Multiperiod Portfolio Optimization in the Presence of Transaction Costs

by

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Abstract

The main focus of the finance literature on portfolio optimization models has been on (a) single period, mean-variance (quadratic utility) models with multiple assets in the 1950s and 1960s, (b) multiperiod models with multiple assets, geometric Brownian motion price dynamics, exponential and power utilities and no transaction costs in the 1970s, and (c) multiperiod models with two assets, geometric Brownian motion price dynamics, power utility and proportional transaction costs in the 1980s and early 1990s. A series of studies in the 1980s produced evidence of predictability in asset returns. In both empirical and theoretical studies multifactor autocorrelated pricing and stochastic volatility models have been proposed. In addition, with the significant growth of the mutual fund industry, large mutual funds can generate trades that can affect the price of the underlying assets. Such price impact effect is best captured by transaction costs that are quadratic, as opposed to linear, functions of the size of the trade.

We study in this thesis multiperiod, discrete time portfolio optimization problems under (a) quadratic transaction costs that model price impact effects, (b) quadratic and exponential utility functions, and (c) multifactor autocorrelated pricing and stochastic volatility models. Using stochastic dynamic programming, we find the optimal investment policy over time in closed form for the case of multiple assets, no transaction costs, exponential utility and multifactor autocorrelated pricing models. We also investigate qualitative properties of the optimal portfolio composition.

In the presence of transaction costs and multiple assets closed form solutions are not achievable. We develop a new approximate dynamic programming (ADP) methodology to find near optimal policies for such high-dimensional problems. We propose several algorithms and compare their performance. In problems of small dimension, where exact dynamic programming is feasible, our approximation produces near optimal solutions. We use our ADP algorithms to understand the qualitative behavior of the optimal investment policy: we examine the effect of transaction costs, time horizon, asset correlations and volatilities on the portfolio composition over time.

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To my parents
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Chapter 1

Introduction

The core of finance theory is the study of the behavior of economic agents in allocating and deploying their resources among assets and across time in an uncertain environment. Time and uncertainty are the central elements that influence the portfolio management process. The complexity of their interaction brings intrinsic excitement to the study of multiperiod portfolio optimization models as it often requires sophisticated analytical tools to capture the effect of this interaction.

Most of the portfolio optimization models that have been used in practice are variants of the mean-variance model that was developed by Markowitz [58]. This theory of portfolio selection provided a tractable model for quantifying the risk-return trade-off for general assets with correlated returns. Building on Markowitz's fundamental work, Sharpe [72] and Litner [53] investigated the equilibrium structure of asset prices and their Capital Asset Pricing Model (CAPM) became the foundational quantitative model for measuring the risk of a security. The CAPM also formed the basis for developing an entire industry to measure the investment performance of professional money managers. Optimal portfolio decisions that follow from a static model may be significantly more expensive than the ones that follow from a model that explicitly includes the possibility for adjustment of the portfolio composition in the future. We therefore consider dynamic portfolio optimization models in this thesis.

In the financial economics literature, continuous-time models for optimum consumption and portfolio selection have been studied. Merton [59], [60], [61] has shown that the continuous time
formulation of portfolio theory provides a powerful analytical framework for extending the standard results of one-period mean-variance portfolio theory to the dynamical case. These models consider an investor who maximizes his expected utility of intertemporal consumption and final wealth. Given some initial level of wealth, the problem is to simultaneously determine an optimal consumption pattern and investment strategy over time as a function of the state of the world. The analytic solution of these optimization models by stochastic dynamic programming necessitates idealized assumptions about the preferences and the behavior of the investor, the structure of asset prices and the functioning of financial markets. For example, it only allows a Wiener Brownian-motion process for the return dynamics, and it does not permit the inclusion of market frictions such as transaction costs, taxes and position limits. Besides, most investors do not want to monitor and revise their portfolio in a continuous matter, but rather only at fixed points in time. Therefore we assume that trading takes place only at discrete intervals, and thus we consider dynamic models in discrete time.

In this thesis, we study the effect of transaction costs on dynamic portfolio strategies in discrete time. We investigate the impact of investor's preferences, payoff functions and asset returns' predictability to the optimal investment decisions. We also answer the question of how the investment horizon influences the portfolio composition under varying assumptions about the asset return dynamics. On the methodological side, we develop effective iterative solution algorithms that approximate the optimal dynamic trading strategy whenever a closed-form solution to the investor's optimization problem is unattainable. We show that in-depth investigation of the underlying dynamic optimization problem at every iteration of the algorithm enables us to capture essential characteristics of the optimal investment policy and level of utility. We also propose and evaluate alternative dynamic trading strategies that arise from the solution of a series of stochastic optimization problems with different initial conditions and successively smaller investment horizon. In the case of complicated return dynamics we extend this idea and present it as a systematic approach to deriving approximate trading policies. One of the clear advantages of this policy-approximation procedure is that it can easily accommodate nonnegativity and any desirable budget constraints.
1.1 Models of Asset Returns

This section provides an overview of time-series models of asset returns. The emphasis is on the models appearing in the portfolio management problems proposed in the literature and used in this thesis.

1.1.1 Multifactor Pricing Models

One of the most important factors that influences the dynamic trading policies followed by an investor is the asset return process. The assumption of independent and identically distributed (IID) returns, which has been used extensively in the pre 1980's literature, has been questioned by a series of studies in the 1980's that produced evidence of predictability in asset returns.¹ The fine structure of securities markets and frictions in the trading process can generate predictability. Time-varying expected returns due to changing business conditions can also generate predictability. Poterba and Summers [66], and Fama and French [31] demonstrate empirically that over long horizons significant mean reversion in annual stock prices occurs, implying negatively autocorrelated stock returns. Using weekly returns, Lo and MacKinlay [54] document the opposite, that stock returns are positively autocorrelated. The volatility tests of Schiller [73] and LeRoy & Porter [52] showed that volatility in dividends was not sufficient to explain the variation in prices which pointed to time-varying expected returns as a way of accounting for the residual variation. Several further papers provide regression evidence for this time-variation ([15], [29], [30], [31], [49].)

Empirical evidence also indicates that the CAPM beta does not completely explain the cross section of expected asset returns. This evidence suggests that one or more additional factors may be required to characterize the behavior of expected returns and naturally leads to consideration of multifactor pricing models. Theoretical arguments also suggest that more than one factor is required, since only under strong assumptions will the CAPM apply period by period. The Arbitrage Pricing Theory (APT) was introduced by Ross [67] as an alternative to the Capital Asset Pricing Model (CAPM). The APT can be more general than the CAPM in that

¹See for example, Bekaert and Hodrick [4], Bessembinder and Chan [10], Engle, Lilien and Robbins [28], Ferson [32], Ferson, Kandel and Stambaugh [33], Gibbons and Ferson [36], Harvey [41]. For a comprehensive review and more references, see Cambell, Lo and MacKinlay [16].
it allows for multiple risk factors and does not require the identification of the market portfolio, but it provides an approximate relation for expected asset returns with an unknown number of unidentified factors. Exact factor pricing can also be derived in an intertemporal asset pricing framework. The Intertemporal Capital Asset Pricing Model, developed by Merton [61], combined with assumptions on the conditional distribution of returns delivers a multifactor model in which the market portfolio serves as one factor and state variables serve as additional factors that arise from investors' demand to hedge uncertainty about future investment opportunities.

In this thesis, we explore the impact of asset return predictability to the dynamic investment decisions and we consider the following return generating process for asset returns $r_t$:

\begin{align}
    r_t &= c_t + A_t f_t + \epsilon_t, \\
    f_t &= d_{t-1} + B_{t-1} f_{t-1} + \eta_t,
\end{align}

where $K$ is the total number of factors, $N$ is the total number of assets, $f_t$ is the $K \times 1$ vector of the factor realizations at time $t$, $A_t$ is the $N \times K$ matrix of the factor sensitivities, $B_{t-1}$ is the $K \times K$ symmetric matrix of the factor correlations, $c_t$ and $d_{t-1}$ are $N \times 1$ and $K \times 1$ vectors of constants respectively, and $\epsilon_t, \eta_t$ are uncorrelated normally distributed random vectors with mean zero and covariance matrices $\Sigma_{\epsilon}$ and $\Sigma_{\eta}$ respectively.

### 1.1.2 Stochastic Volatility Models

Recent advances in dynamical systems theory, nonlinear time-series analysis, stochastic-volatility models and nonparametric statistics have sparked great interest in nonlinearities in financial data. For a comprehensive review on the nonlinearities in financial data see Cambell, Lo and MacKinlay [16]. In this thesis, we explore the application of models that are nonlinear in the variance, concentrating on univariate Generalized Autoregressive Conditionally Heteroskedastic (GARCH) models.

A basic observation about asset return data is that large returns (of either sign) tend to be followed by more large returns (of either sign). In other words, the volatility of asset returns appears to be serially correlated. To capture the serial correlation of volatility, Engle [28] proposed the class of Autoregressive Conditionally Heteroskedastic, or ARCH, models. These
write conditional variance as a distributed lag of past squared innovations in an asset return 
\[ \eta_t^2 \equiv (r_t - \mu)^2: \]

\[
\begin{align*}
  r_{t+1} &= \mu + \sigma_t \epsilon_{t+1}, \\
  \sigma_t^2 &= \alpha_0 + \alpha_1 (L) \eta_t^2, \\
  \epsilon_{t+1} &\sim N(0,1),
\end{align*}
\]

where \( \alpha_1 (L) \) is a polynomial in the lag operator. To keep the conditional variance positive, \( \alpha_0 \) and the coefficients in \( \alpha_1 (L) \) must be nonnegative.

As a way to model persistent movements in volatility without estimating a very large number of coefficients in a high-order polynomial \( \alpha_1 (L) \), Bollerslev [12] suggested the Generalized Autoregressive Conditionally Heteroskedastic, or GARCH, model:

\[ \sigma_t^2 = \alpha_0 + \beta (L) \sigma_{t-1}^2 + \alpha_1 (L) \eta_t^2, \]

where \( \beta (L) \) is also a polynomial in the lag operator. By analogy with ARMA models, this is called a GARCH\((p,q)\) model when the order of the polynomial \( \beta (L) \) is \( p \) and the order of the polynomial \( \alpha_1 (L) \) is \( q \). The most commonly used model in the GARCH class, and the one that we will concentrate on, is the GARCH(1,1) which can be written as:

\[
\begin{align*}
  \sigma_t^2 &= \alpha_0 + \beta \sigma_{t-1}^2 + \alpha_1 \eta_t^2 \\
  &= \alpha_0 + (\alpha_1 + \beta) \sigma_{t-1}^2 + \alpha_1 (\eta_t^2 - \sigma_{t-1}^2) \\
  &= \alpha_0 + (\alpha_1 + \beta) \sigma_{t-1}^2 + \alpha_1 \sigma_{t-1}^2 (\epsilon_t^2 - 1),
\end{align*}
\]

where \( \epsilon_t \) is a normally distributed random variable with mean zero and variance one. In the second equality in (1.6), the term \((\eta_t^2 - \sigma_{t-1}^2)\) has zero mean, conditional on information available at time \( t - 1 \), and can be thought as the shock to the volatility. The coefficient \( \alpha_1 \) measures the extend to which a volatility shock today feeds through into next period’s volatility, while \((\alpha_1 + \beta)\) measures the rate at which this effect dies out over time. The third equality in (1.6) rewrites the volatility shock as \( \sigma_{t-1}^2 (\epsilon_t^2 - 1) \), the square of a standard normal less its
mean, that is a demeaned $\chi^2(1)$ random variable, multiplied by past volatility $\sigma_{t-1}^2$. When $\alpha_1 + \beta < 1$, the unconditional expectation of $\sigma_t^2$ is $E_\cdot = \alpha_0 / (1 - \alpha_1 - \beta)$. The GARCH(1,1) model is an ARMA(1,1) model for squared innovations; but a standard ARMA(1,1) model has homoskedastic shocks, while here the shocks $(\eta_t^2 - \sigma_{t-1}^2)$ are themselves heteroskedastic.

Given that the conditional variance $\sigma_t^2$ is given in Equation (1.6), the rate of return $r_t$ of the single risky asset is given by

$$r_t = \mu + \sigma_{t-1} \epsilon_t,$$

therefore, conditional on time $t - 1$, is normally distributed with mean $\mu$ and variance $\sigma_{t-1}^2$.

The ideas considered in a univariate context translate to the multivariate setting, but the model becomes very complicated very quickly. As a result, much of the literature on multivariate GARCH models seeks to place plausible restrictions to the model in order to reduce the number of parameters and to find restrictions that guarantee that the conditional covariance matrix is positive definite. Since this is an area of ongoing research, we concentrate on the univariate case and its application to dynamic portfolio optimization.

### 1.2 Preferences Representation

The portfolio selection problem in the financial economics literature considers an investor who wants to maximize his expected utility of consumption and final wealth. The HARA (Hyperbolic Absolute Risk Aversion) class of utility functions is most commonly used. Utility functions in this class are

$$U(W) = \frac{1 - \gamma}{\gamma} \left( \frac{a}{1 - \gamma} + b \right)^\gamma, \quad b > 0.$$  

This utility function is defined over the domain $b + aW/(1 - \gamma) > 0$. Special cases of the HARA family are the widely used isoelastic or power utility with $b = 0$ and $\gamma < 1$, and logarithmic with $b = \gamma = 0$, the negative exponential utility with $\gamma = -\infty$ and $b = 1$ and the quadratic utility function with $\gamma = 2$. The power utility function exhibits constant relative risk aversion (CRRA), and therefore decreasing absolute risk aversion. Thus, the proportion of wealth invested in the risky asset is invariant with respect to changes in the initial wealth level. The negative exponential utility function displays constant absolute risk aversion (CARA), and
thus implies that the demand for risky assets is unaffected by changes in initial wealth, with riskless borrowing and lending absorbing all changes in initial wealth. Finally, the quadratic utility function displays increasing absolute risk aversion and thus treats the risky asset as inferior.

In this thesis, we focus our attention on dynamic portfolio optimization models faced by an investor with preferences given by either the quadratic or the negative exponential utility functions. We investigate how the choice of a utility function influences the optimal investment decisions in the presence of transaction costs.

1.3 Transaction Costs Modeling

The tremendous growth in equity trading over the past twenty years has created a renewed interest in the measurement and management of trading costs. Execution costs that include commissions, bid/ask spreads, opportunity costs and price impact from trading\(^2\), are large enough to merit the attention of any investor and their control (or its lack) profoundly affects long-term investment success. For example, Pérol [65] observes that a hypothetical portfolio constructed according to the Value Line rankings outperforms the market by almost 20% per year during the period from 1965 to 1986, whereas the actual portfolio (the Value Line Fund) outperformed the market by only 2.5% per year, the difference arising from execution costs.

There are mainly three forms of transaction costs that are present in the financial economics literature:

1. Portfolio management fees, that is investors pay \textit{fixed} costs whenever a trade is performed. The portfolio management fee is meant to include the cost of adjusting the portfolio and the cost of processing information.

2. Withdrawal costs, that is costs that are generated by the bid-asked spread and investors pay \textit{proportional} costs on the amount transacted.

3. Price impact costs. Institutional investors, such as mutual and pension funds, often

\(^2\text{See Berkowitz, Logue and Noser [9], Hasbrouck and Schwartz [42], Kraus and Stoll [50], Loeb [56], and Wagner [78] for further discussion.}\)
desire to trade large fractions of the average daily volume of many stocks. Their trading activity yields sometimes a noticeable market impact and, as a result, their incurred costs are higher.

In an attempt to capture the effect of price impact, we choose to model transaction costs as a \textit{quadratic} function on the value of the trade; as the size of the transaction increases, the asset price will most likely increase as well, as a result, the incurred costs will be higher. For further motivation and discussion, see also Bertsimas and Lo [7]. We thus study the effect of quadratic transaction costs to the investment behavior over time of an institutional investor who maximizes his expected utility of terminal wealth.

1.4 Optimization Models for Portfolio Management under Uncertainty

This section contains a classification of various optimization models that have been proposed in the literature for portfolio management under uncertainty. The emphasis is on the structure of and assumptions behind the different types of models, the majority of which are in continuous time. We successively discuss dynamic portfolio optimization models where transaction costs are ignored and models that do consider transaction costs.

1.4.1 Portfolio Optimization Models with No Transaction Costs

Previous academic literature in the context of random walks for asset prices has reached various conclusions for the time profile of the risky asset holdings. A host of papers\textsuperscript{3} has provided the benchmark result that, under CRRA preferences and serially uncorrelated returns, the risky share is constant over time until retirement, and that under a CARA utility function the dollar value of the wealth invested in the risky asset is constant. Under the HARA utility specification with a positive subsistence level, Merton [60] derives that the risky share rises with age because early in life an individual needs to secure more periods of subsistence consumption so that more wealth will be locked into the riskless asset. By assuming that an individual saves for a

\textsuperscript{3}See, for example, Hakansson [39], Merton ([59], [60], [61], [63] chapters 4 and 5), Mossin [64], Samuelson [68].
retirement annuity and ignoring consumption before retirement, Samuelson [69] obtains that the risky share falls with age because the wealth that is stowed away to meet the subsistence level of retirement income will be lower initially as it is guaranteed to grow at the risk-free rate over time until the subsistence level is hit at retirement. Cox and Huang [20] propose a method to solve the dynamic portfolio optimization problem when there are short-sale constraints. He and Pearson [43] generalize the Cox and Huang [20] approach to the case of incomplete markets. Grossman and Vila [38] introduce a borrowing limit and a nonnegativity constraint on wealth, but limit their analysis to an investor with constant relative risk aversion who can only choose between one risky and one riskless asset.

There has been surprisingly little work on the implications of time-varying expected returns and variance on long-horizon allocation, partly because of the difficulty of solving the investor's dynamic optimization problem. Fischer and Pennacchi [34] consider a continuous time process with serial correlation and CRRA preferences, and they focus on the effect of the holding-period length on the optimal portfolio composition and do not consider the time path of sequential portfolios. Merton [60] examines two cases with CARA preferences and an infinite horizon. Under negative autocorrelated returns, it is shown that, conditional on the mean returns, investment in the risky asset is not a function of time, but with portfolio managers holding more on the risky asset compared with the no-autocorrelated case. Under positive autocorrelated returns, investment managers always hold less of the risky asset for a given mean return. Samuelson [70] uses a two-point Markov rebound process and derives by numerical example the age effect by showing that the optimal risky share is greater with two periods to go than with one for an investor with relative risk aversion equal to two.

Wang [79] presents a dynamic asset-pricing model in closed-form under asymmetric information assuming correlated return dynamics. Investors have different information concerning the future growth rate of dividends and they rationally extract information from prices as well as dividends and maximize their expected CARA utility. It is shown that information asymmetry among investors can increase price volatility and negative autocorrelation of returns, and that less-informed investors may rationally behave like price chasers. Lo and Wang [55] investigate the effect of asset return predictability on the prices of options on that asset. Since for discretely-sampled data predictability is linked to the parameters that enter the option pric-
ing formula, they construct an adjustment for predictability to the Black-Scholes formula and show that this adjustment can be important even for small levels of predictability, especially for longer maturity options.

Barberis [3] examines the implications of time-varying expected returns for long-horizon portfolio allocation in discrete time. The problem is solved numerically and particular attention is paid to the uncertainty about the true values of the models parameters. It is found that even after incorporating parameter uncertainty, the mean-reversion induced by time variation in expected returns leads long-horizon investors to allocate substantially more to stocks than their short-horizon counterparts. Balvers and Mitchell [2] derive an analytical solution to the dynamic portfolio problem of an individual agent saving for retirement under an ARMA(1,1) process for the single risky asset. They show that with a positive moving average parameter and positive risk-free rates, if first-order serial correlation is nonnegative, then the expected value of the optimal risky investment is increasing over time, while if first-order serial correlation is negative this path can be increasing or decreasing over time. Thus, a necessary but not sufficient condition to obtain the conventional age effect of increasing conservatism over time is that the first-order serial correlation should be negative.

Market incompleteness due to changing volatility and its implication on option pricing is investigated in several papers⁴. In contrast, stochastic volatility models are generally absent from dynamic portfolio optimization problems. We therefore attempt to explore the effects of changing volatility to an individual’s investment behavior over time.

### 1.4.2 Portfolio Optimization Models in the Presence of Transaction Costs

The presence of any friction in financial markets qualitatively changes the nature of the dynamic optimization problem. To the best of our knowledge, the focus of the literature thus far has been in analyzing the investment and consumption behavior of an investor with a portfolio of only two assets, one risky and one riskless, in the presence of proportional transaction costs assuming that prices follow a geometric Brownian motion. Kamin [48], Magill and Constantinides [57], and Constantinides [18] show that transaction costs lead to less frequent trading. They show

---

⁴See, for example, Amin and Ng [1], Bertsimas, Kogan and Lo [8], Hull and White [46], Wiggins [80].
that the optimal investment strategy is to refrain from transacting if portfolio holdings lie within a region (convex cone) characterized by two parameters that are functions of the investment opportunity and time, and to transact to the nearest boundary if portfolio holdings lie outside the region.

The topic of dynamic strategies under transaction costs has recently attracted significant interest. Constantinides [19] proposes an approximate solution to the infinite-horizon portfolio choice problem under proportional transaction costs with intermediate consumption and constant relative risk-averse utility function. Portfolio policies are computed numerically under the assumption that the investor in each period consumes a fixed proportion of his wealth. Duffie and Sun [24] consider the case of fixed plus proportional transaction charges and Tekser, Klass and Assaf [74] study the problem of maximizing the expected growth rate of a portfolio of two assets with no intermediate consumption; they prove that an optimal policy keeps the ratio of funds in risky and nonrisky assets within a certain interval with minimal effort. Davis and Norman [23] solve the infinite horizon model with intermediate consumption in closed form. It is shown that the optimal buying and selling policies are the local times of the two-dimensional process of bank and stock holdings at the boundaries of a wedge-shaped region which is determined by the solution of a nonlinear free boundary problem that is found numerically. Dumas and Luciano [25] derive a closed form solution to the infinite horizon problem with no intermediate consumption and Grossman and Laroque [37] consider fixed transaction costs and a lot size constraint. Gennette and Jung [35] present numerical results for the finite-horizon, discrete-time problem of determining the optimal investment strategy of an investor who maximizes his expected power utility of terminal wealth under a multiplicative binomial stock process.

A study of the effects of transaction costs on option pricing is given by Leland [51], and Boyle and Vorst [13]. A technique for replicating option returns is developed and the resulting strategy depends upon the level of transaction costs and the time period between portfolio revision. However, the strategies considered are not chosen to satisfy some optimality criteria that investors may wish to meet. One optimality criterion is to maximize the expected utility of the difference between the realized cash flow and the desired one at maturity, as suggested by Hodges and Neuberger [45]. Edirisinghe, Naik and Uppal [27] consider the replication of options for an investor facing proportional transaction costs as well as trading restrictions in the
form of lot-size constraints and position limits in order to minimize the initial cost of obtaining a terminal payoff that is at least as large as that from the option being hedged. Finally, Toft [76] analyzes the trade-off between cost and risk of discretely rebalanced option hedges, and Clewiow and Hodges [17] examine the problem of delta-hedging portfolios of options by maximizing expected utility or minimizing a loss function on the replication error.

Transaction costs are also generally absent from asset pricing models. A few papers endogenize asset prices but have to resort to numerical methods, like Heaton and Lucas [44] that also consider short-sale constraints. Vayanos [77] studies the effects of transaction costs, that are proportional to the number of shares traded, on asset prices and develops a general equilibrium model.

Most of the assumptions about the choice of the utility function and the behavior of the state variables and securities are crucial for the use of stochastic dynamic programming and therefore cannot be relaxed. Moreover, the introduction of market imperfections significantly complicates the dynamic optimization problem. The main focus of the literature on optimization models for portfolio management has been the effect of proportional transaction costs to the investment strategies of an investor who maximizes the expected value of a power or logarithmic utility function in the presence of just two assets, one risky and one riskless, and under the assumption that asset prices follow a geometric Brownian motion. Even under this particular setting, numerical methods have to be performed in order to evaluate the boundaries of the no transactions region. Furthermore, due to dimensionality problems existing analyses cannot be extended to the case of multiple risky assets and/or alternative models for the asset return dynamics. This thesis constitutes a proposal in this direction.

1.5 Methods of Dynamic Optimization

Dynamic optimization problems involve systems where decisions are made sequentially in time. Each decision results in some immediate cost, but also affects the cost incurred in future states of the system. The objective is to find decision making policies that minimize the total cost (or maximize the total reward) over a number of different states. Such problems are challenging primarily because of the tradeoff between immediate and future costs. Dynamic programming
(DP) provides a mathematical formalization of this tradeoff. For a comprehensive review on
dynamic programming techniques see Bertsekas [5]. For many important problems, though, the
computational requirements of DP are overwhelming, mainly because of a very large number
of states and controls (the curse of dimensionality).

To address the dimensionality problem, several approaches have been proposed recently.
Bertsekas and Tsitsiklis [6] review under the term neuro-dynamic programming (NDP) several
efforts to use DP approximately. The approximations can be broadly categorized into two
arenas: a) policy approximations and b) value function approximations. Under policy approxi-
mations, particular classes of policies that involve certain parameters are considered, and using
simulation and trial and error the parameter values are chosen. Under value function approx-
imations, a particular function structure is chosen that involves some parameters. Problem
specific insight, heuristics and trial and error are all important ingredients for constructing
value function approximations in DP. Although a promising methodology, NDP has certain
shortcomings: the choice of the value function structure is not always apparent and often a lot
of trial and error is required.

1.6 Problem Definition

We consider an institutional investor, whose trades often comprise a large fraction of the average
daily volume of many stocks, and a discrete-time investment horizon $T$. Investment managers
can invest at times $t = 0, \ldots, T - 1$ in $N + 1$ financial assets that can be bought and sold in
unlimited amounts. The zeroth asset is a bank account (cash) with a constant rate of return
$r_f$, while the rest of the assets are risky. Let $x^0_t$ be the holdings (in dollars) of the zeroth asset
at time $t$, and $x^i_t$ the holdings (in dollars) of the $i$-th risky asset before making any investment
decision. Let $u^i_t$ be the number of account units (dollars) by which the investor increases
(decreases) his holdings of the $i$-th risky asset; after the transaction, the investor’s holdings of
the $i$-th risky asset become $x^i_t + u^i_t$. Transaction costs, that are incurred by the purchase or
sale of an asset, must be financed from the bank account, while the purchase (sale) of assets
reduces (increases) the cash holdings. Moreover, transaction costs are assumed to be quadratic
in the control $u^i_t$. Thus, the transaction cost function $T(u^i_t)$ indicating the cost of buying or
selling asset \(i\) is given by

\[ T(u^i_t) = \tau_i \cdot (u^i_t)^2, \]

where \(\tau_i\) is the transactions cost coefficient and is taken to be between zero and one. Let \(r^i_t\) be the rate of return for the \(i\)-th asset at time \(t\) defined as:

\[ r^i_t = \frac{P^i_t + D^i_t}{P^i_{t-1}} - 1, \]

where \(P^i_t\) is taken to be the ex-dividend price of asset \(i\) at time \(t\), and \(D^i_t\) is the asset's \(i\) dividend assumed to be paid just before the date-\(t\) price \(P^i_t\) is recorded. Then, the investor's holdings at the next time period \(t+1\) and the terminal wealth \(W_T\) are given by

\[ x^0_{t+1} = (1 + r_f) \left[ x^0_t - \sum_{i=1}^{N} u^i_t - \sum_{i=1}^{N} \tau_i \left( u^i_t \right)^2 \right], \quad (1.7) \]

\[ x^i_{t+1} = (1 + r^i_{t+1}) \left[ x^i_t + u^i_t \right], \quad (1.8) \]

\[ W_T = x^T e + x^0_T, \quad (1.9) \]

where \(r_f\) is the risk-free rate of return, \(x_T = (x^1_T, \ldots, x^N_T)'\) is the vector of the asset holdings and \(e\) is the vector of ones. A schematic representation of the wealth dynamics is given in Figure 1-1.

We consider the following assumptions about the structure of the financial market in which the investor operates.

**Assumption 1** Asset prices are considered exogenous and securities are infinitely divisible.

In this thesis, we treat asset prices as exogenous and concentrate on investment policies. Assumption 1 allows us to focus our discussion on the optimization model for portfolio management and to avoid discussion about the implications of our model to the behavior of stock prices, price volatility, risk premia and serial correlation in stock returns.

**Assumption 2** Short sales of assets are allowed.

We impose no nonnegativity constraints on the asset holdings and we exclude the restriction that wealth must be nonnegative. These are necessary conditions for the stochastic dynamic
programming algorithm. On the other hand, the proposed semi-static approach can easily incorporate these constraints.

We finally assume that the investor accumulates wealth without consuming until some terminal time $T$ when he consumes all. This is a natural assumption made from the point of view of an investment manager that is taken in this thesis. Thus, his objective is to maximize the expected utility of terminal wealth.

Let $r_T = (r_T^1, \ldots, r_T^N)'$ be the vector of the rate of returns at time $t$, $u_t = (u_t^1, \ldots, u_t^N)'$ the vector of the investment decisions made at time $t$, and define

$$u_{t-1}^2 = (u_{t-1}^2, \ldots, u_{t-1}^2)'$$

the vector that consists of the squares of the controls. In addition, let
\[
\Gamma = \begin{bmatrix}
\tau_1 & 0 & \cdots & 0 \\
0 & \tau_2 & 0 & \vdots \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \tau_N
\end{bmatrix}
\]

be the \(N \times N\) diagonal cost coefficient matrix. The objective of the optimization problem is to maximize the expected value of the investor's utility function of terminal wealth by sequentially selecting the appropriate investment positions \(u_t\). Therefore, the investment manager faces the following optimization problem

\[
\max_{\{u_0, u_1, \ldots, u_{T-1}\}} \ E_0 \{U(W_T)\}
\]

subject to

\[
W_t = x_t^0 + x'_te
\]

\[
= (1 + \tau_f) \left[ x_{t-1}^0 - \epsilon' u_{t-1} - e' \ \Gamma \ u_{t-1}^2 \right] + (e + r_t)' (x_{t-1} + u_{t-1}),
\]

\[
r_t = f_t (r_{t-1}, Z_t, \epsilon_t), \quad (1.10)
\]

\[
Z_t = g_t (Z_{t-1}, \ldots, Z_{t-k}, \eta_t), \quad (1.11)
\]

where (1.10) and (1.11) are general set of state equations: the return dynamics \(r_t\) are assumed to be a time-varying function \(f_t\) of the return vector \(r_{t-1}\), the state-vector \(Z_t\) that represents the information available at time \(t\), and a random shock \(\epsilon_t\). In addition, \(g_t\) is a function of \(k\)-lag state realizations and a random shock \(\eta_t\) which is assumed to be independent of \(\epsilon_t\). Both functions \(f_t\) and \(g_t\) are not assumed to be linear, so we can capture complex dynamic behavior of the state variables parsimoniously. In this thesis, we focus our attention on the multifactor pricing model given in Equations (1.1)-(1.2) and the stochastic volatility model described in Equations (1.3)-(1.6).
1.7 Thesis Contribution

In this thesis, we study multiperiod, discrete time portfolio optimization problems under (a) quadratic transaction costs that model price impact effects, (b) quadratic and exponential utility functions, and (c) multifactor autocorrelated pricing and stochastic volatility models.

Our contributions are:

1. Under no transaction costs, we find the optimal investment policy over time in closed form for the case of multiple assets, exponential utility and multifactor autocorrelated pricing models using stochastic dynamic programming. We also investigate qualitative properties of the optimal portfolio composition. In all other cases, we propose approximation algorithms that yield near optimal policies for small dimensional problems where exact dynamic programming is feasible. A summary of the models that we consider and our research findings is provided in Table 1.1.

<table>
<thead>
<tr>
<th>Utility/Model of Return</th>
<th>Factor Pricing</th>
<th>Stochastic Volatility</th>
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<tr>
<td>Quadratic</td>
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<td>(Section 2.3)</td>
<td>(Section 2.3)</td>
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<tr>
<td>Exponential</td>
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<tr>
<td>$e^{-\gamma W_T}$</td>
<td>Closed-Form</td>
<td>App. Algorithm</td>
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<td>(Section 5.1)</td>
<td>(Section 5.2.1)</td>
</tr>
</tbody>
</table>

Table 1.1: Summary of research findings when transaction costs are ignored. The abbreviation App. refers to an approximation algorithm.

2. Under transaction costs, we develop approximate dynamic programming algorithms in all cases considered. A summary of the models that we consider and our research findings is provided in Table 1.2.

<table>
<thead>
<tr>
<th>Utility/Model of Return</th>
<th>Factor Pricing</th>
<th>Stochastic Volatility</th>
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<tr>
<td>Exponential</td>
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<tr>
<td>$e^{-\gamma W_T}$</td>
<td>App. Algorithm</td>
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<td>(Section 6.2)</td>
<td>(Section 7.2)</td>
</tr>
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</table>

Table 1.2: Summary of research findings under transaction costs. The abbreviation App. refers to an approximation algorithm.
3. We investigate the qualitative behavior of the optimal investment policy: we examine the effect of transaction costs, time horizon, asset correlations and volatilities on the portfolio composition over time.

4. On the methodological side, we propose two ADP approaches:

   a) A policy approximation (called the semi-static approach) that is capable of including side constraints and is based on mathematical programming techniques.

   b) A value function approximation (called the structured approach) that uses the structure of the optimal single-period solution. Guided by the functional form of the optimal single-period control, we develop iterative approximations that yield computationally efficient algorithms. A suboptimal control policy is derived that identifies a priori the representative states of the system, and captures both the dynamic nature of the investment problem under consideration and the essential characteristics of the optimal dynamic investment policy.

1.8 Thesis Overview

In the next chapter, we consider a generalization of the classical mean-variance model developed by Markowitz to a multiperiod setting. Even though we develop the general DP recursion for arbitrary return dynamics, we are not able to obtain analytical solutions. We illustrate why closed form solutions for the multiperiod optimization problem are not achievable. The optimal risky holdings in the absence of transaction costs are proved to be independent of wealth, but do depend on the risk-aversion parameter and the asset return characteristics.

In Chapter 3, we consider the multiperiod portfolio optimization problem in the presence of transaction costs and multifactor pricing models. We propose structured approximations that utilize characteristics of the optimal investment strategy, and dynamic policies based on stochastic optimization techniques. The proposed approximations outperform existing dynamic trading strategies and produce near optimal policies in problems of small dimension, where exact dynamic programming is feasible. In addition, interesting insights about the qualitative behavior of the optimal control policy are derived. In Chapter 4, we consider the multiperiod
portfolio optimization problem in the presence of transaction costs and stochastic volatility models. The investment behavior, as given by the proposed approximation algorithms, is shown to be quite different relative to the one derived for factor models.

In Chapter 5, we investigate the effect of the asset return process to the investment behavior over time of an investment manager with CARA utility under the assumption that transaction costs can be ignored. We provide a closed-form solution for multifactor pricing models and investigate the effect of autocorrelation on the optimal risky holdings. On the other hand, stochastic volatility models do not produce closed-form solutions. In response, we propose two classes of approximation algorithms and investigate their behavior.

In Chapter 6, we examine the effect of quadratic transaction costs on dynamic portfolio strategies that account for lagged correlations in asset returns under a CARA utility specification. We show that in-depth investigation of the resulting dynamic optimization problem at every point in time enables us to capture essential characteristics of the optimal policy and level of utility. We discuss the dependence of the resulting control policy on the various parameters influencing both the return dynamics and the investors' preferences.

In Chapter 7, we examine the case of stochastic volatility models and their effect on the multiperiod portfolio optimization problem faced by an investor with CARA utility in the presence of quadratic transaction costs.

In Chapter 8, we present a comparative study for large scale portfolios, and in Chapter 9, we present our concluding remarks and discuss possible extensions of the models and the solution methodology in this thesis.
Chapter 2

Quadratic Utility and General Asset Return Dynamics

The stochastic dynamic optimization problem considered in this chapter constitutes a generalization of the mean-variance model that was developed by Markowitz [58] to a multiperiod setting. Mean-variance analysis has become quite popular in practice for the selection of equity portfolios. The selection of such models is sometimes justified by the claim that the expected rate of return on a stock and its variance tend to be fairly constant over short periods of time, as are the covariances between the rates of return on different stocks. The introduction of transaction costs, though, makes the selection of single-period optimization models inadequate, since the anticipation of future events may substantially influence the optimal portfolio composition today. As we show later in our analysis, even in the case when asset returns are assumed to be independent and identically distributed (IID) the optimal investment strategy is not myopic. Therefore, in the chapters that follow we explore the effect of transaction costs on the multiperiod portfolio optimization problem not only under the IID assumption for asset returns, but also under some degree of predictability in both the mean and the volatility of the time-series asset return dynamics.

The remainder of this chapter is organized as follows. In Section 2.1, we present the dynamic optimization problem considered and the recursive algorithm used for its solution under general asset return dynamics. In Section 2.2, we present the solution of the single-period problem and
illustrate why a closed form solution for the multiperiod optimization problem is not achievable. Finally, in Section 2.3, we present the solution when transaction costs are ignored that will serve as our benchmark case for the evaluation of the impact of transaction costs to the multiperiod portfolio optimization problem.

### 2.1 Problem Formulation

Consider an investor who faces the problem of making sequential investment decisions at discrete times \( t = 0, \ldots, T - 1 \). The evolution of the wealth dynamics is described in Equations (1.7)-(1.9). The state of the system at time \( t = 0, 1, \ldots, T - 1 \) consists of the asset holdings \( (x_0^t, x_t) \) before a transaction is made at time \( t \), and \( \xi_t \) the available information at time \( t \), i.e., the vector of the asset returns up to time \( t \) and a vector of state variables \( Z_t, \xi_t = (r_0, \ldots, r_t, Z_0, \ldots, Z_t) \). We denote the conditional expectation given the information at time \( t \) as

\[
E_t \{ \bullet \} = E \{ \bullet | \xi_t \}.
\]

For example, when \( r_t \) is a \( k \)-th order vector autoregression process, \( VAR(k) \), \( \xi_t = (r_t, \ldots, r_{t-k}) \), and when \( r_t \) follow an APT model with \( M \) serially correlated factors that follow a \( VAR(k) \) process, \( \xi_t = (r_t, Z_t, \ldots, Z_{t-k}) \) where \( Z_t \) is the \( m \)-dimensional vector of the factor realizations at time \( t \).

The objective of the optimization problem is to maximize the expected value of the investor’s utility function of terminal wealth by sequentially selecting the appropriate investment positions \( u_t \). Therefore, the portfolio manager faces the following optimization problem

\[
\max_{\{u_0, u_1, \ldots, u_{T-1}\}} \quad E_0 \{ W_T \} - \lambda \ Var_0 \{ W_T \}
\]

subject to

\[
\begin{align*}
W_t &= x_0^t + x_t' \cdot e \\
&= (1 + r_f) \left[ x_{t-1}^0 - e' \cdot u_{t-1} - e' \cdot \Gamma \cdot u_{t-1}^2 \right] + (e + r_t)' \cdot (x_{t-1} + u_{t-1}) \\
r_t &= f_t (r_{t-1}, Z_t, \xi_t)
\end{align*}
\] (2.1)
\[ Z_t = g_t(Z_{t-1}, \ldots, Z_{t-k}, \eta_t), \quad (2.2) \]

where (2.1) and (2.2) are general set of state equations: the return dynamics \( r_t \) are assumed to be a time-varying function \( f_t \) of the return vector \( r_{t-1} \), the state-vector \( Z_t \) and a random shock \( \epsilon_t \). In addition, \( g_t \) is a function of \( k \)-lag state realizations and a random shock \( \eta_t \) which is assumed to be independent of \( \epsilon_t \). Both functions \( f_t \) and \( g_t \) are not assumed to be linear, so we can capture complex dynamic behavior of the state variables parsimoniously. The information vector \( \xi_t \) is thus defined as \( \xi_t = (r_t, Z_t, \ldots, Z_{t-k}) \). Throughout our analyses we use the following operator:

\[
\mathbf{u}^2 = \begin{bmatrix}
(u_1)^2 \\
\vdots \\
(u_N)^2
\end{bmatrix},
\]

where \( \mathbf{u} \) is an arbitrary \( N \times 1 \) vector. We state the dynamic programming (DP) algorithm for the optimization problem considered above.\(^1\)

**Proposition 2.1** For every initial state \( (x_0^0, x_0, \xi_0) \), the optimal reward \( V^*(x_0^0, x_0, \xi_0) \) of the optimization problem is equal to \( V_0(x_0^0, x_0, \xi_0) \), where the function \( V_0 \) is given by the last step of the following algorithm, which proceeds backwards in time from period \( T - 1 \) to period 0:

\[
\begin{align*}
V_T(x_T^0, x_T, \xi_T) &= K_T(x_T^0, x_T, \xi_T) = W_T, \\
V_t(x_t^0, x_t, \xi_t) &= \max_{\{u_t\}} \left\{ V_{t+1}(x_{t+1}^0, x_{t+1}, \xi_{t+1}) \right\} - \lambda \text{ Var}_t \left\{ K_{t+1}(x_{t+1}^0, x_{t+1}, \xi_{t+1}) \right\}, \\
K_t(x_t^0, x_t, \xi_t) &= E_t \left\{ K_{t+1}(x_{t+1}^0, x_{t+1}, \xi_{t+1}) \right\},
\end{align*}
\]

where the quantity \( K_t(x_{t+1}^0, x_{t+1}, \xi_{t+1}) \) is defined as

\[ K_t \equiv E_t \{ W_T^* \}. \]

**Proof.** For any admissible policy \( \pi = \{u_0, u_1, \ldots, u_{T-1}\} \) and each \( t = 0, 1, \ldots, T - 1 \), let \( \pi^t = \{u_t, u_{t+1}, \ldots, u_{T-1}\} \). For \( t = 0, \ldots, T - 1 \), let \( V_t^*(x_t^0, x_t, \xi_t) \) be the optimal reward for

\(^1\)For a comprehensive review on dynamic programming techniques, see Bertsekas [5].
the \((N - t)\)-stage problem that starts at state \((x_t^0, x_t, \xi_t)\) and time \(t\) and ends at time \(T\); that is

\[
V_t^* \left( x_t^0, x_t, \xi_t \right) = \max_{\{\pi_t^i\}} E_t \{ W_T \} - \lambda \, Var_t \{ W_T \} \\
= \max_{\{\pi_t^i\}} E_t \{ W_T \} - \lambda \, E_t \{ W_T^2 \} + \lambda \, [E_t \{ W_T \}]^2.
\] (2.3)

For \(t = T\), we define \(V_T^* ( x_0^0, x_T, \xi_T ) = W_T\). We will show by induction that the functions \(V_t^*\) are equal to the functions \(V_t\) generated by the DP algorithm, so that for \(t = 0\), we will obtain the desired result.

Indeed, we have by definition that \(V_T^* = V_T = W_T\). We assume that the Proposition holds for all \(\tau = t + 1, \ldots, T\); i.e., \(V_{t+1}^* ( x_{t+1}^0, x_{t+1}, \xi_{t+1} ) = V_{t+1} ( x_{t+1}^0, x_{t+1}, \xi_{t+1} )\). Then, since \(\pi_t^i = \{ u_t, \pi_t^{i+1} \}\), we have for all \((x_t^0, x_t, \xi_t)\)

\[
V_t^* \left( x_t^0, x_t, \xi_t \right) = \max_{\{\pi_t^i\}} E_t \{ W_T \} - \lambda \, E_t \{ W_T^2 \} + \lambda \, [E_t \{ W_T \}]^2 \\
= \max_{\{\pi_t^i\}} E_t \{ W_{t+1} \} - \lambda \, E_t \{ E_{t+1} \{ W_T^2 \} \} + \lambda \, [E_t \{ E_{t+1} \{ W_T \} \}]^2 \\
= E_t \{ E_{t+1} \{ W_T^2 \} \} - \lambda \, E_t \{ E_{t+1} \{ (W_T^*)^2 \} \} + \lambda \, [E_t \{ E_{t+1} \{ W_T^* \} \}]^2,
\]

where \(W_T^*\) denotes the wealth under an optimal policy. From Equation (2.3) we have that

\[
E_t \{ V_{t+1}^* ( x_{t+1}^0, x_{t+1}, \xi_{t+1} ) \} = E_t \{ E_{t+1} \{ W_T^2 \} \} - \lambda \, E_t \{ E_{t+1} \{ W_T^* \} \} + \lambda \, [E_t \{ E_{t+1} \{ W_T^* \} \}]^2.
\]

Thus, \(V_t^* ( x_t^0, x_t, \xi_t )\) reduces to

\[
V_t^* \left( x_t^0, x_t, \xi_t \right) = \max_{\{u_t\}} E_t \{ V_{t+1}^* ( x_{t+1}^0, x_{t+1}, \xi_{t+1} ) \} - \lambda \, E_t \{ [E_{t+1} \{ W_T^* \}]^2 + \lambda \, [E_t \{ E_{t+1} \{ W_T^* \} \}]^2.
\]

Since \(K_{t+1} = E_{t+1} \{ W_T^2 \}\), then the term \(E_t \{ K_{t+1}^2 \} - [E_t \{ K_{t+1} \}]^2\) is just \(Var_t \{ K_{t+1} \}\). Therefore,

\[
V_t^* \left( x_t^0, x_t, \xi_t \right) = \max_{\{u_t\}} E_t \{ V_{t+1}^* ( x_{t+1}^0, x_{t+1}, \xi_{t+1} ) \} - \lambda \, Var_t \{ K_{t+1} \},
\]

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and using the induction hypothesis we finally obtain that

\[ V_t^* \left( x_t^0, x_t, \xi_t \right) = \max_{\{u_t\}} E_t \left\{ V_{t+1} \left( x_{t+1}^0, x_{t+1}, \xi_{t+1} \right) \right\} - \lambda \quad \text{Var}_t \{ K_{t+1} \}. \]

The argument of the preceding proof provides an interpretation of \( V_t \left( x_t^0, x_t, \xi_t \right) \) as the optimal reward for an \((T - t)\)-stage problem starting at state \((x_t^0, x_t, \xi_t)\) and time \(t\), and ending at time \(T\). We consequently call \( V_t \left( x_t^0, x_t, \xi_t \right) \) the \textit{cost-to-go} at state \((x_t^0, x_t, \xi_t)\) and time \(t\), and refer to \( V_t \) as the \textit{cost-to-go} function at time \(t\). Ideally, we would like to use the DP algorithm developed in Proposition 2.1 to obtain closed-form expressions for \( V_t \) or an optimal policy. Unfortunately, an analytical solution is only available for the single-period optimization problem as we show in the next section. Nevertheless, we propose a series of recursive algorithms that approximate the \textit{cost-to-go} function at every point in time using characteristics of the optimal control policy and the structure of the DP recursion.

