>
<td>0.15</td>
<td>0.17</td>
</tr>
<tr>
<td>$C_{n_{\delta}}$</td>
<td>0.2</td>
<td>0.1</td>
<td>0.17</td>
<td>0.18</td>
<td>0.28</td>
<td>0.4</td>
<td>0.53</td>
<td>0.59</td>
</tr>
<tr>
<td>$C_{Y_{\delta}}$</td>
<td>-1</td>
<td>-0.77</td>
<td>-0.6</td>
<td>-0.75</td>
<td>-0.75</td>
<td>-0.5</td>
<td>-0.005</td>
<td>-0.35</td>
</tr>
<tr>
<td>$C_{n_{\delta_{\alpha}}}$</td>
<td>-0.07</td>
<td>-0.05</td>
<td>-0.037</td>
<td>-0.013</td>
<td>-0.005</td>
<td>-0.005</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$C_{n_{\delta}}$</td>
<td>-0.07</td>
<td>-0.09</td>
<td>-0.09</td>
<td>-0.15</td>
<td>-0.17</td>
<td>-0.2</td>
<td>-0.28</td>
<td>-0.35</td>
</tr>
<tr>
<td>$C_{l_{\delta}}$</td>
<td>-0.08</td>
<td>-0.06</td>
<td>-0.06</td>
<td>-0.08</td>
<td>-0.094</td>
<td>-0.105</td>
<td>-0.115</td>
<td>-0.12</td>
</tr>
<tr>
<td>$C_{L_{\alpha}}$</td>
<td>0.05</td>
<td>0.035</td>
<td>0.035</td>
<td>0.037</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.05</td>
</tr>
<tr>
<td>$C_{l_{\alpha}}$</td>
<td>0.02</td>
<td>0.015</td>
<td>0.009</td>
<td>0.005</td>
<td>0.005</td>
<td>0.002</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$C_{2\phi}$</td>
<td>-0.09</td>
<td>-0.07</td>
<td>-0.058</td>
<td>-0.06</td>
<td>-0.067</td>
<td>-0.07</td>
<td>-0.072</td>
<td>-0.073</td>
</tr>
</tbody>
</table>

Table B.2: Aerodynamic Stability and Control Derivatives
Appendix C

Incorporating Parametric Uncertainty

This appendix explains how to fit parametric uncertainties into the state-space formulation of Section 4.3.1. We begin by considering a state-space plant with parameter errors [14]:

\[
\begin{align*}
    x_{k+1} &= \left( A_k + \sum_{j=1}^{l} \Delta A_{kj} \delta_j \right) x_k + \left( G_k + \sum_{j=1}^{l} \Delta G_{kj} \delta_j \right) d_k \quad (C.1a) \\
    y_k &= \left( C_k + \sum_{j=1}^{l} \Delta C_{kj} \delta_j \right) x_k + \left( E_k + \sum_{j=1}^{l} \Delta E_{kj} \delta_j \right) d_k \quad (C.1b)
\end{align*}
\]

Each \( \delta_j \) represents a parameter error that is normalized as

\[
-1 < \delta_j < 1 \quad \forall j = 1, \ldots, l \quad (C.2)
\]

For each pair \((k, j)\), we may define the matrix \( N_{kj} \), \( j = 1, \ldots, l \), by

\[
N_{kj} = \begin{bmatrix}
    \Delta A_{kj} & \Delta G_{kj} \\
    \Delta C_{kj} & \Delta E_{kj}
\end{bmatrix} \in \mathbb{R}^{(n_x+n_y) \times (n_x+n_d)}
\]

where \( n_x, n_y, \) and \( n_d \) are the dimensions of the vectors \( x_k, y_k, \) and \( d_k \), respectively. Generally, this matrix will not be of full rank since one parameter rarely
affects all of the states and outputs. Therefore, using a singular value decomposition, $N_{kj}$ may be decomposed as follows:

$$
N_{kj} = \begin{bmatrix} Q_{kj} \\ R_{kj} \end{bmatrix} \begin{bmatrix} S_{kj} & L_{kj} \end{bmatrix}
$$