### 2.2 The Single Period Problem

In this section, we show that there exists a closed-form solution for the single-period optimization problem and obtain the optimal control policy at time \(T - 1\) that is linear in the risky holdings. More specifically, using Proposition 2.1, \( V_{T-1} \) can be expressed as

\[
V_{T-1} \left( x_{T-1}^0, x_{T-1}, \xi_{T-1} \right) = \max_{u_{T-1}} \left\{ (1 + r_f) \left[ x_{T-1}^0 - e' u_{T-1} - e' \Gamma u_{T-1}^2 \right] + (e + r_T)' \left( x_{T-1} + u_{T-1} \right) \right\} - \lambda \quad \text{Var}_{T-1} \left\{ (e + r_T)' \left( x_{T-1} + u_{T-1} \right) \right\} = \max_{u_{T-1}} \left\{ \begin{array}{l} (1 + r_f) \left[ x_{T-1}^0 - e' u_{T-1} - e' \Gamma u_{T-1}^2 \right] + E_{T-1} \left\{ (e + r_T)' \left( x_{T-1} + u_{T-1} \right) \right\} - \lambda \quad \left( x_{T-1} + u_{T-1} \right)' \Lambda_{T-1} \left( x_{T-1} + u_{T-1} \right) \end{array} \right\},
\]

where the symmetric covariance matrix of the rate of returns at time \(T\) conditional on the information available at time \(T - 1\) is

\[
\Lambda_{T-1} \left( \xi_{T-1} \right) = E_{T-1} \left\{ \left[ r_T - E_{T-1} \{ r_T \} \right] \left[ r_T - E_{T-1} \{ r_T \} \right]' \right\}.
\]
This is a concave quadratic maximization problem; the first order conditions are necessary and sufficient, since both $\Lambda_{T-1}$ and $\Gamma$ are positive semi-definite matrices, and given by

$$-(1 + r_f) e - 2(1 + r_f) \Gamma u_{T-1} + E_{T-1} \{e + r_T\} - 2\lambda \Lambda_{T-1} (x_{T-1} + u_{T-1}) = 0.$$ 

If we let

$$Q_{T-1}(\xi_{T-1}) = [2(1 + r_f) \Gamma + 2\lambda \Lambda_{T-1}]^{-1},$$

$$m_{T-1}(\xi_{T-1}) = Q_{T-1}[E_{T-1}\{e + r_T\} - (1 + r_f) e],$$

$$L_{T-1}(\xi_{T-1}) = Q_{T-1}(2\lambda \Lambda_{T-1}),$$

then the optimal control at time $T - 1$ can be written as

$$u^*_{T-1}(x_{T-1}, \xi_{T-1}) = m_{T-1} - L_{T-1} x_{T-1}. \tag{2.4}$$

Substituting for the optimal control given by Equation (2.4), we obtain\(^2\) the value function at time $T - 1$:

$$V_{T-1}(x^0_{T-1}, x_{T-1}, \xi_{T-1}) =$$

$$(1 + r_f) x^0_{T-1} - (1 + r_f) e' [m_{T-1} - L_{T-1} x_{T-1}] -$$

$$(1 + r_f) e' \Gamma [m_{T-1} - L_{T-1} x_{T-1}]^2 +$$

$$E_{T-1} \{e + r_T\}' [m_{T-1} + (I - L_{T-1}) x_{T-1}] -$$

$$\lambda [m_{T-1} + (I - L_{T-1}) x_{T-1}]' \Lambda_{T-1} [m_{T-1} + (I - L_{T-1}) x_{T-1}] .$$

The function $V_{T-1}$ is linear in the state variable $x^0_{T-1}$ and quadratic in $x_{T-1}$. In order to simplify the above expression, we need to concentrate on the term $\mathcal{L} = e'\Gamma[m_{T-1} - L_{T-1} x_{T-1}]^2$, that

---

\(^2\)We denote $I$ the $(N \times N)$ identity matrix.
can be written as

\[
\mathcal{L} = \sum_{i=1}^{N} \tau_i \left[ m_i^2 + \left( \sum_{j=1}^{N} l_{ij} x_j \right)^2 - 2 m_i \left( \sum_{j=1}^{N} l_{ij} x_j \right) \right] = e' \Gamma m^2 - 2 \sum_{i=1}^{N} \tau_i m_i \left( \sum_{j=1}^{N} l_{ij} x_j \right) + \sum_{i=1}^{N} \tau_i \left( \sum_{j=1}^{N} l_{ij} x_j \right)^2 = e' \Gamma m^2 - 2 m' \Gamma L x + x' L \Gamma L x.
\]

(2.5)

Using Equation (2.5), the quantity \( K_{T-1} \equiv E_{T-1} \{ W_T \} \) and value function \( V_{T-1} \) can be expressed as

\[
K_{T-1} (x_{T-1}^0, x_{T-1}, \xi_{T-1}) = h_{T-1} + (1 + r_f) x_{T-1}^0 + a_{T-1} x_{T-1} - \xi_{T-1}^T D_{T-1} x_{T-1},
\]

\[
V_{T-1} (x_{T-1}^0, x_{T-1}, \xi_{T-1}) = z_{T-1} + (1 + r_f) x_{T-1}^0 + b_{T-1} x_{T-1} - \xi_{T-1}^T C_{T-1} x_{T-1},
\]

where

\[
h_{T-1} (\xi_{T-1}) = -(1 + r_f) e' m_{T-1} - (1 + r_f) e' \Gamma m_{T-1}^2 + E_{T-1} \{ e + r_T \}' m_{T-1},
\]

\[
z_{T-1} (\xi_{T-1}) = h_{T-1} - \lambda m_{T-1}' \Lambda_{T-1} m_{T-1},
\]

\[
a_{T-1} (\xi_{T-1}) = (1 + r_f) L_{T-1}' e + 2 (1 + r_f) L_{T-1}' \Gamma m_{T-1} + (I - L_{T-1})' E_{T-1} \{ e + r_T \},
\]

\[
b_{T-1} (\xi_{T-1}) = a_{T-1} - 2 \lambda (I - L_{T-1})' \Lambda_{T-1} m_{T-1},
\]

\[
D_{T-1} (\xi_{T-1}) = (1 + r_f) L_{T-1}' \Gamma L_{T-1},
\]

\[
C_{T-1} (\xi_{T-1}) = D_{T-1} + \lambda (I - L_{T-1})' \Lambda_{T-1} (I - L_{T-1}).
\]

From the above relations it is evident that the matrices \( D_{T-1} \) and \( C_{T-1} \) are symmetric. Moreover, both functions \( K_{T-1} \) and \( V_{T-1} \) are linear in the riskless holdings, \( x_{T-1}^0 \), and quadratic in the holdings in the risky assets, \( x_{T-1} \). In contrast, both depend on the information vector \( \xi_{T-1} \) in a highly nonlinear fashion. Thus, even though we are able to solve the single-period problem in closed-form, we cannot proceed recursively and solve for arbitrary times. In order to illustrate why a closed-form solution is not attainable, consider the value function at time
\[ V_{T-2} \left( x_{T-2}^0, x_{T-2}, \xi_{T-2} \right) = \max_{\{ u_{T-2} \}} \ E_{T-2} \{ V_{T-1} \} - \lambda \ Var_{T-2} \{ K_{T-1} \} = \] 

\[
\max_{\{ u_{T-2} \}} E_{T-2} \begin{cases} 
\varepsilon_{T-1} (\xi_{T-1}) + (1 + r_f)^2 \left[ x_{T-2}^0 - e' u_{T-2} - e' \Gamma u_{T-2}^2 \right] + \\
\left[ b_{T-1} (\xi_{T-1}) \otimes (e + r_{T-1}) \right]' (x_{T-2} + u_{T-2}) - \\
(x_{T-2} + u_{T-2})' \left\{ C_{T-1} (\xi_{T-1}) \otimes (e + r_{T-1}) (e + r_{T-1})' \right\} (x_{T-2} + u_{T-2}) \\
\end{cases} \\
- \lambda \ Var_{T-2} \begin{cases} 
\varepsilon_{T-1} (\xi_{T-1}) + (1 + r_f)^2 \left[ x_{T-2}^0 - e' u_{T-2} - e' \Gamma u_{T-2}^2 \right] + \\
\left[ a_{T-1} (\xi_{T-1}) \otimes (e + r_{T-1}) \right]' (x_{T-2} + u_{T-2}) - \\
(x_{T-2} + u_{T-2})' \left\{ D_{T-1} (\xi_{T-1}) \otimes (e + r_{T-1}) (e + r_{T-1})' \right\} (x_{T-2} + u_{T-2}) \\
\end{cases}, 
\]

where the notation \( \otimes \) denotes the Kronecker product of vectors or matrices; that is the element-by-element multiplication of the corresponding vectors or matrices. In more detail, consider the term \( b_{T-1}' x_{T-1} \) in \( V_{T-1} \). It can be written as

\[
b_{T-1}' x_{T-1} = \sum_{i=1}^{N} b_{T-1,i} x_{T-1,i} = \sum_{i=1}^{N} \left[ b_{T-1,i} \left( 1 + r_{T-1}^i \right) \right] (x_{T-2,i} + u_{T-2,i}) = \\
\left[ b_{T-1} \otimes (e + r_{T-1}) \right]' (x_{T-2} + u_{T-2}).
\]

Similarly, consider the term \( x_{T-1}' C_{T-1} x_{T-1} \) in \( V_{T-1} \) that can be expressed as

\[
x_{T-1}' C_{T-1} x_{T-1} = \sum_{i=1}^{N} x_{T-1,i} \sum_{j=1}^{N} C_{T-1,ij} x_{T-1,j} = \\
\sum_{i=1}^{N} \left( 1 + r_{T-1}^i \right) \sum_{j=1}^{N} \left[ C_{T-1,ij} (x_{T-2,i} + u_{T-2,i})(x_{T-2,j} + u_{T-2,j}) \right] \left( 1 + r_{T-1}^j \right).
\]

But, the term \( (x_{T-2,i} + u_{T-2,i})(x_{T-2,j} + u_{T-2,j}) \) is the \( ij \)-th element of the matrix

\[
(x_{T-2} + u_{T-2})(x_{T-2} + u_{T-2})'.
\]

As a result,

\[
x_{T-1}' C_{T-1} x_{T-1} = (e + r_{T-1})' \left\{ C_{T-1} \otimes (x_{T-2} + u_{T-2})(x_{T-2} + u_{T-2})' \right\} (e + r_{T-1})
\]

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\[
(x_{T-2} + u_{T-2})' \left\{ C_{T-1} \otimes (e + r_{T-1}) (e + r_{T-1})' \right\} (x_{T-2} + u_{T-2}) .
\]

A similar argument can be carried out for the rest of the terms appearing in \( K_{T-1} \).

From Equation (2.6) we observe that the optimization problem at time \( T-2 \) is not quadratic any more, as was the case at time \( T - 1 \), since it involves a maximization of a fourth-order polynomial with respect to the control variable \( u_{T-2} \). The quadratic nature of the optimization problem is only preserved in the case where transaction costs are ignored. Moreover, the coefficients of the control variable \( u_{T-2} \) are expectations of complicated functions of the future return realizations, which can be computed in closed form only in the presence of independent and identically distributed (IID) returns. When transaction costs are incorporated and general asset return dynamics are used, a closed-form solution to the multiperiod portfolio optimization is not attainable. We propose approximation algorithms that preserve the quadratic nature of the single-period problem that can be solved in closed-form.

We next present the case where transaction costs are ignored. We are able to show that with arbitrary return dynamics the optimal investment decision is a linear function of the vector of holdings in the risky assets with coefficients that are, in general, expectations of complicated functions of the future return realizations. Under the assumption of IID returns, a closed-form solution is found; under more complicated return dynamics, we propose several approximation procedures.

### 2.3 Benchmark Problem: No Transaction Costs

In the absence of transaction costs, it suffices to fully characterize the state space of the asset holdings at time \( t \) with the total wealth present at that time, and we need not differentiate between the holdings of the various assets. The wealth at time \( t \) as given by the dynamics of Equations (1.7)-(1.9) is simplified to:

\[
W_t = x_t^0 + e' x_t \\
= (1 + r_f) \left[ x_{t-1}^0 - e' u_{t-1} \right] + (e + r_t)' (x_{t-1} + u_{t-1}) \\
= (1 + r_f) W_{t-1} + (r_t - r_f e)' (x_{t-1} + u_{t-1}) ,
\]

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since $W_{t-1} = x_{t-1}^0 + e' x_{t-1}$. The quantity $\tilde{u}_{t-1} = (x_{t-1} + u_{t-1})$ is the dollar amount invested in the risky assets after a transaction is made at time $t - 1$ and constitutes the new control variable. For convenience, we also let $\tilde{r}_t = (r_t - r_f e)$ denote the excess rate of return at time $t$.

Using Proposition 2.1, we can characterize the optimal value function $V_{T-1}$ by using the boundary condition and then proceed recursively:

$$
V_{T-1} (W_{T-1}, \xi_{T-1}) = \max_{\{\tilde{u}_{T-1}\}} E_{T-1} \{W_T\} - \lambda \ Var_{T-1} \{W_T\}
$$

$$
= \max_{\{\tilde{u}_{T-1}\}} E_{T-1} \{(1 + r_f) W_{T-1} + \tilde{r}_T' \tilde{u}_{T-1}\} - \lambda \ Var_{T-1} \{\tilde{r}_T' \tilde{u}_{T-1}\}
$$

$$
= \max_{\{\tilde{u}_{T-1}\}} \left\{(1 + r_f) W_{T-1} + E_{T-1} \{\tilde{r}_T\}' \tilde{u}_{T-1} - \lambda \tilde{r}_T' A_{T-1} \tilde{u}_{T-1}\right\},
$$

where the symmetric covariance matrix of the excess rate of returns at time $T$ conditional on the information available at time $T - 1$ is

$$
\Lambda_{T-1} (\xi_{T-1}) = E_{T-1} \left\{[\tilde{r}_T - E_{T-1} \{\tilde{r}_T\}] [\tilde{r}_T - E_{T-1} \{\tilde{r}_T\}]'\right\}.
$$

This is a quadratic optimization problem; the first order conditions are necessary and sufficient, since $\Lambda_{T-1}$ is a positive semi-definite matrix, and given by

$$
E_{T-1} \{\tilde{r}_T\} - 2\lambda \Lambda_{T-1} \tilde{u}_{T-1} = 0.
$$

Therefore, the optimal control at time $T - 1$ is given by

$$
\tilde{u}^*_T_{T-1} = \frac{1}{2\lambda} \Lambda^{-1}_{T-1} E_{T-1} \{\tilde{r}_T\},
$$

or in terms of the original control variables is expressed as

$$
u_{T-1} = \frac{1}{2\lambda} \Lambda^{-1}_{T-1} E_{T-1} \{\tilde{r}_T\} - x_{T-1}.$$
The value function $V_{T-1}$ can, therefore, be obtained by

$$V_{T-1} (W_{T-1}, \xi_{T-1}) = (1 + r_f) W_{T-1} + \frac{1}{4\lambda} E_{T-1} \{ \hat{r}_T \} \Lambda_{T-1}^{-1} E_{T-1} \{ \hat{r}_T \},$$

and the quantity $K_{T-1} \equiv E_{T-1} \{ W_T \}$ by

$$K_{T-1} (W_{T-1}, \xi_{T-1}) = (1 + r_f) W_{T-1} + \frac{1}{2\lambda} E_{T-1} \{ \hat{r}_T \} \Lambda_{T-1}^{-1} E_{T-1} \{ \hat{r}_T \}.$$

The following theorem shows that the optimal risky holdings at every point in time are independent of wealth, and that the value function is separable in the state variables in contrast with the case where transaction costs are incorporated. As a result, highly efficient approximation algorithms are deduced.

**Theorem 2.1** The optimal investment decisions $\hat{u}_{T-k}^*$, the value function $V_{T-k}$ and the quantity $K_{T-k}$ for $k = 1, \ldots, T$ are given by the following relations:

\[
\begin{align*}
\hat{u}_{T-k}^* (\xi_{T-k}) &= \frac{1}{2\lambda} \left( \frac{1}{1 + r_f} \right)^{k-1} \Lambda_{T-k}^{-1} \mu_{T-k}, \\
V_{T-k} (W_{T-k}, \xi_{T-k}) &= (1 + r_f)^k W_{T-k} + \frac{1}{4\lambda} z_{T-k}, \\
K_{T-k} (W_{T-k}, \xi_{T-k}) &= (1 + r_f)^k W_{T-k} + \frac{1}{2\lambda} h_{T-k},
\end{align*}
\]

where

\[
\begin{align*}
\Lambda_{T-k} (\xi_{T-k}) &= E_{T-k} \{ [\hat{r}_{T-k+1} - E_{T-k} \{ \hat{r}_{T-k+1} \}] [\hat{r}_{T-k+1} - E_{T-k} \{ \hat{r}_{T-k+1} \}]' \}, \\
\mu_{T-k} (\xi_{T-k}) &= E_{T-k} \{ \hat{r}_{T-k+1} \} - Cov_{T-k} \{ \hat{r}_{T-k+1}, h_{T-k+1} \}, \\
z_{T-k} (\xi_{T-k}) &= \mu_{T-k}' \Lambda_{T-k}^{-1} \mu_{T-k} + E_{T-k} \{ z_{T-k+1} \} - Var_{T-k} \{ h_{T-k+1} \}, \\
h_{T-k} (\xi_{T-k}) &= E_{T-k} \{ \hat{r}_{T-k+1} \}' \Lambda_{T-k}^{-1} \mu_{T-k} + E_{T-k} \{ h_{T-k+1} \},
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
h_T &= 0, \\
z_T &= 0.
\end{align*}
\]
**Proof.** We prove the theorem by induction. We have shown that the relations are valid for \( k = 1 \). Assuming that they are true for arbitrary \( k \), we prove that they hold for \( k + 1 \). Substituting for the wealth dynamics, we have that

\[
V_{T-k-1} (W_{T-k}, \xi_{T-k}) = \max \left\{ \tilde{u}_{T-k-1} \right\} \left( E_{T-k-1} \{ V_{T-k} \} - \lambda \, \text{Var}_{T-k-1} \{ K_{T-k} \} = \right.
\]

\[
\max \left\{ \tilde{u}_{T-k-1} \right\} \left\{ \begin{array}{l}
E_{T-k-1} \{ (1 + r_f)^k W_{T-k} + \frac{1}{2} \lambda z_{T-k} \} - \\
\lambda \, \text{Var}_{T-k-1} \{ (1 + r_f)^k W_{T-k} + \frac{1}{2} \lambda h_{T-k} \}
\end{array} \right\} = 
\]

\[
\max \left\{ \tilde{u}_{T-k-1} \right\} \left\{ \begin{array}{l}
(1 + r_f)^{k+1} W_{T-k-1} + (1 + r_f)^k E_{T-k-1} \{ \tilde{r}_{T-k} \} \tilde{u}_{T-k-1} + \\
\frac{1}{4} \lambda E_{T-k-1} \{ z_{T-k} \} - \lambda (1 + r_f)^{2k} \text{Var}_{T-k-1} \{ \tilde{u}_{T-k-1} \tilde{r}_{T-k} \} - \\
\frac{1}{4} \lambda \text{Var}_{T-k-1} \{ h_{T-k} \} - (1 + r_f)^k \text{Cov}_{T-k-1} \{ h_{T-k}, \tilde{u}_{T-k-1} \tilde{r}_{T-k} \}
\end{array} \right\}.
\]

Let

\[
\Lambda_{T-k-1} (\xi_{T-k-1}) = E_{T-k-1} \{ [\tilde{r}_{T-k} - E_{T-k} \{ \tilde{r}_{T-k} \}] [\tilde{r}_{T-k} - E_{T-k-1} \{ \tilde{r}_{T-k} \}]' \}.
\]

Then, the value function \( V_{T-k-1} \) becomes

\[
V_{T-k-1} (W_{T-k}, \xi_{T-k}) = 
\max \left\{ \tilde{u}_{T-k-1} \right\} \left\{ \begin{array}{l}
(1 + r_f)^{k+1} W_{T-k-1} + (1 + r_f)^k E_{T-k-1} \{ \tilde{r}_{T-k} \} \tilde{u}_{T-k-1} + \\
\frac{1}{4} \lambda E_{T-k-1} \{ z_{T-k} \} - \lambda (1 + r_f)^{2k} \text{Var}_{T-k-1} \{ \tilde{u}_{T-k-1} \tilde{r}_{T-k} \} - \\
\frac{1}{4} \lambda \text{Var}_{T-k-1} \{ h_{T-k} \} - (1 + r_f)^k \text{Cov}_{T-k-1} \{ h_{T-k}, \tilde{u}_{T-k-1} \tilde{r}_{T-k} \}
\end{array} \right\}.
\]

The first order conditions are necessary and sufficient, since the matrix \( \Lambda_{T-k-1} \) is symmetric positive semi-definite, and given by

\[
E_{T-k-1} \{ \tilde{r}_{T-k} \} - 2 \lambda (1 + r_f)^k \text{Cov}_{T-k-1} \{ h_{T-k}, \tilde{r}_{T-k} \} = 0.
\]

By letting

\[
\mu_{T-k-1} (\xi_{T-k-1}) = E_{T-k-1} \{ \tilde{r}_{T-k} \} - \text{Cov}_{T-k-1} \{ h_{T-k}, \tilde{r}_{T-k} \},
\]

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we have shown that

$$\tilde{u}_{T-k-1}^* (\xi_{T-k-1}) = \frac{1}{2\lambda (1 + r_f)^k} \Lambda_{T-k-1}^{-1} \mu_{T-k-1}.$$  

Substituting back into the value function $V_{T-k-1}$ results in

$$V_{T-k-1} (W_{T-k-1}, \xi_{T-k-1}) =$$

$$(1 + r_f)^{k+1} W_{T-k-1} + \frac{1}{2\lambda} E_{T-k-1} \{ \tilde{r}_{T-k} \}' \Lambda_{T-k-1}^{-1} \mu_{T-k-1} + \frac{1}{4\lambda} E_{T-k-1} \{ T_{T-k} \} -$$

$$\frac{1}{4\lambda} \mu_{T-k-1} \Lambda_{T-k-1}^{-1} \mu_{T-k-1} - \frac{1}{4\lambda} Var_{T-k-1} \{ h_{T-k} \} -$$

$$\frac{1}{2\lambda} Cov_{T-k-1} \{ h_{T-k} \} \tilde{r}_{T-k} \}' \Lambda_{T-k-1}^{-1} \mu_{T-k-1} =$$

$$(1 + r_f)^{k+1} W_{T-k-1} + \frac{1}{4\lambda} \mu_{T-k-1} \Lambda_{T-k-1}^{-1} \mu_{T-k-1} + \frac{1}{4\lambda} E_{T-k-1} \{ T_{T-k} \}$$

$$- \frac{1}{4\lambda} Var_{T-k-1} \{ h_{T-k} \},$$

and by setting

$$z_{T-k-1} (\xi_{T-k-1}) = \mu_{T-k-1} \Lambda_{T-k-1}^{-1} \mu_{T-k-1} + E_{T-k-1} \{ T_{T-k} \} - Var_{T-k-1} \{ h_{T-k} \},$$

we prove that

$$V_{T-k-1} (W_{T-k-1}, \xi_{T-k-1}) = (1 + r_f)^{k+1} W_{T-k-1} + \frac{1}{4\lambda} z_{T-k-1}.$$  

Finally,

$$K_{T-k-1} (W_{T-k-1}, \xi_{T-k-1}) \equiv E_{T-k-1} \{ K_{T-k} \} = E_{T-k-1} \left\{ (1 + r_f)^k W_{T-k} + \frac{1}{2\lambda} h_{T-k} \right\} =$$

$$(1 + r_f)^{k+1} W_{T-k-1} + \frac{1}{2\lambda} E_{T-k-1} \{ \tilde{r}_{T-k} \}' \Lambda_{T-k-1}^{-1} \mu_{T-k-1} + \frac{1}{2\lambda} E_{T-k-1} \{ h_{T-k} \},$$

and by letting

$$h_{T-k-1} (\xi_{T-k-1}) = E_{T-k-1} \{ \tilde{r}_{T-k} \}' \Lambda_{T-k-1}^{-1} \mu_{T-k-1} + E_{T-k-1} \{ h_{T-k} \},$$

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we prove that

\[ K_{T-k-1} (W_{T-k-1}, \xi_{T-k-1}) = (1 + r_f)^{k+1} W_{T-k-1} + \frac{1}{2\lambda} h_{T-k-1}. \]

The risky holdings after a transaction is made at time \( t \), \( \tilde{u}_t \), are therefore independent of wealth and they just depend on the state realizations concerning the asset returns. On the other hand, the optimal expected reward is linear in wealth. The dependence between the risky investment at time \( t \) and the information available at \( t \), \( \xi_t \), is not so obvious, since the expectations appearing in the optimal control and value function cannot be calculated in closed-form. The only instance where a closed-form is available is under the assumption that the asset returns are independent and identically distributed, as we show in the next section. For all other asset return dynamics, the optimal control policy has to be derived numerically. Of course, this is only feasible for small dimensions.

### 2.3.1 Special Case: IID Returns

Let the excess return process to be given by

\[ \tilde{r}_t \equiv r_t - r_f \quad \epsilon_t \sim \mathcal{N}(0, \Sigma). \]

Then, the optimal control policy is myopic and equivalent to the strategy resulting from the solution to a series of single-period optimization problems. The coefficients appearing in the optimal control are constant and given by the following result.

**Corollary 2.1** Under IID returns, there is a closed-form solution to the optimal investment policy with Equations (2.7)-(2.10) simplifying to

\[
\begin{align*}
\Lambda_{T-k} &= \Sigma, \\
\mu_{T-k} &= \mu, \\
z_{T-k} &= k \mu' \Sigma^{-1} \mu, \\
h_{T-k} &= k \mu' \Sigma^{-1} \mu.
\end{align*}
\]
As a result, the only time dependence of the optimal control is through the term \((1 + r_f)^{k-1}\) appearing in the denominator of \(\tilde{u}^*_T\), and the optimal sequence of investment decisions is static.

For more complicated return dynamics a numerical procedure is implemented that is efficient for problems of small dimension. For high-dimensional problems it becomes infeasible, but we implement instead the approximation algorithms developed in the chapters that follow. We represent the functions \(\Lambda_{T-k}(\xi), \mu_{T-k}(\xi), z_{T-k}(\xi),\) and \(h_{T-k}(\xi)\) by their values over a spatial grid \(\{\xi^j : j = 1, \ldots, J\}\). For any given \(\xi\), the values of \(\Lambda_{T-k}(\xi), \mu_{T-k}(\xi), z_{T-k}(\xi),\) and \(h_{T-k}(\xi)\) are obtained from the \(\Lambda_{T-k}(\xi^j), \mu_{T-k}(\xi^j), z_{T-k}(\xi^j),\) and \(h_{T-k}(\xi^j)\) using a piecewise quadratic interpolation. The values of \(\Lambda_{T-k}(\xi^j), \mu_{T-k}(\xi^j), z_{T-k}(\xi^j),\) and \(h_{T-k}(\xi^j)\) are updated according to Equations (2.7)-(2.10). In addition, the expectations in (2.7)-(2.10) are evaluated by replacing them with the corresponding integrals. For all models considered, these integrals are obtained numerically by using Gauss-Hermite quadrature formulas.
Chapter 3

Quadratic Utility: Transaction Costs and Factor Models

In this chapter we examine the effect of transaction costs on dynamic portfolio strategies in discrete time of an investor who maximizes the expected terminal wealth of the portfolio, while minimizing its variance. Investors' portfolios are composed of investments in one riskless and $N$ risky assets. Adjustments in the portfolio are subject to transaction costs that are quadratic in the size of the trade.

In the presence of transaction costs, a closed-form solution for the optimal investment policy is unattainable. We propose instead approximation algorithms under the assumption that asset returns follow a multi-factor pricing model with serially correlated factors. We show that for small-dimensional problems, the approximate algorithms result in near optimal solutions and we develop insights about the qualitative behavior of the investment policy and its dependence on the size of transaction costs, time to maturity, asset correlations and volatilities.

We consider an institutional investor who faces the problem of making sequential investment decisions at discrete times $t = 0, \ldots, T - 1$. The asset return dynamics, $r_t$, are given by the multifactor pricing model described in Equations (1.1)-(1.2):

\[
\begin{align*}
    r_t &= c_t + A_t f_t + \epsilon_t, \\
    f_t &= d_{t-1} + B_{t-1} f_{t-1} + \eta_t,
\end{align*}
\]
where $K$ is the total number of factors, $r_t$ is the $N \times 1$ vector of the rate of returns, $f_t$ is the $K \times 1$ vector of the factor realizations at time $t$, $A_t$ is the $N \times K$ matrix of the factor sensitivities, $B_{t-1}$ is the $K \times K$ symmetric matrix of the factor correlations, $c_t$ and $d_{t-1}$ are $N \times 1$ and $K \times 1$ vectors of constants respectively, and $\epsilon_t$, $\eta_t$ are uncorrelated normally distributed random vectors with mean zero and covariance matrices $\Sigma_\epsilon$ and $\Sigma_\eta$ respectively. The state of the system at time $t = 0, 1, \ldots, T - 1$ consists of the asset holdings $(x^0_t, x_t)$ before a transaction is made at time $t$, and $f_t$ the factor realizations at time $t$. The portfolio manager faces the following dynamic optimization problem

$$\max_{\{u_0, u_1, \ldots, u_{T-1}\}} E_0 \{W_T\} - \lambda \Var_0 \{W_T\}$$

subject to

$$W_t = x^0_t + x_t' e$$
$$= (1 + r_f) \left[ x^0_{t-1} - e' u_{t-1} - e' \Gamma u^2_{t-1} \right] + (e + r_t)' (x_{t-1} + u_{t-1})$$
$$r_t = c_t + A_t f_t + \epsilon_t$$
$$f_t = d_{t-1} + B_{t-1} f_{t-1} + \eta_t$$
$$\epsilon_t \sim N(0, \Sigma_\epsilon) \quad \text{and} \quad \eta_t \sim N(0, \Sigma_\eta).$$

The remainder of this chapter is organized as follows. In Section 3.1, we present the single-period optimization problem and its closed-form solution. In Section 3.2, we present a structured approximation that uses characteristics of the optimal cost-to-go function at every point in time. In Section 3.3, we present an alternative approximate policy resulting from the solution of a series of stochastic optimization problems with different initial conditions and successively smaller investment horizon. In Section 3.4, we evaluate the relative performance of the proposed approximation algorithms and obtain insights about the qualitative behavior of the optimal portfolio composition over time.
3.1 The Single Period Problem

In this section, we show that there exists a closed-form solution for the single-period portfolio optimization problem, and obtain the optimal control policy at time $T-1$ that is linear in the risky holdings. The conditional mean of asset returns, $E_{T-1} \{e + r_T\} = e + c_T + A_T d_{T-1} + A_T B_{T-1} f_{T-1}$, is linear in the state variable $f_{T-1}$, and thus the optimal control at time $T-1$ as given by Equation (2.4) can be written as

$$
 u_{T-1}^* (x_{T-1}, f_{T-1}) = m_{T-1} + G_{T-1} f_{T-1} - L_{T-1} x_{T-1}, \quad (3.1)
$$

where the following constants are defined

$$
 Q_{T-1} = [2(1 + r_f) \Gamma + 2\lambda \begin{pmatrix} A_T \Sigma_\eta \Sigma_T & A_T \Sigma_\epsilon \end{pmatrix}]^{-1}, \quad (3.2)
$$

$$
 m_{T-1} = Q_{T-1} [-r_f e + c_T + A_T d_{T-1}],
$$

$$
 G_{T-1} = Q_{T-1} A_T B_{T-1},
$$

$$
 L_{T-1} = Q_{T-1} 2\lambda \begin{pmatrix} A_T \Sigma_\eta \Sigma_T & A_T \Sigma_\epsilon \end{pmatrix}. \quad (3.3)
$$

The function $V_{T-1}$ is linear in $x_{T-1}^0$ and quadratic in the state variables $x_{T-1}, f_{T-1}$. Using the fact that

$$
 e' \Gamma (m + G f - L x)^2 = \sum_{i=1}^N \tau_i \left[ \frac{m_i^2 + \left( \sum_{j=1}^K g_{ij} f_j \right)^2 + \left( \sum_{j=1}^N l_{ij} x_j \right)^2}{2 m_i \left( \sum_{j=1}^K g_{ij} f_j \right) - 2 m_i \left( \sum_{j=1}^N l_{ij} x_j \right)} \right] =
$$

$$
 e' \Gamma m^2 + 2 \sum_{i=1}^N \tau_i m_i \left( \sum_{j=1}^K g_{ij} f_j \right) - 2 \sum_{i=1}^N \tau_i m_i \left( \sum_{j=1}^N l_{ij} x_j \right) -
$$

$$
 2 \sum_{i=1}^N \tau_i \left( \sum_{j=1}^K g_{ij} f_j \right) \left( \sum_{j=1}^N l_{ij} x_j \right) + \sum_{i=1}^N \tau_i \left( \sum_{j=1}^K g_{ij} f_j \right)^2 + \sum_{i=1}^N \tau_i \left( \sum_{j=1}^N l_{ij} x_j \right)^2 =
$$

$$
 e' \Gamma m^2 + 2 m' \Gamma G f - 2 m' \Gamma L x - 2 f' G' \Gamma L x +
$$

$$
 f' G' \Gamma G f + x' L' \Gamma L x, \quad (3.4)
$$

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the quantity \( K_{T-1} \) and the value function \( V_{T-1} \) reduce to

\[
K_{T-1} \left( x_{T-1}^0, x_{T-1}, f_{T-1} \right) = h_{T-1} + (1 + r_f) x_{T-1}^0 + a'_{T-1} x_{T-1} + q'_{T-1} f_{T-1} -
\]
\[
x'_{T-1} D_{T-1} x_{T-1} - f'_{T-1} J_{T-1} f_{T-1} + f'_{T-1} Y_{T-1} x_{T-1},
\]

\[
V_{T-1} \left( x_{T-1}^0, x_{T-1}, f_{T-1} \right) = z_{T-1} + (1 + r_f) x_{T-1}^0 + b'_{T-1} x_{T-1} + p'_{T-1} f_{T-1} -
\]
\[
x'_{T-1} C_{T-1} x_{T-1} - f'_{T-1} H_{T-1} f_{T-1} + f'_{T-1} S_{T-1} x_{T-1},
\]

where the encountered constants are given by

\[
h_{T-1} = - (1 + r_f) e' \ m_{T-1} - (1 + r_f) e' \ \Gamma \ m_{T-1}^2 + (e' c + A_T \ d_{T-1})' \ m_{T-1}, \quad (3.5)
\]

\[
z_{T-1} = h_{T-1} - \lambda \ m'_{T-1} \ (A_T \ \Sigma_{\eta} A_T' + \Sigma_{c}) \ m_{T-1},
\]

\[
a_{T-1} = (1 + r_f) L_{T-1} e + 2 (1 + r_f) L'_{T-1} \ \Gamma \ m_{T-1} - (I - L_{T-1})' \ (e + c_T + A_T \ d_{T-1}),
\]

\[
b_{T-1} = a_{T-1} - 2 \lambda \ (I - L_{T-1})' \ (A_T \ \Sigma_{\eta} A_T' + \Sigma_{c}) \ m_{T-1},
\]

\[
q_{T-1} = - (1 + r_f) G'_{T-1} e - 2 (1 + r_f) G'_{T-1} \ \Gamma \ m_{T-1} + G'_{T-1} \ (e + c_T + A_T \ d_{T-1})
\]
\[
+ B'_{T-1} A_T' \ m_{T-1},
\]

\[
p_{T-1} = q_{T-1} - 2 \lambda \ G'_{T-1} \ (A_T \ \Sigma_{\eta} A_T' + \Sigma_{c}) \ m_{T-1},
\]

\[
D_{T-1} = (1 + r_f) L'_{T-1} \ \Gamma \ L_{T-1},
\]

\[
C_{T-1} = D_{T-1} + \lambda \ (I - L_{T-1})' \ (A_T \ \Sigma_{\eta} A_T' + \Sigma_{c}) \ (I - L_{T-1}),
\]

\[
J_{T-1} = (1 + r_f) G'_{T-1} \ \Gamma \ G_{T-1} - B'_{T-1} A_T' \ G_{T-1},
\]

\[
H_{T-1} = J_{T-1} + \lambda \ G'_{T-1} \ (A_T \ \Sigma_{\eta} A_T' + \Sigma_{c}) \ G_{T-1},
\]

\[
Y_{T-1} = 2 (1 + r_f) G'_{T-1} \ \Gamma \ L_{T-1} + B'_{T-1} A_T' \ (I - L_{T-1}),
\]

\[
S_{T-1} = Y_{T-1} - 2 \lambda \ G'_{T-1} \ (A_T \ \Sigma_{\eta} A_T' + \Sigma_{c}) \ (I - L_{T-1}). \quad (3.6)
\]

In the following section we propose an approximation algorithm that uses characteristics of the optimal cost-to-go function at every point in time and investigate its behavior.
3.2 Approximation A: A Structured Approximation

In order to motivate the algorithm that follows, consider the value function at time $T-2$:

$$V_{T-2}\left(x_{T-2}^0, x_{T-2}, f_{T-2}\right) = \max_{\{u_{T-2}\}} E_{T-2} \left\{ V_{T-1}\left(x_{T-1}^0, x_{T-1}, f_{T-1}\right) - \lambda Var_{T-2}\{K_{T-1}\} = \right.$$  

$$\max_{\{u_{T-2}\}} E_{T-2} \left\{ x_{T-1} + (1+r_f) x_{T-1}^0 + b'_{T-1} x_{T-1} + p'_{T-1} f_{T-1} - \right.$$  

$$\left. C_{T-1} x_{T-1} - f'_{T-1} H_{T-1} f_{T-1} + f'_{T-1} S_{T-1} x_{T-1} \right\} - \lambda Var_{T-2} \left\{ h_{T-1} + (1+r_f) x_{T-1}^0 + a'_{T-1} x_{T-1} + q'_{T-1} f_{T-1} - \right.$$  

$$\left. D_{T-1} x_{T-1} - f'_{T-1} J_{T-1} f_{T-1} + f'_{T-1} Y_{T-1} x_{T-1} \right\}. \right)$$

Substituting for the wealth and asset return dynamics we obtain that

$$V_{T-2}\left(x_{T-2}^0, x_{T-2}, f_{T-2}\right) = \max_{\{u_{T-2}\}} E_{T-2} \{ \Phi_1 \} - \lambda Var_{T-2} \{ \Phi_2 \}, \quad (3.7)$$

where

$$\Phi_1 = x_{T-1} + (1+r_f)^2 \left[ x_{T-2}^0 - e' u_{T-2} - e' \Gamma u_{T-2}^2 \right] +$$

$$\left[ b'_{T-1} (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1}) \right]'$$

$$(x_{T-2} + u_{T-2}) + p'_{T-1} (d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1}) - (x_{T-2} + u_{T-2})'$$

$$\left\{ C_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1}) \right] \right\}$$

$$(x_{T-2} + u_{T-2}) - (d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1})' H_{T-1} (d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1}) +$$

$$(d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1})'$$

$$\left\{ S_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1}) \right\}$$

$$(x_{T-2} + u_{T-2}),$$

that can be written for notation convenience as

$$\Phi_1 (f, \hat{x}) = a + b' f + c' \hat{x} + f' A_1 f + f' A_2 \hat{x} + \hat{x}' A_3 \hat{x} + L_1 (f, \hat{x}) + L_2 (f, \hat{x}),$$

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for \( \tilde{x} = x + u \). Note that \( L_1, L_2 \) are third and fourth order polynomials with respect to the state variables. Moreover,

\[
\phi_2 = h_{T-1} + (1 + r_f)^2 \left[ z_{T-2}^0 - e' u_{T-2} - e' \Gamma u_{T-2}^2 \right] +
\]
\[
\left[ a_{T-1} \otimes \left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1} \right) \right]'
\]
\[
(x_{T-2} + u_{T-2}) + q_{T-1}' \left( d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1} \right) - (x_{T-2} + u_{T-2})'
\]
\[
\left\{ D_{T-1} \otimes \left[ \begin{array}{c}
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1})'
\end{array} \right] \right\}
\]
\[
(x_{T-2} + u_{T-2}) - (d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1})' J_{T-1} \left( d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1} \right) +
\]
\[
(d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1})' Y_{T-1} \otimes e \left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1} \right)'
\]
\[
(x_{T-2} + u_{T-2}),
\]

(3.8)

where the notation \( \otimes \) denotes the Kronecker product of matrices; that is the element-by-element multiplication of the corresponding matrices. For notation convenience we rewrite \( \phi_2 \) as

\[
\phi_2 = g' \eta + q' \epsilon + \eta' A \eta + \epsilon' B \epsilon + \eta' C \eta,
\]

where \( g, q, A, B, C \) are independent of \( \eta, \epsilon \), but depend on \( (f, x + u) \). The DP recursion is solvable when the value function is quadratic in the state variables (see Section 2.2). Unfortunately, when substituting for the wealth and return dynamics in Equation (3.7) we obtain third and fourth order terms with respect to \( f \) and \( \tilde{x} = (x + u) \) of the type:

\[
L_1 (f, \tilde{x}) = (d + B f)' \{ S \otimes e \tilde{x}' \} (c + D f),
\]
\[
L_2 (f, \tilde{x}) = (c + D f)' \{ C \otimes \tilde{x} \tilde{x}' \} (c + D f).
\]

Our proposed approximation is motivated by the need to keep the quadratic nature of the value function, i.e. we want to approximate \( L_1 (f, \tilde{x}), L_2 (f, \tilde{x}) \) by quadratic functions in \( f, \tilde{x} \):
• For $L_1$, we achieve this by performing a Taylor’s series expansion around the expectation of $f$ conditioned on the information available at time $t$, defined as $E_{f,T-k}$

$$E_{f,T-k} = E\{f_{T-k} \mid f_0\} = \sum_{m=1}^{T-k} \left( \prod_{l=1}^{m-1} B_{T-k-l} \right) d_{T-k-m} + \left( \prod_{m=1}^{T-k} B_{T-k-m} \right) f_0,$$

(3.9)

keeping up to second order terms. If $d$ and $B$ do not depend on time, we approximate around the unconditional expectation $(I - B)^{-1} d$.

• For $L_2$, we achieve this by replacing $f$ by $E_{f,T-k}$.

Regarding the terms in Equation (3.8), we perform the following operations:

• We approximate the matrices $A(\bar{x})$, $B(\bar{x})$, $C(\bar{x})$ by $A(x_0)$, $B(x_0)$, $C(x_0)$ replacing $\bar{x}$ by the initial risky holdings $x_0$.

• We approximate the vectors $g(f,\bar{x})$ and $q(f,\bar{x})$ with linear functions in the state variables.

All approximations are performed using Taylor’s series expansion around the expectation of the factor realizations and the initial risky holdings. The choice of the approximating point is not crucial to the performance of the algorithm as manifested by a series of computational experiments. The details of the approximation procedure are presented below.

**Approximations Performed in $\Phi_1$**

The expectation of $\Phi_1$ conditioned on the information available at time $T-2$ can be written as

$$E_{T-2}\{\Phi_1\} = z_{T-1} + (1 + r_f)^2 \left[ x_{T-2}^0 - e' u_{T-2} - e' \Gamma u_{T-2}^2 \right] +$$

$$\left[ b_{T-1} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \right]' (x_{T-2} + u_{T-2}) +$$

$$p_{T-1}' \left( d_{T-2} + B_{T-2} f_{T-2} \right) - (x_{T-2} + u_{T-2})'$$

$$\left\{ C_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \right] + \right\} (x_{T-2} + u_{T-2}) -$$

$$\left\{ C_{T-1} \otimes \left[ A_{T-1} \Sigma_\eta A_{T-1}' + \Sigma_e \right] \right\} (d_{T-2} + B_{T-2} f_{T-2})' H_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - E_{T-2}\{\eta_{T-1} H_{T-1} \eta_{T-1}\} +$$

$$\left( d_{T-2} + B_{T-2} f_{T-2} \right) \left\{ S_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \right\}$$

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\[(\mathbf{x}_{T-2} + \mathbf{u}_{T-2}) + E_{T-2} \left\{ \eta^\prime_{T-1} \left[ S_{T-1} \otimes \mathbf{e} \left( \begin{array}{c} e + c_{T-1} + A T_{-1} \mathbf{d}_{T-2} + A T_{-1} B T_{-2} f_{T-2+} \\ A T_{-1} \eta_{T-1} + \epsilon_{T-1} \end{array} \right) \right] \right\} \right\} = \mathbf{x}_{T-2} + \mathbf{u}_{T-2} \).