(C.3)

where $Q_{kj} \in \mathcal{R}^{n_x \times n_{kj}}$, $R_{kj} \in \mathcal{R}^{n_y \times n_{kj}}$, $S_{kj} \in \mathcal{R}^{n_{kj} \times n_x}$, $L_{kj} \in \mathcal{R}^{n_{kj} \times n_y}$, and $n_{kj}$ is the rank of the matrix $N_{kj}$. Combining Equations (C.1a) and (C.1b) with Equation (C.3), and recalling that $\xi_k = [\eta_k^T \ d_k^T]^T$, the state-space model of the perturbed system may be rewritten as follows:

$$
x_{k+1} = \left[ A_k + \sum_{j=1}^{l} Q_{kj} \delta_j I_{n_{kj}} S_{kj} \right] x_k + \left[ G_k + \sum_{j=1}^{l} Q_{kj} \delta_j I_{n_{kj}} L_{kj} \right] d_k
$$

$$
= A_k x_k + [Q_{k1} \ldots Q_{kl}] \begin{bmatrix} \eta_{k1} \\ \vdots \\ \eta_{kl} \end{bmatrix} + G_k d_k
$$

$$
= A_k x_k + Q_k \xi_k + G_k d_k
$$

$$
= A_k x_k + Q_k \xi_k \quad \text{with} \quad Q_k = [Q_k \ G_k]
$$

(C.4a)

$$
y_k = \left[ C_k + \sum_{j=1}^{l} R_{kj} \delta_j I_{n_{kj}} S_{kj} \right] x_k + \left[ E_k + \sum_{j=1}^{l} R_{kj} \delta_j I_{n_{kj}} L_{kj} \right] d_k
$$

$$
= C_k x_k + [R_{k1} \ldots R_{kl}] \begin{bmatrix} \eta_{k1} \\ \vdots \\ \eta_{kl} \end{bmatrix} + E_k d_k
$$

$$
= C_k x_k + R_k \eta_k + E_k d_k
$$

$$
= C_k x_k + E_k \xi_k \quad \text{with} \quad E_k = [R_k \ E_k]
$$

(C.4b)

$$
\epsilon_k = \begin{bmatrix} \epsilon_{k1} \\ \vdots \\ \epsilon_{kl} \end{bmatrix}
$$
\[
\begin{align*}
&= \begin{bmatrix} S_{k1} \\ \vdots \\ S_{kl} \end{bmatrix} x_k + 0\eta_k + \begin{bmatrix} L_{k1} \\ \vdots \\ L_{kl} \end{bmatrix} d_k \\
&= S_k x_k + T_k \xi_k \\
\end{align*}
\]
(C.4c)

where \( T_k = \begin{bmatrix} 0 & L_{k1} \\ \vdots & \vdots \\ 0 & L_{kl} \end{bmatrix} \)

We have created some sequence \( \epsilon_k \) and \( \eta_k \) as well as matrices \( S_k, T_k, G_k, \) and \( E_k \) to fit into the state-space description of Equation (4.73).

The relationship between \( \eta_k \) and \( \epsilon_k \) is therefore:

\[
\eta = \begin{bmatrix} \eta_{k1} \\ \vdots \\ \eta_{kl} \end{bmatrix} = \begin{bmatrix} \delta_1 & \cdots & \delta_l \\ \vdots & \ddots & \vdots \\ \delta_l & \cdots & \delta_l \end{bmatrix} \begin{bmatrix} \epsilon_{k1} \\ \vdots \\ \epsilon_{kl} \end{bmatrix} = \Delta \epsilon
\]
(C.5)
Appendix D

The Small Gain Theorem

The small gain theorem is useful in stability and performance robustness analysis of control systems with modeling uncertainty. Stability can be guaranteed if the loop gains are less than unity. Performance levels, as measured by the gain from disturbances to error, can be guaranteed by adding a fictitious uncertainty element that feeds back from the output error to the input. As a result, robust performance is guaranteed if this augmented system with the fictitious perturbation is shown robustly stable using the small gain test.