But, since \(\eta_{T-1}\) is a normally distributed random vector with mean 0 and covariance matrix \(\Sigma_\eta\), and uncorrelated with \(\epsilon_{T-1}\), we obtain that

\[E_{T-2} \{ \eta^\prime_{T-1} H_{T-1} \eta_{T-1} \} = e^\prime \{ H_{T-1} \otimes \Sigma_\eta \} e,\]

and

\[E_{T-2} \left\{ \eta^\prime_{T-1} \left[ S_{T-1} \otimes \mathbf{e} \right] \eta^\prime_{T-1} \mathbf{A}_{T-1}^\prime \right\} = E_{T-2} \left\{ [S_{T-1} \otimes \mathbf{e} \eta^\prime_{T-1} \mathbf{A}_{T-1}^\prime] \eta_{T-1} \right\}^\prime.\]

We evaluate the above expression in the following lemma:

**Lemma 3.1** For a normally distributed random vector \(\eta\) of mean 0 covariance matrix \(\Sigma_\eta\), and arbitrary matrices \(S\) and \(A\) of dimensions \((K \times N)\) and \((N \times K)\) respectively, the following relation holds:

\[E \{ [S \otimes e \eta^\prime A^\prime] \eta \} = sa,\]

where the \(i\)-th element of the vector \(sa\) is the \(ii\)-th element of the matrix \(S^\prime \Sigma_\eta A^\prime:\)

\[[sa]_i = [S^\prime \Sigma_\eta A^\prime]_{ii} \quad \text{for} \quad i = 1, \ldots, N.\]

**Proof.** We write \([S \otimes e \eta^\prime A^\prime]\) as

\[
\begin{bmatrix}
S_{11} & \cdots & S_{1N} \\
\vdots & \ddots & \vdots \\
S_{K1} & \cdots & S_{KN}
\end{bmatrix} \otimes \begin{bmatrix}
\eta_1 & \cdots & \eta_K \\
\vdots & \ddots & \vdots \\
\eta_1 & \cdots & \eta_K
\end{bmatrix} \begin{bmatrix}
A_{11} & \cdots & A_{N1} \\
\vdots & \ddots & \vdots \\
A_{1K} & \cdots & A_{NK}
\end{bmatrix} =
\]

66
\[
\begin{bmatrix}
S_{11} \sum_{j=1}^{K} A_{1j} \eta_j & \cdots & S_{1N} \sum_{j=1}^{K} A_{Nj} \eta_j \\
\vdots & \ddots & \vdots \\
S_{K1} \sum_{j=1}^{K} A_{1j} \eta_j & \cdots & S_{KN} \sum_{j=1}^{K} A_{Nj} \eta_j 
\end{bmatrix}.
\]

Thus, the term \([S \otimes e \eta' A']' \eta\) is equal to

\[
\begin{bmatrix}
S_{11} \sum_{j=1}^{K} A_{1j} \eta_j & \cdots & S_{K1} \sum_{j=1}^{K} A_{1j} \eta_j \\
\vdots & \ddots & \vdots \\
S_{1N} \sum_{j=1}^{K} A_{Nj} \eta_j & \cdots & S_{KN} \sum_{j=1}^{K} A_{Nj} \eta_j 
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\vdots \\
\eta_K 
\end{bmatrix} =
\begin{bmatrix}
\sum_{i=1}^{K} \eta_i \sum_{j=1}^{K} A_{1j} \eta_j \\
\sum_{i=1}^{K} \eta_i \sum_{j=1}^{K} A_{Nj} \eta_j 
\end{bmatrix}
\]

and its expectation is equal to

\[
E\{[S \otimes e \eta' A']' \eta\} =
\begin{bmatrix}
\sum_{i=1}^{K} \eta_i \sum_{j=1}^{K} A_{1j} \Sigma_{ij} \\
\vdots \\
\sum_{i=1}^{K} \eta_i \sum_{j=1}^{K} A_{Nj} \Sigma_{ij} 
\end{bmatrix},
\]

where \(\Sigma_{ij}\) is the \(ij\)-th element of the covariance matrix \(\Sigma_{\eta}\). On the other hand, the matrix \(S' \Sigma_{\eta} A'\) is just

\[
\begin{bmatrix}
S_{11} & \cdots & S_{K1} \\
\vdots & \ddots & \vdots \\
S_{1N} & \cdots & S_{KN} 
\end{bmatrix}
\begin{bmatrix}
\sum_{j=1}^{K} (\Sigma_{1j} A_{1j}) & \cdots & \sum_{j=1}^{K} (\Sigma_{1j} A_{Nj}) \\
\vdots & \ddots & \vdots \\
\sum_{j=1}^{K} (\Sigma_{Kj} A_{1j}) & \cdots & \sum_{j=1}^{K} (\Sigma_{Kj} A_{Nj}) 
\end{bmatrix}
= \begin{bmatrix}
\sum_{i=1}^{K} S_{i1} \sum_{j=1}^{K} (\Sigma_{ij} A_{1j}) & \cdots & \sum_{i=1}^{K} S_{i1} \sum_{j=1}^{K} (\Sigma_{ij} A_{Nj}) \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{K} S_{iN} \sum_{j=1}^{K} (\Sigma_{ij} A_{1j}) & \cdots & \sum_{i=1}^{K} S_{iN} \sum_{j=1}^{K} (\Sigma_{ij} A_{Nj}) 
\end{bmatrix},
\]

and thus the result follows. \(\blacksquare\)

For convenience, construct the symmetric matrix \(\mathbf{B} \Theta_{T-1}\) of dimension \((N \times N)\) as follows:

\[
\mathbf{B} \Theta_{T-1} = \text{diag}(\mathbf{b}_{T-1}).
\]
Consequently, using Lemma 3.1 $E_{T-2} \{ \Phi_1 \}$ is provided by

$$E_{T-2} \{ \Phi_1 \} = x_{T-1} + (1 + r_f)^2 \left[ x_{T-2}^2 - e' u_{T-2} - e' \Gamma u_{T-2}^2 \right] + (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \Theta_{T-1} (x_{T-2} + u_{T-2}) + p^{T-1}_{T-2} (d_{T-2} + B_{T-2} f_{T-2}) - (x_{T-2} + u_{T-2})'$$

$$\left\{ C_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \right] + \right\} (x_{T-2} + u_{T-2}) - (d_{T-2} + B_{T-2} f_{T-2})' H_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - e' \{ H_{T-1} \otimes \Sigma_{\eta} \} e + (d_{T-2} + B_{T-2} f_{T-2})' \{ S_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \} \}

\begin{align*}
(x_{T-2} + u_{T-2}) & + s a'_{T-1} (x_{T-2} + u_{T-2})
\end{align*}

with the $i$-th element of vector $s a_{T-1}$ being equal to the $ii$-th element of matrix $S_{T-1}^T \Sigma_{\eta} A_{T-1}'$. Notice that $E_{T-2} \{ \Phi_1 \}$ is a quadratic not-separable function of the state variables $x_{T-2}$ and $f_{T-2}$. The first approximation performed involves making $E_{T-2} \{ \Phi_1 \}$ separable in the state variables $x_{T-2}$ and $f_{T-2}$, by concentrating on the terms $L_1$ and $L_2$ appearing in $E_{T-2} \{ \Phi_1 \}$:

$$L_1 = (d_{T-2} + B_{T-2} f_{T-2})' \{ S_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \} \}
\begin{align*}
(x_{T-2} + u_{T-2}) ,
\end{align*}

$$L_2 = (x_{T-2} + u_{T-2})'$$

$$\left\{ C_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \right] \right\} (x_{T-2} + u_{T-2}).$$

Thus, we perform the following operations:

1. Approximate $L_1$ with a linear function in $f_{T-2}$ by using the first-order Taylor's expansion around the expectation of $f_{T-2}$ conditioned on the information available at time 0, defined as $E_{f_{T-2}}$:

$$E_{f_{T-2}} \equiv E \{ f_{T-2} | f_0 \} = \sum_{m=1}^{T-2} \left( \prod_{l=1}^{m-1} B_{T-2-l} \right) d_{T-2-m} + \left( \prod_{m=1}^{T-2} B_{T-2-m} \right) f_0.$$

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If \( d \) and \( B \) do not depend on time, we approximate around the unconditional expectation \( E_f = (I - B)^{-1} d \).

2. Approximate \( \mathcal{L}_2 \) with a quadratic function in \((x_{T-2} + u_{T-2})\) by replacing the state variable \( f_{T-2} \) with its expectation \( E_{f,T-2} \).

In response, the term \( \mathcal{L}_1 \) can be written as

\[
\mathcal{L}_1 = d'_{T-2} \left\{ S_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \right] \right\} (x_{T-2} + u_{T-2}) + \\
E'_{f,T-2} B'_{T-2} \left\{ S_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2})' \right] \right\} (x_{T-2} + u_{T-2}) + \\
E'_{f,T-2} B'_{T-2} \left\{ S_{T-1} \otimes \left[ (x_{T-2} + u_{T-2})' \right] \right\} A_{T-1} B_{T-2} f_{T-2},
\]

and we concentrate on the only part of \( \mathcal{L}_1 \) that needs to be approximated

\[
f'_{T-2} B'_{T-2} \left\{ S_{T-1} \otimes \left[ (x_{T-2} + u_{T-2})' \right] \right\} A_{T-1} B_{T-2} f_{T-2}.
\]

Let

\[
\Delta = B'_{T-2} \left\{ S_{T-1} \otimes \left[ (x_{T-2} + u_{T-2})' \right] \right\} A_{T-1} B_{T-2}.
\]

Since \( \Delta \) is not a symmetric matrix, the approximation yields

\[
f'_{T-2} \Delta f_{T-2} \approx E'_{f,T-2} \Delta E_{f,T-2} + [(\Delta + \Delta') E_{f,T-2}]' (f_{T-2} - E_{f,T-2}) = -E'_{f,T-2} \Delta' E_{f,T-2} + E'_{f,T-2} (\Delta + \Delta') f_{T-2}.
\]

As a result, \( \mathcal{L}_1 \) is approximated by

\[
\mathcal{L}_1 \approx d'_{T-2} \left\{ S_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \right] \right\} \left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} \right) + \\
f'_{T-2} B'_{T-2} \left\{ S_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2})' \right] \right\} \left( e + c_{T-1} + A_{T-1} d_{T-2} \right) - \\
E'_{f,T-2} B'_{T-2} A_{T-1} \left\{ S_{T-1} \otimes \left[ (x_{T-2} + u_{T-2})' \right] \right\} B_{T-2} E_{f,T-2} + \\
E'_{f,T-2} \left\{ B_{T-2} A_{T-1} \left\{ S_{T-1} \otimes \left[ (x_{T-2} + u_{T-2})' \right] \right\} A_{T-1} B_{T-2} + \\
B_{T-2} A_{T-1} \left\{ S_{T-1} \otimes \left[ (x_{T-2} + u_{T-2})' \right] \right\} B_{T-2} \right\} f_{T-2}.
\]
In order to simplify $\mathcal{L}_1$ we prove the following lemma.

**Lemma 3.2** For arbitrary vectors $d$, $x$, $c$ and matrices $S$ of dimensions $(K \times 1)$, $(N \times 1)$, $(N \times 1)$ and $(K \times N)$ respectively, the following relation is valid

$$d' \{S \otimes [e \ x']\} \ c = x' \{S' d \otimes c\}.$$

**Proof.** We can write $d' \{S \otimes [e \ x']\} \ c$ as

$$\begin{bmatrix} d_1 & \ldots & d_K \end{bmatrix} \begin{bmatrix} S_{11} & \ldots & S_{1N} \\ \vdots & \ddots & \vdots \\ S_{K1} & \ldots & S_{KN} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} x_1 & \ldots & x_N \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} =$$

$$\begin{bmatrix} d_1 & \ldots & d_K \end{bmatrix} \begin{bmatrix} S_{11} x_1 & \ldots & S_{1N} x_N \\ \vdots & \ddots & \vdots \\ S_{K1} x_1 & \ldots & S_{KN} x_N \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} =$$

$$d_1 (S_{11} x_1 c_1 + \ldots + S_{1N} x_N c_N) + \ldots + d_K (S_{K1} x_1 c_1 + \ldots + S_{KN} x_N c_N) =$$

$$x_1 (S_{11} d_1 c_1 + \ldots + S_{1N} d_K c_1) + \ldots + x_N (S_{1N} d_1 c_N + \ldots + S_{KN} d_K c_N) =$$

$$x_1 c_1 \sum_{i=1}^{K} S_{i1} d_i + \ldots + x_N c_N \sum_{i=1}^{K} S_{iN} d_i.$$ 

But since

$$S' d \otimes c = \begin{bmatrix} \sum_{i=1}^{K} S_{i1} d_i \\ \vdots \\ \sum_{i=1}^{K} S_{iN} d_i \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} c_1 \sum_{i=1}^{K} S_{i1} d_i \\ \vdots \\ c_N \sum_{i=1}^{K} S_{iN} d_i \end{bmatrix},$$

$d' \{S \otimes [e \ x']\} \ c$ reduces to

$$x' \{S' d \otimes c\}.$$ 

Using Lemma 3.2, $\mathcal{L}_1$ reduces to

$$\mathcal{L}_1 \approx (x_{T-2} + u_{T-2})' \{S'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})\} + (x_{T-2} + u_{T-2})' \{S'_{T-1} B_{T-2} f_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2})\} - (x_{T-2} + u_{T-2})' \{S'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} E_{f,T-2}\} +$$

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\[(x_{T-2} + u_{T-2})' \{ S'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} f_{T-2} \} + \]
\[(x_{T-2} + u_{T-2})' \{ S'_{T-1} B_{T-2} f_{T-2} \otimes A_{T-1} B_{T-2} E_{f,T-2} \} . \]

But since the following property holds for Kronecker products: \( c \otimes A d + c \otimes A f = c \otimes A (d + f) \),
we obtain that

\[ L_1 \approx (x_{T-2} + u_{T-2})' \{ S'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \} + \]
\[ (x_{T-2} + u_{T-2})' \{ S'_{T-1} B_{T-2} f_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \} + \]
\[ (x_{T-2} + u_{T-2})' \{ S'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} (f_{T-2} - E_{f,T-2}) \} . \quad (3.10) \]

The second approximating operation performed involves the term \( L_2 \). We replace \( f_{T-2} \) with its conditional expectation \( E_{f,T-2} \) and thus approximate \( L_2 \) with

\[ L_2 \approx (x_{T-2} + u_{T-2})' \left\{ C_{T-1} \otimes \left( \begin{array}{c}
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})'
\end{array} \right) \right\} \]
\[ (x_{T-2} + u_{T-2}) . \quad (3.11) \]

Using Equations (3.10)-(3.11), the conditional expectation of \( \Phi_1 \) can finally be approximated by

\[ E_{T-2} \{ \hat{\Phi}_1 \} = z_{T-1} + (1 + r_f)^2 \left[ x_{T-2}^2 + e' u_{T-2} - e' \Gamma u_{T-2}^2 \right] + \]
\[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' B \Theta_{T-1} (x_{T-2} + u_{T-2}) + \]
\[ p'_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - (x_{T-2} + u_{T-2})' \]
\[ \left\{ C_{T-1} \otimes \left( \begin{array}{c}
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})'
\end{array} \right) \right\} (x_{T-2} + u_{T-2}) - \]
\[ + C_{T-1} \otimes [A_{T-1} \Sigma A_{T-1}' + \Sigma e] \]
\[ (d_{T-2} + B_{T-2} f_{T-2})' H_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - e' \{ H_{T-1} \otimes \Sigma e \} e + \]
\[ (x_{T-2} + u_{T-2})' \{ S'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \} + \]
\[ (x_{T-2} + u_{T-2})' \{ S'_{T-1} B_{T-2} f_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \} + \]
\[ 71 \]
\[ (x_{T-2} + u_{T-2})' \{ S_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} (f_{T-2} - E_{f,T-2}) \} + (x_{T-2} + u_{T-2})' \sigma a_{T-1}. \] (3.12)

**Approximations Performed in \( \Phi_2 \)**

The variance of \( \Phi_2 \) conditioned on the information available at time \( T-2 \) can be written as

\[
\text{Var}_{T-2} \{ \Phi_2 \} = \begin{bmatrix}
(x_{T-2} + u_{T-2})' A \Theta_{T-1} (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) + a_{T-1} \eta_{T-1} \\
(x_{T-2} + u_{T-2})' \\
\{ D_{T-1} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) (A_{T-1} \eta_{T-1} + \epsilon_{T-1})' \} \\
(x_{T-2} + u_{T-2}) - (x_{T-2} + u_{T-2})' \\
\{ D_{T-1} \otimes (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \} \\
(x_{T-2} + u_{T-2}) - (x_{T-2} + u_{T-2})' \\
\{ D_{T-1} \otimes (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) (A_{T-1} \eta_{T-1} + \epsilon_{T-1})' \} (x_{T-2} + u_{T-2}) - 2 (d_{T-2} + B_{T-2} f_{T-2})' J_{T-1} \eta_{T-1} - \eta_{T-1} J_{T-1} \eta_{T-1} + \\
(d_{T-2} + B_{T-2} f_{T-2})' \{ Y_{T-1} \otimes e (A_{T-1} \eta_{T-1} + \epsilon_{T-1})' \} (x_{T-2} + u_{T-2}) - \\
\eta_{T-1}' \{ Y_{T-1} \otimes e \left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + \right) \} \\
(x_{T-2} + u_{T-2})
\end{bmatrix}
\]

where the diagonal matrix \( A \Theta_{T-1} \) is defined as follows

\[
A \Theta_{T-1} = \text{diag}(a_{T-1}).
\]

We observe that \( \Phi_2 \) is quadratic in \( (x_{T-2} + u_{T-2}) \) and thus its conditional variance is a fourth-order polynomial in the control variable \( u_{T-2} \). As a result, the optimization problem cannot be solved in closed-form. In response, we perform the following operation:

1. Approximate \( \Phi_2 \) with a linear and separable function in \( (x_{T-2} + u_{T-2}) \) and \( f_{T-2} \), by using the first-order Taylor’s expansion around the initial risky holdings \( x_0 \) and the conditional expectation \( E_{f,T-2} \).
We concentrate on the term $\mathcal{L}_3$ that is defined as

\[
\mathcal{L}_3 = 2 \left( A_{T-1} \eta_{T-1} + \epsilon_{T-1} \right)' \left\{ D_{T-1} \otimes (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right\} \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) + \\
\left( A_{T-1} \eta_{T-1} + \epsilon_{T-1} \right)' \left\{ D_{T-1} \otimes (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right\} \left( A_{T-1} \eta_{T-1} + \epsilon_{T-1} \right) - \\
f'_{T-2} B_{T-2} \left\{ Y_{T-1} \otimes e \left( A_{T-1} \eta_{T-1} + \epsilon_{T-1} \right)' \right\} (x_{T-2} + u_{T-2}) - \\
\eta'_{T-1} \left\{ Y_{T-1} \otimes e \left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1} \right)' \right\} \\
(x_{T-2} + u_{T-2}).
\]

Next we prove some formulae necessary for the approximation procedure.

**Lemma 3.3** For arbitrary matrices $D$, $A$ and vectors $x$, $f$ of dimensions $(N \times N)$, $(N \times K)$, $(N \times 1)$ and $(K \times 1)$ respectively, the first-order Taylor's expansion of $\{D \otimes x \ x'\} A f$ around $x_0$ and $E_f$ can be written as

\[
\left[ \left\{ D \otimes e \ (A \ E_f)' \otimes x_0 \ e' \right\} + DE \right] (x - x_0) + \{D \otimes x_0 \ x_0'\} A f,
\]

where the matrix $DE$ is constructed as follows:

\[
DE = \text{diag} \left[ \{D \otimes e \ x_0'\} \ A \ E_f \right].
\]

**Proof.** We first derive the effect of the differentiation of $\{D \otimes x \ x'\} A f$ with respect to $x$. We can write $\{D \otimes x \ x'\} A f$ as

\[
\left[
D_{11} x_1^2 (A_{11} f_1 + \ldots + A_{1K} f_K) + \ldots + D_{1N} x_1 x_N (A_{N1} f_1 + \ldots + A_{NK} f_K)
\right.
\]

\[
\vdots
\]

\[
\left[
D_{N1} x_1 x_N (A_{11} f_1 + \ldots + A_{1K} f_K) + \ldots + D_{NN} x_N^2 (A_{N1} f_1 + \ldots + A_{NK} f_K)
\right].
\]

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Consider for convenience the first element of the above vector. Its partial derivative with respect to \(x\) estimated at \((x_0, E_f)\) multiplied with \((x - x_0)\) is

\[
\begin{bmatrix}
2D_{11} x_0^1 (A_{11} E_{f,1} + \ldots + A_{1K} E_{f,K}) + \ldots + \\
D_{1N} x_0^N (A_{N1} E_{f,1} + \ldots + A_{NK} E_{f,K}) \\
D_{1N} x_0^1 (A_{N1} E_{f,1} + \ldots + A_{NK} E_{f,K}) (x_N - x_0^N)
\end{bmatrix} (x_1 - x_0^1) + \ldots +
\]

Therefore, we must construct a matrix that carries the above expression as its first row. Consider the matrix \(T_1 = \{D \otimes e (A E_f)' \otimes x_0 e'\}\); it can be written as

\[
T_1 = \begin{bmatrix}
D_{11} x_0^1 (A_{11} E_{f,1} + \ldots + A_{1K} E_{f,K}) & \ldots & D_{1N} x_0^1 (A_{N1} E_{f,1} + \ldots + A_{NK} E_{f,K}) \\
\vdots & \ddots & \vdots \\
D_{N1} x_0^N (A_{11} E_{f,1} + \ldots + A_{1K} E_{f,K}) & \ldots & D_{NN} x_0^N (A_{N1} E_{f,1} + \ldots + A_{NK} E_{f,K})
\end{bmatrix}.
\]

In addition, let \(T_2 = DE\). The \(ii\)-th element of \(T_2\) is the \(i\)-th element of the vector

\[
\{D \otimes e x_0^i\} A E_f
\]

that is just

\[
\begin{bmatrix}
D_{11} x_0^1 (A_{11} E_{f,1} + \ldots + A_{1K} E_{f,K}) + \ldots + D_{1N} x_0^N (A_{N1} E_{f,1} + \ldots + A_{NK} E_{f,K}) \\
\vdots \\
D_{N1} x_0^1 (A_{11} E_{f,1} + \ldots + A_{1K} E_{f,K}) + \ldots + D_{NN} x_0^N (A_{N1} E_{f,1} + \ldots + A_{NK} E_{f,K})
\end{bmatrix}.
\]

Therefore, \(\{D \otimes x x'\} A f\) can be approximated with

\[
\{D \otimes x x'\} A f \approx \{D \otimes x_0 x_0'\} A E_f + [T_1 + T_2] (x - x_0) + \{D \otimes x_0 x_0'\} A (f - E_f).
\]

**Lemma 3.4** For arbitrary matrices \(Y\), \(A\) and vectors \(x\), \(f\) of dimensions \((K \times N)\), \((N \times K)\), \((N \times 1)\) and \((K \times 1)\) respectively, the first-order Taylor's expansion of \(\{Y \otimes e x'\} Af\) around
\( x_0 \) and \( E_f \) can be written as

\[
\left\{ Y \otimes e \ (A \ E_f)' \right\} (x - x_0) + \left\{ Y \otimes e \ x'_0 \right\} A f.
\]

**Proof.** We proceed similarly as before. If we express \( \left\{ Y \otimes e \ x' \right\} A f \) as

\[
\begin{bmatrix}
Y_{11} \ x_1 (A_{11} f_1 + \ldots + A_{1K} f_K) + \ldots + Y_{1N} \ x_N (A_{N1} f_1 + \ldots + A_{NK} f_K)
\vdots \\
Y_{K1} \ x_1 (A_{11} f_1 + \ldots + A_{1K} f_K) + \ldots + Y_{KN} \ x_N (A_{N1} f_1 + \ldots + A_{NK} f_K)
\end{bmatrix},
\]

the result follows easily. \( \blacksquare \)

Using the above formulae, we can now approximate \( L_3 \) with

\[
L_3 \approx \begin{bmatrix}
2 (A_{T-1} \eta_{T-1} + \epsilon_{T-1})'
\left[ \left\{ D_{T-1} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \otimes x_0 \ e' \right\} + D E_{T-2} \right] \\
(x_{T-2} + u_{T-2} - x_0) + 2 (A_{T-1} \eta_{T-1} + \epsilon_{T-1})' \left\{ D_{T-1} \otimes x_0 \ x'_0 \right\} \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \\
\end{bmatrix} + \\
\begin{bmatrix}
-x_0 \left\{ D_{T-1} \otimes (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) (A_{T-1} \eta_{T-1} + \epsilon_{T-1})' \right\} x_0 + \\
2 x_0 \left\{ D_{T-1} \otimes (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) (A_{T-1} \eta_{T-1} + \epsilon_{T-1})' \right\} (x_{T-2} + u_{T-2}) \\
-x_0 \left\{ Y_{T-1}' \otimes (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) e' \right\} B_{T-2} E_{f,T-2} \\
E_{f,T-2}' B_{T-2} \left\{ Y_{T-1}' \otimes e \left( A_{T-1} \eta_{T-1} + \epsilon_{T-1} \right)' \right\} (x_{T-2} + u_{T-2}) + \\
x_0 \left\{ Y_{T-1}' \otimes (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) e' \right\} B_{T-2} f_{T-2} \\
\end{bmatrix}
\begin{bmatrix}
(\eta_{T-1}' \{- Y_{T-1} \otimes e \left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2} + \right) \} \right)' \\
(x_{T-2} + u_{T-2} - x_0) + \\
\eta_{T-1}' \{ Y_{T-1} \otimes e \ x'_0 \} \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1}) \\
\end{bmatrix},
\]

where the matrix \( DE_{T-2} \) is diagonal and given by

\[
DE_{T-2} = \text{diag} \left( \left\{ D_{T-1} \otimes e \ x'_0 \right\} \ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \right).
\]
Let

\[ \Psi_{T-2} = \{ D_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \otimes x_0 e'\} + D E_{T-2}, \]

\[ YD_{T-2} = \text{diag} \ (Y'_{T-1} d_{T-2}), \]

\[ YB_{T-2} = \text{diag} \ (Y'_{T-1} B_{T-2} E_{f,T-2}). \]

Then, the conditional expectation of \( \Phi_2 \) can be approximated by

\[
\begin{align*}
\text{Var}_{T-2} \{ \hat{\Phi}_2 \} = 
\text{Var}_{T-2} \left( \begin{array}{c}
(x_{T-2} + u_{T-2})' \ A \Theta_{T-1} \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) + \\
q'_{T-1} \eta_{T-1} - 2 (d_{T-2} + B_{T-2} f_{T-2})' \ J_{T-1} \eta_{T-1} + \\
(x_{T-2} + u_{T-2})' \ YD_{T-2} \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) - \\
\end{array} \right)
\approx \\
\left( \begin{array}{c}
(x_{T-2} + u_{T-2})' \ A \Theta_{T-1} \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) + q'_{T-1} \eta_{T-1} - \\
2 (d_{T-2} + B_{T-2} f_{T-2})' \ J_{T-1} \eta_{T-1} - \eta'_{T-1} J_{T-1} \eta_{T-1} + \\
(x_{T-2} + u_{T-2})' \ YD_{T-2} \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) - \\
2 (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \\
\{ D_{T-1} \otimes x_0 x_0' \} \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) + \\
(A_{T-1} \eta_{T-1} + \epsilon_{T-1})' \{ D_{T-1} \otimes x_0 x_0' \} \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) - \\
2 (A_{T-1} \eta_{T-1} + \epsilon_{T-1})' \{ D_{T-1} \otimes x_0 (x_{T-2} + u_{T-2})' \} (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) - \\
E'_{f,T-2} B_{T-2} \ { Y_{T-1} \otimes e x_0' } \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) + \\
(x_{T-2} + u_{T-2})' \ YB_{T-2} \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) + \\
f'_{T-2} B'_{T-2} \ { Y_{T-1} \otimes e x_0' } \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) + \\
(x_{T-2} + u_{T-2} - x_0') \\
\{ Y'_{T-1} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' e' \} \ \eta_{T-1} + \\
\eta'_{T-1} \ { Y_{T-1} \otimes e (x_{T-2} + u_{T-2} - x_0') } \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1}) + \\
\eta'_{T-1} \ { Y_{T-1} \otimes e x_0' } \ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) + \\
\eta'_{T-1} \ { Y_{T-1} \otimes e x_0' } \ (A_{T-1} \eta_{T-1} + \epsilon_{T-1})
\end{array} \right) \}
\end{align*}
\]
In order to evaluate the variance of $\Phi_2$ conditioned on the information available at time $T - 2$, we present the following property of normally distributed random vectors:

**Lemma 3.5** Let $\eta$ and $\epsilon$ be two normally distributed uncorrelated random vectors with mean 0 and covariance matrices $\Sigma_\eta$ and $\Sigma_\epsilon$ respectively. For arbitrary vectors $a$ and $b$ of dimensions $(K \times 1)$ and $(N \times 1)$ respectively, and arbitrary matrices $C$ and $D$ of dimensions $(K \times K)$ and $(K \times N)$ respectively, the following relation holds:

$$
\text{Var} \left( a' \eta + b' \epsilon + \eta' D \epsilon + \epsilon' F \epsilon \right) = \\
\ a' \Sigma_\eta a + b' \Sigma_\epsilon b + e' \left[ C \otimes (\Sigma_\eta C \Sigma_\eta) \right] e + e' \left[ C \otimes (\Sigma_\eta C' \Sigma_\eta) \right] e + \\
\ e' \left[ D \otimes (\Sigma_\eta D \Sigma_\eta) \right] e + e' \left[ F \otimes (\Sigma_\epsilon F \Sigma_\epsilon) \right] e + e' \left[ F \otimes (\Sigma_\epsilon F' \Sigma_\epsilon) \right] e.
$$

**Proof.** By definition we know that

$$
\text{Var} \left( a' \eta + b' \epsilon + \eta' D \epsilon + \epsilon' F \epsilon \right) = \\
\text{Var} (a' \eta) + \text{Var} (b' \epsilon) + \text{Var} (\eta' D \epsilon) + \text{Var} (\epsilon' F \epsilon) + \\
2 \text{Cov} (a' \eta, b' \epsilon) + 2 \text{Cov} (a' \eta, \eta' C \eta) + 2 \text{Cov} (a' \eta, \eta' D \epsilon) + \\
2 \text{Cov} (a' \eta, \epsilon' F \epsilon) + 2 \text{Cov} (b' \epsilon, \eta' C \eta) + 2 \text{Cov} (b' \epsilon, \eta' D \epsilon) + \\
2 \text{Cov} (b' \epsilon, \epsilon' F \epsilon) + 2 \text{Cov} (\eta' C \eta, \eta' D \epsilon) + 2 \text{Cov} (\eta' C \eta, \epsilon' F \epsilon) + \\
2 \text{Cov} (\eta' D \epsilon, \epsilon' F \epsilon).
$$

Since $\eta$ and $\epsilon$ are uncorrelated and their corresponding mean is zero, all covariance terms are zero. For example

$$
\text{Cov} (a' \eta, \eta' D \epsilon) = E (a' \eta \eta' D \epsilon) - E (a' \eta) E (\eta' D \epsilon) = \\
E (a' \eta \eta') E (D \epsilon) = 0.
$$

In addition,

$$
\text{Cov} (a' \eta, \eta' C \eta) = E (a' \eta \eta' C \eta) - E (a' \eta) E (\eta' C \eta) = E (a' \eta \eta' C \eta)
$$

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\[ E \left\{ \sum_{i=1}^{K} a_i \eta_i \sum_{j=1}^{K} \eta_j \sum_{m=1}^{K} C_{jm} \eta_m \right\} = \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{m=1}^{K} a_i C_{jm} E \left( \eta_i \eta_j \eta_m \right) = 0. \]

Therefore, we only need to concentrate on the variance terms. It is also known that \( \text{Var}(a' \eta) = a' \Sigma_a a \) and \( \text{Var}(b' \varepsilon) = b' \Sigma_e b \), since \( \eta \) and \( \varepsilon \) are normally distributed. By definition, we also have that

\[ \text{Var}(\eta' C \eta) = E(\eta' C \eta \eta' C \eta) - [E(\eta' C \eta)]^{2} \]

\[ = E(\eta' C \eta \eta' C \eta) - [e' (C \otimes \Sigma_\eta) e]^{2}. \]

Concentrating on the expectation of the above relation we obtain that

\[ E(\eta' C \eta \eta' C \eta) = E \left\{ \sum_{i=1}^{K} \eta_i \sum_{j=1}^{K} C_{ij} \eta_j \sum_{m=1}^{K} \eta_m \sum_{l=1}^{K} C_{ml} \eta_l \right\} = \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{m=1}^{K} \sum_{l=1}^{K} C_{ij} C_{ml} E \left( \eta_i \eta_j \eta_m \eta_l \right) = \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{m=1}^{K} \sum_{l=1}^{K} C_{ij} C_{ml} \left( \Sigma_{ij}^{\eta} \Sigma_{ml}^{\eta} + \Sigma_{im}^{\eta} \Sigma_{jl}^{\eta} + \Sigma_{il}^{\eta} \Sigma_{jm}^{\eta} \right), \]

where \( \Sigma_{ij}^{\eta} \) is the \( ij \)-th element of the covariance matrix \( \Sigma_\eta \). Moreover, \( C_{ij} \Sigma_{ij}^{\eta} \) is the \( ij \)-th element of the matrix \( C \otimes \Sigma_\eta \). Therefore,

\[ \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{m=1}^{K} \sum_{l=1}^{K} C_{ij} C_{ml} \Sigma_{ij}^{\eta} \Sigma_{ml}^{\eta} = \sum_{i=1}^{K} \sum_{j=1}^{K} C_{ij} \Sigma_{ij}^{\eta} \sum_{m=1}^{K} \sum_{l=1}^{K} C_{ml} \Sigma_{ml}^{\eta} = [e' (C \otimes \Sigma_\eta) e]^{2}, \]

\[ \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{m=1}^{K} \sum_{l=1}^{K} C_{ij} C_{ml} \Sigma_{im}^{\eta} \Sigma_{jl}^{\eta} = \sum_{i=1}^{K} \sum_{j=1}^{K} C_{ij} \sum_{m=1}^{K} \sum_{l=1}^{K} \Sigma_{im}^{\eta} C_{ml} \Sigma_{ij}^{\eta} = e' [C \otimes (\Sigma_\eta C \Sigma_\eta)] e, \]

\[ \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{m=1}^{K} \sum_{l=1}^{K} C_{ij} C_{ml} \Sigma_{il}^{\eta} \Sigma_{jm}^{\eta} = \sum_{i=1}^{K} \sum_{j=1}^{K} C_{ij} \sum_{m=1}^{K} \sum_{l=1}^{K} \Sigma_{il}^{\eta} (C_{im})' \Sigma_{mj}^{\eta} = e' [C \otimes (\Sigma_\eta C' \Sigma_\eta)] e. \]

As a result,

\[ E(\eta' C \eta \eta' C \eta) = [e' (C \otimes \Sigma_\eta) e]^{2} + e' [C \otimes (\Sigma_\eta C \Sigma_\eta)] e + e' [C \otimes (\Sigma_\eta C' \Sigma_\eta)] e. \]

Similarly,

\[ \text{Var}(\eta' D \varepsilon) = E(\eta' D \varepsilon \eta' D \varepsilon) - [E(\eta' D \varepsilon)]^{2} = \]

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\[
E(\eta' D \epsilon \eta' D \epsilon) = E\left\{ \sum_{i=1}^{K} \eta_i \sum_{j=1}^{N} D_{ij} \epsilon_j \sum_{m=1}^{K} \eta_m \sum_{l=1}^{N} D_{ml} \epsilon_l \right\} = \\
\sum_{i=1}^{K} \sum_{j=1}^{N} \sum_{m=1}^{K} \sum_{l=1}^{N} D_{ij} D_{ml} E(\eta_i \eta_m \epsilon_j \epsilon_l) = \sum_{i=1}^{K} \sum_{j=1}^{N} \sum_{m=1}^{K} \sum_{l=1}^{N} D_{ij} D_{ml} \Sigma_{im}^{\eta} \Sigma_{jl}^{\epsilon} = \\
\sum_{i=1}^{K} \sum_{j=1}^{N} D_{ij} \sum_{m=1}^{K} \sum_{l=1}^{N} \Sigma_{im}^{\eta} D_{ml} \Sigma_{lj}^{\epsilon} = e' [D \otimes (\Sigma_{\eta} D \Sigma_{\epsilon})] e.
\]

Finally, \( \text{Var}(\epsilon' F \epsilon) \) can be computed similarly to \( \text{Var}(\eta' C \eta) \). \( \blacksquare \)

Using Lemma 3.1 we can now evaluate the conditional variance \( \text{Var}_{T-2} \{ \Phi_2 \} \) appearing in the optimization problem. For convenience, define the following parameters

\[
\beta_{T-2} = 2\Psi_{T-2} \mathbf{x}_0 - 2 \{ \mathbf{D}_{T-1} \otimes \mathbf{x}_0 \mathbf{x}_0' \} (e + c_{T-1} + A_{T-1} d_{T-2}) - \\
\{ \mathbf{Y}_{T-1} \otimes \mathbf{x}_0 \mathbf{e}' \} \mathbf{B}_{T-2} \mathbf{E}_{f,T-2}, \\
\delta_{T-2} = A'_{T-1} \beta_{T-2} + q_{T-1} - 2J_{T-1} d_{T-2} - \\
\{ \mathbf{Y}_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + \mathbf{A}_{T-1} \mathbf{B}_{T-2} \mathbf{E}_{f,T-2})' \} \mathbf{x}_0 + \\
\{ \mathbf{Y}_{T-1} \otimes \mathbf{e} \mathbf{x}_0' \} (e + c_{T-1} + A_{T-1} d_{T-2}), \\
\Theta_{T-2} = A \Theta_{T-1} + \mathbf{Y} D_{T-2} - 2\Psi_{T-2} + \mathbf{Y} B_{T-2}, \\
\Pi_{T-2} = A'_{T-1} \Theta_{T-2} + \{ \mathbf{Y}_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + \mathbf{A}_{T-1} \mathbf{B}_{T-2} \mathbf{E}_{f,T-2})' \}, \\
\Xi_{T-2} = -2 \{ \mathbf{D}_{T-1} \otimes \mathbf{x}_0 \mathbf{x}_0' \} \mathbf{A}_{T-1} \mathbf{B}_{T-2} + \{ \mathbf{Y}_{T-1} \otimes \mathbf{e} \mathbf{x}_0' \} \mathbf{B}_{T-2}, \\
\Omega_{T-2} = A'_{T-1} \Xi_{T-2} - 2J_{T-1} \mathbf{B}_{T-2} + \{ \mathbf{Y}_{T-1} \otimes \mathbf{e} \mathbf{x}_0' \} \mathbf{A}_{T-1} \mathbf{B}_{T-2}.
\]
Then, \( \text{Var}_{T-2} \{ \Phi_2 \} \) can be expressed as

\[
\begin{bmatrix}
[\delta_{T-2} + \Pi_{T-2} (x_{T-2} + u_{T-2}) + \Omega_{T-2} f_{T-2}]' \eta_{T-1} + \\
[\beta_{T-2} + \Theta_{T-2} (x_{T-2} + u_{T-2}) + \Xi_{T-2} f_{T-2}]' \epsilon_{T-1} + \\
\eta_{T-1}' - J_{T-1} + A'_{T-1} \{ D_{T-1} \otimes x_0 x_0' \} A_{T-1} + \\
2A'_{T-1} \{ D_{T-1} \otimes x_0 (x_{T-2} + u_{T-2})' \} A_{T-1} + \\
\{ Y_{T-1} \otimes e (x_{T-2} + u_{T-2})' \} A_{T-1} + \\
2A'_{T-1} \{ D_{T-1} \otimes x_0 (x_{T-2} + u_{T-2})' \} - \\
2A'_{T-1} \{ D_{T-1} \otimes x_0 (x_{T-2} + u_{T-2})' \} \}
\end{bmatrix}
\]

Even though we can directly apply Lemma 3.1 to the above expression, we choose to perform an additional approximation for simplification purposes:

1. We approximate \( (x_{T-2} + u_{T-2}) \) with the constant initial risky holdings \( x_0 \) in the terms of \( \Phi_2 \) that are nonlinear in the normally distributed random vectors \( \eta_{T-1} \) and \( \epsilon_{T-1} \). This approximation is equivalent to assuming that changes in the asset holdings have only a first-order effect to the portfolio variance.

As a result, \( \text{Var}_{T-2} \{ \Phi_2 \} \) can finally be approximated by

\[
\begin{bmatrix}
[\delta_{T-2} + \Pi_{T-2} (x_{T-2} + u_{T-2}) + \Omega_{T-2} f_{T-2}]' \eta_{T-1} + \\
[\beta_{T-2} + \Theta_{T-2} (x_{T-2} + u_{T-2}) + \Xi_{T-2} f_{T-2}]' \epsilon_{T-1} + \\
\eta_{T-1}' - J_{T-1} + A'_{T-1} \{ D_{T-1} \otimes x_0 x_0' \} A_{T-1} + \\
\{ Y_{T-1} \otimes e x_0' \} A_{T-1} + \\
\{ Y_{T-1} \otimes e x_0' \} - 2A'_{T-1} \{ D_{T-1} \otimes x_0 x_0' \} \}
\end{bmatrix}
\]

By letting

\[
\Delta_{T-2} = -J_{T-1} - A'_{T-1} \{ D_{T-1} \otimes x_0 x_0' \} A_{T-1} + \{ Y_{T-1} \otimes e x_0' \} A_{T-1},
\]

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\[ \Lambda_{T-2} = \{ Y_{T-1} \otimes e \ x_0' \} - 2A'_{T-1} \{ D_{T-1} \otimes x_0 \ x_0' \}, \]
\[ F_{T-2} = -\{ D_{T-1} \otimes x_0 \ x_0' \}, \]

and applying Lemma 3.1, we obtain that

\[
\text{Var}_{T-2} \{ \hat{\Phi}_2 \} = \xi_{T-2} + [\delta_{T-2} + \Pi_{T-2} \ (x_{T-2} + u_{T-2}) + \Omega_{T-2} \ f_{T-2}]' \ \Sigma_\eta
\]
\[ [\beta_{T-2} + \Theta_{T-2} \ (x_{T-2} + u_{T-2}) + \Xi_{T-2} \ f_{T-2}]' \ \Sigma_\epsilon
\]
\[ [\beta_{T-2} + \Theta_{T-2} \ (x_{T-2} + u_{T-2}) + \Xi_{T-2} \ f_{T-2}], \quad (3.13) \]

where

\[
\xi_{T-2} = e' [\Delta_{T-2} \otimes (\Sigma_\eta \ \Delta_{T-2} \ \Sigma_\eta)] e + e' [\Delta_{T-2} \otimes (\Sigma_\eta \ \Delta'_{T-2} \ \Sigma_\eta)] e +
\]
\[ e' [\Lambda_{T-2} \otimes (\Sigma_\eta \ \Lambda_{T-2} \ \Sigma_\eta)] e + e' [F_{T-2} \otimes (\Sigma_\epsilon \ F_{T-2} \ \Sigma_\epsilon)] e +
\]
\[ e' [F_{T-2} \otimes (\Sigma_\epsilon \ F'_{T-2} \ \Sigma_\epsilon)] e. \]

Using Equations (3.12) and (3.13) we can now state the quadratic optimization problem. The cost-to-function \( V_{T-2} \) is approximated by

\[
\hat{V}_{T-2} \left( x_{T-2}^0, x_{T-2}, f_{T-2} \right) = \max_{\{u_{T-2} \}} E_{T-2} \{ \hat{\Phi}_1 \} - \lambda \ \text{Var}_{T-2} \{ \hat{\Phi}_2 \} =
\]
The solution to the above optimization problem is given by the first-order conditions that are necessary and sufficient:
\[ 2\lambda \Pi_{T-2} \Sigma_{\eta} [\delta_{T-2} + \Pi_{T-2} (x_{T-2} + u_{T-2}) + \Omega_{T-2} f_{T-2} - \\
2\lambda \Theta_{T-2} \Sigma_{\epsilon} [\beta_{T-2} + \Theta_{T-2} (x_{T-2} + u_{T-2}) + \Xi_{T-2} f_{T-2}] = 0. \]

The approximate optimal control at time $T-2$, $\hat{u}_{T-2}$, is therefore expressed as a linear function of the state variables $x_{T-2}$ and $f_{T-2}$. If we let

\[
Q_{T-2} = \begin{bmatrix}
2(1 + r_f)^2 \Gamma + \\
\left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2} \right) \\
\left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}' \right)
\end{bmatrix} \left[X_{T-2} \right] +
\]

\[
\begin{bmatrix}
2 \Sigma_{\eta} \Pi_{T-2} X_{T-2} + 2\lambda \Theta_{T-2} \Sigma_{\epsilon} \Theta_{T-2} \\
2 \Sigma_{\eta} \Pi_{T-2} X_{T-2} + 2\lambda \Theta_{T-2} \Sigma_{\epsilon} \Theta_{T-2} \\
\end{bmatrix}
\]

\[
m_{T-2} = Q_{T-2} \begin{bmatrix}
- (1 + r_f)^2 e + B \Theta_{T-1} \left( e + c_{T-1} + A_{T-1} d_{T-2} \right) + \\
\{ S_{T-1} d_{T-2} \} \\
\{ S_{T-1} B_{T-2} B_{T-2} e \} \\
\{ 2 \Sigma_{\eta} \Pi_{T-2} X_{T-2} + 2\lambda \Theta_{T-2} \Sigma_{\epsilon} \Theta_{T-2} \}
\end{bmatrix},
\]

\[
G_{T-2} = Q_{T-2} \begin{bmatrix}
B \Theta_{T-1} A_{T-1} B_{T-2} + \{ S_{T-1} d_{T-2} e' \} A_{T-1} B_{T-2} + \\
\{ S_{T-1} B_{T-2} e \} A_{T-1} B_{T-2} \\
\{ 2 \Sigma_{\eta} \Pi_{T-2} X_{T-2} + 2\lambda \Theta_{T-2} \Sigma_{\epsilon} \Theta_{T-2} \}
\end{bmatrix},
\]

\[
L_{T-2} = Q_{T-2} \begin{bmatrix}
C_{T-1} \left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2} \right) \\
\{ S_{T-1} d_{T-2} \} \\
\{ S_{T-1} B_{T-2} e \} A_{T-1} B_{T-2} \\
\{ 2 \Sigma_{\eta} \Pi_{T-2} X_{T-2} + 2\lambda \Theta_{T-2} \Sigma_{\epsilon} \Theta_{T-2} \}
\end{bmatrix},
\]

we obtain that

\[
\hat{u}_{T-2} (x_{T-2}, f_{T-2}) = m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}.
\]

The approximated value function $\hat{V}_{T-2}$ becomes

\[
\hat{V}_{T-2} \left( x_{T-2}, x_{T-2}, f_{T-2} \right) = z_{T-2} + (1 + r_f)^2 x_{T-2}^0 + b'_{T-2} x_{T-2} + p'_{T-2} f_{T-2} -
\]

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\[ x_{T-2} C_{T-2} \times_{T-2} - f'_{T-2} H_{T-2} f_{T-2} + f'_{T-2} S_{T-2} x_{T-2}, \]

where the encountered constants are given by

\[ z_{T-2} = z_{T-1} - (1 + r_f)^2 e' \Gamma m_{T-2} - (1 + r_f)^2 e' \Gamma m^2_{T-2} + \]

\[ (e + c_{T-1} + A_{T-1} d_{T-2})' B \Theta_{T-1} m_{T-2} + p'_{T-1} d_{T-2} - \]

\[ d'_{T-2} H_{T-1} d_{T-2} - e' \{ H_{T-1} \otimes \Sigma \} e - \]

\[ m'_{T-2} = \begin{cases} C_{T-1} \otimes \left( (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \right) \\ + C_{T-1} \otimes \left[ A_{T-1} \Sigma \eta A_{T-1} + \Sigma \right] \end{cases} m_{T-2} + \]

\[ \{ S'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2})' \} m_{T-2} - \]

\[ \{ S'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} E_{f,T-2} \}' m_{T-2} + \]

\[ s a'_{T-1} m_{T-2} - \lambda \xi_{T-2} - \lambda (\delta_{T-2} + \Pi_{T-2} m_{T-2})' \Sigma \eta (\delta_{T-2} + \Pi_{T-2} m_{T-2}) - \]

\[ \lambda (\beta_{T-2} + \Theta_{T-2} m_{T-2})' \Sigma \epsilon (\beta_{T-2} + \Theta_{T-2} m_{T-2}), \]

\[ b_{T-2} = (1 + r_f)^2 L'_{T-2} e + 2 (1 + r_f)^2 L'_{T-2} \Gamma m_{T-2} + \]

\[ (I - L_{T-2})' B \Theta_{T-1} (e + c_{T-1} + A_{T-1} d_{T-2}) - 2 (I - L_{T-2})' \]

\[ \left\{ C_{T-1} \otimes \left( (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \right) \\ + C_{T-1} \otimes \left[ A_{T-1} \Sigma \eta A_{T-1} + \Sigma \right] \end{cases} m_{T-2} + \]

\[ (I - L_{T-2})' \{ S'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2}) \} - \]

\[ (I - L_{T-2})' \{ S'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} E_{f,T-2} \} + \]

\[ (I - L_{T-2})' s a_{T-1} - 2 \lambda (I - L_{T-2})' \Pi_{T-2} \Sigma \eta (\delta_{T-2} + \Pi_{T-2} m_{T-2}) - \]

\[ 2 \lambda (I - L_{T-2})' \Theta_{T-2} \Sigma \epsilon (\beta_{T-2} + \Theta_{T-2} m_{T-2}), \]

\[ p_{T-2} = -(1 + r_f)^2 G'_{T-2} e - 2 (1 + r_f)^2 G'_{T-2} \Gamma m_{T-2} + B'_{T-2} A'_{T-1} B \Theta_{T-1} m_{T-2} + \]

\[ G'_{T-2} B \Theta_{T-1} (e + c_{T-1} + A_{T-1} d_{T-2}) + B'_{T-2} p_{T-1} - \]

\[ 2 G'_{T-2} \left\{ C_{T-1} \otimes \left( (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \right) \\ + C_{T-1} \otimes \left[ A_{T-1} \Sigma \eta A_{T-1} + \Sigma \right] \end{cases} m_{T-2} - \]

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2B'_{T-2} H_{T-1} d_{T-2} + G'_{T-2} \left\{ S'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2}) \right\} + \\
\left\{ B'_{T-2} A'_{T-1} \otimes e d'_{T-2} S_{T-1} \right\} m_{T-2} + \\
\left\{ B'_{T-2} S_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \right\} m_{T-2} + \\
\left\{ B'_{T-2} A'_{T-1} \otimes e E'_{f,T-2} B'_{T-2} S_{T-1} \right\} m_{T-2} - \\
G'_{T-2} \left\{ S'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} E_{f,T-2} \right\} + G'_{T-2} s a_{T-1} - \\
2\lambda (\Pi_{T-2} G_{T-2} + \Omega_{T-2})' \Sigma_\eta (\delta_{T-2} + \Pi_{T-2} m_{T-2}) - \\
2\lambda (\Theta_{T-2} G_{T-2} + \Xi_{T-2})' \Sigma_\epsilon (\beta_{T-2} + \Theta_{T-2} m_{T-2}),

\text{and}

C_{T-2} = (1 + r_f)^2 L'_{T-2} \Gamma L_{T-2} + \\
(I - L_{T-2})' \left\{ C_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \right] \right\} \\
(I - L_{T-2})' + \lambda (I - L_{T-2})' \Pi'_{T-2} \Sigma_\eta \Pi_{T-2} (I - L_{T-2}) + \\
\lambda (I - L_{T-2})' \Theta'_{T-2} \Sigma_\epsilon \Theta_{T-2} (I - L_{T-2}),

H_{T-2} = (1 + r_f)^2 G'_{T-2} \Gamma G_{T-2} - B'_{T-2} A'_{T-1} B \Theta_{T-1} G_{T-2} + \\
G'_{T-2} \left\{ C_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \right] \right\} G_{T-2} + \\
(I - L_{T-2})' \left\{ A_{T-1} \Sigma_\eta A'_{T-1} + \Sigma_\epsilon \right\} B_{T-2} H_{T-1} B_{T-2} - \left\{ B'_{T-2} A'_{T-1} \otimes e d'_{T-2} S_{T-1} \right\} G_{T-2} - \\
\left\{ B'_{T-2} S_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \right\} G_{T-2} - \\
\left\{ B'_{T-2} A'_{T-1} \otimes e E'_{f,T-2} B'_{T-2} S_{T-1} \right\} G_{T-2} + \\
\lambda (\Pi_{T-2} G_{T-2} + \Omega_{T-2})' \Sigma_\eta (\Pi_{T-2} G_{T-2} + \Omega_{T-2}) + \\
\lambda (\Theta_{T-2} G_{T-2} + \Xi_{T-2})' \Sigma_\epsilon (\Theta_{T-2} G_{T-2} + \Xi_{T-2}),

S_{T-2} = 2(1 + r_f)^2 G'_{T-2} \Gamma L_{T-2} + B'_{T-2} A'_{T-1} B \Theta_{T-1} (I - L_{T-2}) - 2G'_{T-2}

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\[
\left\{ \begin{array}{l}
C_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} \ d_{T-2} + A_{T-1} \ B_{T-2} \ E_{f,T-2}) \\
+ C_{T-1} \otimes \left[ A_{T-1} \ \Sigma_\eta \ A_{T-1}^t + \Sigma_\epsilon \right] \right] \end{array} \right\} (I - L_{T-2}) + \\
\left\{ B_{T-2}^t \ A_{T-1}^t \otimes e \ d_{T-2}^t \ S_{T-1} \right\} (I - L_{T-2}) + \\
\left\{ B_{T-2}^t \ S_{T-1} \otimes e \ (e + c_{T-1} + A_{T-1} \ d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \right\} (I - L_{T-2}) + \\
2 \lambda \ (\Pi_{T-2} \ G_{T-2} + \Omega_{T-2})^t \ \Sigma_\eta \ \Pi_{T-2} \ (I - L_{T-2}) - \\
2 \lambda \ (\Theta_{T-2} \ G_{T-2} + \Xi_{T-2})^t \ \Sigma_\epsilon \ \Theta_{T-2} \ (I - L_{T-2}).
\]

In addition, the quantity \( K_{T-2} \) can be expressed, using its definition, as follows

\[
K_{T-2} \equiv E_{T-2} \{ K_{T-1} \} \\
= h_{T-1} + (1 + r_f)^2 x_{T-2}^0 - (1 + r_f)^2 e^t \ [m_{T-2} + G_{T-2} \ f_{T-2} - L_{T-2} \ x_{T-2}] - \\
(1 + r_f)^2 e^t \ \Gamma \ [m_{T-2} + G_{T-2} \ f_{T-2} - L_{T-2} \ x_{T-2}]^2 + \\
E_{T-2} \{ a_{T-1} \otimes (e + r_{T-1}) \}^t \ [m_{T-2} + G_{T-2} \ f_{T-2} - L_{T-2} \ x_{T-2}] + \\
q_{T-1}^t \ (d_{T-2} + B_{T-2} \ f_{T-2}) - [m_{T-2} + G_{T-2} \ f_{T-2} - L_{T-2} \ x_{T-2}]^t \\
E_{T-2} \{ D_{T-1} \otimes (e + r_{T-1}) (e + r_{T-1})^t \} \ [m_{T-2} + G_{T-2} \ f_{T-2} - L_{T-2} \ x_{T-2}] - \\
(d_{T-2} + B_{T-2} \ f_{T-2})^t \ J_{T-1} (d_{T-2} + B_{T-2} \ f_{T-2}) - \\
e^t \ \{ J_{T-1} \otimes \Sigma_\eta \} \ e + [m_{T-2} + G_{T-2} \ f_{T-2} - L_{T-2} \ x_{T-2}]^t \\
E_{T-2} \left\{ \begin{array}{l}
(d_{T-2} + B_{T-2} \ f_{T-2} + \eta_{T-1}) \\
Y_{T-1} \otimes e \ \left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + \right) \\
A_{T-1} \ \eta_{T-1} + \epsilon_{T-1}
\end{array} \right\}.
\]

By constructing the vector \( y a_{T-1} \) using Lemma 3.1

\[
[y a_{T-1}]_i = [Y_{T-1}^t \ \Sigma_\eta \ A_{T-1}^t]_{ii} \quad \text{for } i = 1, \ldots, N,
\]

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and using similar approximations as the ones performed in $E_{T-2} \{ \Phi_1 \}$ by replacing $S_{T-1}$ with $Y_{T-1}$, and $C_{T-1}$ with $D_{T-1}$ in Equations (3.10)-(3.11), we approximate $K_{T-2}$ with

$$ \hat{K}_{T-2} = h_{T-1} + (1 + r_f)^2 x_{T-2}^0 - (1 + r_f)^2 e' [m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}] - $$

$$ (1 + r_f)^2 e' \Gamma [m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}]^2 + $$

$$ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' A \Theta_{T-1} $$

$$ [m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}] + q'_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - $$

$$ [m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}]' \left\{ \begin{array}{l}
D_{T-1} \otimes \left[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \end{array} \right]
+ \right. $$

$$ D_{T-1} \otimes \left[ A_{T-1} \Sigma_{T-1} A_{T-1} + \Sigma_{T-1} \right] $$

$$ [m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}] - $$

$$ (d_{T-2} + B_{T-2} f_{T-2})' J_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - e' \{ J_{T-1} \otimes \Sigma_{T-1} \} e^T $$

$$ [m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}]' $$

$$ \{ Y'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \} + $$

$$ [m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}]' $$

$$ \{ Y'_{T-1} B_{T-2} f_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \} + $$

$$ [m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}]' $$

$$ \{ Y'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} (f_{T-2} - E_{f,T-2}) \} + $$

$$ y a'_{T-1} [m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}] \}.$$

As a result, $\hat{K}_{T-2}$ reduces to

$$ \hat{K}_{T-2} \left( x_{T-2}^0, x_{T-2}, f_{T-2} \right) = h_{T-2} + (1 + r_f)^2 x_{T-2}^0 + a'_{T-2} x_{T-2} + q'_{T-2} f_{T-2} - $$

$$ x'_{T-2} D_{T-2} x_{T-2} - f'_{T-2} J_{T-2} f_{T-2} + f'_{T-2} Y_{T-2} x_{T-2}, $$

where the encountered constants are obtained by the following relations:

$$ h_{T-2} = h_{T-1} - (1 + r_f)^2 e' m_{T-2} - (1 + r_f)^2 e' \Gamma m_{T-2}^2 + $$

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\[(e + cT_{-1} + AT_{-1} \, d_{T_{-2}})' \, A\Theta_{T_{-1}} \, m_{T_{-2}} + q'_{T_{-1}} \, d_{T_{-2}} - d'_{T_{-2}} \, J_{T_{-1}} \, d_{T_{-2}} - e' \, \{J_{T_{-1}} \otimes \Sigma_{\eta} \} \, e - m'_{T_{-2}} \)
\[
\begin{align*}
D_{T_{-1}} & \otimes \left[ \begin{array}{c}
(e + cT_{-1} + AT_{-1} \, d_{T_{-2}} + AT_{-1} \, B_{T_{-2}} \, E_{f,T_{-2}}) \\
(e + cT_{-1} + AT_{-1} \, d_{T_{-2}} + AT_{-1} \, B_{T_{-2}} \, E_{f,T_{-2}})' \end{array} \right] + m_{T_{-2}} + \\
D_{T_{-1}} & \otimes \left[ \begin{array}{c}
AT_{-1} \, \Sigma_{\eta} \, A'_{T_{-1}} + \Sigma_{\epsilon} \end{array} \right] \\
m'_{T_{-2}} & \{Y'_{T_{-1}} \, d_{T_{-2}} \otimes (e + cT_{-1} + AT_{-1} \, d_{T_{-2}}) \} - \\
m'_{T_{-2}} & \{Y'_{T_{-1}} \, B_{T_{-2}} \, E_{f,T_{-2}} \otimes AT_{-1} \, B_{T_{-2}} \, E_{f,T_{-2}} \} + ya'_{T_{-1}} \, m_{T_{-2}},
\end{align*}
\]

\[
a_{T_{-2}} = (1 + r_f)^2 L'_{T_{-2}} \, e + 2(1 + r_f)^2 L'_{T_{-2}} \, \Gamma \, m_{T_{-2}} +
\]

\[
(I - L_{T_{-2}})' \, A\Theta_{T_{-1}} \, (e + cT_{-1} + AT_{-1} \, d_{T_{-2}}) - 2(I - L_{T_{-2}})' 
\]

\[
\begin{align*}
D_{T_{-1}} & \otimes \left[ \begin{array}{c}
(e + cT_{-1} + AT_{-1} \, d_{T_{-2}} + AT_{-1} \, B_{T_{-2}} \, E_{f,T_{-2}}) \\
(e + cT_{-1} + AT_{-1} \, d_{T_{-2}} + AT_{-1} \, B_{T_{-2}} \, E_{f,T_{-2}})' \end{array} \right] + m_{T_{-2}} + \\
D_{T_{-1}} & \otimes \left[ \begin{array}{c}
AT_{-1} \, \Sigma_{\eta} \, A'_{T_{-1}} + \Sigma_{\epsilon} \end{array} \right] \\
(I - L_{T_{-2}})' & \{Y'_{T_{-1}} \, d_{T_{-2}} \otimes (e + cT_{-1} + AT_{-1} \, d_{T_{-2}}) \} - \\
(I - L_{T_{-2}})' & \{Y'_{T_{-1}} \, B_{T_{-2}} \, E_{f,T_{-2}} \otimes AT_{-1} \, B_{T_{-2}} \, E_{f,T_{-2}} \} + (I - L_{T_{-2}})' \, ya'_{T_{-1}},
\end{align*}
\]

\[
q_{T_{-2}} = -(1 + r_f)^2 G'_{T_{-2}} \, e - 2(1 + r_f)^2 G'_{T_{-2}} \, \Gamma \, m_{T_{-2}} + B'_{T_{-2}} \, A'_{T_{-1}} \, A\Theta_{T_{-1}} \, m_{T_{-2}} + \\
G'_{T_{-2}} \, A\Theta_{T_{-1}} \, (e + cT_{-1} + AT_{-1} \, d_{T_{-2}}) + B'_{T_{-2}} \, q_{T_{-1}} -
\]

\[
\begin{align*}
2G'_{T_{-2}} & \left[ \begin{array}{c}
D_{T_{-1}} \otimes \left[ \begin{array}{c}
(e + cT_{-1} + AT_{-1} \, d_{T_{-2}} + AT_{-1} \, B_{T_{-2}} \, E_{f,T_{-2}}) \\
(e + cT_{-1} + AT_{-1} \, d_{T_{-2}} + AT_{-1} \, B_{T_{-2}} \, E_{f,T_{-2}})' \end{array} \right] \right] m_{T_{-2}} - \\
+ & D_{T_{-1}} \otimes \left[ \begin{array}{c}
AT_{-1} \, \Sigma_{\eta} \, A'_{T_{-1}} + \Sigma_{\epsilon} \end{array} \right]
\end{align*}
\]

\[
2B'_{T_{-2}} \, J_{T_{-1}} \, d_{T_{-2}} + G'_{T_{-2}} \, \{Y'_{T_{-1}} \, d_{T_{-2}} \otimes (e + cT_{-1} + AT_{-1} \, d_{T_{-2}}) \} + \\
\{B'_{T_{-2}} \otimes d'_{T_{-2}} \, Y_{T_{-1}} \} \, m_{T_{-2}} +
\]

\[
\begin{align*}
\{B_{T_{-2}} \otimes e'_{T_{-2}} \, Y_{T_{-1}} \} & \, m_{T_{-2}} + \\
\{B'_{T_{-2}} \otimes e'_{T_{-2}} \, B_{T_{-2}} \, Y_{T_{-1}} \} & \, m_{T_{-2}} - \\
G'_{T_{-2}} & \{Y'_{T_{-1}} \, B_{T_{-2}} \, E_{f,T_{-2}} \otimes AT_{-1} \, B_{T_{-2}} \, E_{f,T_{-2}} \} + G'_{T_{-2}} \, ya'_{T_{-1}},
\end{align*}
\]

and

\[
D_{T_{-2}} = (1 + r_f)^2 L'_{T_{-2}} \, \Gamma \, L_{T_{-2}} +
\]

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\[
(I - L_{T-2})' \left\{ \begin{array}{l}
D_{T-1} \otimes \begin{bmatrix}
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})'
\end{bmatrix} \\
+ D_{T-1} \otimes \begin{bmatrix}
A_{T-1} \Sigma_{\eta} A'_{T-1} + \Sigma_{\epsilon}
\end{bmatrix}
\end{array} \right\}
\]

\[
(I - L_{T-2}),
\]

\[
J_{T-2} = (1 + r_f)^2 G'_{T-2} \Gamma G_{T-2} - B_{T-2}' A'_{T-1} A\Theta_{T-1} G'_{T-2} +
\]

\[
G'_{T-2} \left\{ \begin{array}{l}
D_{T-1} \otimes \begin{bmatrix}
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})'
\end{bmatrix} \\
+ D_{T-1} \otimes \begin{bmatrix}
A_{T-1} \Sigma_{\eta} A'_{T-1} + \Sigma_{\epsilon}
\end{bmatrix}
\end{array} \right\} G'_{T-2} +
\]

\[
B'_{T-2} J_{T-1} B_{T-2} - \{ B'_{T-2} A'_{T-1} \otimes e d'_{T-2} Y_{T-1} \} G'_{T-2} -
\]

\[
\{ B'_{T-2} Y_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \} G'_{T-2} -
\]

\[
\{ B'_{T-2} A'_{T-1} \otimes e E'_{f,T-2} B'_{T-2} Y_{T-1} \} G'_{T-2},
\]

\[
Y_{T-2} = 2 (1 + r_f)^2 G'_{T-2} \Gamma L_{T-2} + B'_{T-2} A'_{T-1} A\Theta_{T-1} (I - L_{T-2}) - 2 G'_{T-2}
\]

\[
\left\{ \begin{array}{l}
D_{T-1} \otimes \begin{bmatrix}
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})'
\end{bmatrix} \\
+ D_{T-1} \otimes \begin{bmatrix}
A_{T-1} \Sigma_{\eta} A'_{T-1} + \Sigma_{\epsilon}
\end{bmatrix}
\end{array} \right\} (I - L_{T-2}) +
\]

\[
\{ B'_{T-2} A'_{T-1} \otimes e d'_{T-2} Y_{T-1} \} (I - L_{T-2}) +
\]

\[
\{ B'_{T-2} Y_{T-1} \otimes e (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \} (I - L_{T-2}) +
\]

\[
\{ B'_{T-2} A'_{T-1} \otimes e E'_{f,T-2} B'_{T-2} Y_{T-1} \} (I - L_{T-2}).
\]

Based on the previous analysis we present the theorem that yields the proposed approximation algorithm:

**Theorem 3.1** The optimal investment decisions \( u_{T-k}^* \), the value function \( V_{T-k} \) and the quantity \( K_{T-k} \) are approximated for \( k = 2, \ldots , T \) by the following relations:

\[
\hat{u}_{T-k} (x_{T-k}, f_{T-k}) = m_{T-k} + G_{T-k} f_{T-k} - L_{T-k} x_{T-k},
\]

\[
\hat{V}_{T-k} (x^0_{T-k}, x_{T-k}, f_{T-k}) = z_{T-k} + (1 + r_f)^k x^0_{T-k} + b'_{T-k} x_{T-k} + p'_{T-k} f_{T-k} -
\]

\[
x_{T-k} C_{T-k} x_{T-k} - f'_{T-k} H_{T-k} f_{T-k} + f'_{T-k} S_{T-k} x_{T-k},
\]

\[
\hat{K}_{T-k} (x^0_{T-k}, x_{T-k}, f_{T-k}) = h_{T-k} + (1 + r_f)^k x^0_{T-k} + a'_{T-k} x_{T-k} + q'_{T-k} f_{T-k} -
\]

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\[ x'_{T-k} D_{T-k} x_{T-k} - f'_{T-k} J_{T-k} f_{T-k} + f'_{T-k} Y_{T-k} x_{T-k}, \]

where the following vectors

\[ [s a_{T-k}]_i = [S'_{T-k} \Sigma_\eta A'_{T-k}]_{ii}, \]
\[ [y a_{T-k}]_i = [Y'_{T-k} \Sigma_\eta A'_{T-k}]_{ii}, \]

and diagonal matrices are constructed recursively for \( i = 1, \ldots, N: \)

\[ B T_{T-k} = \text{diag}(b_{T-k}), \]
\[ A T_{T-k} = \text{diag}(a_{T-k}), \]
\[ D E_{T-k} = \text{diag} \left( \{D_{T-k+1} \otimes e x'_0\} \left( \begin{array}{c} e + c_{T-k+1} + A_{T-k+1} d_{T-k} + \\
A_{T-k+1} B_{T-k} E_{f,T-k} \end{array} \right) \right), \]
\[ Y D_{T-k} = \text{diag}(Y'_{T-k+1} d_{T-k}), \]
\[ Y B_{T-k} = \text{diag}(Y'_{T-k+1} B_{T-k} E_{f,T-k}). \]

In addition, the following constant parameters are defined:

\[ \Psi_{T-k} = \left\{ D_{T-k+1} \otimes e \left( \begin{array}{c} e + c_{T-k+1} + A_{T-k+1} d_{T-k} + \\
A_{T-k+1} B_{T-k} E_{f,T-k} \end{array} \right) \right\} x_0 e' + \text{DE}_{T-k}, \]
\[ \beta_{T-k} = 2\Psi_{T-k} x_0 - 2 \{D_{T-k+1} \otimes x_0 x'_0\} (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) - \]
\[ \{Y'_{T-k+1} \otimes x_0 e'\} B_{T-k} E_{f,T-k}, \]
\[ \delta_{T-k} = A'_{T-k+1} \beta_{T-k} + q_{T-k+1} - 2J_{T-k+1} d_{T-k} - \]
\[ \{Y_{T-k+1} \otimes e (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})\} x_0 + \]
\[ \{Y'_{T-k+1} \otimes e x'_0\} (e + c_{T-k+1} + A_{T-k+1} d_{T-k}), \]
\[ \Theta_{T-k} = A T_{T-k+1} + Y D_{T-k} - 2\Psi_{T-k} + Y B_{T-k}, \]
\[ \Pi_{T-k} = A'_{T-k+1} \Theta_{T-k} + \left\{ Y_{T-k+1} \otimes e \left( \begin{array}{c} e + c_{T-k+1} + A_{T-k+1} d_{T-k} + \\
A_{T-k+1} B_{T-k} E_{f,T-k} \end{array} \right) \right\}, \]
\[ \Xi_{T-k} = -2 \{D_{T-k+1} \otimes x_0 x'_0\} A_{T-k+1} B_{T-k} + \{Y'_{T-k+1} \otimes x_0 e'\} B_{T-k}, \]
\[ \Omega_{T-k} = A'_{T-k+1} \Xi_{T-k} - 2J_{T-k+1} B_{T-k} + \{Y_{T-k+1} \otimes e x'_0\} A_{T-k+1} B_{T-k}, \]

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\[
\Delta T_{-k} = -J_{T-k+1} - A'_{T-k+1} \{ D_{T-k+1} \otimes x_0 \} A_{T-k+1} + \{ Y_{T-k+1} \otimes e \} A_{T-k+1},
\]
\[
\Lambda T_{-k} = \{ Y_{T-k+1} \otimes e \} x_0' - 2A'_{T-k+1} \{ D_{T-k+1} \otimes x_0 \} x_0',
\]
\[
F_{T-k} = - \{ D_{T-k+1} \otimes x_0 \} x_0',
\]
\[
\xi_{T-k} = e'[\Delta T_{-k} \otimes (\Sigma \Delta_{T-k} \Sigma_{T-k})] e + e'[\Delta T_{-k} \otimes (\Sigma \Delta'_{T-k} \Sigma_{T-k})] e +
\]
\[
e'[\Lambda T_{-k} \otimes (\Sigma \Lambda_{T-k} \Sigma_{T-k})] e + e'[F_{T-k} \otimes (\Sigma \delta_{T-k} \Sigma_{T-k})] e +
\]
\[
e'[F_{T-k} \otimes (\Sigma \delta_{T-k} \Sigma_{T-k})] e.
\]

Moreover

\[
Q_{T-k} = \begin{pmatrix}
2(1 + r_f)^k \Gamma + & \left( \begin{array}{c}
e + c_{T-k+1} + \\
e + c_{T-k+1} + A_{T-k+1} d_{T-k} + \\
A_{T-k+1} B_{T-k} E_{f,T-k} + \\
A_{T-k+1} B_{T-k} E_{f,T-k} + \\
A_{T-k+1} \Sigma_{T-k} A'_{T-k+1} + \Sigma_{T-k} + \\
\Sigma_{T-k} A'_{T-k+1} + \Sigma_{T-k} + \end{array} \right)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
2 & \left( \begin{array}{c}
e + c_{T-k+1} + \\
e + c_{T-k+1} + A_{T-k+1} d_{T-k} + \\
A_{T-k+1} B_{T-k} E_{f,T-k} + \\
A_{T-k+1} B_{T-k} E_{f,T-k} + \\
A_{T-k+1} \Sigma_{T-k} A'_{T-k+1} + \Sigma_{T-k} + \\
\Sigma_{T-k} A'_{T-k+1} + \Sigma_{T-k} + \end{array} \right)
\end{pmatrix}^{-1}
\]

\[
m_{T-k} = Q_{T-k}
\]

\[
G_{T-k} = Q_{T-k}
\]
\[ L_{T-k} = Q_{T-k} \begin{bmatrix} 2 & \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) \\
A_{T-k+1} B_{T-k} E_{f,T-k} \\
e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \\
+ c_{T-k+1} \otimes \left[ A_{T-k+1} \Sigma \eta \ A'_{T-k+1} + \Sigma_e \right] \\
+ 2 \lambda \Pi'_{T-k} \Sigma \eta \Pi_{T-k} + 2 \lambda \Theta'_{T-k} \Sigma \epsilon \Theta_{T-k} \end{bmatrix} \].

Also,

\[ z_{T-k} = z_{T-k+1} - (1 + r'_j)^k e' m_{T-k} - (1 + r_j)^k e' \Gamma m^2_{T-k} + \\
(e + c_{T-k+1} + A_{T-k+1} d_{T-k})' B \Theta_{T-k+1} m_{T-k} + p'_{T-k+1} d_{T-k} - \\
d_{T-k} H_{T-k+1} d_{T-k} - e' \left( H_{T-k+1} \otimes \Sigma \eta \right) e - \\
m'_{T-k} \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right) \\
+ c_{T-k+1} \otimes \left[ A_{T-k+1} \Sigma \eta \ A'_{T-k+1} + \Sigma_e \right] \\
m_{T-k} + \left( S'_{T-k+1} d_{T-k} \otimes \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) \right)' m_{T-k} - \\
\left( S'_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right)' m_{T-k} + \\
s a_{T-k+1} m_{T-k} - \lambda \xi_{T-k} - \lambda (\delta_{T-k} + \Pi_{T-k} m_{T-k})' \Sigma \eta \left( \delta_{T-k} + \Pi_{T-k} m_{T-k} \right) - \\
\lambda (\beta_{T-k} + \Theta_{T-k} m_{T-k})' \Sigma \epsilon \left( \beta_{T-k} + \Theta_{T-k} m_{T-k} \right), \\
b_{T-k} = (1 + r_j)^k L'_{T-k} e + 2 (1 + r_j)^k L'_{T-k} \Gamma m_{T-k} + \\
(I - L_{T-k})' B \Theta_{T-k+1} \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) - 2 (I - L_{T-k})' \\
\left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right) \\
+ c_{T-k+1} \otimes \left[ A_{T-k+1} \Sigma \eta \ A'_{T-k+1} + \Sigma_e \right] \\
(I - L_{T-k})' \left( S'_{T-k+1} d_{T-k} \otimes \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) \right) - \\
(I - L_{T-k})' \left( S'_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right) + \\
(1 - L_{T-k})' sa_{T-k+1} - 2 \lambda (I - L_{T-k})' \Pi'_{T-k} \Sigma \eta \left( \delta_{T-k} + \Pi_{T-k} m_{T-k} \right) - \\
2 \lambda (I - L_{T-k})' \Theta'_{T-k} \Sigma \epsilon \left( \beta_{T-k} + \Theta_{T-k} m_{T-k} \right), \]
\[ p_{T-k} = -(1+r_f)^k G'_{T-k} \mathbf{e} - 2(1+r_f)^k G'_{T-k} \Gamma m_{T-k} + B'_{T-k} A'_{T-k+1} B \Theta_{T-k+1} m_{T-k} + \]
\[ G'_{T-k} \Theta_{T-k+1} \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) + B'_{T-k} p_{T-k+1} - 2G'_{T-k} \]
\[ \left\{ C_{T-k+1} \otimes \left[ \begin{array}{c}
(e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}) \\
(e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})'
\end{array} \right] \\
+ C_{T-k+1} \otimes \left[ A_{T-k+1} \Sigma \eta A'_{T-k+1} + \Sigma \epsilon \right] \right\} \]
\[ m_{T-k} - 2B'_{T-k} H_{T-k+1} d_{T-k} + \]
\[ G'_{T-k} \left\{ S'_{T-k+1} d_{T-k} \otimes \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) \right\} + \]
\[ \left\{ B'_{T-k} A'_{T-k+1} \otimes e d'_{T-k} S_{T-k+1} \right\} m_{T-k} + \]
\[ \left\{ B'_{T-k} S_{T-k+1} \otimes e \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right) \right\} m_{T-k} + \]
\[ \left\{ B'_{T-k} A'_{T-k+1} \otimes e E'_{f,T-k} B'_{T-k} S_{T-k+1} \right\} m_{T-k} - \]
\[ G'_{T-k} \left\{ S'_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right\} + G'_{T-k} \left\{ S_{T-k+1} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right\} + G'_{T-k} \left\{ S_{T-k+1} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right\} + G'_{T-k} \left\{ S_{T-k+1} \otimes \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) \right\} \]
\[ + \lambda \left( \Pi_{T-k} G_{T-k} + \Omega_{T-k} \right) \left( \delta_{T-k} + \Pi_{T-k} m_{T-k} \right) - \]
\[ 2\lambda \left( \Omega_{T-k} G_{T-k} + \Xi_{T-k} \right) \left( \beta_{T-k} + \Theta_{T-k} m_{T-k} \right), \]

and

\[ C_{T-k} = (1+r_f)^k L'_{T-k} \Gamma L_{T-k} + \]
\[ (I - L_{T-k})' \]
\[ \left\{ C_{T-k+1} \otimes \left[ \begin{array}{c}
(e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}) \\
(e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})'
\end{array} \right] \\
+ C_{T-k+1} \otimes \left[ A_{T-k+1} \Sigma \eta A'_{T-k+1} + \Sigma \epsilon \right] \right\} \]
\[ (I - L_{T-k}) + \lambda \left( I - L_{T-k} \right) \Pi_{T-k} \Sigma \eta \Pi_{T-k} \left( I - L_{T-k} \right) + \]
\[ \lambda \left( I - L_{T-k} \right) \Theta_{T-k} \Sigma \epsilon \Theta_{T-k} \left( I - L_{T-k} \right), \]

\[ H_{T-k} = (1+r_f)^k G'_{T-k} \Gamma G_{T-k} - B'_{T-k} A'_{T-k+1} B \Theta_{T-k+1} G_{T-k} + \]
\[ G'_{T-k} \left\{ C_{T-k+1} \otimes \left[ \begin{array}{c}
(e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}) \\
(e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})'
\end{array} \right] \\
+ C_{T-k+1} \otimes \left[ A_{T-k+1} \Sigma \eta A'_{T-k+1} + \Sigma \epsilon \right] \right\} \]
\[ G_{T-k} + B'_{T-k} H_{T-k+1} B_{T-k} - \left\{ B'_{T-k} A'_{T-k+1} \otimes e d'_{T-k} S_{T-k+1} \right\} G_{T-k} = \]

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\[
\begin{align*}
S_{T-k} &= 2(1 + r_f)^k G'_{T-k} \Gamma L_{T-k} + B'_{T-k} A'_{T-k+1} B \Theta'_{T-k+1} (I - L_{T-k}) - 2G'_{T-k} \\
&\quad + \left\{ C_{T-k+1} \otimes \left[ (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}) + (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})' \right] + C_{T-k+1} \otimes [A_{T-k+1} \Sigma \eta A'_{T-k+1} + \Sigma \epsilon] \right\} (I - L_{T-k}) + \\
&\quad + \left\{ B'_{T-k} S_{T-k+1} \otimes e (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})' \right\} (I - L_{T-k}) - \\
&\quad + 2\lambda (\Pi_{T-k} G_{T-k} + \Omega_{T-k})' \Sigma \eta \Pi_{T-k} (I - L_{T-k}) - \\
&\quad + 2\lambda (\Theta_{T-k} G_{T-k} + \Xi_{T-k})' \Sigma \epsilon \Theta_{T-k} (I - L_{T-k}) .
\end{align*}
\]

Finally,

\[
\begin{align*}
h_{T-k} &= h_{T-k+1} - (1 + r_f)^k e' m_{T-k} - (1 + r_f)^k e' \Gamma m^2_{T-k} + \\
&\quad (e + c_{T-k+1} + A_{T-k+1} d_{T-k})' A \Theta_{T-k+1} m_{T-k} + q'_{T-k+1} d_{T-k} - \\
&\quad d'_{T-k} J_{T-k+1} d_{T-k} - e' \left\{ J_{T-k+1} \otimes \Sigma \eta \right\} e - m'_{T-k} \\
&\quad + \left\{ D_{T-k+1} \otimes \left[ (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}) + (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})' \right] + \right\} \\
&\quad \left\{ D_{T-k+1} \otimes [A_{T-k+1} \Sigma \eta A'_{T-k+1} + \Sigma \epsilon] \right\} m_{T-k} + m'_{T-k} \left\{ Y_{T-k+1} d_{T-k} \otimes (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) \right\} - \\
&\quad m'_{T-k} \left\{ Y_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right\} + y a'_{T-k+1} m_{T-k},
\end{align*}
\]

\[
\begin{align*}
a_{T-k} &= (1 + r_f)^k L'_{T-k} e + 2(1 + r_f)^k L'_{T-k} \Gamma m_{T-k} + \\
&\quad (I - L_{T-k})' A \Theta_{T-k+1} (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) - 2(I - L_{T-k})'
\end{align*}
\]
\[
D_{T-k+1} \otimes \left[ (e + c_{T-k+1} + A_{T-k+1} \ B_{T-k} \ E_{f,T-k}) \right] + \\
D_{T-k+1} \otimes \left[ \left( e + c_{T-k+1} + A_{T-k+1} \ B_{T-k} \ E_{f,T-k} \right)' \right] + \\
\Sigma_{\eta} \ A_{T-k+1}' + \Sigma_{\epsilon}
\]

\[m_{T-k} + (I - L_{T-k})' \{ Y_{T-k}' d_{T-k} \otimes (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) \} - \]

\[ (I - L_{T-k})' \{ Y_{T-k+1}' B_{T-k} \ E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \} + \]

\[ (I - L_{T-k})' \ y_{a_{T-k+1}}, \]

\[q_{T-k} = - (1 + r_f)^k G_{T-k}' e - 2 (1 + r_f)^k G_{T-k}' \Gamma m_{T-k} + B_{T-k}' A_{T-k+1}' A_{T-k+1} A_{T-k+1} m_{T-k} + \]

\[G_{T-k}' A_{T-k+1} (e + c_{T-k+1} + A_{T-k+1} \ d_{T-k}) + B_{T-k}' q_{T-k+1} - \]

\[2G_{T-k}' \cdot 2 \ D_{T-k+1} \otimes \left[ \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right) \right] \]

\[+ D_{T-k+1} \otimes \left[ \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right)' \right] \]

\[m_{T-k} - 2B_{T-k}' J_{T-k+1} d_{T-k} + \]

\[G_{T-k}' \{ Y_{T-k+1}' d_{T-k} \otimes (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) \} + \]

\[\{ B_{T-k}' \ A_{T-k+1}' \otimes e d_{T-k}' Y_{T-k+1} \} \ m_{T-k} + \]

\[\{ B_{T-k}' Y_{T-k+1}' \otimes e (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})' \} \ m_{T-k} + \]

\[\{ B_{T-k}' \ A_{T-k+1}' \otimes e E_{f,T-k}' B_{T-k}' Y_{T-k+1} \} \ m_{T-k} - \]

\[G_{T-k}' \{ Y_{T-k+1}' B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \} + G_{T-k}' \ y_{a_{T-k+1}}, \]

and

\[D_{T-k} = (1 + r_f)^k L_{T-k}' \Gamma L_{T-k} + (I - L_{T-k})' \]

\[D_{T-k+1} \otimes \left[ \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right) \right] \]

\[+ D_{T-k+1} \otimes \left[ \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right)' \right] \]

\[\Sigma_{\eta} \ A_{T-k+1}' + \Sigma_{\epsilon} \]

\[I - L_{T-k}, \]

\[J_{T-k} = (1 + r_f)^k G_{T-k}' \Gamma G_{T-k} - B_{T-k}' A_{T-k+1}' A_{T-k+1} G_{T-k} + \]

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\[ G'_{T-k} \left\{ \begin{aligned} &D_{T-k+1} \otimes \left[ (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}) \\ &+ D_{T-k+1} \otimes \left[ A_{T-k+1} \Sigma \eta \ A'_{T-k+1} + \Sigma \epsilon \right] \right] \right. \\
&= B'_{T-k} J_{T-k+1} \ B_{T-k} - \{ B'_{T-k} A'_{T-k+1} \otimes e \ d'_{T-k} \ Y_{T-k+1} \} \ G_{T-k} - \\
&\left. \{ B'_{T-k} Y_{T-k+1} \otimes e (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})' \} \ G_{T-k} - \\
&\left. \{ B'_{T-k} A'_{T-k+1} \otimes e \ E'_{f,T-k} \ B'_{T-k} Y_{T-k+1} \} \ G_{T-k}, \right. \\
\end{aligned} \]

\[ Y_{T-k} = 2 (1 + r_f)^k G'_{T-k} \Gamma L_{T-k} + B'_{T-k} A'_{T-k+1} A \Theta_{T-k+1} (I - L_{T-k}) - 2G'_{T-k} \]

\[ \left\{ \begin{aligned} &D_{T-k+1} \otimes \left[ (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}) \\ &+ D_{T-k+1} \otimes \left[ A_{T-k+1} \Sigma \eta \ A'_{T-k+1} + \Sigma \epsilon \right] \right] \\
&= (I - L_{T-k}) + \{ B_{T-k} A'_{T-k+1} \otimes e \ d'_{T-k} \ Y_{T-k+1} \} (I - L_{T-k}) + \\
&\left. \{ B_{T-k} Y_{T-k+1} \otimes e (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})' \} \right. \\
&\left. (I - L_{T-k}) + \{ B'_{T-k} A'_{T-k+1} \otimes e \ E'_{f,T-k} \ B'_{T-k} Y_{T-k+1} \} (I - L_{T-k}), \right. \\
\end{aligned} \]

with the boundary conditions for \( Q_{T-1}, m_{T-1}, G_{T-1}, L_{T-1}, z_{T-1}, b_{T-1}, p_{T-1}, C_{T-1}, H_{T-1}, \\ S_{T-1}, h_{T-1}, a_{T-1}, q_{T-1}, D_{T-1}, J_{T-1}, \) and \( Y_{T-1} \) given by Equations (3.2)-(3.3) and (3.5)-(3.6).

**Proof.** We have shown the expressions to be valid for \( k = 2 \). By following the same steps and changing the indices considered we can prove that they hold for arbitrary \( k \); therefore we omit the details of the calculations. ■

Summarying, the algorithm proposed in this section provides

1. An approximation to the optimal sequence of investment in the risky assets. The derived investment policy is linear in the risky holdings and the realizations of the factor variables, while is independent of the holdings in the riskless asset. By inspecting the equations of Theorem 3.1, it is apparent that the higher the transaction costs, the smaller the change in the investment positions, since it is costly to transact. In the next section, we analyze how the risk-free rate, characteristics of the underlying return process, time to maturity and investor's risk aversion affect the investment decisions.
2. An approximation of the value function at every point in time. Using characteristics of
the optimal cost-to-function we are able to derive an efficient way of measuring investors'
utility levels as a function of time.

3. An approximation of the optimal expected value of terminal wealth, and thus the ability
to monitor the evolution of expected wealth as time to maturity decreases.

The proposed approximation algorithm does not only provide the means to explore the
impact of the various parameters influencing the pricing model and of risk preferences to the
investment activity, but also performs better relative to other dynamic trading strategies, as we
illustrate in Section 3.4. The advantages of the approximation scheme are summarized below:

1. It is a polynomial-time algorithm. Moreover, in contrast with the usual quadratic approx-
imation algorithm, that approximates the cost-to-go function at every time period with
a quadratic solving a least squares problem in order to estimate the parameters involved,
we avoid discretization of the state space and thus are able to expand our analysis to the
case of multiple assets and/or factors and to include lagged correlations in asset returns.

2. It provides near optimal policies for small-dimensional problems where exact dynamic
programming is feasible.

3. It outperforms both the "myopic" policy, where investors follow a sequence of optimal
single-period policies, and the "semi-static" trading strategy (see next section) that arises
from the solution to a series of stochastic optimization problems with different initial
conditions and decreasing investment horizon.

In the next section, we present an alternative dynamic trading strategy that arises from the
solution to a series of stochastic dynamic optimization problems, and we evaluate its perfor-
man in Section 3.4.

3.3 Approximation B: Semi-Static Approach

In this section, we present the dynamic investment policy followed by an investor who is allowed
to optimally rebalance his portfolio holdings given the new state realizations and the remaining
time to maturity.

![Figure 3-1: A binomial lattice.](image)

Figure 3-1 shows an event tree in the form of a binomial lattice, which will be used to illustrate the idea behind the proposed approximate dynamic strategy. The exact dynamic programming algorithm evaluates the optimal investment decision for all states of the world (1 through 6). For problems of realistic size and complexity though, complete enumeration of the possible outcomes is impossible. The proposed approximation algorithm performs *state aggregation* at every point in time: at $t = 1$ it aggregates the information available at the nodes 2 and 3, and at $t = 2$ it aggregates the information available at the nodes 4, 5 and 6. Consequently, the approximate description of the state space, as given by the new nodes A, B and C, provides an efficient way of characterizing the event space and the corresponding investment decisions. Even though the portfolio manager evaluates the future aggregate investment decisions (at the nodes B and C), he does not apply them; instead he realizes the new true state of the world at the next time period and resolves the portfolio optimization problem with the new realized initial conditions and investment horizon decreased by one period. Therefore, according to this new approximate procedure the portfolio manager solves a series of optimization problems with different initial conditions and successively decreasing investment horizon.
In contrast with the dynamic programming optimization problem of the previous section, that is an exact problem solved approximately, this is an approximate dynamic optimization problem that is solved exactly moving forward in time. In contrast with the myopic strategy, this is an optimization problem that considers the entire investment horizon and solves optimally for the single-period investment decision. This is the reason why we refer to this new dynamic trading strategy as the semi-static policy.

More specifically, consider an investor who solves at time \( t \) for future rebalancing decisions but only applies the optimal control produced at time \( t \). Then, realizing the state of the system at the next time period \( t + 1 \), he resolves the same optimization problem having as initial conditions the new realized asset holdings and factor returns with a time window that is decreased by one. All encountered optimization problems are convex, with an objective function that is quadratic in the controls, and can be solved efficiently using standard packages. The semi-static policy is thus obtained using the following algorithm:

**Algorithm 3.1** To find the semi-static control policy at each time period \( t = 0, \ldots, T - 1 \). do the following:

1. Solve the quadratic optimization problem with investment horizon \( T - t \):

\[
V_t \left( x^0_t, x_t, f_t \right) = \max_{\{u_t, \ldots, u_{t-1}\}} E_t \left( x^0_T + e' x_T \right) - \lambda \ Var_t \left( x^0_T + e' x_T \right)
\]

subject to

\[
x^0_T = (1 + r_f)^{T-t} \left[ x^0_t - e' u_t - e' \Gamma u^2_t \right] - \sum_{k=t+1}^{T-1} (1 + r_f)^{T-k} \left[ e' u_k + e' \Gamma u^2_k \right]
\]
\[
x_T = \{(e + r_T) \otimes \ldots \otimes (e + r_{t+1})\} \otimes (x_t + u_t) + \sum_{k=t+1}^{T-1} \left( \prod_{m=k+1}^{T} (e + r_m) \right) \otimes u_k,
\]

where the symbol \( \prod \) denotes the Kronecker multiplication of the corresponding vectors.

2. Apply the resulted optimal decision \( u_t \). Realize the new asset holdings and factor realizations that constitute the new state of the system at time \( t + 1 \), and resolve.

Notice that at time \( T - 1 \) the optimization problem solved by the above algorithm is the same
with the one encountered in the DP recursion. Even though the optimization problems solved with the above algorithm are well defined, they are computationally very intense since they require the calculation of expectations and covariances of correlated multivariate normals. We therefore propose an alternative algorithm that is computationally more efficient.

More specifically, consider a portfolio manager who always follows an optimize-and-hold strategy, i.e. he makes an investment decision for every time period and then assumes that he does not transact for the remaining time to expiration. Therefore, at time $t$ he decides the optimal risky rebalancing of his portfolio holdings $u_t$, and for times $t+1, \ldots, T-1$ he assumes that no rebalancing takes place, setting $u_{t+1} = \ldots = u_{T-1} = 0$. Hence, he solves the following dynamic optimization problem at every time period $t = 0, 1, \ldots, T-1$

$$\max_{u_t} E_t (W_T) - \lambda \; Var_t (W_T)$$

subject to

$$W_T = x_T^0 + e' x_T$$
$$x_T^0 = (1 + r_f)^{T-t} \left[ x_t^0 - e' u_t - e' \Gamma u_t^2 \right]$$
$$x_T = \{ (e + r_T) \otimes \ldots \otimes (e + r_{t+1}) \} \otimes (x_t + u_t).$$

The above problem is equivalent to

$$\max_{u_t} \left\{ (1 + r_f)^{T-t} \left[ x_t^0 - e' u_t - e' \Gamma u_t^2 \right] + (x_t + u_t)' E_t [(e + r_T) \otimes \ldots \otimes (e + r_{t+1})] - \lambda (x_t + u_t)' \Phi_t (x_t + u_t) \right\},$$

where

$$\mu_t = E_t [(e + r_T) \otimes \ldots \otimes (e + r_{t+1})],$$
$$\Phi_t = E_t \left\{ \left[ (e + r_T) \otimes \ldots \otimes (e + r_{t+1}) \right] - \mu_t \right\} \left[ (e + r_T) \otimes \ldots \otimes (e + r_{t+1}) - \mu_t \right].$$

This optimization problem is solvable in closed-form with the resulting investment decision being linear in the risky holdings. The optimize-and-hold policy is thus provided by the following algorithm:
Algorithm 3.2  The investment decisions according to the optimize-and-hold policy are given by the following relations for $t = 0, \ldots, T - 1$:

$$u_t^{opt-h} (x_t, f_t) = m_t^{opt-h} (f_t) - L_t^{opt-h} (f_t) x_t,$$

(3.14)

where the

$$
\begin{align*}
\mu_t (f_t) &= E_t [(e + r_T) \otimes \ldots \otimes (e + r_{t+1})], \\
\Phi_t (f_t) &= E_t \left\{ \left[ [(e + r_T) \otimes \ldots \otimes (e + r_{t+1})] - \mu_t \right] \left[ [(e + r_T) \otimes \ldots \otimes (e + r_{t+1})] - \mu_t \right]' \right\}, \\
Q_t (f_t) &= \left[ 2 (1 + r_f)^{T-t} \Gamma + 2\lambda \Phi_t \right]^{-1}, \\
m_t^{opt-h} (f_t) &= Q_t \left[ (1 + r_f)^{T-t} e + \mu_t \right], \\
L_t^{opt-h} (f_t) &= 2\lambda Q_t \Phi_t.
\end{align*}
$$

Notice that the control policy as given by Equation (3.14) depends on the factor realizations but not in a linear fashion, in contrast with the approximate dynamic policy derived in the previous section. In the section that follows, we present the performance of the proposed optimize-and-hold policy relative to the other approximate dynamic policies considered.