Since robust closed-loop stability is essentially independent of estimator designs, performance robustness is the main focus of estimator design with modeling uncertainty. The estimator's performance is measured by the largest gain in the closed-loop transfer function between the process and measurement noises and the estimation error over the range of plant model uncertainty.

Most feedback systems can be represented as shown in Figure D-1 where $u_1$ and $u_2$ are exogenous inputs, and $e_1$ and $e_2$ are error signals [1]. This system is bounded-input, bounded-output stable if $e_1$ and $e_2$ are bounded given bounded inputs $u_1$ and $u_2$. The errors can be found from

$$ e_1 = u_1 + G_2(u_2 + G_1e_1) $$

$$ = (I - G_2G_1)^{-1}(u_1 + G_2u_2) $$

(D.1)
\[ e_2 = u_2 + G_1(u_1 + G_2 e_2) \]
\[ = (I - G_1 G_2)^{-1}(G_1 u_1 + u_2) \quad \text{(D.2)} \]

The norms of the errors are given by

\[ \|e_1\| = \|(I - G_2 G_1)^{-1}(u_1 + G_2 u_2)\| \]
\[ \leq \|(I - G_2 G_1)^{-1}\| \left(\|u_1\| + \|G_2\| \|u_2\|\right) \quad \text{(D.3)} \]

\[ \|e_2\| = \|(I - G_1 G_2)^{-1}(G_1 u_1 + u_2)\| \]
\[ \leq \|(I - G_1 G_2)^{-1}\| \left(\|G_1\| \|u_1\| + \|u_2\|\right) \quad \text{(D.4)} \]

where the norms of the transfer functions are induced norms defined as

\[ \|G\| \equiv \max_{u \neq 0} \frac{\|y\|}{\|u\|} \quad \text{for the system} \quad y = Gu \quad \text{(D.5)} \]

Assuming \( G_1 \) and \( G_2 \) are stable, i.e. have bounded norms, then necessary and sufficient conditions for bounded-input, bounded-output stability are

\[ \|(I - G_1 G_2)^{-1}\| < \infty \quad \text{and} \quad \|(I - G_2 G_1)^{-1}\| < \infty \quad \text{(D.6)} \]
These conditions are satisfied if

$$\|G_1 G_2\| < 1 \quad \text{and} \quad \|G_2 G_1\| < 1 \quad (D.7)$$

For induced norms, the inequalities

$$\|G_1 G_2\| \leq \|G_1\| \|G_2\| \quad \text{and} \quad \|G_2 G_1\| \leq \|G_1\| \|G_2\| \quad (D.8)$$

can be used. Therefore, a sufficient condition for closed-loop stability is

$$\|G_1\| \|G_2\| < 1 \quad (D.9)$$

This is the small gain test applied to bounded operators. For linear systems, bounded-input, bounded-output stability for a particular choice of norm implies bounded-input, bounded-output stability for all other norms and also implies exponential stability. Therefore, any induced norm may be used with the small gain theorem to guarantee stability of a linear system with linear perturbations.

It was shown in Section 4.1.1, that the steady-state Kalman filter is the $H_2$ optimal filter. Unfortunately, since the $H_2$ norm is defined directly from a transfer function, it is not induced from the input-output norms and cannot be used with the small gain theorem. The $H_\infty$ norm, on the other hand, is an induced norm and may be used as an alternative. This is the main reason why the $H_\infty$ norm is used for robust control design. The $H_\infty$ norm is defined by the following:

$$\|G(s)\|_{H_\infty} \equiv \sup_{\omega} \sigma_{\text{max}} (G(j\omega)) = \sup_{\omega \neq 0} \frac{\|y\|_2}{\|u\|_2} \quad (D.10)$$

Therefore, the $H_\infty$ norm is the induced 2-norm.
Bibliography