3.4 Computational Experiments for Multifactor Pricing Models

In this section, we examine the impact of transaction costs, risk aversion, return autocorrelation and volatility on the investor's portfolio composition over time. In our comparative analysis we present:

1. A comparison of the performance of the following dynamic trading strategies:

   - The approximate structured policy described in Section 3.2 (Structured).
   - The approximate optimize-and-hold policy described in Section 3.3 (Opt-and-Hold).
   - The static policy that derives as the solution to a series of single-period optimization problems (Static).
2. An investigation of the following parameters to the investment behavior of an investment manager who trades according to the structured policy:

- Transaction Costs
- Risk Aversion
- Time Horizon
- Asset Volatilities

For all policies considered, 1,000 independent sample paths of the asset returns are simulated and for each path the approximate dynamic policies are implemented. We denote as $\hat{V}_0$ the expected utility of terminal wealth and $\hat{K}_0$ the expected final wealth derived analytically from the theorems presented in the previous sections, and as $\overline{V}_0$, $\overline{K}_0$ the simulated expected utility of terminal wealth and expected wealth defined as the averages (over 1,000 sample paths) of the manager’s utility. We also plot the average risky investments (over the 1,000 sample paths) as a function of time in order to explore the evolution of the portfolio composition as time to expiration decreases. We assume that asset returns are (1) independent and normally distributed and (2) are given by a 1-factor pricing model.

3.4.1 IID Returns

In this section we present a numerical example under the assumption that asset returns are independent and normally distributed. In this case, the problem where transaction costs are ignored can be solved in closed-form as indicated in Section 2.3.1, and it serves as our benchmark problem for evaluating the impact of transaction costs on dynamic investment policies and utility levels.

We consider an example of $N = 5$ risky assets with returns that are normally distributed with mean $\mu = [0.07, 0.10, 0.12, 0.15, 0.30]'$, vector of volatilities $\sigma = [0.10, 0.25, 0.35, 0.40, 0.70]'$ and correlation coefficients assumed to be $\rho_{ij} = 0.3$ for all assets. We assume that changing the portfolio position in all assets is equally costly and that the transaction costs coefficient is $\tau = 0.01$. In addition, we consider that $r_f = 5\%$, $\lambda = 0.001$, $x_0^0 = 1$, $x_0 = [1, 1, 1, 1, 1]'$ and $T = 10$.  

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Relative Performance of Dynamic Policies

In Table 3.1 we present the resulted utility levels and final wealth from the application of the different dynamic policies considered. The structured approximate policy outperforms the optimize-and-hold policy by 10% for the chosen parameters in our simulation experiment. We also notice that both its expected utility level and final wealth are close to the ones obtained through our simulation example. This is evidence that the proposed policy is quite robust. There is also a significant increase in the utility level in the absence of transaction costs. Indeed, when trading costs are ignored risky holdings are significantly higher and costless portfolio rebalancing allows investors a more effective asset allocation resulting in higher utility levels. As shown in Table 3.2, where we report the initial risky investment, all policies suggest buying all assets in the beginning of the investment horizon, with the static policy being the most conservative and the no-costs policy the most aggressive as expected.

<table>
<thead>
<tr>
<th>Policies</th>
<th>$\bar{V}_0$</th>
<th>$\bar{V}_0$</th>
<th>$\bar{K}_0$</th>
<th>$\bar{K}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structured</td>
<td>159.9887</td>
<td>144.7454</td>
<td>243.5282</td>
<td>248.7150</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>144.1147</td>
<td>–</td>
<td>224.5521</td>
<td>–</td>
</tr>
<tr>
<td>Static</td>
<td>–385.2110</td>
<td>–</td>
<td>442.5620</td>
<td>–</td>
</tr>
<tr>
<td>No Costs</td>
<td>417.0614</td>
<td>407.4261</td>
<td>797.5796</td>
<td>805.0788</td>
</tr>
</tbody>
</table>

Table 3.1: Monte Carlo simulation of the investment policies under investigation under the assumption of IID returns. 1,000 independent sample paths were simulated, each path containing 10 periods.

<table>
<thead>
<tr>
<th>Policies</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Structured</td>
<td>5.8282</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>7.4685</td>
</tr>
<tr>
<td>Static</td>
<td>0.9132</td>
</tr>
<tr>
<td>No Costs</td>
<td>161.9461</td>
</tr>
</tbody>
</table>

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<td>10.6913</td>
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<tr>
<td>Opt-and-Hold</td>
<td>14.3388</td>
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<tr>
<td>Static</td>
<td>2.2726</td>
</tr>
<tr>
<td>No Costs</td>
<td>64.1785</td>
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<tr>
<td>Structured</td>
<td>9.9121</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>15.7282</td>
</tr>
<tr>
<td>Static</td>
<td>3.1667</td>
</tr>
<tr>
<td>No Costs</td>
<td>45.5560</td>
</tr>
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</table>

<table>
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</tr>
</thead>
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<td>Structured</td>
<td>11.8646</td>
</tr>
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<td>Opt-and-Hold</td>
<td>18.9514</td>
</tr>
<tr>
<td>Static</td>
<td>4.5512</td>
</tr>
<tr>
<td>No Costs</td>
<td>97.2909</td>
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</tbody>
</table>

<table>
<thead>
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<th>Policies</th>
<th>$\bar{u}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structured</td>
<td>9.9213⁺</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>1.6081⁺</td>
</tr>
<tr>
<td>Static</td>
<td>11.2398⁺</td>
</tr>
<tr>
<td>No Costs</td>
<td>125.6410⁺</td>
</tr>
</tbody>
</table>

Table 3.2: The initial investment decision for IID returns.

In order to investigate the investment behavior and the evolution of the risky holdings, we plot the expected risky investment $E \{u_t\}$ (Figure 3-2) and expected risky holdings $E \{x_t\}$ (Figure 3-3) as a function of time.

The expected risky investment does not exhibit the same behavior for the two (out of the five) assets considered. The policy from the structured approximation suggests that in the beginning of the investment horizon both assets are being bought favoring the more risky Asset
5. As time to maturity approaches, risky investments for all but Asset 5 decrease monotonically. For the most risky Asset 5 the policy suggests selling, once investment has been underway, and buying towards the end. The investor is willing to take more risk in the beginning of the investment horizon in gain of higher expected return. Later on, due to variance considerations he alters his initial position favoring the rest of the assets. We also notice that the proposed policy significantly reduces investment near the maturity date, because the expected return earned over the remaining time period decreases. Therefore, near maturity, portfolio rebalancing is minimal.

The optimize-and-hold policy exhibits similar behavior for the less risky assets and somewhat different for the most risky Asset 5. According to this policy, the investor is conservative with respect to Asset 5 in the beginning of the horizon and starts favoring the asset later

Figure 3-2: The expected risky investment plotted as a function of time for the different policies considered. In Panel (a) we present the investment decision for Asset 1 and in Panel (b) for Asset 5 under IID returns.
Figure 3-3: The expected risky holdings plotted as a function of time for the different policies considered. In Panel (a) we present the holdings for Asset 1 and in Panel (b) for Asset 5 under IID returns.

on. Therefore, he is willing to take most of his risk and return from Asset 5 roughly half-way through the investment window. As a result, this policy is more conservative with regards to the riskier Asset 5. Expected holdings in both assets increase over time for both policies, always being below the holdings in the case when transaction costs are ignored.

According to the static policy, investment in all assets decrease over time but always favoring the most risky Asset 5. As a result, this policy provides a poor way to bound the final portfolio variance, resulting in unrealistically excessive risky holdings for Asset 5 and very low utility levels.
3.4.2 1-Factor Pricing Model

In this section, we turn our attention to factor pricing models and the effect of autocorrelation to dynamic investment strategies. We consider again a portfolio of five risky and one riskless asset. The risky asset return dynamics are given by

\[ r_t = c + A f_t + \epsilon_t, \]
\[ f_t = d + B f_{t-1} + \eta_t, \]

where it is assumed that \( c = [0.01, 0.04, 0.06, 0.09, 0.24]' \), \( A = [0.2, 0.2, 0.2, 0.2, 0.2]' \), \( d = 0.24 \), \( B = 0.2 \), and all correlation coefficients are set to \( \rho_{ij} = 0.3 \). Thus, all assets have the same expected return as in the previous example. We also assume that the vector of volatilities is \( \sigma = [0.10, 0.25, 0.35, 0.40, 0.70]' \) and that \( \sigma_\eta = 0.25 \). The transaction cost coefficient is set to \( \tau = 0.01 \), and we finally consider that \( r_f = 5\% \), \( \lambda = 0.001 \), \( x_0^0 = 1 \), \( x_0 = [1, 1, 1, 1, 1]' \) and \( T = 10 \).

Relative Performance of Dynamic Policies (Positive Factor Correlation \( B = 0.2 \))

In Table 3.3 we present the resulted utility levels and final wealth from the application of the different dynamic policies considered in the case of positive return autocorrelation. The structured approximate policy outperforms again the optimize-and-hold policy by 1\% for the chosen parameters in our simulation experiment. Concentrating on the initial risky investment as given by Table 3.4, we observe that all policies suggest an increase in the portfolio risky positions. Under the optimize-and-hold policy the initial investment is higher, compared to the one proposed by the structured policy, for all assets but Asset 5. Since Asset 5 is the riskier portfolio asset, its contribution to the portfolio variance is greater and thus, the optimize-and-hold policy chooses a minimal change in the holdings of this particular asset. On the other hand, the static policy significantly favors the riskier asset resulting in a policy with high variability. As a result, the static policy is clearly inappropriate for asset allocation when investors have quadratic preferences.

For positive factor correlation \( B \), the expected investment decisions are qualitatively similar to the ones obtained in the previous section under the IID assumption (see Figures 3-4 and
Table 3.3: Monte Carlo simulation of the investment policies under investigation for positive factor correlation $B = 0.2$. 1,000 independent sample paths were simulated, each path containing 10 periods.

<table>
<thead>
<tr>
<th>Policies</th>
<th>$V_0$</th>
<th>$\tilde{V}_0$</th>
<th>$\tilde{K}_0$</th>
<th>$K_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structured</td>
<td>143.6610</td>
<td>140.3960</td>
<td>249.8312</td>
<td>242.4893</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>127.7209</td>
<td>--</td>
<td>233.8985</td>
<td>--</td>
</tr>
<tr>
<td>Static</td>
<td>-571.5677</td>
<td>--</td>
<td>463.2715</td>
<td>--</td>
</tr>
</tbody>
</table>

Table 3.4: The initial investment decision for positive autocorrelation $B = 0.2$.

3-5). Expected investment in all assets but Asset 5 is decreasing in time. The closer we are to expiration, the smaller the changes in the investment positions due to the fact that trading costs offset the predictability effect of returns.

**Relative Performance of Dynamic Policies (Negative Factor Correlation $B = -0.2$)**

In Table 3.5 we present the resulted utility levels and final wealth from the application of the different dynamic policies considered in the case of negative serial correlation. We observe that, all else equal, the expected utility is lower in the presence of negative serial correlation. This is due to the fact that the magnitude of the change in both risky positions is smaller resulting in lower asset holdings, as it is also shown in Table 3.6. In the presence of transaction costs, the investor does not react as much to anticipated movements in future returns, as it is shown in Figure 3-6.

<table>
<thead>
<tr>
<th>Policies</th>
<th>$\hat{V}_0$</th>
<th>$\hat{V}_0$</th>
<th>$\bar{K}_0$</th>
<th>$\bar{K}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static</td>
<td>0.9068</td>
<td>2.2663</td>
<td>3.1605</td>
<td>4.5449</td>
</tr>
</tbody>
</table>

Table 3.5: Monte Carlo simulation of the investment policies under investigation for negative factor correlation $B = -0.2$. 1,000 independent sample paths were simulated, each path containing 10 periods.

In the presence of negative autocorrelation, the investor is suggested, by both the structured and the optimize-and-hold policy, to follow an investment strategy that favors the riskier assets.
Figure 3-4: The expected risky investment plotted as a function of time for the different policies considered. In Panel (a) we present the investment decision for Asset 1 and in Panel (b) for Asset 5 for positive factor correlation $B = 0.2$.

Table 3.6: The initial investment decision for negative autocorrelation $B = -0.2$.

<table>
<thead>
<tr>
<th>Policies</th>
<th>$\hat{u}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structured</td>
<td>$[-2.9940, 5.0229, 7.0233, 10.3795, 9.7396]'$</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>$[-1.9965, 6.4474, 13.5249, 17.6321, 2.8165]'$</td>
</tr>
<tr>
<td>Static</td>
<td>$[-0.0391, 1.3306, 2.2332, 3.6223, 10.3444]'$</td>
</tr>
</tbody>
</table>

in the beginning of the investment horizon, thus selling the less risky Asset 1. Indeed, negative autocorrelation makes it more likely that bad returns will be offset by good future returns, more so when many future periods remain, making risky assets more attractive farther from the horizon. Thus, the riskier Asset 5 provides a hedging effect: if the current risky return is low, the opportunity set improves canceling some of its risk and making it more attractive compared to Asset 1. Consequently, the investor chooses to decrease his holdings in the less risky Asset 1 and invest in the riskier assets.

**Effect of Transaction Costs and Risk Aversion on the Structured Policy**
Figure 3-5: The expected risky holdings plotted as a function of time for the different policies considered. In Panel (a) we present the holdings for Asset 1 and in Panel (b) positive factor correlation $B = 0.2$.

As expected, increasing the size of transaction costs reduces the size of the trades as illustrated in Figure 3-8. The structured policy suggests a gradual decrease in the investment position for both assets as time to expiration decreases. In addition, increasing the investor's risk aversion parameter decreases the magnitude of the risky investments. As the investor's aversion to risk increases, he becomes less tolerant to fluctuations in the value of his portfolio. Therefore, he moves out of the risky stocks and into the riskless bond.
Figure 3-6: The expected risky investment plotted as a function of time for the different policies considered. In Panel (a) we present the investment decision for Asset 1 and in Panel (b) for Asset 5 for negative factor correlation $B = -0.2$. 
Figure 3-7: The expected risky holdings plotted as a function of time for the different policies considered. In Panel (a) we present the holdings for Asset 1 and in Panel (b) negative factor correlation $B = -0.2$. 
Figure 3-8: The expected risky investment plotted as a function of time for $T = 10$. In Panels (a)-(b) we show the dependence on the risk aversion parameter $\lambda$ and in Panels (c)-(d) the dependence on the transaction cost coefficient $\tau$. The solid line corresponds to Asset 1 and the dotted line to Asset 5.
Chapter 4

Quadratic Utility: Transaction Costs and Stochastic Volatility Models

In the previous chapter we have examined the effect of transaction costs on dynamic portfolio strategies in the presence of models that account for lagged correlations in asset returns. In this chapter we explore the application of models that are nonlinear in the variance, concentrating on univariate Generalized Autoregressive Conditionally Heteroskedastic (GARCH) models. As we have mentioned in Section 1.1.2, we concentrate on univariate models since multivariate models are harder to validate and become very complicated very quickly. The ideas presented in this chapter, though, can be readily expanded to the multivariate setting.

Changes in the variance are quite important to understanding financial markets, since investors require higher expected returns as compensation for holding riskier assets. The terms of many financial contracts such as options and other derivative securities are nonlinear as well. The strategic interactions among market participants, the process by which information is incorporated into security prices, and the dynamics of economy-wide fluctuations are all inherently nonlinear. Therefore, economists started concentrating and creating new models and tools that can capture nonlinearities in the financial phenomena.

Research into time series models of changing variance and covariance has exploded in the last ten years. This activity has been driven by two major factors. First, out of the growing realization that much of modern theoretical finance is related to volatility has emerged the need
to develop empirically reasonable models to test, apply and deepen this theoretical work. Second, volatility models provide an excellent testing ground for the development of new nonlinear and non-Gaussian time series techniques. There has been surprisingly little work on portfolio management in the presence of stochastic volatility models. The bulk of the research efforts is in the context of valuing options and other derivative securities\(^1\). In what follows, we explore the effect of transaction costs on dynamic investment in the presence of stochastic volatilities.

The asset return dynamics are given by the stochastic volatility model of Equations (1.3)-(1.6)

\[
\begin{align*}
    r_t &= \mu + \sigma_{t-1} \epsilon_t, \\
    \sigma_t^2 &= \alpha_0 + \beta \sigma_{t-1}^2 + \alpha_1 \sigma_{t-1}^2 \epsilon_t^2, \\
    \epsilon_t &\sim N(0,1),
\end{align*}
\]

and therefore the rate of return at time \(t\) conditioned on the information available at time \(t-1\) is normally distributed with mean \(\mu\) and variance \(\sigma_t^2\). The parameters \(\alpha_0, \beta,\) and \(\alpha_1\) are assumed to be nonnegative to keep the conditional variance positive. Moreover, we assume that \(\alpha_1 + \beta < 1\), and thus the unconditional expectation of \(\sigma_t^2\) is \(E_s = \alpha_0 / (1 - \alpha_1 - \beta)\). Therefore, the investment manager faces the following dynamic optimization problem:

\[
\begin{align*}
    \text{maximize}_{\{u_0, \ldots, u_{T-1}\}} & \quad E_0 \left\{ x_T^0 + x_T \right\} - \lambda \text{Var}_0 \left\{ x_T^0 + x_T \right\} \\
\text{subject to} & \quad x_t^0 = (1+r_f) \left[ x_{t-1}^0 - u_{t-1} - \tau \, u_{t-1}^2 \right] \\
& \quad x_t = (1+r_t) \left[ x_{t-1} + u_{t-1} \right] \\
& \quad r_t = \mu + \sigma_{t-1} \epsilon_t \\
& \quad \sigma_t^2 = \alpha_0 + \beta \sigma_{t-1}^2 + \alpha_1 \sigma_{t-1}^2 \epsilon_t^2 \\
& \quad \epsilon_t \sim N(0,1).
\end{align*}
\]

\(^1\)See for example, Hull and White [46] and Wiggins [80].
The remainder of this chapter is organized as follows. In Section 4.1, we present the solution to the single-period optimization problem and in Section 4.2, we propose a structured approximation algorithm that uses characteristics of the optimal investment policy and in-depth analysis of the DP recursion. Finally, in Section 4.3, we present numerical examples that compare the performance of the proposed dynamic policies and illustrate the impact of transaction costs, risk aversion and return characteristics to the investment behavior over time.

4.1 The Single Period Problem

In this section we show that there exists a closed-form solution for the single period problem that results in an investment policy that is inversely proportional to the variance realizations.

The state of the system at time \( t = 0, 1, \ldots, T - 1 \) consists of the asset holdings \((x_t^0, x_t)\) before a transaction is made at time \( t \), and \( \sigma_t^2 \), the conditional variance of the risky asset at time \( t \). Using the relations of Section 2.2, the optimal control at time \( T - 1 \) can be written as

\[
 u_{T-1}^* (x_{T-1}, \sigma_{T-1}^2) = m_{T-1} \left( \sigma_{T-1}^2 \right) - L_{T-1} \left( \sigma_{T-1}^2 \right) x_{T-1},
\]

where

\[
 Q_{T-1} \left( \sigma_{T-1}^2 \right) = 2\tau \ (1 + r_f) + 2\lambda \ \sigma_{T-1}^2,
\]

\[
 m_{T-1} \left( \sigma_{T-1}^2 \right) = \frac{\mu - r_f}{Q_{T-1}},
\]

\[
 L_{T-1} \left( \sigma_{T-1}^2 \right) = \frac{2\lambda \ \sigma_{T-1}^2}{Q_{T-1}}.
\]

The optimal investment in the risky asset is inversely proportional to the state variable \( \sigma_{T-1}^2 \), in contrast with the one obtained in Equation (3.1) under the assumption of a factor pricing model for the return dynamics, where it is linear in the corresponding state variable. The value function \( V_{T-1} \) and the quantity \( K_{T-1} \) are both linear in \( x_{T-1}^0 \), quadratic in \( x_{T-1} \) and nonlinear in the other state variable \( \sigma_{T-1}^2 \):

\[
 V_{T-1} \left( x_{T-1}^0, x_{T-1}, \sigma_{T-1}^2 \right) = \frac{(\mu - r_f)^2}{4\tau \ (1 + r_f) + 4\lambda \ \sigma_{T-1}^2} + (1 + r_f) \ x_{T-1}^0 +
\]

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\[
\begin{align*}
\frac{2\lambda \sigma^2_{T-1} + 2\tau (1 + \mu)}{2\tau (1 + r_f) + 2\lambda \sigma^2_{T-1}} (1 + r_f) x_{T-1} - \frac{\lambda \sigma^2_{T-1} \tau (1 + r_f)}{\tau (1 + r_f) + \lambda \sigma^2_{T-1}} x_{T-1}^2,
\end{align*}
\] (4.1)

\[
K_{T-1} \left( x^0_{T-1}, x_{T-1}, \sigma^2_{T-1} \right) = \frac{(\mu - r_f)^2}{4 \left[ \frac{\tau (1 + r_f) + \lambda \sigma^2_{T-1}}{\tau (1 + r_f) + \lambda \sigma^2_{T-1}} \right]^2} + (1 + r_f) x^0_{T-1} +
\]

\[
\frac{(1 + r_f)}{\tau (1 + r_f) + \lambda \sigma^2_{T-1}} \left[ \frac{\tau (1 + \mu) + \lambda \sigma^2_{T-1} + \frac{\tau (\mu - r_f) \lambda \sigma^2_{T-1}}{\tau (1 + r_f) + \lambda \sigma^2_{T-1}}} {\left[ \tau (1 + r_f) + \lambda \sigma^2_{T-1} \right]^2} \right] x_{T-1} -
\]

\[
\frac{\lambda^2 \sigma^2_{T-1} \tau (1 + r_f)}{\left[ \tau (1 + r_f) + \lambda \sigma^2_{T-1} \right]^2} x_{T-1}^2.
\] (4.2)

In the following section we propose an approximation algorithm that uses characteristics of the optimal cost-to-go function at every point in time and investigate its behavior.

### 4.2 A Structured Approximation

In this section, we propose an approximation algorithm that utilizes characteristics of the optimal cost-to-go function at every time period. Since a closed-form solution for the DP algorithm is not achievable, we perform the following operations for \( k = 1, \ldots, T \):

1. Approximate the value function \( V_{T-k} \) with a quadratic in the state variables \( x_{T-k}, \sigma^2_{T-k} \) by using Taylor's expansion around the initial risky holdings \( x_0 \) and the unconditional expectation of \( \sigma^2_{T-k}, E_s \).

2. Approximate the quantity \( K_{T-k} \) with a function that is linear the state variables by using the first-order Taylor's expansion around \( x_0 \) and \( E_s \).

More specifically, ignoring the term involving the riskless holdings we can approximate \( V_{T-1} \) with

\[
V_{T-1} \left( x_{T-1}, \sigma^2_{T-1} \right) \approx
V_{T-1} \left( x_0, E_s \right) + \frac{\partial V_{T-1}}{\partial x_{T-1}} \left( x_{T-1} - x_0 \right) + \frac{\partial V_{T-1}}{\partial \sigma^2_{T-1}} \left( \sigma^2_{T-1} - E_s \right) +
\frac{1}{2} \left[ \frac{\partial^2 V_{T-1}}{\partial x^2_{T-1}} \right] \left( x_{T-1} - x_0 \right)^2 + \frac{1}{2} \left[ \frac{\partial^2 V_{T-1}}{\partial \sigma^2_{T-1}} \right] \left( \sigma^2_{T-1} - E_s \right)^2 +
\]

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\[
\left[ \frac{\partial^2 V_{T-1}}{\partial x_{T-1} \partial \sigma_{T-1}^2} \right]_{(x_0, E_s)} (x_{T-1} - x_0) \left( \sigma_{T-1}^2 - E_s \right) + \\
\frac{1}{2} \left[ \frac{\partial^3 V_{T-1}}{\partial x_{T-1}^2 \partial \sigma_{T-1}^2} \right]_{(x_0, E_s)} (x_{T-1} - x_0)^2 \left( \sigma_{T-1}^2 - E_s \right),
\]

and \( K_{T-1} \) with

\[
K_{T-1} \left( x_{T-1}, \sigma_{T-1}^2 \right) \approx (1 + r_f) x_{T-1}^0 + K_{T-1} (x_0, E_s) + \left[ \frac{\partial K_{T-1}}{\partial x_{T-1}} \right]_{(x_0, E_s)} (x_{T-1} - x_0) + \\
\left[ \frac{\partial K_{T-1}}{\partial \sigma_{T-1}^2} \right]_{(x_0, E_s)} \left( \sigma_{T-1}^2 - E_s \right).
\]

Thus, if we let

\[
\begin{align*}
v_{0,T-1} &= \frac{(\mu - r_f)^2}{4 \tau (1 + r_f) + 4 \lambda E_s} + \frac{2 \lambda E_s + 2 \tau (1 + \mu)}{2 \tau (1 + r_f) + 2 \lambda E_s} (1 + r_f) x_0 - \frac{\lambda E_s \tau (1 + r_f)}{\tau (1 + r_f) + \lambda E_s} x_0^2, \\
v_{1,T-1} &= \frac{2 \lambda E_s + 2 \tau (1 + \mu)}{2 \tau (1 + r_f) + 2 \lambda E_s} (1 + r_f) - \frac{2 \lambda E_s \tau (1 + r_f)}{\tau (1 + r_f) + \lambda E_s} x_0, \\
v_{11,T-1} &= -\frac{2 \lambda E_s \tau (1 + r_f)}{\tau (1 + r_f) + \lambda E_s}, \\
v_{2,T-1} &= -\frac{\lambda \left[ (\mu - r_f)^2 + 4 \tau (1 + r_f) (\mu - r_f) x_0 + 4 \tau^2 (1 + r_f)^2 x_0^2 \right]}{4 \left[ \tau (1 + r_f) + \lambda E_s \right]^2}, \\
v_{12,T-1} &= -\frac{\lambda \left[ 4 \tau (1 + r_f) (\mu - r_f) + 8 \tau^2 (1 + r_f)^2 x_0 \right]}{4 \left[ \tau (1 + r_f) + \lambda E_s \right]^2}, \\
v_{22,T-1} &= \frac{\lambda^2 \left[ (\mu - r_f)^2 + 4 \tau (1 + r_f) (\mu - r_f) x_0 + 4 \tau^2 (1 + r_f)^2 x_0^2 \right]}{2 \left[ \tau (1 + r_f) + \lambda E_s \right]^4}, \\
v_{13,T-1} &= \frac{-2 \lambda \tau^2 (1 + r_f)^2}{\left[ \tau (1 + r_f) + \lambda E_s \right]^2},
\end{align*}
\]

we obtain the following approximation of \( V_{T-1}, \tilde{V}_{T-1} \):

\[
\tilde{V}_{T-1} = (1 + r_f) x_{T-1}^0 + v_{0,T-1} + v_{1,T-1} (x_{T-1} - x_0) + v_{2,T-1} \left( \sigma_{T-1}^2 - E_s \right) + \\
\frac{1}{2} v_{11,T-1} (x_{T-1} - x_0)^2 + \frac{1}{2} v_{22,T-1} \left( \sigma_{T-1}^2 - E_s \right)^2 + \\
v_{12,T-1} (x_{T-1} - x_0) \left( \sigma_{T-1}^2 - E_s \right) + \frac{1}{2} v_{13,T-1} (x_{T-1} - x_0)^2 \left( \sigma_{T-1}^2 - E_s \right).
\]
Finally, by denoting

\[
\begin{align*}
  z_{T-1} &= v_{0,T-1} - v_{1,T-1}x_0 - v_{2,T-1}E_s + \frac{1}{2}v_{11,T-1}x_0^2 + \frac{1}{2}v_{22,T-1}E_s^2 + \\
  &\quad + v_{12,T-1}x_0E_s - \frac{1}{2}v_{13,T-1}x_0^2E_s, \\
  b_{T-1} &= v_{1,T-1} - v_{11,T-1}x_0 - v_{12,T-1}E_s + v_{13,T-1}x_0E_s, \\
  p_{T-1} &= v_{2,T-1} - v_{22,T-1}E_s - v_{12,T-1}x_0 + \frac{1}{2}v_{13,T-1}x_0^2, \\
  C_{T-1} &= -\frac{1}{2}v_{11,T-1} + \frac{1}{2}v_{13,T-1}E_s, \\
  H_{T-1} &= \frac{1}{2}v_{22,T-1}, \\
  S_{T-1} &= v_{12,T-1} - v_{13,T-1}x_0, \\
  \Pi_{T-1} &= \frac{1}{2}v_{13,T-1},
\end{align*}
\]

\(\tilde{V}_{T-1}\) results in

\[
\tilde{V}_{T-1} (x_{T-1}^0, x_{T-1}, \sigma_{T-1}^2) = z_{T-1} + (1 + \tau_f) x_{T-1}^0 + b_{T-1} x_{T-1} + p_{T-1} \sigma_{T-1}^2 - C_{T-1} x_{T-1}^2 + \\
\quad + H_{T-1} \sigma_{T-1}^4 + S_{T-1} x_{T-1} \sigma_{T-1}^2 + \Pi_{T-1} x_{T-1}^2 \sigma_{T-1}^2.
\]

Similarly, by letting

\[
\begin{align*}
  k_{0,T-1} &= \frac{(\mu - \tau_f)^2 \left[ \tau (1 + \tau_f) + 2\lambda E_s \right]}{4 \left[ \tau (1 + \tau_f) + \lambda E_s \right]^2} - \frac{\lambda^2 E_s^2 \tau (1 + \tau_f)}{\left[ \tau (1 + \tau_f) + \lambda E_s \right]^2} x_0^2 + \\
  &\quad + \frac{1}{\tau (1 + \tau_f) + \lambda E_s} \left[ \tau (1 + \mu) + \lambda E_s + \frac{\tau (\mu - \tau_f) \lambda E_s}{\tau (1 + \tau_f) + \lambda E_s} \right] x_0, \\
  k_{1,T-1} &= \frac{(1 + \tau_f)}{\tau (1 + \tau_f) + \lambda E_s} \left[ \tau (1 + \mu) + \lambda E_s + \frac{\tau (\mu - \tau_f) \lambda E_s}{\tau (1 + \tau_f) + \lambda E_s} \right] - \frac{2\lambda^2 E_s^2 \tau (1 + \tau_f)}{\left[ \tau (1 + \tau_f) + \lambda E_s \right]^2} x_0, \\
  k_{2,T-1} &= -\frac{(\mu - \tau_f)^2 \lambda^2 E_s}{2 \left[ \tau (1 + \tau_f) + \lambda E_s \right]^3} - \frac{2\lambda^2 \tau^2 (1 + \tau_f)^2 E_s^2}{\left[ \tau (1 + \tau_f) + \lambda E_s \right]^3} x_0^2 + \\
  &\quad + \frac{1 + \tau_f}{\tau (1 + \tau_f) + \lambda E_s} \left\{ \frac{\tau (1 + \tau_f) + \lambda E_s}{\lambda + \frac{\tau (\mu - \tau_f) \lambda [\tau (1 + \tau_f) + \lambda E_s] - \tau (\mu - \tau_f) \lambda^2 E_s}{\left( \tau (1 + \tau_f) + \lambda E_s \right)^2}} - \frac{\tau (\mu - \tau_f) \lambda E_s}{\tau (1 + \tau_f) + \lambda E_s} \right\} x_0,
\end{align*}
\]
we approximate \( K_{T-1} \) with \( \widehat{K}_{T-1} \):

\[
\widehat{K}_{T-1} = (1 + \gamma_f) x_{T-1}^0 + k_{0,T-1} + k_{1,T-1} (x_{T-1} - x_0) + k_{2,T-1} (\sigma_{T-1}^2 - E_t).
\]

Therefore, by letting

\[
h_{T-1} = k_{0,T-1} - k_{1,T-1} x_0 - k_{2,T-1} E_t,
\]

\[
a_{T-1} = k_{1,T-1},
\]

\[
q_{T-1} = k_{2,T-1},
\]

we derive the desired approximation for \( K_{T-1} \):

\[
\widehat{K}_{T-1} (x_{T-1}^0, x_{T-1}, \sigma_{T-1}^2) = h_{T-1} + (1 + \gamma_f) x_{T-1}^0 + a_{T-1} x_{T-1} + q_{T-1} \sigma_{T-1}^2.
\]

Consequently, the value function at time \( T-2 \) is approximated by

\[
V_{T-2} (x_{T-2}^0, x_{T-2}, \sigma_{T-2}^2) \approx \max_{\{w_{T-2}\}} \left\{ \mathbb{E}_{T-2} \left\{ \tilde{V}_{T-1} (x_{T-1}^0, x_{T-1}, \sigma_{T-1}^2) \right\} - \lambda \Var_{T-2} \{ \widehat{K}_{T-1} \} \right\} =
\]

\[
\max_{\{w_{T-2}\}} \left\{ x_{T-1} + (1 + \gamma_f)^2 \left[ x_{T-2}^0 - u_{T-2} - \tau u_{T-2}^2 \right] + b_{T-1} (1 + \mu) (x_{T-2} + u_{T-2}) +
\]

\[
p_{T-1} \left( \alpha_0 + \beta \sigma_{T-2}^2 + \alpha_1 \sigma_{T-2}^4 \right) - C_{T-1} \left[ (1 + \mu)^2 + \sigma_{T-2}^2 \right] (x_{T-2} + u_{T-2})^2 +
\]

\[
H_{T-1} \left[ \left( \alpha_0 + \beta \sigma_{T-2}^2 \right)^2 + 3 \alpha_1 \sigma_{T-2}^4 + 2 \left( \alpha_0 + \beta \sigma_{T-2}^2 \right) \alpha_1 \sigma_{T-2}^4 \right] +
\]

\[
S_{T-1} (1 + \mu) \left( \alpha_0 + \beta \sigma_{T-2}^2 + \alpha_1 \sigma_{T-2}^4 \right) (x_{T-2} + u_{T-2}) +
\]

\[
P_{T-1} (x_{T-2} + u_{T-2})^2 \left[ (1 + \mu)^2 \left( \alpha_0 + \beta \sigma_{T-2}^2 + \alpha_1 \sigma_{T-2}^4 \right) + \right.
\]

\[
\left. \sigma_{T-2}^2 \left( \alpha_0 + \beta \sigma_{T-2}^2 \right) + 3 \alpha_1 \sigma_{T-2}^4 \right] -
\]

\[
\lambda \Var_{T-2} \left\{ a_{T-1} (1 + \mu + \sigma_{T-2} \epsilon_{T-1}) (x_{T-2} + u_{T-2}) +
\]

\[
q_{T-1} \left( \alpha_0 + \beta \sigma_{T-2}^2 + \alpha_1 \sigma_{T-2}^4 \right) \right\}.
\]

Using Lemma 3.5 for the univariate case we can evaluate the conditional variance \( \mathcal{L} \) appearing in the above optimization problem

\[
\mathcal{L} = (x_{T-2} + u_{T-2})^2 \sigma_{T-2}^2 + 2 \alpha_1 \sigma_{T-2}^4,
\]

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and thus approximate the value function by

\[
V_{T-2} \left( x_{T-2}^0, x_{T-2}, \sigma_{T-2}^2 \right) \approx \max_{\{u_{T-2}\}} \left\{ \begin{array}{l}
  z_{T-1} + (1 + r_f)^2 \left[ x_{T-2}^0 - u_{T-2} - \tau u_{T-2}^2 \right] + b_{T-1} (1 + \mu) (x_{T-2} + u_{T-2}) + \\
  \rho_{T-1} \left( \alpha_0 + \beta \sigma_{T-2}^2 + \alpha_1 \sigma_{T-2}^2 \right) - C_{T-1} \left( (1 + \mu)^2 + \sigma_{T-2}^2 \right) (x_{T-2} + u_{T-2})^2 + \\
  H_{T-1} \left( \alpha_0 + \beta \sigma_{T-2}^2 \right)^2 + 3 \alpha_1^2 \sigma_{T-2}^4 + 2 \left( \alpha_0 + \beta \sigma_{T-2}^2 \right) \alpha_1 \sigma_{T-2}^2 + \\
  \Pi_{T-1} (x_{T-2} + u_{T-2})^2 \left[ \left( 1 + \mu \right)^2 \left( \alpha_0 + \beta \sigma_{T-2}^2 + \alpha_1 \sigma_{T-2}^2 \right) + \sigma_{T-2}^2 \left( \alpha_0 + \beta \sigma_{T-2}^2 \right) + 3 \alpha_1 \sigma_{T-2}^4 \right] - \\
  \lambda (x_{T-2} + u_{T-2})^2 \sigma_{T-1}^2 \sigma_{T-2} - 2 \lambda \sigma_{T-1}^2 \sigma_{T-2}^4 \end{array} \right\}.
\]

This is a quadratic optimization problem and its solution is given by

\[
- (1 + r_f)^2 - 2 \tau (1 + r_f)^2 u_{T-2} + b_{T-1} (1 + \mu) - 2 C_{T-1} \left( (1 + \mu)^2 + \sigma_{T-2}^2 \right) (x_{T-2} + u_{T-2}) + \\
S_{T-1} (1 + \mu) \left( \alpha_0 + \beta \sigma_{T-2}^2 + \alpha_1 \sigma_{T-2}^2 \right) + 2 (x_{T-2} + u_{T-2}) \Pi_{T-1} \\
\left[ (1 + \mu)^2 \left( \alpha_0 + \beta \sigma_{T-2}^2 + \alpha_1 \sigma_{T-2}^2 \right) + \sigma_{T-2}^2 \left( \alpha_0 + \beta \sigma_{T-2}^2 \right) + 3 \alpha_1 \sigma_{T-2}^4 \right] - \\
2 \lambda \sigma_{T-1}^2 \sigma_{T-2}^2 (x_{T-2} + u_{T-2}) = 0.
\]

The optimal investment in the risky asset is therefore approximated by

\[
\tilde{w}_{T-1} \left( x_{T-2}, \sigma_{T-2}^2 \right) = m_{T-2} (\sigma_{T-2}^2) - L_{T-2} (\sigma_{T-2}^2) x_{T-2},
\]

where

\[
\begin{align*}
\gamma_{T-2} &= 2 C_{T-1} (1 + \mu)^2 - 2 \Pi_{T-1} (1 + \mu)^2 \alpha_0, \\
\pi_{T-2} &= - (1 + r_f)^2 + b_{T-1} (1 + \mu) + S_{T-1} (1 + \mu) \alpha_0, \\
\delta_{T-2} &= 2 C_{T-1} - 2 \Pi_{T-1} (1 + \mu)^2 (\beta + \alpha_1) - 2 \Pi_{T-1} \alpha_0 + 2 \lambda \sigma_{T-1}^2, \\
\varphi_{T-2} &= S_{T-1} (1 + \mu) (\beta + \alpha_1), \\
\vartheta_{T-2} &= -2 \Pi_{T-1} \beta - 6 \Pi_{T-1} \alpha_1,
\end{align*}
\]
\[ Q_{T-2} \left( \sigma^2_{T-2} \right) = 2\tau \ (1 + r_f)^2 + \gamma_{T-2} + \delta_{T-2} \sigma^2_{T-2} + \vartheta_{T-2} \sigma^4_{T-2}, \]
\[ m_{T-2} \left( \sigma^2_{T-2} \right) = \frac{\pi_{T-2} + \varphi_{T-2} \sigma^2_{T-2}}{Q_{T-2}}, \]
\[ L_{T-2} \left( \sigma^2_{T-2} \right) = \frac{\gamma_{T-2} + \delta_{T-2} \sigma^2_{T-2} + \vartheta_{T-2} \sigma^4_{T-2}}{Q_{T-2}}. \]

Substituting the approximate control policy in the value function and quantity \( K_{T-2} \) yields the approximated cost-to-go function \( \hat{V}_{T-2} \)

\[
\hat{V}_{T-2} \left( x^0_{T-2}, x_{T-2}, \sigma^2_{T-2} \right) = (1 + r_f)^2 x^0_{T-2} + \\
\left[ \begin{array}{c}
\left( 1 + r_f \right)^2 \alpha_0 + \frac{\beta \sigma^2_{T-2} + \alpha_1 \sigma^4_{T-2}}{\sigma^2_{T-2}} - 2\lambda \varphi_{T-1} \alpha^2 \sigma^2_{T-2} + \\
H_{T-1} \left[ \alpha_0^2 + 2\alpha_0 \beta \alpha_1 \sigma^2_{T-2} + \left( \beta^2 + 3\alpha_1^2 + 2\beta \alpha_1 \right) \sigma^4_{T-2} \right] + \\
\left\{ -\left( 1 + r_f \right)^2 + b_{T-1} \left( 1 + \mu \right) + S_{T-1} \left( 1 + \mu \right) \left( \alpha_0 + \beta \sigma^2_{T-2} + \alpha_1 \sigma^4_{T-2} \right) \right\} m_{T-2} + \\
\tau \left( 1 + r_f \right)^2 \left( 1 + \mu \right)^2 \sigma^2_{T-2} + \left( 1 + \mu \right)^2 \sigma^2_{T-2} \right) m_{T-2} + \\
\Pi_{T-1} \left[ (1 + \mu)^2 \alpha_0 + \left( 1 + \mu \right)^2 \left( \beta + \alpha_1 \right) + \alpha_0 \right] \sigma^2_{T-2} + \left( \beta + 3\alpha_1 \right) \sigma^4_{T-2} \right] m_{T-2} - \\
\lambda a^2_{T-1} \sigma^2_{T-2} m_{T-2}
\end{array} \right] + \\
\left[ (1 + r_f)^2 L_{T-2} + 2\tau \left( 1 + r_f \right)^2 m_{T-2} L_{T-2} + b_{T-1} \left( 1 + \mu \right) \left( 1 - L_{T-2} \right) - \\
2C_{T-1} \left( 1 + \mu \right)^2 \sigma^2_{T-2} m_{T-2} \left( 1 - L_{T-2} \right) + \\
S_{T-1} \left( 1 + \mu \right) \left( \alpha_0 + \beta \sigma^2_{T-2} + \alpha_1 \sigma^4_{T-2} \right) \left( 1 - L_{T-2} \right) + \\
2\Pi_{T-1} \left[ (1 + \mu)^2 \alpha_0 + \left( 1 + \mu \right)^2 \left( \beta + \alpha_1 \right) + \alpha_0 \right] \sigma^2_{T-2} + \\
\left( \beta + 3\alpha_1 \right) \sigma^4_{T-2} \right] m_{T-2} \left( 1 - L_{T-2} \right) - \\
2\lambda a^2_{T-1} \sigma^2_{T-2} m_{T-2} \left( 1 - L_{T-2} \right)
\right]
\]
and the approximated quantity

\[
\hat{K}_{T-k} \left( x_{T-k}, \sigma_{T-k}^2 \right) = (1 + r_f)^2 x_{T-k}^0 + \\
\left[ h_{T-k} - (1 + r_f)^2 m_{T-k} - \tau (1 + r_f)^2 m_{T-k}^2 + o_{T-k} \left( 1 + \sigma_{T-k}^2 \right) \right] + \\
\left[ (1 + r_f)^2 L_{T-k} + 2 \tau (1 + r_f)^2 m_{T-k} L_{T-k} + \sigma_{T-k} \left( 1 + \sigma_{T-k}^2 \right) \right] x_{T-k} - \\
\tau (1 + r_f)^2 L_{T-k}^2 \sigma_{T-k}^2.
\]

We prove the following theorem that yields the proposed approximation algorithm:

**Theorem 4.1** The optimal investment decisions \( u_{T-k}^* \), the value function \( V_{T-k} \) and the quantity \( K_{T-k} \) are approximated for \( k = 1, \ldots, T \) by the following relations:

\[
\hat{u}_{T-k} \left( x_{T-k}, \sigma_{T-k}^2 \right) = m_{T-k} - L_{T-k} x_{T-k},
\]

and

\[
\hat{V}_{T-k} \left( x_{T-k}, \sigma_{T-k}^2 \right) = (1 + r_f)^2 x_{T-k}^0 + \\
\left[ x_{T-k+1} + p_{T-k+1} \left( o_{0} + \beta \sigma_{T-k}^2 + \alpha_1 \sigma_{T-k}^2 \right) - 2 \lambda q_{T-k+1} \sigma_{T-k}^2 + \\
H_{T-k+1} \left[ o_{0} + 2 \alpha_0 (\beta + \alpha_1) \sigma_{T-k}^2 \right] + \left( \beta^2 + 3 \alpha_1^2 + 2 \beta \alpha_1 \right) \sigma_{T-k}^4 \right] + \\
\left\{ - (1 + r_f)^k + b_{T-k+1} \left( 1 + \mu \right) + S_{T-k+1} \left( 1 + \mu \right) \left( o_{0} + \beta \sigma_{T-k}^2 + \alpha_1 \sigma_{T-k}^2 \right) \right\} m_{T-k} - \\
\tau (1 + r_f)^k m_{T-k}^2 - C_{T-k+1} \left[ (1 + \mu)^2 + \sigma_{T-k}^2 \right] m_{T-k}^2 + \\
\Pi_{T-k+1} \left[ (1 + \mu)^2 o_{0} + \left( (1 + \mu)^2 (\beta + \alpha_1) + o_{0} \right) \sigma_{T-k}^2 \right] \sigma_{T-k}^2 \left( \beta + 3 \alpha_1 \right) \sigma_{T-k}^4 \right] m_{T-k}^2 - \\
\lambda a_{T-k+1} \sigma_{T-k}^2 m_{T-k}^2 + \\
\left[ (1 + r_f)^k L_{T-k} + 2 \tau (1 + r_f)^k m_{T-k} L_{T-k} + b_{T-k+1} \left( 1 + \mu \right) \left( 1 - L_{T-k} \right) - \\
2C_{T-k+1} \left[ (1 + \mu)^2 + \sigma_{T-k}^2 \right] m_{T-k} \left( 1 - L_{T-k} \right) + \\
S_{T-k+1} \left( 1 + \mu \right) \left( o_{0} + \beta \sigma_{T-k}^2 + \alpha_1 \sigma_{T-k}^2 \right) \left( 1 - L_{T-k} \right) + \\
2 \Pi_{T-k+1} \left[ (1 + \mu)^2 o_{0} + \left( (1 + \mu)^2 (\beta + \alpha_1) + o_{0} \right) \sigma_{T-k}^2 \right] \left( \beta + 3 \alpha_1 \right) \sigma_{T-k}^4 \right] m_{T-k} \left( 1 - L_{T-k} \right) - \\
2 \lambda a_{T-k+1} \sigma_{T-k}^2 m_{T-k} \left( 1 - L_{T-k} \right)
\]
\[
\begin{bmatrix}
\tau (1 + r_f)^k L^2_{T-k} + C_{T-k+1} \left[ (1 + \mu)^2 + \sigma^2_{T-k} \right] (1 - L_{T-k})^2 - \\
\Pi_{T-k+1} \left[ (1 + \mu)^2 \alpha_0 + \left[ (1 + \mu)^2 \beta + \alpha_1 \right] \sigma^2_{T-k} + (\beta + 3 \alpha_1) \sigma^4_{T-k} \right] \\
(1 - L_{T-k})^2 + \lambda a^2_{T-k+1} \sigma^2_{T-k} (1 - L_{T-k})^2
\end{bmatrix} x^2_{T-k},
\]

and

\[
\overline{K}_{T-k} \left( x^0_{T-k}, x_{T-k}, \sigma_{T-k}^2 \right) = (1 + r_f)^k x^0_{T-k} + \\
\begin{bmatrix}
h_{T-k+1} - (1 + r_f)^k m_{T-k} - \tau (1 + r_f)^k m^2_{T-k} + a_{T-k+1} (1 + \mu) m_{T-k} + \end{bmatrix} + \\
\begin{bmatrix}
q_{T-k+1} \left( \alpha_0 + \beta \sigma^2_{T-k} + \alpha_1 \sigma^2_{T-k} \right) \\
(1 + r_f)^k L_{T-k} + 2 \tau (1 + r_f)^k m_{T-k} L_{T-k} + a_{T-k+1} (1 + \mu) (1 - L_{T-k}) x_{T-k} - \\
\tau (1 + r_f)^k L^2_{T-k} x^2_{T-k},
\end{bmatrix}
\]

where

\[
m_{T-k} \left( \sigma^2_{T-k} \right) = \frac{\pi_{T-k} + \varphi_{T-k} \sigma^2_{T-k}}{2 \tau (1 + r_f)^k + \gamma_{T-k} + \delta_{T-k} \sigma^2_{T-k} + \vartheta_{T-k} \sigma^4_{T-k}},
\]

\[
L_{T-k} \left( \sigma^2_{T-k} \right) = \frac{\gamma_{T-k} + \delta_{T-k} \sigma^2_{T-k} + \vartheta_{T-k} \sigma^4_{T-k}}{2 \tau (1 + r_f)^k + \gamma_{T-k} + \delta_{T-k} \sigma^2_{T-k} + \vartheta_{T-k} \sigma^4_{T-k}},
\]

\[
\gamma_{T-k} = \frac{2C_{T-k+1} (1 + \mu)^2 - 2 \Pi_{T-k+1} (1 + \mu)^2 \alpha_0,}
\]

\[
\pi_{T-k} = \frac{- (1 + r_f)^k + b_{T-k+1} (1 + \mu) + S_{T-k+1} (1 + \mu) \alpha_0,}
\]

\[
\delta_{T-k} = \frac{2C_{T-k+1} - 2 \Pi_{T-k+1} (1 + \mu)^2 \beta + \alpha_1) - 2 \Pi_{T-k+1} \alpha_0 + 2 \lambda a^2_{T-k+1},}
\]

\[
\varphi_{T-k} = \frac{S_{T-k+1} (1 + \mu) (\beta + \alpha_1),}
\]

\[
\vartheta_{T-k} = -2 \Pi_{T-k+1} (\beta + 3 \alpha_1).
\]

In addition, we define the following parameters

\[
z_{T-k} = v_0_{T-k} - v_{1, T-k} x_0 - v_{2, T-k} E_s + \frac{1}{2} v_{11, T-k} x_0^2 + \frac{1}{2} v_{22, T-k} E_s^2 + \\
v_{12, T-k} x_0 E_s - \frac{1}{2} v_{13, T-k} x_0^2 E_s,
\]

\[
h_{T-k} = v_{1, T-k} - v_{11, T-k} x_0 - v_{12, T-k} E_s + v_{13, T-k} x_0 E_s,
\]

\[
p_{T-k} = v_{2, T-k} - v_{22, T-k} E_s - v_{12, T-k} x_0 + \frac{1}{2} v_{13, T-k} x_0^2,
\]

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\[ C_{T-k} = \frac{1}{2} v_{11,T-k} + \frac{1}{2} v_{13,T-k} E_s, \]
\[ H_{T-k} = \frac{1}{2} v_{22,T-k}, \]
\[ S_{T-k} = v_{12,T-k} - v_{13,T-k} x_0, \]
\[ \Pi_{T-k} = \frac{1}{2} v_{13,T-k} , \]

and

\[ h_{T-k} = k_{0,T-k} - k_{1,T-k} x_0 - k_{2,T-k} E_s, \]
\[ a_{T-k} = k_{1,T-k}, \]
\[ q_{T-k} = k_{2,T-k}, \]

with the boundary conditions

\[ z_T = h_T = p_T = q_T = C_T = H_T = S_T = \Pi_T = 0, \]
\[ a_T = b_T = 1. \]

**Proof.** We prove the theorem by induction. We have shown the relations to be true for \( k = 1 \). Assuming that they hold for arbitrary \( k \), we show that they are also valid for \( k + 1 \). Using Proposition 2.1 we can express the value function at time \( T - k - 1 \) as

\[ V_{T-k-1} \left( x_{T-k-1,1}^0, x_{T-k-1,2}, \sigma_{T-k-1}^2 \right) = \max_{\{u_{T-k-1}\}} \left\{ E_{T-k-1} \left\{ \tilde{V}_{T-k} \left( x_{T-k}^0, x_{T-k,2}, \sigma_{T-k-1}^2 \right) \right\} - \lambda \text{Var}_{T-k-1} \{ \tilde{K}_{T-k} \} \right\}. \]

The above optimization problem is a fourth-order polynomial in the control variable \( u_{T-k-1} \) and involves expectations of complicated functions of the asset’s variance. In response, we perform the following approximation steps:

1. Approximate \( V_{T-k} \) with a quadratic function in the state variables \( (x_{T-k-1}, \sigma_{T-k-1}^2) \) by using Taylor’s expansion around the initial risky holdings \( x_0 \) and the unconditional expectation of \( \sigma_{T-k-1}^2 , E_s \).
2. Approximate $K_{T-k}$ with a linear function in the state variables by using the first-order Taylor's expansion around $x_0$ and $E_s$.

For convenience, denote

\[
\begin{align*}
\tilde{Q}_{T-k} &= 2\tau (1 + r_f)^k + \gamma_{T-k} + \delta_{T-k} E_s + \vartheta_{T-k} E_s^2, \\
\tilde{m}_{T-k} &= \frac{\pi_{T-k} + \varphi_{T-k} E_s}{\tilde{Q}_{T-k}}, \\
\tilde{L}_{T-k} &= \frac{\gamma_{T-k} + \delta_{T-k} E_s + \vartheta_{T-k} E_s^2}{\tilde{Q}_{T-k}}, \\
\tilde{m}_1_{T-k} &= \frac{\varphi_{T-k} \tilde{Q}_{T-k} - (\pi_{T-k} + \varphi_{T-k} E_s) (\delta_{T-k} + 2\vartheta_{T-k} E_s)}{(\tilde{Q}_{T-k})^2}, \\
\tilde{L}_1_{T-k} &= \frac{2\tau (1 + r_f)^k (\delta_{T-k} + 2\vartheta_{T-k} E_s)}{(\tilde{Q}_{T-k})^2},
\end{align*}
\]

\[
\tilde{m}_2_{T-k} = -2 \left( \tilde{Q}_{T-k} \right)^{-3} \left[ \tilde{Q}_{T-k} \vartheta_{T-k} (\pi_{T-k} + \varphi_{T-k} E_s) + (\delta_{T-k} + 2\vartheta_{T-k} E_s) \right],
\]

\[
\tilde{L}_2_{T-k} = \frac{4\tau (1 + r_f)^k \vartheta_{T-k} \tilde{Q}_{T-k} - (\delta_{T-k} + 2\vartheta_{T-k} E_s)^2}{(\tilde{Q}_{T-k})^3},
\]

where $\tilde{m}_1_{T-k}$ is the first derivative of $m_{T-k}$ with respect to $\sigma_{T-k}^2$ evaluated at $E_s$ and $\tilde{m}_2_{T-k}$ is the second derivative evaluated at the same point. Similar notation prevails for the derivatives of $L_{T-k}$. The value function $\tilde{V}_{T-k}$ is then approximated by

\[
\tilde{V}_{T-k} = (1 + r_f)^k x_{T-k}^0 + v_{0,T-k} + v_{1,T-k} (x_{T-k} - x_0) + v_{2,T-k} \left( \sigma_{T-k}^2 - E_s \right) + \frac{1}{2} v_{11,T-k} (x_{T-k} - x_0)^2 + \frac{1}{2} v_{22,T-k} \left( \sigma_{T-k}^2 - E_s \right)^2 + \frac{1}{2} v_{13,T-k} (x_{T-k} - x_0)^2 \left( \sigma_{T-k}^2 - E_s \right) + \frac{1}{2} v_{23,T-k} (x_{T-k} - x_0)^2 \left( \sigma_{T-k}^2 - E_s \right),
\]

where $v_{0,T-k}$ is the value of $\left( \tilde{V}_{T-k} - (1 + r_f)^k x_{T-k}^0 \right)$ at the point $(x_0, E_s)$ given by

\[
v_{0,T-k} = z_{T-k+1} + p_{T-k+1} (\alpha_0 + \beta E_s + \alpha_1 E_s) - 2\lambda_0 \q_{T-k+1}^2 \alpha_1^2 E_s^2 + \ldots
\]
\[ H_{T-k+1} \left[ \alpha_0^2 + 2\omega_0 (\beta + \alpha_1) E_s + (\beta^2 + 3\alpha_1^2 + 2\beta\alpha_1) E_s^2 \right] + \]
\[ \{ - (1 + r_f)^k + b_{T-k+1} (1 + \mu) + S_{T-k+1} (1 + \mu) (\alpha_0 + \beta E_s + \alpha_1 E_s) \} \tilde{m}_{T-k} - \]
\[ \tau (1 + r_f)^k (\tilde{m}_{T-k})^2 - C_{T-k+1} \left[ (1 + \mu)^2 + E_s \right] (\tilde{m}_{T-k})^2 - \lambda a_{T-k+1}^2 E_s (\tilde{m}_{T-k})^2 + \]
\[ \Pi_{T-k+1} \left[ (1 + \mu)^2 \alpha_0 + \left[ (1 + \mu)^2 (\beta + \alpha_1) + \alpha_0 \right] E_s + (\beta + 3\alpha_1) E_s^2 \right] (\tilde{m}_{T-k})^2 + \]
\[ \left\{ (1 + r_f)^k \tilde{L}_{T-k} + 2\tau (1 + r_f)^k \tilde{m}_{T-k} \tilde{L}_{T-k} + b_{T-k+1} (1 + \mu) \left( 1 - \tilde{L}_{T-k} \right) - \right. \]
\[ 2C_{T-k+1} \left[ (1 + \mu)^2 + E_s \right] \tilde{m}_{T-k} \left( 1 - \tilde{L}_{T-k} \right) + \]
\[ S_{T-k+1} (1 + \mu) (\alpha_0 + \beta E_s + \alpha_1 E_s) \left( 1 - \tilde{L}_{T-k} \right) + \]
\[ 2\Pi_{T-k+1} \left[ (1 + \mu)^2 \alpha_0 + \left[ (1 + \mu)^2 (\beta + \alpha_1) + \alpha_0 \right] E_s + (\beta + 3\alpha_1) E_s^2 \right] \]
\[ \tilde{m}_{T-k} \left( 1 - \tilde{L}_{T-k} \right) - 2\lambda a_{T-k+1}^2 E_s \tilde{m}_{T-k} \left( 1 - \tilde{L}_{T-k} \right) \right\} x_0 - \]
\[ \left\{ \tau (1 + r_f)^k \left( \tilde{L}_{T-k} \right)^2 + C_{T-k+1} \left[ (1 + \mu)^2 + E_s \right] \left( 1 - \tilde{L}_{T-k} \right)^2 - \right. \]
\[ \Pi_{T-k+1} \left[ (1 + \mu)^2 \alpha_0 + \left[ (1 + \mu)^2 (\beta + \alpha_1) + \alpha_0 \right] E_s + (\beta + 3\alpha_1) E_s^2 \right] \]
\[ \left( 1 - \tilde{L}_{T-k} \right)^2 + \lambda a_{T-k+1}^2 E_s \left( 1 - \tilde{L}_{T-k} \right)^2 \right\} x_0^2, \]

and

\[ v_{1,T-k} = \left[ \frac{\partial V_{T-k}}{\partial x_{T-k}} \right]_{(x_0,E_s)}, \quad v_{2,T-k} = \left[ \frac{\partial V_{T-k}}{\partial \sigma_{T-k}^2} \right]_{(x_0,E_s)}, \quad v_{11,T-k} = \left[ \frac{\partial^2 V_{T-k}}{\partial x_{T-k} \partial \sigma_{T-k}^2} \right]_{(x_0,E_s)}, \quad v_{12,T-k} = \left[ \frac{\partial^2 V_{T-k}}{\partial x_{T-k} \partial x_{T-k}} \right]_{(x_0,E_s)}, \quad v_{13,T-k} = \left[ \frac{\partial^2 V_{T-k}}{\partial \sigma_{T-k}^2 \partial \sigma_{T-k}^2} \right]_{(x_0,E_s)}. \]

If we let

\[ \mathcal{L}_{1,T-k} = \tau (1 + r_f)^k \left( \tilde{L}_{T-k} \right)^2 + C_{T-k+1} \left[ (1 + \mu)^2 + E_s \right] \left( 1 - \tilde{L}_{T-k} \right)^2 - \]
\[ \Pi_{T-k+1} \left[ (1 + \mu)^2 \alpha_0 + \left[ (1 + \mu)^2 (\beta + \alpha_1) + \alpha_0 \right] E_s + (\beta + 3\alpha_1) E_s^2 \right] \]
\[ \left( 1 - \tilde{L}_{T-k} \right)^2 + \lambda a_{T-k+1}^2 E_s \left( 1 - \tilde{L}_{T-k} \right)^2, \]

\[ \mathcal{L}_{2,T-k} = (1 + r_f)^k \tilde{L}_{T-k} + 2\tau (1 + r_f)^k \left( \tilde{m}_{T-k} \tilde{L}_{T-k} + \tilde{m}_{T-k} \tilde{L}_{T-k} \right) - \]
\[ b_{T-k+1} (1 + \mu) \tilde{L}_{T-k} - 2C_{T-k+1} (1 + \mu)^2 \left[ \tilde{m}_{T-k} \left( 1 - \tilde{L}_{T-k} \right) - \tilde{m}_{T-k} \tilde{L}_{T-k} \right] - \]
\[ b_{T-k+1} (1 + \mu) \tilde{L}_{T-k} + S_{T-k+1} (1 + \mu) (\alpha_0 + \beta) \left[ \left( 1 - \tilde{L}_{T-k} \right) - E_s \tilde{L}_{T-k} \right] + \]
\[ 2\Pi_{T-k+1} \left[ (1 + \mu)^2 \alpha_0 \left[ \tilde{m}_{T-k} \left( 1 - \tilde{L}_{T-k} \right) - \tilde{m}_{T-k} \tilde{L}_{T-k} \right] + \right. \]
\[ 2\Pi_{T-k+1} \left[ (1 + \mu)^2 (\beta + \alpha_1) + \alpha_0 \right] \]
\[
[\bar{m}_{T-k} (1 - \bar{L}_{T-k}) + E_s \bar{m}_{1T-k} (1 - \bar{L}_{T-k}) - E_s \bar{m}_{T-k} \bar{L}_{1T-k}] + \\
2\Pi_{T-k+1} (\beta + 3\alpha_1) \\
[2E_s \bar{m}_{T-k} (1 - \bar{L}_{T-k}) + E_s^2 \bar{m}_{1T-k} (1 - \bar{L}_{T-k}) - E_s^2 \bar{m}_{T-k} \bar{L}_{1T-k}] - \\
2\lambda a_{T-k+1}^2 \left[ \bar{m}_{T-k} (1 - \bar{L}_{T-k}) + E_s \bar{m}_{1T-k} (1 - \bar{L}_{T-k}) - E_s \bar{m}_{T-k} \bar{L}_{1T-k} \right].
\]

\[
L_{3,T-k} = 2\tau (1 - r_f)^k \bar{L}_{T-k} \bar{L}_{1T-k} - 2C_{T-k+1} (1 + \mu)^2 (1 - \bar{L}_{T-k}) \bar{L}_{1T-k} + \\
C_{T-k+1} \left[ (1 - \bar{L}_{T-k})^2 - 2E_s (1 - \bar{L}_{T-k}) \bar{L}_{1T-k} \right] + \\
2\Pi_{T-k+1} (1 + \mu)^2 a_0 (1 - \bar{L}_{T-k}) \bar{L}_{1T-k} + \Pi_{T-k+1} \left[ (1 + \mu)^2 (\beta + \alpha_1) + \alpha_0 \right] \\
\left[ (1 - \bar{L}_{T-k})^2 - 2E_s (1 - \bar{L}_{T-k}) \bar{L}_{1T-k} \right] - \Pi_{T-k+1} (\beta + 3\alpha_1) \\
\left[ 2E_s (1 - \bar{L}_{T-k})^2 - 2E_s^2 (1 - \bar{L}_{T-k}) \bar{L}_{1T-k} \right] + \lambda a_{T-k+1}^2 \left[ (1 - \bar{L}_{T-k})^2 - 2E_s (1 - \bar{L}_{T-k}) \bar{L}_{1T-k} \right],
\]

then, evaluating the partial derivatives of \( \tilde{V}_{T-k} \) results in

\[
v_{1,T-k} = (1 + r_f)^k \bar{L}_{T-k} + 2\tau (1 + r_f)^k \bar{m}_{T-k} \bar{L}_{T-k} + b_{T-k+1} (1 + \mu) (1 - \bar{L}_{T-k}) - \\
2C_{T-k+1} \left[ (1 + \mu)^2 + E_s \right] \bar{m}_{T-k} (1 - \bar{L}_{T-k}) + S_{T-k+1} (1 + \mu) (a_0 + \beta E_s + \alpha_1 E_s) \\
(1 - \bar{L}_{T-k}) + 2\Pi_{T-k+1} \\
\left[ (1 + \mu)^2 a_0 + \left[ (1 + \mu)^2 (\beta + \alpha_1) + \alpha_0 \right] E_s + (\beta + 3\alpha_1) E_s^2 \right] \\
\bar{m}_{T-k} (1 - \bar{L}_{T-k}) - 2\lambda a_{T-k+1}^2 E_s \bar{m}_{T-k} \left( 1 - \bar{L}_{T-k} \right) - 2L_{1,T-k} x_0,
\]

\[
v_{11,T-k} = -2 L_{1,T-k},
\]

\[
v_{2,T-k} = -(1 + r_f)^k \bar{m}_{1T-k} - 2\tau (1 + r_f)^k \bar{m}_{T-k} \bar{m}_{1T-k} + b_{T-k+1} (1 + \mu) \bar{m}_{1T-k} + \\
p_{T-k+1} (\alpha_1 + \beta) - C_{T-k+1} (1 + \mu)^2 \bar{m}_{T-k} \bar{m}_{1T-k} - \\
C_{T-k+1} \left[ (\bar{m}_{T-k})^2 + 2E_s \bar{m}_{T-k} \bar{m}_{1T-k} \right] + \\
H_{T-k+1} \left[ 2a_0 (\beta + \alpha_1) + 2 (\beta^2 + 3\alpha_1^2 + 2\beta \alpha_1) E_s \right] + S_{T-k+1} (1 + \mu) a_0 \bar{m}_{T-k} + \\
S_{T-k+1} (1 + \mu) (\alpha_1 + \beta) (\bar{m}_{T-k} + E_s \bar{m}_{1T-k}) + \Pi_{T-k+1} (1 + \mu)^2 a_0 \\
\bar{m}_{T-k} \bar{m}_{1T-k} + \Pi_{T-k+1} \left[ (1 + \mu)^2 (\beta + \alpha_1) + a_0 \right] \left[ (\bar{m}_{T-k})^2 + 2E_s \bar{m}_{T-k} \bar{m}_{1T-k} \right] + \\
\Pi_{T-k+1} (\beta + 3\alpha_1) E_s \bar{m}_{T-k} \left[ \bar{m}_{T-k} + E_s \bar{m}_{1T-k} \right] - 4\lambda q_{T-k+1}^2 a_1^2 E_s -
\]

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\[
\lambda a_{T-k+1}^2 \left[ (\bar{m}_{T-k})^2 + 2 E_s \bar{m}_{T-k} \bar{m}_{1T-k} \right] + \\
\mathcal{L}_{2,T-k} x_0 - \mathcal{L}_{3,T-k} x_0^2,
\]

and

\[
v_{12,T-k} = \mathcal{L}_{2,T-k} - 2 x_0 \mathcal{L}_{3,T-k},
\]

\[
v_{13,T-k} = -2 \mathcal{L}_{3,T-k},
\]

\[
v_{22,T-k} = \left[ b_{T-k+1} (1 + \mu) - (1 + r_f)^k \right] \bar{m}_{2T-k} - \\
2 \tau (1 + r_f)^k \left[ \left( \bar{m}_{1T-k} \right)^2 + \bar{m}_{T-k} \bar{m}_{2T-k} \right] - \\
2 C_{T-k+1} (1 + \mu)^2 \left[ \left( \bar{m}_{1T-k} \right)^2 + \bar{m}_{T-k} \bar{m}_{2T-k} \right] - \\
2 C_{T-k+1} \left[ 2 \bar{m}_{T-k} \bar{m}_{1T-k} + E_s \left( \bar{m}_{1T-k} \right)^2 + E_s \bar{m}_{T-k} \bar{m}_{2T-k} \right] + \\
2 H_{T-k+1} \left( \beta^2 + 3 \alpha_1^2 + 2 \beta \alpha_1 \right) + S_{T-k+1} (1 + \mu) a_0 \bar{m}_{2T-k} + \\
S_{T-k+1} (1 + \mu) \left( \alpha_1 + \beta \right) \left[ (1 + E_s) \bar{m}_{2T-k} + \bar{m}_{1T-k} \right] + \\
2 \Pi_{T-k+1} (1 + \mu)^2 a_0 \left[ \left( \bar{m}_{1T-k} \right)^2 + \bar{m}_{T-k} \bar{m}_{2T-k} \right] + 2 \Pi_{T-k+1} \left[ (1 + \mu)^2 (\beta + \alpha_1) + a_0 \right] \\
\left[ 2 \bar{m}_{T-k} \bar{m}_{1T-k} + E_s \left( \bar{m}_{1T-k} \right)^2 + E_s \bar{m}_{T-k} \bar{m}_{2T-k} \right] + \\
2 \Pi_{T-k+1} (\beta + 3 \alpha_1) \left[ (\bar{m}_{T-k})^2 + 4 E_s \bar{m}_{T-k} \bar{m}_{1T-k} + E_s^2 \left( \bar{m}_{1T-k} \right)^2 + E_s^2 \bar{m}_{T-k} \bar{m}_{2T-k} \right] - \\
4 \lambda q_{T-k+1}^2 \alpha_1^2 - 2 \lambda a_{T-k+1}^2 \left[ 2 \bar{m}_{T-k} \bar{m}_{1T-k} + E_s \left( \bar{m}_{1T-k} \right)^2 + E_s \bar{m}_{T-k} \bar{m}_{2T-k} \right] +
\]

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\[
(1 + r_f)^k \bar{L}_2 T_{-k} + 2\tau (1 + r_f)^k \left[ m_2 T_{-k} \bar{L}_2 T_{-k} + 2 m_1 T_{-k} \bar{L}_1 T_{-k} + \bar{m}_T T_{-k} \bar{L}_2 T_{-k} \right] -
\]
\[
b_{T-k+1} (1 + \mu) \bar{L}_2 T_{-k} - 2C_{T-k+1} (1 + \mu)^2 \]
\[
\left[ m_2 T_{-k} \left( 1 - \bar{L}_T T_{-k} \right) - 2 m_1 T_{-k} \bar{L}_1 T_{-k} - \bar{m}_T T_{-k} \bar{L}_2 T_{-k} \right] - 2C_{T-k+1} \]
\[
\left[ 2 m_1 T_{-k} + E_s m_2 T_{-k} \left( 1 - \bar{L}_T T_{-k} \right) - 2 m_1 T_{-k} \bar{L}_1 T_{-k} - 2 \bar{m}_T T_{-k} \bar{L}_1 T_{-k} \right] -
\]
\[
2 E_s m_1 T_{-k} \bar{L}_1 T_{-k} - E_s \bar{m}_T T_{-k} \bar{L}_2 T_{-k} \]
\[
S_{T-k+1} (1 + \mu) \alpha_0 \bar{L}_2 T_{-k} - S_{T-k+1} (1 + \mu) (\alpha_1 + \beta) \left[ 2 \bar{L}_1 T_{-k} + E_s \bar{L}_2 T_{-k} \right] +
\]
2\Pi_{T-k+1} (1 + \mu)^2 \alpha_0 \left[ m_2 T_{-k} \left( 1 - \bar{L}_T T_{-k} \right) - 2 m_1 T_{-k} \bar{L}_1 T_{-k} - \bar{m}_T T_{-k} \bar{L}_2 T_{-k} \right] +
2\Pi_{T-k+1} \left[ \left( 1 + \mu \right)^2 (\beta + \alpha_1) + \alpha_0 \right]
\]
\[
2 m_1 T_{-k} + E_s m_2 T_{-k} \left( 1 - \bar{L}_T T_{-k} \right) - 2 m_1 T_{-k} \bar{L}_1 T_{-k} - 2 \bar{m}_T T_{-k} \bar{L}_1 T_{-k} \right] +
\]
\[
2 E_s m_1 T_{-k} \bar{L}_1 T_{-k} - E_s \bar{m}_T T_{-k} \bar{L}_2 T_{-k} \]
\[
2\Pi_{T-k+1} (\beta + 3\alpha_1)
\]
\[
2 m_1 T_{-k} + E_s m_2 T_{-k} \left( 1 - \bar{L}_T T_{-k} \right) - 2 m_1 T_{-k} \bar{L}_1 T_{-k} \right] -
\]
\[
2 E_s m_1 T_{-k} \bar{L}_1 T_{-k} - E_s \bar{m}_T T_{-k} \bar{L}_2 T_{-k} \]
\[
2\lambda a_2^2 \Pi_{T-k+1}
\]
\[
2 \left[ \tau (1 + r_f)^k - C_{T-k+1} (1 + \mu)^2 \right] \left[ (\bar{L}_1 T_{-k})^2 + \bar{L}_T T_{-k} \bar{L}_2 T_{-k} \right] -
\]
\[
2 C_{T-k+1} \left[ 2 \left( 1 - \bar{L}_T T_{-k} \right) \bar{L}_1 T_{-k} + E_s \left( 1 - \bar{L}_T T_{-k} \right) \bar{L}_2 T_{-k} - E_s \left( \bar{L}_1 T_{-k} \right)^2 \right] -
\]
\[
2\Pi_{T-k+1} (1 + \mu)^2 \alpha_0 \left[ \left( \bar{L}_1 T_{-k} \right)^2 - (1 - \bar{L}_T T_{-k}) \bar{L}_2 T_{-k} \right] +
\]
\[
2\Pi_{T-k+1} \left[ \left( 1 + \mu \right)^2 (\beta + \alpha_1) + \alpha_0 \right]
\]
\[
2 \left( 1 - \bar{L}_T T_{-k} \right) \bar{L}_1 T_{-k} + E_s \left( 1 - \bar{L}_T T_{-k} \right) \bar{L}_2 T_{-k} - E_s \left( \bar{L}_1 T_{-k} \right)^2 \right] -
\]
\[
2 E_s \left( 1 - \bar{L}_T T_{-k} \right) \bar{L}_2 T_{-k} + E_s^2 \left( \bar{L}_1 T_{-k} \right)^2 \right] -
\]
\[
2\lambda a_2^2 \Pi_{T-k+1} \left[ 2 \left( 1 - \bar{L}_T T_{-k} \right) \bar{L}_1 T_{-k} + E_s \left( 1 - \bar{L}_T T_{-k} \right) \bar{L}_2 T_{-k} - E_s \left( \bar{L}_1 T_{-k} \right)^2 \right] \]
\[
\]
So, now by letting
\[
\gamma_{T-k} = v_{10,T-k} - v_{11,T-k} x_0 - v_{12,T-k} E_s + \frac{1}{2} v_{111,T-k} x_0^2 + \frac{1}{2} v_{122,T-k} E_s^2 +
\]
\[
v_{12,T-k} x_0 E_s - \frac{1}{2} v_{13,T-k} x_0^2 E_s \]
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\[ b_{T-k} = v_{1,T-k} - v_{11,T-k} x_{0} - v_{12,T-k} E_{s} + v_{13,T-k} x_{0} E_{s}, \]

\[ p_{T-k} = v_{2,T-k} - v_{22,T-k} E_{s} - v_{12,T-k} x_{0} + \frac{1}{2} v_{13,T-k} x_{0}^{2}, \]

\[ C_{T-k} = -\frac{1}{2} v_{11,T-k} + \frac{1}{2} v_{13,T-k} E_{s}, \]

\[ H_{T-k} = \frac{1}{2} v_{22,T-k}, \]

\[ S_{T-k} = v_{12,T-k} - v_{13,T-k} x_{0}, \]

\[ \Pi_{T-k} = \frac{1}{2} v_{13,T-k}, \]

we approximate the value function at time \( T - k \) with

\[ \hat{V}_{T-k} \left( x_{T-k}, x_{T-k}, \sigma_{T-k}^{2} \right) = z_{T-k} + (1 + r_{f})^{k} x_{T-k}^{0} + b_{T-k} x_{T-k} + p_{T-k} \sigma_{T-k}^{2} - C_{T-k} x_{T-k}^{2} + \]

\[ H_{T-k} \sigma_{T-k}^{2} + S_{T-k} x_{T-k} \sigma_{T-k}^{2} + \Pi_{T-k} x_{T-k} \sigma_{T-k}^{2}. \]

Similarly, the other quantity of interest \( K_{T-k} \) is approximated by

\[ \hat{K}_{T-k} = (1 + r_{f})^{k} x_{T-k}^{0} + k_{0,T-k} + k_{1,T-k} (x_{T-k} - x_{0}) + k_{2,T-k} (\sigma_{T-k}^{2} - E_{s}), \]

where the value of \( \hat{K}_{T-k} - (1 + r_{f})^{k} x_{T-k}^{0} \) at the point \((x_{0}, E_{s})\) is given by

\[ k_{0,T-k} = h_{T-k+1} - (1 + r_{f})^{k} \hat{m}_{T-k} - \tau (1 + r_{f})^{k} (\hat{m}_{T-k})^{2} + a_{T-k+1} (1 + \mu) \hat{m}_{T-k} + \]

\[ q_{T-k+1} (\alpha_{0} + \beta E_{s} + \alpha_{1} E_{s}) - \tau (1 + r_{f})^{k} (\tilde{L}_{T-k})^{2} x_{0}^{2} + \]

\[ \left\{ (1 + r_{f})^{k} \tilde{L}_{T-k} + 2 \tau (1 + r_{f})^{k} \hat{m}_{T-k} \tilde{L}_{T-k} + a_{T-k+1} (1 + \mu) \left( 1 - \tilde{L}_{T-k} \right) \right\} x_{0}, \]

and its partial derivatives by

\[ k_{1,T-k} = (1 + r_{f})^{k} \tilde{L}_{T-k} + 2 \tau (1 + r_{f})^{k} \hat{m}_{T-k} \tilde{L}_{T-k} + a_{T-k+1} (1 + \mu) \left( 1 - \tilde{L}_{T-k} \right) - \]

\[ 2 \tau (1 + r_{f})^{k} (\tilde{L}_{T-k})^{2} x_{0}, \]

\[ k_{2,T-k} = - (1 + r_{f})^{k} \tilde{m}_{T-k} - 2 \tau (1 + r_{f})^{k} \hat{m}_{T-k} \tilde{m}_{T-k} + a_{T-k+1} (1 + \mu) \tilde{m}_{T-k} + \]

\[ q_{T-k+1} (\alpha_{1} + \beta) + \]

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\[
\begin{align*}
\left\{ (1 + r_f)^k \hat{L}_{T-k} + 2\tau (1 + r_f)^k \left( \overline{m}_{1,T-k} \hat{L}_{T-k} + \hat{m}_{T-k} \hat{L}_{1,T-k} \right) - \right. \\
\left. a_{T-k+1} (1 + \mu) \hat{L}_{1,T-k} \\
2\tau (1 + r_f)^k \hat{L}_{T-k} \hat{L}_{1,T-k} x_0^2 \right\} x_0 - \\
\end{align*}
\]

So, by letting

\[
\begin{align*}
&h_{T-k} = k_{0,T-k} - k_{1,T-k} x_0 - k_{2,T-k} E_s, \\
a_{T-k} = k_{1,T-k}, \\
q_{T-k} = k_{2,T-k},
\end{align*}
\]

we derive the desired approximation for \( K_{T-k} \):

\[
\overline{K}_{T-k} (x_{T-k}^0, \omega_{T-k}, \sigma_{T-k}^2) = h_{T-k} + (1 + r_f)^k x_{T-k}^0 + a_{T-k} x_{T-k} + q_{T-k} \sigma_{T-k}^2.
\]

The optimization problem therefore can be stated through

\[
V_{T-k-1} (x_{T-k-1}^0, x_{T-k-1}, \sigma_{T-k-1}^2) =
\max_{\{w_{T-k-1}\}} E_{T-k-1} \left\{ z_{T-k} + (1 + r_f)^k x_{T-k}^0 + b_{T-k} x_{T-k} + p_{T-k} \sigma_{T-k}^2 - C_{T-k} x_{T-k}^2 + \\
H_{T-k} \sigma_{T-k}^2 + S_{T-k} x_{T-k} \sigma_{T-k}^2 + \Pi_{T-k} x_{T-k}^2 \sigma_{T-k}^2 \right\} - \\
\lambda \ Var_{T-k-1} \left\{ h_{T-k} + (1 + r_f)^k x_{T-k}^0 + a_{T-k} x_{T-k} + q_{T-k} \sigma_{T-k}^2 \right\}.
\]

Substituting for the wealth and asset return dynamics, \( V_{T-k-1} \) is given by

\[
V_{T-k-1} (x_{T-k-1}^0, x_{T-k-1}, \sigma_{T-k-1}^2) =
\]

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\[
\max_{\{u_{T-k-1}\}} \begin{cases}
-z_{T-k} + (1 + r_f)^{k+1} \left[ x_{T-k-1}^0 - u_{T-k-1} - \tau u_{T-k-1}^2 \right] + b_{T-k} (1 + \mu) \\
(x_{T-k-1} + u_{T-k-1}) + p_{T-k} \left( \alpha_0 + \beta \sigma_{T-k-1}^2 + \alpha_1 \sigma_{T-k-1}^4 \right) - \\
C_{T-k} \left[ (1 + \mu)^2 + \sigma_{T-k-1}^2 \right] (x_{T-k-1} + u_{T-k-1})^2 + \\
H_{T-k} \left[ (\alpha_0 + \beta \sigma_{T-k-1}^2) + 3 \alpha_1 \sigma_{T-k-1}^4 \right] + 2 \left( \alpha_0 + \beta \sigma_{T-k-1}^2 \right) \alpha_1 \sigma_{T-k-1}^4 \\
S_{T-k} (1 + \mu) \left( \alpha_0 + \beta \sigma_{T-k-1}^2 + \alpha_1 \sigma_{T-k-1}^4 \right) (x_{T-k-1} + u_{T-k-1}) + \\
\Pi_{T-k} (x_{T-k-1} + u_{T-k-1})^2 \left[ (1 + \mu)^2 \left( \alpha_0 + \beta \sigma_{T-k-1}^2 + \alpha_1 \sigma_{T-k-1}^4 \right) + \\
\sigma_{T-k-1}^2 \left( \alpha_0 + \beta \sigma_{T-k-1}^2 + \alpha_1 \sigma_{T-k-1}^4 \right) + 3 \alpha_1 \sigma_{T-k-1}^4 \right] - \\
\lambda \sigma_{T-k-1}^2 \sigma_{T-k-1}^2 (x_{T-k-1} + u_{T-k-1})^2 - 2 \lambda q_{T-k}^2 \alpha_1 \sigma_{T-k-1}^4.
\end{cases}
\]

This is now a quadratic optimization problem and its solution is given by

\[-(1 + r_f)^{k+1} - 2 \tau (1 + r_f)^{k+1} u_{T-k-1} + b_{T-k} (1 + \mu) - \\
2C_{T-k} \left[ (1 + \mu)^2 + \sigma_{T-k-1}^2 \right] (x_{T-k-1} + u_{T-k-1}) + S_{T-k} (1 + \mu) \\
\left( \alpha_0 + \beta \sigma_{T-k-1}^2 + \alpha_1 \sigma_{T-k-1}^4 \right) + 2 \Pi_{T-k} (x_{T-k-1} + u_{T-k-1}) \\
\left[ (1 + \mu)^2 \left( \alpha_0 + \beta \sigma_{T-k-1}^2 + \alpha_1 \sigma_{T-k-1}^4 \right) + \sigma_{T-k-1}^2 \left( \alpha_0 + \beta \sigma_{T-k-1}^2 + \alpha_1 \sigma_{T-k-1}^4 \right) + 3 \alpha_1 \sigma_{T-k-1}^4 \right] - \\
2 \lambda \sigma_{T-k-1}^2 \sigma_{T-k-1}^2 (x_{T-k-1} + u_{T-k-1}) = 0.
\]

The optimal investment decision at time \(T - k - 1\) is thus approximated by

\[\hat{u}_{T-k-1} (x_{T-k-1}, \sigma_{T-k-1}^2) = m_{T-k-1} - L_{T-k-1} x_{T-k-1},\]

where

\[m_{T-k-1} = \frac{\pi_{T-k-1} + \varphi_{T-k-1} \sigma_{T-k-1}^2}{2 \tau (1 + r_f)^{k+1} + \gamma_{T-k-1} + \delta_{T-k-1} \sigma_{T-k-1}^2 + \vartheta_{T-k-1} \sigma_{T-k-1}^4},\]

\[L_{T-k-1} = \frac{\gamma_{T-k-1} + \delta_{T-k-1} \sigma_{T-k-1}^2 + \vartheta_{T-k-1} \sigma_{T-k-1}^4}{2 \tau (1 + r_f)^{k+1} + \gamma_{T-k-1} + \delta_{T-k-1} \sigma_{T-k-1}^2 + \vartheta_{T-k-1} \sigma_{T-k-1}^4},\]

\[\gamma_{T-k-1} = 2C_{T-k} (1 + \mu)^2 - 2 \Pi_{T-k} (1 + \mu)^2 \alpha_0,\]

\[\pi_{T-k-1} = -(1 + r_f)^{k+1} + b_{T-k} (1 + \mu) + S_{T-k} (1 + \mu) \alpha_0,\]

\[\delta_{T-k-1} = 2C_{T-k} - 2 \Pi_{T-k} (1 + \mu)^2 (\beta + \alpha_1) - 2 \Pi_{T-k} \alpha_0 + 2 \lambda \sigma_{T-k}^2,\]

\[\varphi_{T-k-1} = S_{T-k} (1 + \mu) (\beta + \alpha_1),\]

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\[ \vartheta_{T-k-1} = -2 \Pi_{T-k} (\beta + 3 \alpha_1). \]

Substituting back into the expressions of \( V_{T-k-1} \) and \( K_{T-k-1} \), we finally obtain that

\[
\begin{align*}
\hat{V}_{T-k-1} (x^0_{T-k-1}, x_{T-k-1}, \sigma^2_{T-k-1}) & = \\
x_{T-k} + (1 + r_f)^{k+1} x^0_{T-k-1} - (1 + r_f)^{k+1} (m_{T-k-1} - L_{T-k-1} x_{T-k-1}) - \\
\tau (1 + r_f)^{k+1} (m_{T-k-1} - L_{T-k-1} x_{T-k-1})^2 + \\
b_{T-k} (1 + \mu) [m_{T-k-1} + (1 - L_{T-k-1}) x_{T-k-1}] + p_{T-k} \left( \alpha_0 + \beta \sigma^2_{T-k-1} + \alpha_1 \sigma^2_{T-k-1} \right) - \\
C_{T-k} \left[ (1 + \mu)^2 + \sigma^2_{T-k-1} \right] [m_{T-k-1} + (1 - L_{T-k-1}) x_{T-k-1}]^2 + \\
H_{T-k} \left[ (\alpha_0 + \beta \sigma^2_{T-k-1})^2 + 3 \alpha^2_0 \sigma^4_{T-k-1} + 2 (\alpha_0 + \beta \sigma^2_{T-k-1}) \alpha_1 \sigma^2_{T-k-1} \right] + \\
S_{T-k} (1 + \mu) \left( \alpha_0 + \beta \sigma^2_{T-k-1} + \alpha_1 \sigma^2_{T-k-1} \right) [m_{T-k-1} + (1 - L_{T-k-1}) x_{T-k-1}] + \\
\Pi_{T-k} \left[ (1 + \mu)^2 \left( \alpha_0 + \beta \sigma^2_{T-k-1} + \alpha_1 \sigma^2_{T-k-1} \right) + \\
\sigma^2_{T-k-1} \left( \alpha_0 + \beta \sigma^2_{T-k-1} \right) + 3 \alpha_1 \sigma^4_{T-k-1} \right] [m_{T-k-1} + (1 - L_{T-k-1}) x_{T-k-1}]^2 - \\
\lambda \left[ a^2_{T-k} \sigma^2_{T-k-1} [m_{T-k-1} + (1 - L_{T-k-1}) x_{T-k-1}]^2 - 2 \alpha q_{T-k}^2 \alpha_1 \sigma^4_{T-k-1}, \right]
\end{align*}
\]

and

\[
\begin{align*}
\hat{K}_{T-k-1} (x^0_{T-k-1}, x_{T-k-1}, \sigma^2_{T-k-1}) & \equiv E_{T-k-1} \left\{ \hat{K}_{T-k} \right\} = \\
h_{T-k} + (1 + r_f)^{k+1} x^0_{T-k-1} - (1 + r_f)^{k+1} (m_{T-k-1} - L_{T-k-1} x_{T-k-1}) - \\
\tau (1 + r_f)^{k+1} (m_{T-k-1} - L_{T-k-1} x_{T-k-1})^2 + \\
q_{T-k} (1 + \mu) [m_{T-k-1} + (1 - L_{T-k-1}) x_{T-k-1}] + q_{T-k} \left( \alpha_0 + \beta \sigma^2_{T-k-1} + \alpha_1 \sigma^2_{T-k-1} \right). \blacksquare
\end{align*}
\]

From the above theorem we conclude that the approximate investment in the risky asset is linear in the risky holdings and independent of the holdings in the riskless asset. It is also apparent that the higher the transaction costs, the smaller the change in the investment positions. In the next section, we analyze how the risk-free rate, characteristics of the underlying return process, time to maturity and investor's risk aversion affect the investment decisions.
The proposed structured approximation algorithm described in this section yields an investment decision at time $T - k$ of the following form:

$$ u_{T-k} \left( x_{T-k}, \sigma^2_{T-k} \right) = m_{T-k} \left( \sigma^2_{T-k} \right) - L_{T-k} \left( \sigma^2_{T-k} \right) \cdot x_{T-k}, $$

$$ m_{T-k} \left( \sigma^2_{T-k} \right) = \frac{\pi_{T-k} + \varphi_{T-k} \sigma^2_{T-k}}{2\tau (1 + r_f)^k + \gamma_{T-k} + \delta_{T-k} \sigma^2_{T-k} + \vartheta_{T-k} \sigma^2_{T-k}^2}, $$

$$ L_{T-k} \left( \sigma^2_{T-k} \right) = \frac{\gamma_{T-k} + \delta_{T-k} \sigma^2_{T-k} + \vartheta_{T-k} \sigma^2_{T-k}^2}{2\tau (1 + r_f)^k + \gamma_{T-k} + \delta_{T-k} \sigma^2_{T-k} + \vartheta_{T-k} \sigma^2_{T-k}^2}. $$

The optimal control at time $T - 1$ is a special case of the above representation for $\gamma_{T-1} = \vartheta_{T-1} = \varphi_{T-1} = 0$. Consequently, a natural simplification of the Structured investment policy is the strategy that results from the generalization of the single-period result to the multiperiod framework and that is given by

$$ \tilde{u}_{T-k} \left( x_{T-k}, \sigma^2_{T-k} \right) = \tilde{m}_{T-k} \left( \sigma^2_{T-k} \right) - \tilde{L}_{T-k} \left( \sigma^2_{T-k} \right) \cdot x_{T-k}, \quad (4.3) $$

$$ \tilde{m}_{T-k} \left( \sigma^2_{T-k} \right) = \frac{\tilde{\pi}_{T-k}}{2\tau (1 + r_f)^k + \tilde{\delta}_{T-k} \sigma^2_{T-k}}, $$

$$ \tilde{L}_{T-k} \left( \sigma^2_{T-k} \right) = \frac{\tilde{\delta}_{T-k} \sigma^2_{T-k}}{2\tau (1 + r_f)^k + \tilde{\delta}_{T-k} \sigma^2_{T-k}}. $$

The coefficients $\tilde{m}_{T-k}$ and $\tilde{L}_{T-k}$ result from approximating the value function $V_{T-k}$ with a linear function instead of a quadratic one, and result as a special case of $m_{T-k}$ and $L_{T-k}$ by setting the second order derivatives of $V_{T-k}$ and the first order derivative of $K_{T-k}$ with respect to $\sigma^2_{T-k}$ equal to zero. The policy described by Equation (4.3) constitutes a reasonable alternative approximate dynamic strategy under special circumstances that we describe in the next section.

### 4.3 Computational Experiments for Stochastic Volatility Models

In this section, we analyze the effect of stochastic volatility models on long-horizon asset allocation and quantify the impact of having some degree of predictability in the volatility of
the time-series asset return dynamics to the manager's investment behavior over time. In our comparative analysis we present a comparison of the performance of the following dynamic trading strategies:

- The approximate structured policy described in Section 4.2 (Structured).
- The simplified version of the structured policy given by Equation (4.3) (S-Structured).
- The approximate optimize-and-hold policy described in Section 3.3 (Opt-and-Hold).
- The static policy that derives as the solution to a series of single-period optimization problems (Static).

We consider the case where $\lambda = 0.001$, $\mu = 0.15$, $\alpha_1 = 0.15$, $\beta = 0.4$, $r_f = 0.05$, $\tau = 0.01$ and $x_0^0 = 1$, $x_0 = 1$. Thus, the single risky asset is assumed to have a constant expected return of 15%. In what follows, we analyze the effect of the following parameters to the investment behavior over time:

1. The unconditional expected variance of the risky asset $E_s$, through the parameter $\alpha_0$.
2. The time horizon $T$.
3. The transaction costs coefficient $\tau$.
4. The risk aversion parameter $\lambda$.

**Relative Performance of Dynamic Policies**

In Table 4.1 we report the expected utility of final wealth for the different policies considered, for $T = 10$ and various values of $\alpha_0$. When the asset's volatility is small, then the best policies found are the ones that invest *inversely* proportional to the variance realization (s-structured and optimize-and-hold). On the other hand, when the asset is significantly riskier, then the structured policy outperforms all other investment strategies considered by almost 10%. In the presence of higher uncertainty about future asset returns, it is advantageous to invest not just inversely proportional to the current realization of the asset's variance, but according to a way that is "more" nonlinear.

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Table 4.1: The expected utility of final wealth for $T = 10$ as given by the simulation experiment. The expectations are taken over 10,000 simulated paths of the risky asset return.

<table>
<thead>
<tr>
<th>$\alpha_0$</th>
<th>Structured</th>
<th>S-Structured</th>
<th>Opt-and-Hold</th>
<th>Static</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.0045$ ($E_s = 0.1^2$)</td>
<td>261.3180</td>
<td>265.1740</td>
<td>265.4014</td>
<td>50.7753</td>
</tr>
<tr>
<td>$0.0405$ ($E_s = 0.3^2$)</td>
<td>165.2196</td>
<td>223.6842</td>
<td>224.8940</td>
<td>49.1237</td>
</tr>
<tr>
<td>$0.2205$ ($E_s = 0.7^2$)</td>
<td>33.6994</td>
<td>$-90.7615$</td>
<td>29.3295</td>
<td>26.4313</td>
</tr>
</tbody>
</table>

Table 4.2: The expected utility of final wealth for $T = 20$ as given by the simulation experiment. The expectations are taken over 10,000 simulated paths of the risky asset return.

<table>
<thead>
<tr>
<th>$\alpha_0$</th>
<th>Structured</th>
<th>S-Structured</th>
<th>Opt-and-Hold</th>
<th>Static</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.0045$ ($E_s = 0.1^2$)</td>
<td>$2.496 \times 10^3$</td>
<td>$6.316 \times 10^3$</td>
<td>$6.720 \times 10^3$</td>
<td>403.0170</td>
</tr>
<tr>
<td>$0.0405$ ($E_s = 0.3^2$)</td>
<td>382.0816</td>
<td>$1.414 \times 10^4$</td>
<td>$1.319 \times 10^4$</td>
<td>352.4409</td>
</tr>
<tr>
<td>$0.2205$ ($E_s = 0.7^2$)</td>
<td>54.1514</td>
<td>$-3.201 \times 10^4$</td>
<td>32.6168</td>
<td>$-1.175 \times 10^4$</td>
</tr>
</tbody>
</table>

For longer investment horizons, the levels of utility are increased as shown in Table 4.2. The investor is able to better allocate his asset holdings, due to the improved opportunity set, and thus to reduce his portfolio variance. Once again, the riskier the asset, the better the performance of the structured approximate policy.

In Figures 4-1, 4-2 and 4-3 we plot the expected risky investment as a function of time for different time horizons. When the asset's volatility is small, the decrease of the expected risky investment over time is almost linear, since the optimal investment decision is inversely proportional to the variance realization. In contrast, as the asset's volatility increases the structured approximate policy, which is the best found, exhibits a different pattern: at the beginning and towards the end of the time horizon risky investment decreases, but in between the investor increases his holdings in the risky asset, thus increasing his utility level.

**Effect of Transaction costs and Risk Aversion**

As expected, both the structured and the optimize-and-hold policies decrease the risky investment as the transaction costs and risk aversion parameter increase, as shown in Figure 4-4.
Figure 4-1: The expected investment decision, plotted as a function of time for $\alpha_0 = 0.0045$. In Panel (a) we consider an investment horizon of $T = 10$ and in Panel (b) of $T = 20$. The expectations are taken over 10,000 simulated paths of the asset return.
Figure 4-2: The expected investment decision, plotted as a function of time for $\alpha_0 = 0.0405$. In Panel (a) we consider an investment horizon of $T = 10$ and in Panel (b) of $T = 20$. 
Figure 4-3: The expected investment decision, plotted as a function of time for $\alpha_0 = 0.2205$. In Panel (a) we consider an investment horizon of $T = 10$ and in Panel (b) of $T = 20$. 
Figure 4-4: The expected risky investment plotted as a function of time. In Panels (a) and (c) we consider the structured approximate policy and in Panels (b) and (d) the optimize-and-hold policy. In Panels (a)-(b) we show the dependence on the risk aversion parameter $\lambda$ and in Panels (c)-(d) the dependence on the transaction costs coefficient $\tau$. 
Chapter 5

Exponential Utility: No Transaction Costs

In the previous chapter, we investigated the effect of transaction costs on investment policies followed by an investment manager who maximizes the expected final wealth of a portfolio while minimizing its variance. We now turn our attention to negative exponential utility functions that display constant absolute risk aversion (CARA), and thus imply that the demand for risky assets is unaffected by changes in initial wealth. In this chapter, we study the effect of the asset return process to the investment behavior over time of a portfolio manager with a CARA utility function under the assumption that transaction costs can be ignored.

So far we have examined the effect of different pricing models on dynamic policies followed by a mean-variance investor. However, investors generally do not value upside and downside risk equally, and thus the variance as a measure of risk might be inappropriate. In addition, if all investors choose mean-variance efficient portfolios, then the prices of all assets have to follow the Capital Asset Pricing Model (CAPM) in equilibrium. However, Dybvig and Ingersoll [26] show that arbitrage opportunities exist if all assets are priced according to the CAPM and if markets are also complete. Investors will take advantage of these opportunities, thus violating the equilibrium. The only possible exception is when all investors have quadratic utility and have reached their point of satiation. Motivated by these objections to the use of mean-variance analysis, we consider the negative exponential utility function as a viable alternative. We
consider the case when transaction costs are ignored that serves as our benchmark problem in order to evaluate the impact of costs to the trading decisions. In the next chapter we discuss the impact of trading costs to the investment behavior.

The remainder of this chapter is organized as follows. In Section 5.1, we present a closed-form solution of the dynamic portfolio optimization problem under the assumption that asset returns follow a multi-factor pricing model (application of the Arbitrage Pricing Theory.) In Section 5.2, we consider the case of stochastic volatility models; even though a closed-form solution is not available, we propose an approximation algorithm that significantly outperforms the “myopic” strategy where investors follow a sequence of optimal single-period policies.

5.1 Return Dynamics: Factor Models

In this section we investigate the impact in the optimal investment decisions of having some predictive power in the asset return dynamics by introducing models that account for lagged correlations in asset returns. We present a closed-form solution for the case where investors' portfolios are comprised of investments in one riskless and \( N \) risky assets. Under the existence of a single risky asset, Wang [79] presents a dynamic asset-pricing model in closed-form under asymmetric information and CARA utility assuming correlated return dynamics. Balvers and Mitchell [2] also derive an analytical solution to the dynamic portfolio problem of an individual agent saving for retirement under a ARMA(1,1) process for the single risky asset. To the best of our knowledge this is the first time when a closed-form solution for the multivariate case is presented.

Consider an investor who faces the problem of making sequential investment decisions at discrete times \( t = 0, 1, \ldots, T \), and is only concerned with his wealth at the end of the investment period \( T \). In the absence of transaction costs, it suffices to fully characterize the state space of the asset holdings at time \( t \) with the total wealth present at that time. The evolution of the wealth dynamics is described in Section 1.6. The wealth at time \( t \) as given by Equations (1.7)-(1.9) is simplified to

\[
W_t = (1 + r_f) W_{t-1} + (r_t - r_f \ e)' (x_{t-1} + u_{t-1}) ,
\]
as shown in Section 2.3. The quantity \( \tilde{u}_t = (x_t - u_t) \) is the holdings in the risky assets after a transaction is made at time \( t - 1 \) and constitutes the new control variable. For convenience, we also let \( \tilde{r}_t = (r_t - r_f e) \) denote the excess rate of return at time \( t \), that follows the multifactor pricing model:

\[
\tilde{r}_t = c_t + A_t f_t + \epsilon_t,
\]

\[
f_t = d_{t-1} + B_{t-1} f_{t-1} + \eta_t,
\]

where \( K \) is the total number of factors, \( \tilde{r}_t \) is the \( N \times 1 \) vector of the excess rate of returns, \( f_t \) is the \( K \times 1 \) vector of the factor realizations at time \( t \), \( A_t \) is the \( N \times K \) matrix of the factor sensitivities, \( B_{t-1} \) is the \( K \times K \) symmetric matrix of the factor correlations, \( c_t \) and \( d_{t-1} \) are \( N \times 1 \) and \( K \times 1 \) vectors of constants respectively, and \( \epsilon_t, \eta_t \) are uncorrelated normally distributed random vectors with mean zero and covariance matrices \( \Sigma_{\epsilon} \) and \( \Sigma_{\eta} \) respectively. The techniques developed in this chapter can easily be extended to include \( p \)-th order vector autoregression models (VAR(\( p \))) for both the asset returns and the factors:

\[
\tilde{r}_t = c_t + A_t f_t + \epsilon_t,
\]

\[
f_t = d_{t-1} + B_{1,t-1} f_{t-1} + B_{2,t-1} f_{t-2} + \ldots + B_{p,t-1} f_{t-p} + \eta_t.
\]

The investor has a CARA utility function with absolute risk aversion parameter equal to \( \gamma \). Therefore, the portfolio manager faces the following dynamic optimization problem

\[
\begin{align*}
\text{maximize} & \quad E_0 \left\{ -e^{-\gamma W_T} \right\} \\
\text{subject to} & \quad W_t = (1 + r_f) W_{t-1} + \tilde{u}_{t-1} \tilde{r}_t \\
& \quad \tilde{r}_t = c_t + A_t f_t + \epsilon_t \\
& \quad f_t = d_{t-1} + B_{t-1} f_{t-1} + \eta_t \\
& \quad \epsilon_t \sim N(0, \Sigma_{\epsilon}) \text{ and } \eta_t \sim N(0, \Sigma_{\eta}).
\end{align*}
\]

Our point of departure is the following proposition:
Proposition 5.1 Let \( a \) be a \( N \times 1 \) vector and \( Q \) be a \( N \times N \) positive definite and symmetric matrix. If \( \epsilon \) is normally distributed random variable with mean 0 and covariance matrix \( \Sigma \), \( \epsilon \sim \mathcal{N}(0, \Sigma) \), then

1. \( E \{ \exp[ a' \epsilon] \} = \exp \left[ \frac{1}{2} a' \Sigma a \right] \),
2. \( E \{ \exp[ a' \epsilon + \epsilon' Q \epsilon] \} = \sqrt{\frac{\Lambda}{|\Sigma|}} \exp \left[ \frac{1}{2} a' \Lambda a \right] \),

where

\[
\Lambda = \left( \Sigma^{-1} - 2Q \right)^{-1}.
\]

Proof. We know by definition that

\[
E \{ \exp \left[ a' \epsilon + \epsilon' Q \epsilon \right] \} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{N/2} \sqrt{\Sigma}} \exp \left[ a' \epsilon + \epsilon' Q \epsilon \right] \exp \left[ -\frac{\epsilon' \Sigma^{-1} \epsilon}{2} \right] d\epsilon =
\]

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{N/2} \sqrt{\Sigma}} \exp \left[ -\frac{1}{2} \epsilon' \left( \Sigma^{-1} - 2Q \right) \epsilon + a' \epsilon \right] d\epsilon. \quad (5.1)
\]

For convenience, let

\[
\mathcal{L} = \exp \left[ -\frac{1}{2} \epsilon' \left( \Sigma^{-1} - 2Q \right) \epsilon + a' \epsilon \right].
\]

If we choose

\[
\Lambda^{-1} = \Sigma^{-1} - 2Q, \quad \mu = \left( \Sigma^{-1} - 2Q \right)^{-1} a = \Lambda a,
\]

we have that \( \Lambda \) is an invertible matrix and:

\[
\mathcal{L} = \exp \left[ -\frac{1}{2} \epsilon' \Lambda^{-1} \epsilon + \mu' \Lambda^{-1} \epsilon \right] = \exp \left[ -\frac{1}{2} (\epsilon - \mu)' \Lambda^{-1} (\epsilon - \mu) + \frac{1}{2} \mu' \Lambda^{-1} \mu \right].
\]
So, Equation (5.1) becomes
\[
\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^N \sqrt{\det(\Sigma)}} \exp \left[ -\frac{1}{2} (\epsilon - \mu)' \Lambda^{-1} (\epsilon - \mu) \right] \exp \left[ \frac{1}{2} \mu' \Lambda^{-1} \mu \right] d\epsilon = \\
\exp \left[ \frac{1}{2} \mu' \Lambda^{-1} \mu \right] \sqrt{\frac{\det(\Lambda)}{\det(\Sigma)}}. \\
\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^N \sqrt{\det(\Lambda)}} \exp \left[ -\frac{1}{2} (\epsilon - \mu)' \Lambda^{-1} (\epsilon - \mu) \right] d\epsilon = \\
\sqrt{\frac{\det(\Lambda)}{\det(\Sigma)}} \exp \left[ \frac{1}{2} \mu' \Lambda^{-1} \mu \right].
\]

But, since \( \Lambda \) is a symmetric matrix we have that
\[
\frac{1}{2} \mu' \Lambda^{-1} \mu = \frac{1}{2} (\Lambda \ a)' \Lambda^{-1} \Lambda \ a = \frac{1}{2} a' \Lambda \ a.
\]

Therefore,
\[
E \{ \exp [a' \epsilon + \epsilon' \mathbf{Q} \epsilon] \} = \sqrt{\frac{\det(\Lambda)}{\det(\Sigma)}} \exp \left[ \frac{1}{2} a' \Lambda \ a \right]. \quad \blacksquare
\]

The state at time \( t = 0, 1, \ldots, T - 1 \) consists of the manager’s wealth at time \( t \), \( W_t \) and the factor realizations at time \( t \), \( f_t \). We denote the conditional expectation given the information at time \( t \) as
\[
E_t \{ \cdot \} = E \{ \cdot \mid W_t, f_t \}.
\]

The control at time \( t \) is the vector of holdings in the risky assets after a transaction is made at time \( t \), \( \bar{u}_t \). The dynamic optimization problem can be stated through Bellman’s equation as follows for \( t = 0, 1, \ldots, T - 1 \):
\[
V_t (W_t, f_t) = \max_{\bar{u}_t} E_t \{ V_{t+1} (W_{t+1}, f_{t+1}) \},
\]

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with the boundary conditions

\[ V_T(W_T, f_T) = -\exp[-\gamma W_T]. \]

We begin by characterizing the optimal value function \( V_{T-1} \) by using the boundary condition and then proceed recursively.

\[
V_{T-1}(W_{T-1}, f_{T-1}) = \max_{\tilde{u}_{T-1}} E_{T-1} \{ -\exp[-\gamma W_T] \} =
\max_{\tilde{u}_{T-1}} E_{T-1} \{ -\exp[-\gamma (1 + r_f) W_{T-1} - \gamma \tilde{u}_{T-1} \tilde{r}_T] \} =
\exp[-\gamma (1 + r_f) W_{T-1}] \max_{\tilde{u}_{T-1}} E_{T-1} \{ -\exp[-\gamma \tilde{u}_{T-1} (c_T + A_T f_T + \epsilon_T)] \} =
\exp[-\gamma (1 + r_f) W_{T-1}] \max_{\tilde{u}_{T-1}} \left\{ -\exp \left[ -\gamma \tilde{u}_{T-1}^T c_T \right] \right. \\
- \gamma \tilde{u}_{T-1}^T A_T \left( d_{T-1} + B_{T-1} f_{T-1} \right) \left[ -\gamma \tilde{u}_{T-1}^T A_T \eta_T - \gamma \tilde{u}_{T-1}^T \epsilon_T \right] \right\} =
\exp[-\gamma (1 + r_f) W_{T-1}] \max_{\tilde{u}_{T-1}} \left\{ -\exp \left[ -\gamma \tilde{u}_{T-1}^T (c_T + A_T (d_{T-1} + B_{T-1} f_{T-1})) \right] \right. \\
- \gamma \tilde{u}_{T-1}^T A_T \eta_T - \gamma \tilde{u}_{T-1}^T \epsilon_T \left\} \right. \\
E_{T-1} \left\{ -\exp \left[ -\gamma \tilde{u}_{T-1}^T A_T \eta_T - \gamma \tilde{u}_{T-1}^T \epsilon_T \right] \right\}.
\]

Since \( \eta_T \) and \( \epsilon_T \) are assumed to be uncorrelated, we have

\[
E_{T-1} \{ -\exp[-\gamma \tilde{u}_{T-1}^T A_T \eta_T - \gamma \tilde{u}_{T-1}^T \epsilon_T] \} =
E_{T-1} \{ -\gamma \tilde{u}_{T-1}^T A_T \eta_T \} E_{T-1} \{ -\gamma \tilde{u}_{T-1}^T \epsilon_T \} =
\exp \left[ \frac{1}{2} \gamma^2 \tilde{u}_{T-1}^T A_T \Sigma \eta \tilde{u}_{T-1} \right] \exp \left[ \frac{1}{2} \gamma^2 \tilde{u}_{T-1}^T \Sigma \epsilon \tilde{u}_{T-1} \right].
\]

So,

\[
V_{T-1}(W_{T-1}, f_{T-1}) = \exp[-\gamma (1 + r_f) W_{T-1}]
\max_{\tilde{u}_{T-1}} \left\{ -\exp \left[ -\gamma \tilde{u}_{T-1}^T (c_T + A_T \left( d_{T-1} + B_{T-1} f_{T-1} \right)) \right] \right. \\
\left. + \frac{1}{2} \gamma^2 \tilde{u}_{T-1}^T \left[ A_T \Sigma \eta A_T^T + \Sigma \epsilon \right] \tilde{u}_{T-1} \right\}.
\]
The maximization problem is equivalent to

\[
\max_{\tilde{u}_{T-1}} \left\{ \gamma \tilde{u}_{T-1}^T (c_T + A_T (d_{T-1} + B_{T-1} f_{T-1})) - \frac{1}{2} \gamma^2 \tilde{u}_{T-1}^T [A_T \Sigma_\eta A_T' + \Sigma_\epsilon] \tilde{u}_{T-1} \right\}.
\]

This is a convex optimization problem since \( A_T \Sigma_\eta A_T' + \Sigma_\epsilon \) is a symmetric positive semi-definite matrix. The first order conditions are necessary and sufficient and given by

\[
c_T + A_T d_{T-1} + A_T B_{T-1} f_{T-1} - \gamma [A_T \Sigma_\eta A_T' + \Sigma_\epsilon] \tilde{u}_{T-1}^* = 0.
\]

The optimal control therefore at time \( T-1 \) is linear in \( f_{T-1} \) and given by

\[
\tilde{u}_{T-1}^* = \frac{1}{\gamma} [A_T \Sigma_\eta A_T' + \Sigma_\epsilon]^{-1} (A_T B_{T-1} f_{T-1} + c_T + A_T d_{T-1}), \quad (5.2)
\]

that can be rewritten as follows:

\[
\tilde{u}_{T-1}^* = \frac{1}{\gamma} Q_{T-1} (G_{T-1} f_{T-1} + h_{T-1}),
\]

where

\[
\Lambda_{T-1} = \Sigma_\eta,
\]
\[
Q_{T-1} = [A_T \Lambda_{T-1} A_T' + \Sigma_\epsilon]^{-1},
\]
\[
G_{T-1} = A_T B_{T-1},
\]
\[
h_{T-1} = c_T + A_T d_{T-1}.
\]

The optimal value function \( V_{T-1} \) is

\[
V_{T-1}(W_{T-1}, f_{T-1}) = -\exp \left[ -\gamma (1 + r_f) W_{T-1} \right]
\exp \left[ \left( G_{T-1} f_{T-1} + h_{T-1} \right)' Q_{T-1} (c_T + A_T d_{T-1} + A_T B_{T-1} f_{T-1}) \right]
\exp \left[ \frac{1}{2} \left( G_{T-1} f_{T-1} + h_{T-1} \right)' Q_{T-1} \Sigma_\epsilon Q_{T-1} \left( G_{T-1} f_{T-1} + h_{T-1} \right) \right]
\exp \left[ \frac{1}{2} \left( G_{T-1} f_{T-1} + h_{T-1} \right)' Q_{T-1} A_T \Sigma_\eta A_T' Q_{T-1} \left( G_{T-1} f_{T-1} + h_{T-1} \right) \right].
\]

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This leads to

\[
V_{T-1}(W_{T-1}, f_{T-1}) = -\exp \left[ -\gamma \left( 1 + r_f \right) W_{T-1} \right] \\
\exp \left[ - (G_{T-1} f_{T-1} + h_{T-1})' Q_{T-1} (c_T + A_T \ b_{T-1} + A_T \ B_{T-1} \ f_{T-1}) \right] \\
\exp \left[ \frac{1}{2} (G_{T-1} f_{T-1} + h_{T-1})' Q_{T-1} (G_{T-1} f_{T-1} + h_{T-1}) \right].
\]

Let

\[
S_{T-1} = G_{T-1}' Q_{T-1} G_{T-1}, \\
y_{T-1} = G_{T-1}' Q_{T-1} h_{T-1}, \\
x_{T-1} = h_{T-1}' Q_{T-1} h_{T-1}.
\]

Since \(\Lambda_{T-1}, Q_{T-1}\) and \(S_{T-1}\) are all symmetric matrices, the optimal value function at \(T-1\) is given by

\[
V_{T-1}(W_{T-1}, f_{T-1}) = -\exp \left[ -\gamma \left( 1 + r_f \right) W_{T-1} \right] \\
\exp \left[ \frac{1}{2} f_{T-1}' S_{T-1} f_{T-1} - y_{T-1}' f_{T-1} - \frac{1}{2} x_{T-1} \right].
\]

We prove the following theorem:

**Theorem 5.1** The optimal investment decisions \(\hat{\alpha}_{T-k}^*\) and the value function \(V_{T-k}\) for \(k = 1, \ldots, T\) are given by the following relations:

\[
\hat{\alpha}_{T-k}^* (f_{T-k}) = \frac{1}{\gamma \left( 1 + r_f \right)^{k-1}} Q_{T-k} (G_{T-k} f_{T-k} + h_{T-k}) \quad (5.3)
\]

\[
V_{T-k}(W_{T-k}, f_{T-k}) = -\left[ \prod_{m=2}^{k} \sqrt{\frac{|\Lambda_{T-m}|}{|\Sigma_{\eta}|}} \right] \exp \left[ -\gamma \left( 1 + r_f \right)^k W_{T-k} \right] \\
\exp \left[ \frac{1}{2} f_{T-k}' S_{T-k} f_{T-k} - y_{T-k}' f_{T-k} - \frac{1}{2} x_{T-k} \right] \quad (5.4)
\]

where

\[
\Lambda_{T-k} = \left[ \Sigma_{\eta}^{-1} + S_{T-k+1} \right]^{-1},
\]

\[
Q_{T-k} = \left[ A_{T-k+1} \ \Lambda_{T-k} \ A_{T-k+1}' + \Sigma_{\eta} \right]^{-1},
\]

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\[ G_{T-k} = A_{T-k+1} B_{T-k} - A_{T-k+1} \Lambda_{T-k} S_{T-k+1} B_{T-k}, \]
\[ h_{T-k} = c_{T-k+1} + A_{T-k+1} d_{T-k} - A_{T-k+1} \Lambda_{T-k} (S_{T-k+1} d_{T-k} + y_{T-k+1}), \]
\[ S_{T-k} = B'_{T-k} S_{T-k+1} B_{T-k} + 2 G'_{T-k} Q_{T-k} A_{T-k+1} B_{T-k} - \]
\[ G'_{T-k} Q_{T-k} \Sigma_{\epsilon} Q_{T-k} G_{T-k} - \]
\[ (G'_{T-k} Q_{T-k} A_{T-k+1} + B'_{T-k} S_{T-k+1}) A_{T-k} \]
\[ (A'_{T-k+1} Q_{T-k} B_{T-k} + S_{T-k+1} B_{T-k}), \]
\[ y_{T-k} = B'_{T-k} y_{T-k+1} + G'_{T-k} Q_{T-k} (c_{T-k+1} + A_{T-k+1} d_{T-k}) + \]
\[ B'_{T-k} A'_{T-k+1} Q_{T-k} h_{T-k} - G'_{T-k} Q_{T-k} \Sigma_{\epsilon} Q_{T-k} h_{T-k} + \]
\[ B'_{T-k} S_{T-k+1} d_{T-k} - \]
\[ (G'_{T-k} Q_{T-k} A_{T-k+1} + B'_{T-k} S_{T-k+1}) A_{T-k} \]
\[ (A'_{T-k+1} Q_{T-k} h_{T-k} + S_{T-k+1} d_{T-k} + y_{T-k+1}), \]
\[ x_{T-k} = x_{T-k+1} + 2 y'_{T-k+1} d_{T-k} + 2 h'_{T-k} Q_{T-k} (c_{T-k+1} + A_{T-k+1} d_{T-k}) - \]
\[ h'_{T-k} Q_{T-k} \Sigma_{\epsilon} Q_{T-k} h_{T-k} + d'_{T-k} S_{T-k+1} d_{T-k} - \]
\[ (h'_{T-k} Q_{T-k} A_{T-k+1} + d'_{T-k} S_{T-k+1} + y'_{T-k+1}) A_{T-k} \]
\[ (A'_{T-k+1} Q_{T-k} h_{T-k} + S_{T-k+1} d_{T-k} + y_{T-k+1}), \]

with the boundary conditions

\[ S_T = 0, \]
\[ y_T = 0, \]
\[ x_T = 0. \]

**Proof.** We prove the theorem by induction. We have showed that the above relations are valid for \( k = 1 \). Assume that they are true for \( k \); we will show that they hold for \( k + 1 \). Substituting for the wealth and return dynamics we have that

\[ V_{T-k-1} (W_{T-k-1}, f_{T-k-1}) = \max_{\bar{u}_{T-k-1}} E_{T-k-1} \{ V_{T-k} (W_{T-k}, f_{T-k}) \} = \]

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\[
\max_{\tilde{u}_{T-k-1}} \prod_{m=2}^{k} \sqrt{\frac{|\Delta_{T-m}|}{|\Sigma_{m}|}} \ E_{T-k-1} \left\{ \begin{array}{l}
- \exp \left[ -\gamma (1 + r_f)^k W_{T-k-1} \right] \\
\exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} \tilde{r}_{T-k} \right] \\
\exp \left[ -\frac{1}{2} f_{T-k}^{\prime} S_{T-k} f_{T-k} - \frac{1}{2} x_{T-k} \right] \end{array} \right\} =
\left[ \prod_{m=2}^{k} \sqrt{\frac{|\Delta_{T-m}|}{|\Sigma_{m}|}} \right] \exp \left[ -\gamma (1 + r_f)^k W_{T-k-1} \right] \\
\max_{\tilde{u}_{T-k-1}} E_{T-k-1} \left\{ \begin{array}{l}
- \exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} \left( c_{T-k} + A_{T-k} f_{T-k-1} \right) \right] \\
\exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} e_{T-k} \right] \\
\exp \left[ -\frac{1}{2} f_{T-k}^{\prime} S_{T-k} f_{T-k} - \frac{1}{2} x_{T-k} \right] \end{array} \right\}.
\]

The maximization problem then becomes

\[
\max_{\tilde{u}_{T-k-1}} E_{T-k-1} \left\{ \begin{array}{l}
- \exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} \left( c_{T-k} + A_{T-k} d_{T-k-1} + \right) \right] \\
\exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} A_{T-k} \eta_{T-k} \right] \\
\exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} e_{T-k} \right] \\
- \frac{1}{2} (d_{T-k-1} + B_{T-k-1} f_{T-k-1} + \eta_{T-k})^{\prime} S_{T-k} \right] \\
\exp \left[ (d_{T-k-1} + B_{T-k-1} f_{T-k-1} + \eta_{T-k}) - \right] \\
y_{T-k}^{\prime} (d_{T-k-1} + B_{T-k-1} f_{T-k-1} + \eta_{T-k}) - \frac{1}{2} x_{T-k} \end{array} \right\}.
\]

The random variables \(e_{T-k}\) and \(\eta_{T-k}\) are assumed to be uncorrelated; therefore the optimization problem becomes:

\[
\max_{\tilde{u}_{T-k-1}} E_{T-k-1} \left\{ \begin{array}{l}
- \exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} \left( c_{T-k} + A_{T-k} d_{T-k-1} + \right) \right] \\
\exp \left[ \frac{1}{2} \gamma^2 (1 + r_f)^{2k} \tilde{u}_{T-k-1}^{\prime} \Sigma_{f} \tilde{u}_{T-k-1} \right] \\
\exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} A_{T-k} \eta_{T-k} \right] \\
\exp \left[ \frac{1}{2} (d_{T-k-1} + B_{T-k-1} f_{T-k-1} + \eta_{T-k})^{\prime} S_{T-k} \right] \\
\exp \left[ (d_{T-k-1} + B_{T-k-1} f_{T-k-1} + \eta_{T-k}) - \right] \\
y_{T-k}^{\prime} (d_{T-k-1} + B_{T-k-1} f_{T-k-1} + \eta_{T-k}) - \frac{1}{2} x_{T-k} \end{array} \right\} =
\left[ \prod_{m=2}^{k} \sqrt{\frac{|\Delta_{T-m}|}{|\Sigma_{m}|}} \right] \exp \left[ -\gamma (1 + r_f)^k W_{T-k-1} \right] \\
\max_{\tilde{u}_{T-k-1}} E_{T-k-1} \left\{ \begin{array}{l}
- \exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} \left( c_{T-k} + A_{T-k} d_{T-k-1} + \right) \right] \\
\exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} A_{T-k} \eta_{T-k} \right] \\
\exp \left[ -\gamma (1 + r_f)^k \tilde{u}_{T-k-1}^{\prime} e_{T-k} \right] \\
\exp \left[ -\frac{1}{2} (d_{T-k-1} + B_{T-k-1} f_{T-k-1} + \eta_{T-k})^{\prime} S_{T-k} \right] \\
\exp \left[ (d_{T-k-1} + B_{T-k-1} f_{T-k-1} + \eta_{T-k}) - \right] \\
y_{T-k}^{\prime} (d_{T-k-1} + B_{T-k-1} f_{T-k-1} + \eta_{T-k}) - \frac{1}{2} x_{T-k} \end{array} \right\}.
\]

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Using Proposition 5.1 we obtain that the above expectation is just

\[
\Phi = \sqrt{\frac{|\Lambda_{T-k-1}|}{|\Sigma_\eta|}} \exp \left\{ \frac{1}{2} \begin{bmatrix} \gamma (1 + r_f)^k \, \tilde{u}_{T-k-1}^\prime \, A_{T-k-1}^+ \\ (d_{T-k-1} + B_{T-k-1} \, f_{T-k-1})^\prime \, S_{T-k} + y_{T-k}^\prime \\ \gamma (1 + r_f)^k \, \tilde{u}_{T-k-1}^\prime \end{bmatrix} \, \Lambda_{T-k-1} \right\},
\]

where

\[
\Lambda_{T-k-1} = \left[ \Sigma_\eta^{-1} + S_{T-k} \right]^{-1}.
\]

The value function is then

\[
V_{T-k-1} (W_{T-k-1}, f_{T-k-1}) = \left[ \prod_{m=2}^{k+1} \sqrt{\frac{|\Lambda_{T-m-1}|}{|\Sigma_\eta|}} \right] \exp \left[ -\gamma (1 + r_f)^{k+1} \, W_{T-k-1} \right] \exp \left[ \begin{bmatrix} -\frac{1}{2} (d_{T-k-1} + B_{T-k-1} \, f_{T-k-1})^\prime \, S_{T-k} \, (d_{T-k-1} + B_{T-k-1} \, f_{T-k-1}) \\ -y_{T-k}^\prime \, (d_{T-k-1} + B_{T-k-1} \, f_{T-k-1}) - \frac{1}{2} \, x_{T-k} \end{bmatrix} \right].
\]
\[
\begin{align*}
\max_{\hat{u}_{T-k-1}} & \quad -\exp\left[ -\gamma (1+r_f)^k \hat{u}_{T-k-1}' \begin{bmatrix} (c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1}) \end{bmatrix} \right] \\
& \exp\left[ \frac{1}{2} \gamma^2 (1+r_f)^{2k} \hat{u}_{T-k-1}' \Sigma_{\epsilon} \hat{u}_{T-k-1} \right] \\
& \exp\left[ \frac{1}{2} \begin{bmatrix} \gamma (1+r_f)^k \hat{u}_{T-k-1}' A_{T-k} + (d_{T-k-1} + B_{T-k-1} f_{T-k-1})' S_{T-k} + y_{T-k} \end{bmatrix} A_{T-k-1} \right].
\end{align*}
\]

The maximization problem is equivalent to
\[
\begin{align*}
\max_{\hat{u}_{T-k-1}} & \quad \frac{1}{2} \begin{bmatrix} \gamma (1+r_f)^k \hat{u}_{T-k-1}' (c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1}) - \end{bmatrix} \\
& \frac{1}{2} \gamma^2 (1+r_f)^{2k} \hat{u}_{T-k-1}' \Sigma_{\epsilon} \hat{u}_{T-k-1} - \\
& \frac{1}{2} \begin{bmatrix} \gamma (1+r_f)^k \hat{u}_{T-k-1}' A_{T-k} + (d_{T-k-1} + B_{T-k-1} f_{T-k-1})' S_{T-k} + y_{T-k} \end{bmatrix} A_{T-k-1} \\
& \frac{1}{2} \begin{bmatrix} \gamma (1+r_f)^k A_{T-k}' \hat{u}_{T-k-1} + S_{T-k} (d_{T-k-1} + B_{T-k-1} f_{T-k-1}) + y_{T-k} \end{bmatrix}.
\end{align*}
\]

This is a convex optimization problem since the matrix $\Sigma_{\epsilon} + A_{T-k} A_{T-k-1} A_{T-k}'$ is symmetric positive semi-definite. The first order condition is
\[
\begin{align*}
\gamma (1+r_f)^k & \left( c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1} \right) - \\
\gamma^2 (1+r_f)^{2k} & \Sigma_{\epsilon} \hat{u}_{T-k-1} - \\
\gamma (1+r_f)^k & A_{T-k} A_{T-k-1} \begin{bmatrix} \gamma (1+r_f)^k A_{T-k}' \hat{u}_{T-k-1} + S_{T-k} (d_{T-k-1} + B_{T-k-1} f_{T-k-1}) + y_{T-k} \end{bmatrix} = 0.
\end{align*}
\]

So, the optimal control at time $T - k - 1$ is given by
\[
\hat{u}_{T-k-1}^* = \frac{1}{\gamma (1+r_f)^k} \left[ A_{T-k} A_{T-k-1} A_{T-k}' + \Sigma_{\epsilon} \right]^{-1} \begin{bmatrix} (A_{T-k} B_{T-k-1} - A_{T-k} A_{T-k-1} S_{T-k} B_{T-k-1}) f_{T-k-1} + \end{bmatrix} \begin{bmatrix} c_{T-k} + A_{T-k} d_{T-k-1} - A_{T-k} A_{T-k-1} (S_{T-k} d_{T-k-1} + y_{T-k}) \end{bmatrix}.
\]

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If we let

\[ Q_{T-k-1} = \left[ A_{T-k} A_{T-k-1} A'_{T-k} + \Sigma_e \right]^{-1}, \]
\[ G_{T-k-1} = A_{T-k} B_{T-k-1} - A_{T-k} A_{T-k-1} S_{T-k} B_{T-k-1}, \]
\[ h_{T-k-1} = c_{T-k} + A_{T-k} d_{T-k-1} - A_{T-k} A_{T-k-1} (S_{T-k} d_{T-k-1} + y_{T-k}), \]

we have shown that

\[ \bar{u}_{T-k-1}^* (f_{T-k-1}) = \frac{1}{\gamma (1 + r_f)^k} Q_{T-k-1} (G_{T-k-1} f_{T-k-1} + h_{T-k-1}). \]

The value function now becomes

\[ V_{T-k-1}(W_{T-k-1}, f_{T-k-1}) = -\left( \prod_{m=2}^{k+1} \frac{\sqrt{A_{T-m-1}}}{|\Sigma_m|} \right) \exp \left[ -\gamma (1 + r_f)^{k+1} W_{T-k-1} \right] \]
\[ \exp \left[ -\frac{1}{2} (d_{T-k-1} + B_{T-k-1} f_{T-k-1})' S_{T-k} (d_{T-k-1} + B_{T-k-1} f_{T-k-1}) \right] \]
\[ \exp \left[ -y'_{T-k} (d_{T-k-1} + B_{T-k-1} f_{T-k-1}) - \frac{1}{2} z_{T-k} \right] \]
\[ \exp \left[ - (G_{T-k-1} f_{T-k-1} + h_{T-k-1})' Q_{T-k-1} \right] \]
\[ \exp \left[ (c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1}) \right] \]
\[ \exp \left[ \frac{1}{2} (G_{T-k-1} f_{T-k-1} + h_{T-k-1})' \Sigma_e Q_{T-k-1} (G_{T-k-1} f_{T-k-1} + h_{T-k-1}) \right] \]
\[ \exp \left[ \frac{1}{2} \left[ (G_{T-k-1} f_{T-k-1} + h_{T-k-1})' Q_{T-k-1} A_{T-k} + (d_{T-k-1} + B_{T-k-1} f_{T-k-1})' S_{T-k} + y'_{T-k} \right] A_{T-k-1} \right] \]
\[ \exp \left[ A'_{T-k} Q_{T-k-1} (G_{T-k-1} f_{T-k-1} + h_{T-k-1}) + S_{T-k} (d_{T-k-1} + B_{T-k-1} f_{T-k-1}) + y_{T-k} \right] \]

Thus, if we let

\[ S_{T-k-1} = B'_{T-k-1} S_{T-k} B_{T-k-1} + 2 G'_{T-k-1} Q_{T-k-1} A_{T-k} B_{T-k-1} - \]
\[ G'_{T-k-1} Q_{T-k-1} \Sigma_e Q_{T-k-1} G_{T-k-1} - \]
\[ (G_{T-k-1} Q_{T-k-1} A_{T-k} + B'_{T-k-1} S_{T-k}) A_{T-k-1} \]
\[ (A'_{T-k} Q_{T-k-1} G_{T-k-1} + S_{T-k} B_{T-k-1}), \]

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\[ y_{T-k-1} = B'_{T-k-1} y_{T-k} + B'_{T-k-1} S_{T-k} d_{T-k-1} + \\
G'_{T-k-1} Q_{T-k-1} (c_{T-k} + A_{T-k} d_{T-k-1}) + \\
B'_{T-k-1} A'_{T-k} Q_{T-k-1} h_{T-k-1} - G'_{T-k-1} Q_{T-k-1} \Sigma_{c} Q_{T-k-1} h_{T-k-1} - \\
(G'_{T-k-1} Q_{T-k-1} A_{T-k} + B'_{T-k-1} S_{T-k}) A_{T-k-1} \\
(A'_{T-k} Q_{T-k-1} h_{T-k-1} + S_{T-k} d_{T-k-1} + y_{T-k}) , \\
x_{T-k-1} = x_{T-k} + d'_{T-k-1} S_{T-k} d_{T-k-1} + 2 y'_{T-k} d_{T-k-1} + \\
2 h'_{T-k-1} Q_{T-k-1} (c_{T-k} + A_{T-k} d_{T-k-1}) - \\
h'_{T-k-1} Q_{T-k-1} \Sigma_{c} Q_{T-k-1} h_{T-k-1} - \\
(h'_{T-k-1} Q_{T-k-1} A_{T-k} + d'_{T-k-1} S_{T-k} + y'_{T-k}) A_{T-k-1} \\
(A'_{T-k} Q_{T-k-1} h_{T-k-1} + S_{T-k} d_{T-k-1} + y_{T-k}) , \\
\]

we finally have

\[ V_{T-k-1}(W_{T-k-1}, f_{T-k-1}) = \left[ -\left( \prod_{m=2}^{k+1} \sqrt{\frac{|A_{T-m}|}{\Sigma_{\eta}}} \right) \exp \left[ -\gamma \left( 1 + r_{f} \right)^{k+1} W_{T-k-1} \right] \exp \left[ -\frac{1}{2} f'_{T-k-1} S_{T-k-1} f_{T-k-1} - y'_{T-k-1} f_{T-k-1} - \frac{1}{2} x_{T-k-1} \right] \right]. \]

From Equation (5.3) we conclude that the optimal investment decisions are linear in the factor realizations and independent of the level of wealth. They depend on the risk-aversion parameter \( \gamma \), the risk-free rate \( r_{f} \), the time to maturity \( k \) and the return process characteristics: volatility, mean and matrix of autoregressive coefficients. By inspecting Equation (5.3), it is apparent that the higher \( \gamma \) and \( r_{f} \) are, the smaller the investment in the risky assets: the more risk averse the investment manager is, the less he is willing to invest in the risky asset, and the higher the risk-free rate the smaller the risk-premium is, therefore, the less attractive the risky opportunities are. In what follows, we analyze how the underlying return process and time to maturity affect the investment decisions.
5.1.1 A Numerical Illustration

To provide an illustration of the possible impact of the return autocorrelation on risky portfolio holdings, we consider the case of a portfolio consisting of one riskless and two risky assets. The excess return dynamics are given by

\[
\tilde{r}_t = c + A f_t + \epsilon_t,
\]

\[
f_t = d + B f_{t-1} + \eta_t,
\]

where it is assumed that \(c = [-0.01, 0.09]'\), \(A = [0.2, 0.2]'\), \(d = 0.24\), \(B = 0.2\), the correlation coefficient is \(\rho_{12} = 0.4\), the volatilities are \(\sigma_{\epsilon,11} = 0.1\), \(\sigma_{\epsilon,22} = 0.3\), \(\sigma_\eta = 0.25\). Notice, that the factor realizations follow an AR(1) process with mean 0.3. Asset 1 has mean excess return and volatility equal to 5% and 10% respectively, while Asset 2 has mean excess return and volatility of 15% and 30% respectively. Therefore, Asset 2 is more risky than and positively correlated with Asset 1. In addition, we consider that \(r_f = 0.05\), \(\gamma = 0.01\), \(W_0 = 1\) and \(T = 24\).

The optimal investment on Assets 1 and 2 exhibit different behavior as time to maturity, first-order autocorrelation and factor realization changes, as we show in Figure 5-1.

For positive correlation coefficient \(B\), the holdings in the less risky Asset 1 increase as a function of both time and factor realization. The investment on Asset 2, though, increases over time, but decreases slightly as a function of \(f_t\). In all cases, the investment in the less risky asset is higher than the investment in the more risky asset, since the investor is risk averse.

For negative correlation coefficient \(B\), the investment on Asset 1 is a decreasing function of \(f_t\): the higher \(f_t\) is, the lower \(r_{t+1}\) is expected to be, and therefore, we reduce our holdings in Asset 1. The decrease in the factor realizations is more acute closer to the terminal date. Indeed, negative autocorrelation makes it more likely that bad returns will be offset by good future returns, more so when the remaining investment time window is larger, making risky assets more attractive farther in the horizon. Thus, the risky asset (Asset 1 in this case) provides a hedging effect: if the current risky return is low, the opportunity set improves canceling some of the risk. This time-diversification intuition of the monotonic decline in risky investment over time, is valid only for negative serial correlation. On the other hand, the holdings on the most risky Asset 2 are always increasing in time and \(f_t\), and is positive. Even for high realized returns
Figure 5-1: The optimal risky holdings (in thousands), plotted as a function of the factor realization for various time periods. In Panels (a)-(b), we consider positive factor correlation \((B = 0.2)\), while in Panels (c)-(d), we consider negative correlation \((B = -0.2)\).

At time \(t\), when we expect their future decrease at time \(t + 1\), the holdings in Asset 2 increase, for diversification purposes since the optimal decision indicates a significant decrease in the holdings of Asset 1.

Two other quantities of interest are the expected investment, defined as the unconditional expectation of the optimal amount invested in the risky assets that serves as a way to examine the optimal portfolio choice over time,

\[
E\left\{ G^*_T \right\} = \frac{1}{\gamma} \frac{1}{(1 + r_f)^{k-1}} Q_{T-k} \left( G_{T-k} \frac{d}{1 - B} + h_{T-k} \right),
\]

and the extrapolated expected risky investment, defined as the expected risky investment at
time $t$ extrapolated forward to time $T$ by the risk-free investment

$$E \{ \hat{u}_{T-k}^* \} (1 + r_f)^{k-1}.$$

The extrapolation of the risky investment at the risk-free rate also appears in Mossin [64], where it is shown that for HARA preferences but under serial independence investment managers make decisions as if all proceeds from the current risky investment are henceforth to be invested at the risk-free rate. The intuition is that, evaluated at the optimal portfolio composition, the marginal contribution of each asset is identical and equal to the risk-free return, so that the return on the risk-free asset is appropriate for extrapolating to the future.

![Graphs showing expected risky holdings as a function of time](image)

Figure 5-2: The expected risky holdings, plotted as a function of time. In Panel (a) we consider a positive autocorrelation coefficient for the factor dynamics ($B = 0.2$), and in Panel (b) a negative one ($B = -0.2$).

As it is shown in Figure 5-2, for positively correlated returns, the optimal expected investment for both assets increases monotonically over time. Observe that the rate of change in the risky holdings for Asset 1 is higher than the one for the more risky Asset 2. As we
approach the end of the investment horizon, the portfolio manager is willing to tolerate less relative risk, therefore he invests more on the less risky asset. The contrary is true for negative serial correlation. In this case, the value of the expected investment in Asset 1 is lower, but also increases over time, up until time $T - 2$, when it falls. The increase of the risky holdings is attributed to the decrease of the investor's risk aversion with age. At the beginning of the investment horizon potential gain and losses are magnified through the riskless rate of return, and thus investors exhibit a higher aversion to risk. On the contrary, in the presence of IID returns and intermediate consumption risky holdings decrease over time due to the fact that investors exhibit an increasing aversion to risk with age.

Figure 5-3: The extrapolated risky investments, plotted as a function of time. In Panels (a)-(b) we consider positive autocorrelation ($B = 0.2$) and in (c)-(d) negative ($B = -0.2$). The solid line represents asset 1, and the dotted asset 2.
Finally, in Figure 5-3, we plot the extrapolated expected risky investment, expressed as a fraction of the expected risky investment in the last period, over time. We focus our attention on the age effect on risky investments, as discussed by Samuelson [70] and Merton [60]. Samuelson derived numerically that the optimal risky share is greater two periods prior to expiration than with one for an investor with relative risk aversion equal to two. On the other hand, Merton showed that, under CARA preferences and an infinite horizon, risky investment is not a function of time, but with managers holding more on the risky asset compared with the no-autocorrelated case. From Figure 5-3 it is evident that the extrapolated risky investment stays constant over time until the final period for both assets. The occurrence of a jump in the terminal investment date is due to the boundary conditions introduced to the dynamic optimization problem and the minor saw-tooth pattern arises due to the negative correlation coefficient. Thus, this behavior of the extrapolated risky investments is in accordance with Merton’s claim under the specific return dynamics considered. However, the behavior of the two assets is different. Under positive serial correlation, the expected extrapolated risky investment rises for Asset 1 and decreases for Asset 2 at the end of the investment horizon. The contrary is true for negative autocorrelation.

5.1.2 Vector Autoregressive Process VAR(1)

In this section, we present a special instance of the encountered return dynamics and justify the discontinuity appearing in the optimal risky investment one period prior to expiration. Consider the case where the return dynamics are given by (1.1)-(1.2) with $c_t = 0$, $A_t = I$, $\Sigma_{\epsilon} = 0$, $\eta_{t-1} = \mu$, $B_{t-1} = B$. Then, the excess return is given by the relation:

$$\hat{r}_t = \mu + B \hat{r}_{t-1} + \eta_t$$

where $\mu$ is a $N \times 1$ vector of constants and $B$ an $N \times N$ matrix of autoregressive coefficients. Similarly as before, $\eta_t$ is a vector generalization of white noise with mean 0 and covariance matrix $\Sigma_{\eta}$, which is assumed to be symmetric and positive definite. Now, the unconditional mean of the return process is $(I - B)^{-1} \mu$.

Under the VAR(1) dynamics, we prove that the coefficients appearing in the expression

---

$^1$I is the $N \times N$ identity matrix.
of the optimal investment decisions $\hat{u}_{t-k}^*$ are independent of time for all $k$ but $k = 1$. Due to the boundary conditions, the optimization problem solved at time $T - 1$ has not the same form with the optimization problems encountered at times $T - k$ for $k \neq 1$. As a result, there is a discontinuity in the coefficients of the optimal investment, and risky holdings appear to decrease (for negative autocorrelation) or increase (for positive autocorrelation) at time $T - 1$ as it is manifested in Figure 5-4.

**Proposition 5.2** If the excess return is governed by a Vector Autoregressive Process, the coefficients appearing in (5.3) are given by the following relations:

For $k = 1$,

\[
\begin{align*}
\Lambda_{T-1} &= \Sigma_{\eta}, \\
Q_{T-1} &= \Sigma_{\eta}^{-1}, \\
G_{T-1} &= B, \\
h_{T-1} &= \mu, \\
S_{T-1} &= B' \Sigma_{\eta}^{-1} B, \\
y_{T-1} &= B' \Sigma_{\eta}^{-1} \mu, \\
x_{T-1} &= \mu' \Sigma_{\eta}^{-1} \mu.
\end{align*}
\]

For $2 \leq k \leq T$,

\[
\begin{align*}
\Lambda_{T-k} &= \left[\Sigma_{\eta}^{-1} + S_{T-1}\right]^{-1}, \\
Q_{T-k} &= \Lambda_{T-k}^{-1}, \\
G_{T-k} &= (I - \Lambda_{T-k} S_{T-1}) B, \\
h_{T-k} &= \mu - \Lambda_{T-k} (S_{T-1} \mu + y_{T-1}), \\
S_{T-k} &= S_{T-1}, \\
y_{T-k} &= y_{T-1}, \\
x_{T-k} &= k x_{T-1}.
\end{align*}
\]

**Proof.** Since this is a special case of the dynamics given by (1.1)-(1.2), the coefficients are given by the relations present in Theorem 5.1. Therefore, for $A_{T-k+1} = I$, $\Sigma_c = 0$, $d_{T-k} = \mu$.  

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\( B_{T-k} = B \), we obtain that

\[
\begin{align*}
\Lambda_{T-k} &= \left[ \Sigma^{-1}_\eta + S_{T-k+1} \right]^{-1}, \\
Q_{T-k} &= \Lambda_{T-k}^{-1}, \\
G_{T-k} &= \left( I - \Lambda_{T-k} S_{T-k+1} \right) B, \\
h_{T-k} &= \mu - \Lambda_{T-k} \left( S_{T-k+1} \mu + y_{T-k+1} \right).
\end{align*}
\]

Using the boundary conditions of Theorem 5.1, we deduce the desired expressions for \( \Lambda_{T-1} \), \( Q_{T-1} \), \( G_{T-1} \) and \( h_{T-1} \). For \( S_{T-1} \) we have

\[
S_{T-1} = B' S_T B + 2 G'_{T-1} Q_{T-1} B - \left( G'_{T-1} Q_{T-1} + B' S_T \right) \Lambda_{T-1} \left( Q_{T-1} G_{T-1} + S_T B \right)
\]

\[
= 2 B' \Sigma^{-1}_\eta B - B' \Sigma^{-1}_\eta B
\]

\[
= B' \Sigma^{-1}_\eta B.
\]

Similarly, substituting for the boundary conditions and the new dynamics into the expressions given by Theorem 5.1, we obtain the result for \( y_{T-1} \) and \( x_{T-1} \).

In order to prove the desired expressions, we need to show is that, for \( k = 2, \ldots, T \) the following holds:

\[
S_{T-k} = S_{T-1}, \\
y_{T-k} = y_{T-1}.
\]

For \( S_{T-k} \) we have that

\[
S_{T-k} = B' S_{T-k+1} B + 2 G'_{T-k} Q_{T-k} B - \left( G'_{T-k} Q_{T-k} + B' S_{T-k+1} \right) \Lambda_{T-k} \left( Q_{T-k} G_{T-k} + S_{T-k+1} B \right)
\]

\[
= B' S_{T-k+1} B + 2 \left( B - \Lambda_{T-k} S_{T-k+1} B \right)' Q_{T-k} B - \left[ \left( B - \Lambda_{T-k} S_{T-k+1} B \right)' Q_{T-k} + B' S_{T-k+1} \right] \Lambda_{T-k}.
\]

\[
= B' S_{T-k+1} B + 2 B' Q_{T-k} B - 2 B' S_{T-k+1} \Lambda_{T-k} Q_{T-k} B - 
\]

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But $Q_{T-k} \Lambda_{T-k} = I$. Thus,

\[
S_{T-k} = B' S_{T-k+1} B + 2 B' Q_{T-k} B - 2 B' S_{T-k+1} B - B' (I - S_{T-k+1} \Lambda_{T-k}) Q_{T-k} (I - \Lambda_{T-k} S_{T-k+1}) B - 2 B' (I - S_{T-k+1} \Lambda_{T-k}) S_{T-k+1} B - B' S_{T-k+1} \Lambda_{T-k} S_{T-k+1} B = B' Q_{T-k} B - B' S_{T-k+1} B = B' (Q_{T-k} - S_{T-k+1}) B.
\]

But we also know that,

\[
Q_{T-k} = \Lambda_{T-k}^{-1} = \Sigma_{\eta}^{-1} + S_{T-k+1}.
\]

Therefore,

\[
S_{T-k-1} = B' \Sigma_{\eta}^{-1} B = S_{T-1}.
\]

Similarly for $y_{T-k}$ we have

\[
y_{T-k} = B' y_{T-k+1} + G'_{T-k} Q_{T-k} \mu + B' Q_{T-k} h_{T-k} + B' S_{T-k+1} \mu - (G'_{T-k} Q_{T-k} + B' S_{T-k+1}) \Lambda_{T-k} (Q_{T-k} h_{T-k} + S_{T-k+1} \mu + y_{T-k+1}) = B' y_{T-k+1} + B' (I - S_{T-k+1} \Lambda_{T-k}) Q_{T-k} \mu + B' Q_{T-k} [\mu - \Lambda_{T-k} (S_{T-k+1} \mu + y_{T-k+1})] + B' S_{T-k+1} \mu - G'_{T-k} Q_{T-k} \Lambda_{T-k} Q_{T-k} h_{T-k} - G'_{T-k} Q_{T-k} h_{T-k} - G'_{T-k} Q_{T-k} y_{T-k+1} + B' S_{T-k+1} \Lambda_{T-k} Q_{T-k} h_{T-k} - B' S_{T-k+1} \Lambda_{T-k} S_{T-k+1} \mu - B' S_{T-k+1} \Lambda_{T-k} y_{T-k+1}.
\]
Cancelling the term $Q_{T-k} \Lambda_{T-k}$ we obtain

$$y_{T-k} = 2 B' Q_{T-k} \mu - B' S_{T-k+1} \mu - (G'_{T-k} Q_{T-k} + B' S_{T-k+1}) h_{T-k} -$$

$$(G'_{T-k} + B' S_{T-k+1} \Lambda_{T-k}) (S_{T-k+1} \mu + y_{T-k+1}).$$

Substituting for $G_{T-k}$, the coefficients of $h_{T-k}$ and $(S_{T-k+1} \mu + y_{T-k+1})$ become

$$B' (I - S_{T-k+1} \Lambda_{T-k}) Q_{T-k} + B' S_{T-k+1} = B' Q_{T-k},$$

and

$$B' (I - S_{T-k+1} \Lambda_{T-k}) + B' S_{T-k+1} \Lambda_{T-k} = B',$$

respectively. Therefore, $y_{T-k}$ results in

$$y_{T-k} = 2 B' Q_{T-k} \mu - B' S_{T-k+1} \mu - B' Q_{T-k} [\mu - \Lambda_{T-k} (S_{T-k+1} \mu + y_{T-k+1})] -$$

$$B' (S_{T-k+1} \mu + y_{T-k+1})$$

$$= B' Q_{T-k} \mu - B' S_{T-k+1} \mu$$

$$= B' (Q_{T-k} - S_{T-k+1}) \mu$$

$$= B' \Sigma_{\eta}^{-1} \mu = y_{T-1}.$$

Finally, for $x_{T-k}$ we have

$$x_{T-k} = x_{T-k+1} + 2 y_{T-k+1}' \mu + 2 h_{T-k}' Q_{T-k} \mu + \mu' S_{T-k+1} \mu -$$

$$(h_{T-k}' Q_{T-k} + \mu' S_{T-k+1} + y_{T-k+1}') \Lambda_{T-k} (Q_{T-k} h_{T-k} + S_{T-k+1} \mu + y_{T-k+1})$$

$$= x_{T-k+1} + 2 y_{T-k+1}' \mu +$$

$$2 [\mu - \Lambda_{T-k} (S_{T-k+1} \mu + y_{T-k+1})]' Q_{T-k} \mu + \mu' S_{T-k+1} \mu -$$

$$[\mu - \Lambda_{T-k} (S_{T-k+1} \mu + y_{T-k+1})]' Q_{T-k} [\mu - \Lambda_{T-k} (S_{T-k+1} \mu + y_{T-k+1})] -$$

$$2 [\mu - \Lambda_{T-k} (S_{T-k+1} \mu + y_{T-k+1})]' (S_{T-k+1} \mu + y_{T-k+1}) -$$

$$\mu' S_{T-k+1} \Lambda_{T-k} S_{T-k+1} \mu - 2 \mu' S_{T-k+1} \Lambda_{T-k} y_{T-k+1} - y_{T-k+1}' \Lambda_{T-k} y_{T-k+1}$$

$$= x_{T-k+1} + \mu' Q_{T-k} \mu - \mu' S_{T-k+1} \mu$$

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= x_{T-k+1} + \mu' \left( Q_{T-k} - S_{T-k+1} \right) \mu \\
= x_{T-k+1} + \mu' \Sigma^{-1}_\eta \mu = x_{T-k+1} + x_{T-1}.

As a result,

\[ x_{T-k} = k x_{T-1}. \]

Since the optimal control coefficients are independent of time for all but the last period, the extrapolated expected risky investment, defined for \( k \neq 1 \) as

\[
(1 + r_f)^{k-1} E\{ \hat{u}_{T-k}^* \} = \\
\frac{1}{\gamma} \left( \Sigma^{-1}_\eta + B' \Sigma^{-1}_\eta B \right) \left\{ \begin{array}{c}
I - \left( \Sigma^{-1}_\eta + B' \Sigma^{-1}_\eta B \right)^{-1} B' \Sigma^{-1}_\eta B \\
\mu - \left( \Sigma^{-1}_\eta + B' \Sigma^{-1}_\eta B \right)^{-1} \left( B' \Sigma^{-1}_\eta B \mu + B' \Sigma^{-1}_\eta \mu \right)
\end{array} \right\}
\]

and for \( k = 1 \) as

\[
E\{ \hat{u}_{T-1}^* \} = \frac{1}{\gamma} \Sigma^{-1}_\eta \left\{ B B (I - B)^{-1} \mu + \mu \right\},
\]

is constant over time until the final period, when it falls for negatively correlated and rises for positively correlated returns. In addition, for negative first-order autocorrelation, the expected risky investment \( E\{ \hat{u}_{T-k}^* \} \) rises monotonically until the last period when it may rise or fall. On the contrary, for positively correlated returns, \( E\{ \hat{u}_{T-k}^* \} \) rises monotonically in all periods. We note that, if the risk-free rate is zero, then the extrapolated expected risky investment is identical to the expected risky investment.

To provide an numerical illustration, we consider the case of a portfolio with only one risky and one riskless asset. The excess risky return follows a AR(1) process. We investigate the effect of negative or positive autocorrelation, the nature of the age effects and the role of positive risk-free interest rates and risk aversion parameter.

For yearly excess returns of the CRSP Equal-Weighted Index over the 1926-1985 period, Fama and French [31] obtain an AR(1) coefficient of \( \beta = -0.07 \). We also assume that the mean is 12%. For weekly excess returns of the CRSP Equal-Weighted Index for 1962-1985, Lo and MacKinlay [54] obtain a positive AR(1) coefficient of \( \beta = 0.30 \) with a mean of 0.35%. Finally, Cambell, Lo and MacKinlay [16] report that for the daily excess returns of the CRSP
Equal-Weighted Index for 1978-1994 a positive AR(1) coefficient of $\beta = 0.35$ with a mean of 0.07%. We assume that the risk-free rate is set at 0.03, the volatility is $\sigma_\epsilon = 0.125$ and that the time horizon is 24 periods.

![Graphs](image)

Figure 5-4: The expected risky holdings and the extrapolated risky investment plotted as a function of time. In Panels (a)-(b) we use the yearly data (negative autocorrelation) and in Panels (c)-(d) the weekly data (positive autocorrelation.)

As it is shown in Figure 5-4, for negatively correlated returns the expected optimal risky investment increases monotonically up to the next to last period when it drops, and the extrapolated expected risky investment is constant up to time $T-1$, when it falls. For positively correlated returns, the expected investment rises monotonically in time and the extrapolated is constant up to time $T-1$ and then increases.

In Figure 5-5, we show the dependence of the optimal investment decision on the level of
the realized return for different time periods. The optimal control is a decreasing function of return for negative autocorrelation and increasing in time up until the next to last period, when it drops. The reverse is true for positive autocorrelation. In addition, the level of the risky investment is higher when using the yearly data, because the mean of the excess return process is higher. But, its rate of change (the absolute value of the slope in Panel (a)) is lower, since the correlation coefficient is smaller in absolute terms.

Figure 5-5: The optimal risky holdings plotted as a function of the return realization. We consider different values for the time to maturity. In Panel (a) we use the annual data (negative correlation) and in Panel (b) the weekly data (positive correlation.)

Finally, in Figure 5-6 we present the dependance of the risky investment to the risk-free rate and the volatility parameter $\sigma_e$. The higher $r_f$ and $\sigma_e$ are, the less attractive the risky opportunity becomes and, as a result, the risky holdings decrease.
Figure 5-6: The optimal risky holdings plotted as a function of the return realization for daily data (positive correlation.) We consider different values for the riskfree rate \( r_f \) and the volatility parameter \( \sigma_x \).

5.2 Return Dynamics: Stochastic Volatility Models

In this section we investigate the impact of stochastic volatility models to the investment behavior over time of an investment manager with CARA utility. The excess asset return dynamics are given by the stochastic volatility model of Section 1.1.2

\[
\tilde{r}_t = \mu + \sigma_{t-1} \epsilon_t, \\
\sigma_t^2 = \alpha_0 + \beta \sigma_{t-1}^2 + \alpha_1 \sigma_{t-1}^2 \epsilon_t^2, \\
\epsilon_t \sim N(0,1),
\]

and therefore the rate of return at time \( t \) conditioned on the information available at time \( t-1 \) is normally distributed with mean \( \mu \) and variance \( \sigma_{t-1}^2 \). Once again, the parameters \( \alpha_0, \beta, \) and \( \alpha_1 \) are assumed to be nonnegative in order to keep the conditional variance positive. Moreover, we
assume that $\alpha_1 + \beta < 1$, and thus the unconditional expectation of $\sigma_t^2$ is $E_s = \alpha_0 / (1 - \alpha_1 - \beta)$.

The state of the system at time $t = 0, 1, \ldots, T - 1$ consists of the wealth $W_t$ at time $t$, and $\sigma_t^2$, the conditional variance of the single risky asset at time $t$. As a result, the investment manager faces the following optimization problem

$$\max_{\{\tilde{u}_0, \ldots, \tilde{u}_{T-1}\}} E_0 \left\{ -e^{-\gamma W_T} \right\}$$

subject to

$$W_t = (1 + r_f) W_{t-1} + \tilde{u}_{t-1} \tilde{r}_t$$

$$\tilde{r}_t = \mu + \sigma_{t-1} \epsilon_t$$

$$\sigma_t^2 = \alpha_0 + \beta \sigma_{t-1}^2 + \alpha_1 \sigma_{t-1}^2 \epsilon_t^2$$

$$\epsilon_t \sim N(0,1).$$

This problem does not produce a closed-form solution for the optimal investment behavior. Nevertheless, we propose an approximate dynamic policy that utilizes characteristics of the optimal cost-to-go function at every point in time.

### 5.2.1 An Approximate Dynamic Programming Algorithm

In order to motivate the approximation algorithm that follows, consider the optimal value function $V_{T-1}$:

$$V_{T-1} \left( W_{T-1}, \sigma_{T-1}^2 \right) = \max_{\tilde{u}_{T-1}} E_{T-1} \left\{ -\exp \left[ -\gamma \left( 1 + r_f \right) W_{T-1} - \gamma \tilde{u}_{T-1} \tilde{r}_T \right] \right\} =$$

$$= \max_{\tilde{u}_{T-1}} E_{T-1} \left\{ -\exp \left[ -\gamma \left( 1 + r_f \right) W_{T-1} - \gamma \tilde{u}_{T-1} \left( \mu + \sigma_{T-1} \epsilon_T \right) \right] \right\} =$$

$$= \max_{\tilde{u}_{T-1}} \left\{ -\exp \left[ -\gamma \left( 1 + r_f \right) W_{T-1} - \gamma \mu \tilde{u}_{T-1} + \frac{1}{2} \gamma^2 \sigma_{T-1}^2 \tilde{u}_{T-1}^2 \right] \right\}.$$ 

This problem can be solved in closed-form since it is equivalent to

$$\max_{\tilde{u}_{T-1}} \left\{ \gamma \mu \tilde{u}_{T-1} - \frac{1}{2} \gamma^2 \sigma_{T-1}^2 \tilde{u}_{T-1}^2 \right\}.$$
The first order condition is necessary and sufficient and is given by

$$\gamma \mu - \gamma^2 \sigma^2_{T-1} \hat{u}_{T-1} = 0.$$ 

The optimal investment decision at time $T - 1$, therefore, is

$$\hat{u}_{T-1}^* = \frac{\mu}{\gamma \sigma^2_{T-1}}$$

and $V_{T-1}$ is just

$$V_{T-1} \left( W_{T-1}, \sigma^2_{T-1} \right) = -\exp \left[ -\gamma (1 + r_f) \, W_{T-1} - \frac{\mu^2}{2 \sigma^2_{T-1}} \right].$$

The optimal investment in the risky asset is inversely proportional to the state variable $\sigma^2_{T-1}$, in contrast with the one obtained in Equation (5.2) under the assumption of a factor model for the return dynamics, where it is linear in the corresponding state variable. This is the reason why a closed-form solution for the dynamic optimization problem is unattainable. Instead, we propose an algorithm that performs the following operations for $k = 1, \ldots, T$:

1. Linearization of the exponent in $V_{T-k}$ with respect to $\sigma^2_{T-k}$, using the first-order Taylor's expansion around the unconditional expectation of $\sigma^2_{T-k}$, defined as $E_s = \alpha_0 / (1 - \alpha_1 - \beta)$.

2. Approximation of the coefficient of the term $\sigma^2_{T-k}$ with a constant, replacing $\sigma^2_{T-k}$ with its unconditional expectation $E_s$.

More specifically, in $V_{T-1}$, we approximate the term $-\frac{\mu^2}{2 \sigma^2_{T-1}}$ with a linear function in $\sigma^2_{T-1}$

$$-\frac{\mu^2}{2 \sigma^2_{T-1}} \approx -\frac{\mu^2}{2 E_s} + \frac{\mu^2}{2 E_s^2} \left( \sigma^2_{T-1} - E_s \right)$$

$$\approx -\frac{\mu^2}{E_s} + \frac{\mu^2}{2 E_s^2} \sigma^2_{T-1},$$

and we obtain the approximated value function $\hat{V}_{T-1}$

$$\hat{V}_{T-1} \left( W_{T-1}, \sigma^2_{T-1} \right) = -\exp \left[ -\gamma (1 + r_f) \, W_{T-1} - \frac{\mu^2}{E_s} + \frac{\mu^2}{2 E_s^2} \sigma^2_{T-1} \right].$$

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Consequently, the value function at $T - 2$ is given by

$$V_{T-2}(W_{T-2}, \sigma_{T-2}^2) = \max_{\hat{u}_{T-2}} E_{T-2} \left\{ \hat{V}_{T-1}(W_{T-1}, \sigma_{T-1}^2) \right\}$$

$$= \max_{\hat{u}_{T-2}} E_{T-2} \left\{ -\exp \left[ -\gamma (1 + r_f) W_{T-1} - \frac{\mu^2}{E_s} + \frac{\mu^2}{2 E_s^2} \sigma_{T-1}^2 \right] \right\}$$

$$= \max_{\hat{u}_{T-2}} E_{T-2} \left\{ -\exp \left[ -\gamma (1 + r_f)^2 W_{T-2} \left( \mu + \sigma_{T-2} \epsilon_{T-1} \right) \right] \right\}.$$

Using the dynamics given in Equation (1.6), we obtain

$$V_{T-2}(W_{T-2}, \sigma_{T-2}^2) = \max_{\hat{u}_{T-2}} E_{T-2} \left\{ -\exp \left[ -\gamma (1 + r_f)^2 W_{T-2} - \frac{\mu^2}{E_s} \right] \right\}$$

$$= \max_{\hat{u}_{T-2}} E_{T-2} \left\{ -\exp \left[ -\gamma (1 + r_f)^2 W_{T-2} - \frac{\mu^2}{E_s} + \frac{\mu^2}{2 E_s^2} \left[ \alpha_0 + \beta \sigma_{T-2}^2 + \alpha_1 \sigma_{T-2}^2 \epsilon_{T-1}^2 \right] \right] \right\}$$

$$= \max_{\hat{u}_{T-2}} E_{T-2} \left\{ -\exp \left[ -\gamma (1 + r_f)^2 W_{T-2} - \frac{\mu^2}{E_s} + \frac{\mu^2}{2 E_s^2} \left[ \alpha_0 + \beta \sigma_{T-2}^2 \right] \right] \right\}.$$

The second step of the approximation procedure involves approximating the coefficient of $\epsilon_{T-1}^2$ with a constant:

$$\left[ \frac{\mu^2}{2 E_s^2} \alpha_1 \sigma_{T-2}^2 \right] \approx \frac{\alpha_1 \mu^2}{2 E_s}.$$

Now, by applying Proposition 5.1 and letting

$$\Lambda_{T-2} = \left( 1 - \frac{\alpha_1 \mu^2}{E_s} \right)^{-1},$$

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we have

\[ V_{T-2} \left( W_{T-2}, \sigma^2_{T-2} \right) = \sqrt{\Lambda_{T-2}} \max_{\tilde{u}_{T-2}} \exp \begin{bmatrix} -\gamma (1 + r_f)^2 W_{T-2} - \frac{\mu^2}{\tilde{E}_*} \\ + \frac{\mu^2}{\tilde{E}_*} [\alpha_0 + \beta \sigma^2_{T-2}] \\ -\gamma (1 + r_f) \tilde{u}_{T-2} \mu \\ + \frac{1}{2} \Lambda_{T-2} \gamma^2 (1 + r_f)^2 \tilde{u}^2_{T-2} \sigma^2_{T-2} \end{bmatrix} . \]

As a result, the approximate optimal investment in the risky asset at time \( T - 2 \) is

\[ \tilde{u}_{T-2} = \frac{\mu}{\gamma (1 + r_f) \Lambda_{T-2} \sigma^2_{T-2}} , \]

and the approximated value function is

\[ \tilde{V}_{T-2} \left( W_{T-2}, \sigma^2_{T-2} \right) = -\sqrt{\Lambda_{T-2}} \exp \begin{bmatrix} -\gamma (1 + r_f)^2 W_{T-2} - \frac{\mu^2}{\tilde{E}_*} \\ + \frac{\mu^2}{\tilde{E}_*} [\alpha_0 + \beta \sigma^2_{T-2}] \\ -\gamma (1 + r_f) \tilde{u}_{T-2} \mu \\ + \frac{1}{2} \Lambda_{T-2} \gamma^2 (1 + r_f)^2 \tilde{u}^2_{T-2} \sigma^2_{T-2} \end{bmatrix} . \]

We prove the following theorem that yields the proposed approximation algorithm:

**Theorem 5.2** The optimal investment decisions and the value function \( V_{T-k} \) for \( k = 1, \ldots, T \) can be approximated by the following relations:

\[
\tilde{u}_{T-1} = \frac{\mu}{\gamma \sigma^2_{T-1}},
\]

(5.5)

\[
V_{T-1} \left( W_{T-1}, \sigma^2_{T-1} \right) = -\exp \left[ -\gamma (1 + r_f) W_{T-1} - \frac{\mu^2}{2 \sigma^2_{T-1}} \right] \quad \text{(exact formula)}
\]

and

\[
\tilde{u}_{T-k} = \frac{\mu}{\gamma (1 + r_f)^{k-1} \Lambda_{T-k} \sigma^2_{T-k}},
\]

(5.6)

\[
\tilde{V}_{T-k} \left( W_{T-k}, \sigma^2_{T-k} \right) = -\prod_{m=2}^{k} \sqrt{\Lambda_{T-m}} \exp \begin{bmatrix} -\gamma (1 + r_f)^k W_{T-k} + d_{T-k} \\ + \frac{\mu^2}{\tilde{E}_*} \sigma^2_{T-k} \\ -\gamma (1 + r_f) \tilde{u}_{T-k} \mu \\ + \frac{1}{2} \Lambda_{T-k} \gamma^2 (1 + r_f)^2 \tilde{u}^2_{T-k} \sigma^2_{T-k} \end{bmatrix} ,
\]

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where

\[ c_{T-k} = \frac{\mu^2}{2 \Lambda_{T-k+1} E_s^2} + b_{T-k+1}, \]
\[ \Lambda_{T-k} = (1 - 2 c_{T-k} \alpha_1 E_s)^{-1}, \]
\[ d_{T-k} = d_{T-k+1} - \frac{\mu^2}{\Lambda_{T-k+1} E_s} + c_{T-k} \alpha_0, \]
\[ b_{T-k} = c_{T-k} \beta, \]

with the boundary conditions

\[ \Lambda_{T-1} = 1 \]
\[ c_{T-1} = d_{T-1} = b_{T-1} = 0. \]

**Proof.** We prove the theorem by induction. We have shown that the result is valid for \( k = 2. \) Assume that it is true for \( k; \) we shall show that it holds for \( k + 1. \) The value function at time \( T - k - 1 \) is just

\[ V_{T-k-1} \left( W_{T-k-1}, \sigma_{T-k-1}^2 \right) = \max_{u_{T-k-1}} E_{T-k-1} \left\{ \tilde{V}_{T-k} \left( W_{T-k}, \sigma_{T-k}^2 \right) \right\} = \]

\[ - \prod_{m=2}^{k} \sqrt{\Lambda_{T-m}} \max_{u_{T-k-1}} E_{T-k-1} \left\{ \exp \left[ \begin{array}{c}
-\gamma (1 + rf)^k W_{T-k} + d_{T-k} \\
+ b_{T-k} \sigma_{T-k}^2 \\
- \frac{\mu^2}{2 \Lambda_{T-k} \sigma_{T-k}^2}
\end{array} \right] \right\}. \]

The first approximation is to linearize the term \(-\frac{\mu^2}{2 \Lambda_{T-k} \sigma_{T-k}^2}\) using the first-order Taylor's expansion around \( E_s: \)

\[ -\frac{\mu^2}{2 \Lambda_{T-k} \sigma_{T-k}^2} \approx \frac{\mu^2}{2 \Lambda_{T-k} E_s} + \frac{\mu^2}{2 \Lambda_{T-k} E_s^2} \left( \sigma_{T-k}^2 - E_s \right) \]

\[ \approx \frac{\mu^2}{\Lambda_{T-k} E_s} + \frac{\mu^2}{2 \Lambda_{T-k} E_s^2} \sigma_{T-k}^2. \]

Therefore, by substituting for the wealth and return dynamics we have that

\[ V_{T-k-1} \left( W_{T-k-1}, \sigma_{T-k-1}^2 \right) \]
\[ \approx - \prod_{m=2}^{k} \sqrt{\Lambda_{T-m}} \max_{\tilde{u}_{T-k-1}} E_{T-k-1} \left\{ \begin{array}{l}
\exp \left[ -\gamma (1 + r f)^k W_{T-k} + d_{T-k} \right. \\
+ b_{T-k} \sigma_{T-k}^2 \\
\left. - \frac{\mu^2}{\Lambda_{T-k} E_s} + 2 \frac{\mu^2}{\Lambda_{T-k} E_s^2} \sigma_{T-k}^2 \right] \end{array} \right\} \]

\[ \approx - \prod_{m=2}^{k} \sqrt{\Lambda_{T-m}} \max_{\tilde{u}_{T-k-1}} E_{T-k-1} \left\{ \begin{array}{l}
\exp \left[ -\gamma (1 + r f)^{k+1} W_{T-k-1} + d_{T-k} - \frac{\mu^2}{\Lambda_{T-k} E_s} \right. \\
+ \left( \frac{\mu^2}{2 \Lambda_{T-k} E_s^2} + b_{T-k} \right) \left[ \alpha_0 + \beta \sigma_{T-k-1}^2 \right] \\
\left. + \epsilon_{T-k} \left[ -\gamma (1 + r f)^k \tilde{u}_{T-k-1} \sigma_{T-k-1} \right] \right. \\
\left. + \epsilon_{T-k}^2 \left[ \alpha_1 \sigma_{T-k-1}^2 \left( \frac{\mu^2}{2 \Lambda_{T-k} E_s^2} + b_{T-k} \right) \right] \right\} \right\}. \]

The second approximation is to replace the coefficient of \( \epsilon_{T-k}^2 \) with a constant by approximating \( \sigma_{T-k-1}^2 \) with its unconditional expectation:

\[ V_{T-k-1} \left( W_{T-k-1}, \sigma_{T-k-1}^2 \right) \approx \]

\[ - \prod_{m=2}^{k} \sqrt{\Lambda_{T-m}} \max_{\tilde{u}_{T-k-1}} E_{T-k-1} \left\{ \begin{array}{l}
\exp \left[ -\gamma (1 + r f)^{k+1} W_{T-k-1} + d_{T-k} - \frac{\mu^2}{\Lambda_{T-k} E_s} \right. \\
+ \left( \frac{\mu^2}{2 \Lambda_{T-k} E_s^2} + b_{T-k} \right) \left[ \alpha_0 + \beta \sigma_{T-k-1}^2 \right] \\
\left. + \epsilon_{T-k} \left[ -\gamma (1 + r f)^k \tilde{u}_{T-k-1} \sigma_{T-k-1} \right] \right. \\
\left. + \epsilon_{T-k}^2 \left[ \alpha_1 \sigma_{T-k-1}^2 \left( \frac{\mu^2}{2 \Lambda_{T-k} E_s^2} + b_{T-k} \right) \right] \right\} \right\}. \]

Using Proposition 5.1, we obtain

\[ \hat{V}_{T-k-1} \left( W_{T-k-1}, \sigma_{T-k-1}^2 \right) = \]

\[ - \prod_{m=2}^{k+1} \sqrt{\Lambda_{T-m}} \max_{\tilde{u}_{T-k-1}} E_{T-k-1} \left\{ \begin{array}{l}
\exp \left[ -\gamma (1 + r f)^{k+1} W_{T-k-1} + d_{T-k} - \frac{\mu^2}{\Lambda_{T-k} E_s} \right. \\
+ \left( \frac{\mu^2}{2 \Lambda_{T-k} E_s^2} + b_{T-k} \right) \left[ \alpha_0 + \beta \sigma_{T-k-1}^2 \right] \\
\left. + \gamma^2 (1 + r f)^{2k} \tilde{u}_{T-k-1}^2 \sigma_{T-k-1}^2 \right] \right\}, \]

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where
\[
c_{T-k-1} = \frac{\mu^2}{2 \Lambda_{T-k} E_s^2} + b_{T-k},
\]
\[
\Lambda_{T-k-1} = [1 - 2 c_{T-k-1} \alpha_1 E_s]^{-1}.
\]

The first order condition of the above optimization problem is necessary and sufficient and thus
\[
\bar{u}_{T-k-1} = \frac{\mu}{\gamma (1 + r_f)^k \Lambda_{T-k-1} \sigma_{T-k-1}^2}.
\]

Substituting in the value function, we have
\[
\hat{V}_{T-k-1} \left( W_{T-k-1}, \sigma_{T-k-1}^2 \right) =
\]
\[
\prod_{m=2}^{k+1} \sqrt{\Lambda_{T-k-1}} \exp \left[ -\gamma (1 + r_f)^{k+1} W_{T-k-1} + d_{T-k} - \frac{\mu^2}{\Lambda_{T-k} E_s} \right. \\
\left. + c_{T-k-1} \left[ \alpha_0 + \beta \sigma_{T-k-1}^2 \right] - \frac{\mu^2}{2 \Lambda_{T-k-1} \sigma_{T-k-1}^2} \right].
\]

If we let
\[
d_{T-k-1} = d_{T-k} - \frac{\mu^2}{\Lambda_{T-k} E_s} + c_{T-k-1} \alpha_0,
\]
\[
b_{T-k-1} = c_{T-k-1} \beta,
\]
we have shown that
\[
\hat{V}_{T-k-1} \left( W_{T-k-1}, \sigma_{T-k-1}^2 \right) = \prod_{m=2}^{k+1} \sqrt{\Lambda_{T-k-1}} \exp \left[ -\gamma (1 + r_f)^{k+1} W_{T-k-1} + d_{T-k-1} - b_{T-k-1} + c_{T-k-1} \sigma_{T-k-1}^2 \right. \\
\left. - \frac{\mu^2}{2 \Lambda_{T-k-1} \sigma_{T-k-1}^2} \right]
\]

and the result is true.

From Equations (5.5)-(5.6) we conclude that the approximate dynamic policy is inversely proportional to the variance realization \( \sigma_{T-k}^2 \) and independent of wealth. The approximate investment decisions depend on the risk-aversion parameter \( \gamma \), the risk-free rate \( r_f \), the time to maturity \( k \) and the return process characteristics. By inspecting Equations (5.5)-(5.6), it is apparent that the higher \( \gamma \) and \( r_f \) are, the smaller the investment in the risky asset. In
addition, the higher the mean of the return series \( \mu \) is, the higher the risky investment since the risk premium increases. On the other hand, an increase in the parameters \( \alpha_0 \), \( \alpha_1 \) and \( \beta \) results in a decrease in the amount invested in the risky asset, since now the investment opportunity is riskier. Finally, the longer the time to maturity, the higher the investment.

To illustrate the above qualitative insights, we present a numerical example. We assume that \( T = 24 \), \( r_f = 0.05 \), \( W_0 = 1 \), \( \gamma = 0.01 \), \( \mu = 0.1 \), \( \alpha_0 = 0.01875 \), \( \alpha_1 = 0.2 \), \( \beta = 0.5 \). Therefore, the unconditional mean of the asset's variance is \( E_s = (0.25)^2 \). In the graphs that follow, we concentrate on the expected investment, defined as the unconditional expectation of the amount invested in the risky assets, as a function of time and the various parameters that influence the decisions made.

![Graphs showing expected risky holdings over time](image)

Figure 5-7: The expected risky holdings, plotted as a function of time. In Panel (a), we consider two values of \( \mu \) (0.1 and 0.2) while keeping the rest of the parameters the same. In Panels (b), (c) and (d) we change the values of \( \alpha_0 \), \( \alpha_1 \) and \( \beta \) respectively.
Figure 5-8: The expected risky holdings, plotted as a function of time for different time horizons $T$.

In Figure 5-7, the expected amount invested in the risky asset rises with $\mu$ and falls with $\alpha_0$, $\alpha_1$ and $\beta$. In addition, we observe that the expected risky holdings increase monotonically with time for all values of the parameters involved. The higher the mean of the return process, the more attractive the risky opportunity becomes, and thus investment in the risky asset increases. On the other hand, the higher the asset’s volatility, the less the risk-averse investor is willing to lock wealth in the risky asset and thus prefers the riskless opportunity for investment.

In Figure 5-8, it is shown that the longer the time horizon an investor faces, the smaller the investment made in the risky asset early on. As a result, an investor with $T = 5$ will invest differently at time 0 than an investor with $T = 50$. But, both will react the same for, let’s say, 5 periods to expiration. In addition, the rate of change in the risky holdings increases as the expiration date approaches. Therefore, we conclude that, given the time $T - k$ until expiration, the time since investment began, $k$, has no effect on the holdings of the risky asset and no effect on its expected value.
5.2.2 A Quadratic Approximation

In this section, we propose a different suboptimal control policy by approximating the utility function with a quadratic. More specifically, by using Taylor’s expansion around zero we have that

\[-e^{-\gamma W_T} \approx -1 + \gamma W_T - \frac{1}{2} \gamma^2 W_T^2.\]

Therefore, the investor’s dynamic optimization problem can be approximated by the optimal control problem described by:

\[-1 + \gamma \max_{\{\hat{u}_0, \ldots, \hat{u}_{T-1}\}} E_0 \left\{ W_T - \frac{1}{2} \gamma W_T^2 \right\}\]

subject to

\[W_t = (1 + r_f) W_{t-1} + \hat{u}_{t-1} \hat{r}_t\]

\[\hat{r}_t = \mu + \sigma_{t-1} \epsilon_t\]

\[\sigma_t^2 = \alpha_0 + \beta \sigma_{t-1}^2 + \alpha_1 \sigma_{t-1}^2 \epsilon_t^2\]

\[\epsilon_t \sim N(0,1)\]

This problem still does not produce a closed-form solution for the optimal investment behavior. Nevertheless, we propose an approximate dynamic policy that utilizes characteristics of this new optimal cost-to-go function at every point in time.

We begin by characterizing the optimal value function \(V_{T-1}\) by using the boundary condition and then proceed recursively.

\[V_{T-1}(W_{T-1}, \sigma_{T-1}^2) = \max_{\hat{u}_{T-1}} E_{T-1} \left\{ W_T - \frac{1}{2} \gamma W_T^2 \right\}\]

\[= \max_{\hat{u}_{T-1}} \left\{ \begin{array}{c} (1 + r_f) W_{T-1} + \hat{u}_{T-1} W_{T-1}^2 - \frac{1}{2} \gamma W_{T-1}^2 \hat{u}_{T-1}^2 - \frac{1}{2} \gamma W_{T-1}^2 r_{T-1}^2 \\ \gamma (1 + r_f) W_{T-1} \hat{u}_{T-1} r_T \end{array} \right\}\]

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\[ = \max_{\tilde{u}_{T-1}} \begin{cases} 
(1 + r_f) W_{T-1} + \tilde{u}_{T-1} \mu - \frac{1}{2} \gamma (1 + r_f)^2 W_{T-1}^2 - \\
\frac{1}{2} \gamma \tilde{u}_{T-1}^2 (\mu^2 + \sigma_{T-1}^2) - \gamma (1 + r_f) W_{T-1} \tilde{u}_{T-1} \mu \end{cases}. \]

If we let
\[ Q_{T-1} = \mu^2 + \sigma_{T-1}^2, \]
the optimal control at time \( T - 1 \) is given by
\[ \tilde{u}_{T-1}^* = \frac{\mu}{\gamma Q_{T-1}} - \frac{(1 + r_f) \mu}{Q_{T-1}} W_{T-1}. \]

Notice that the investment decision under a quadratic utility function is linear on the level of wealth, while the risky holdings under an exponential utility are independent of wealth. This consists the fundamental difference between the two proposed approximate dynamic policies.

By letting
\[ S_{T-1} = \frac{\mu^2}{Q_{T-1}}, \]
the value function \( V_{T-1} \) becomes
\[ V_{T-1} \left( W_{T-1}, \sigma_{T-1}^2 \right) = \frac{1}{2 \gamma} S_{T-1} + (1 + r_f) (1 - S_{T-1}) \left[ W_{T-1} - \frac{1}{2} \gamma (1 + r_f) W_{T-1}^2 \right]. \]

The value function is quadratic in wealth at time \( T - 1 \), \( W_{T-1} \), and inversely proportional to the asset's variance at time \( T - 1 \), \( \sigma_{T-1}^2 \). This is the reason why a closed-form solution is unattainable. Instead, we propose an algorithm that approximates \( S_{T-k} \) with a linear function in \( \sigma_{T-k}^2 \), for all \( k = 1, \ldots, T \).

Using the first-order Taylor's expansion around the unconditional expected value of the asset's variance, \( E_s \), we can approximate \( S_{T-1} \) with
\[ \tilde{S}_{T-1} = \frac{\mu^2}{\mu^2 + E_s} - \frac{\mu^2}{(\mu^2 + E_s)^2} \left[ \sigma_{T-1}^2 - E_s \right]. \]

Let
\[ \lambda_{T-1} = \frac{\mu^2}{(\mu^2 + E_s)^2}, \]

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\[ \vartheta_{T-1} = \frac{\mu^2}{\mu^2 + E_s} + \lambda_{T-1} E_s. \]

Then,
\[ \hat{S}_{T-1} = \vartheta_{T-1} - \lambda_{T-1} \sigma^2_{T-1} \]

and \( V_{T-1} \) can be approximated by
\[ \hat{V}_{T-1} = \frac{1}{2\gamma} \hat{S}_{T-1} + (1 + r_f) \left( 1 - \hat{S}_{T-1} \right) \left[ W_{T-1} - \frac{1}{2} \gamma (1 + r_f) W_{T-1}^2 \right]. \]

We prove the following theorem that yields the proposed approximation algorithm:

**Theorem 5.3** The optimal investment decisions \( u_{T-k} \) and the value function \( V_{T-k} \) for \( k = 1, \ldots, T \) can be approximated by the following relations:

\[
\begin{align*}
\hat{u}_{T-k} \left( W_{T-k}, \sigma^2_{T-k} \right) & = \frac{(1 - \xi_{T-k}) \mu}{\gamma (1 + r_f)^{k-1} Q_{T-k}} - \frac{(1 + r_f) (1 - \xi_{T-k}) \mu}{Q_{T-k}} W_{T-k}, \\
\hat{V}_{T-k} \left( W_{T-k}, \sigma^2_{T-k} \right) & = \frac{1}{2\gamma} \hat{S}_{T-k} + (1 + r_f)^k \left( 1 - \hat{S}_{T-k} \right) \left[ W_{T-k} - \frac{1}{2} \gamma (1 + r_f)^k W_{T-k}^2 \right],
\end{align*}
\]

where

\[
\begin{align*}
\xi_{T-k} \left( \sigma^2_{T-k} \right) & = \vartheta_{T-k+1} - \lambda_{T-k+1} \alpha_0 - \lambda_{T-k+1} \left( \alpha_1 + \beta \right) \sigma^2_{T-k}, \\
Q_{T-k} \left( \sigma^2_{T-k} \right) & = \left( 1 - \xi_{T-k} \right) \left( \mu^2 + \sigma^2_{T-k} \right) + 2\lambda_{T-k+1} \alpha_1 \sigma^4_{T-k}, \\
S_{T-k} \left( \sigma^2_{T-k} \right) & = \xi_{T-k} + \frac{(1 - \xi_{T-k})^2 \mu^2}{Q_{T-k}}, \\
\delta_{T-k} & = 1 - \vartheta_{T-k+1} + \lambda_{T-k+1} \alpha_0 + \lambda_{T-k+1} \left( \alpha_1 + \beta \right) E_s, \\
\lambda_{T-k} & = \lambda_{T-k+1} \left( \alpha_1 + \beta \right) + \frac{\mu^2 \delta^2_{T-k} \left[ 4\lambda_{T-k+1} \alpha_1 E_s + \delta_{T-k} + (\mu^2 + E_s) \lambda_{T-k+1} \left( \alpha_1 + \beta \right) \right]}{\left[ (\mu^2 + E_s) \delta_{T-k} + 2\lambda_{T-k+1} \alpha_1 E_s^2 \right]^2} - \frac{2\mu^2 \delta_{T-k} \lambda_{T-k+1} \left( \alpha_1 + \beta \right)}{(\mu^2 + E_s) \delta_{T-k} + 2\lambda_{T-k+1} \alpha_1 E_s^2}, \\
\hat{\vartheta}_{T-k} & = \frac{\mu^2 \delta^2_{T-k}}{(\mu^2 + E_s) \delta_{T-k} + 2\lambda_{T-k+1} \alpha_1 E_s^2} + \lambda_{T-k} E_s, \\
\hat{S}_{T-k} \left( \sigma^2_{T-k} \right) & = \hat{\vartheta}_{T-k} - \lambda_{T-k} \sigma^2_{T-k},
\end{align*}
\]

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with the boundary conditions

\[ \vartheta_T = 0, \]
\[ \lambda_T = 0. \]

**Proof.** We prove the theorem by induction. We have shown that the result is valid for \( k = 1 \). Assume that it is true for \( k \); we shall show that it holds for \( k + 1 \). The value function at time \( T - k - 1 \) is just

\[
V_{T-k-1} \left( W_{T-k-1}, \sigma_{T-k-1}^2 \right) = \max_{\bar{u}_{T-k-1}} E_{T-k-1} \left\{ \frac{1}{2\gamma} \, \hat{S}_{T-k} + \left( 1 + r_f \right)^k \left( 1 - \hat{S}_{T-k} \right) \, W_{T-k} - \frac{1}{2\gamma} \, (1 + r_f)^{2k} \left( 1 - \hat{S}_{T-k} \right) \, W_{T-k}^2 \right\} = \\
\max_{\bar{u}_{T-k-1}} E_{T-k-1} \left\{ \frac{1}{2\gamma} \left( \vartheta_{T-k} - \lambda_{T-k} \, \sigma_{T-k}^2 \right) + \left( 1 + r_f \right)^{k+1} \left( 1 - \vartheta_{T-k} + \lambda_{T-k} \, \sigma_{T-k}^2 \right) \, W_{T-k-1} + \right. \\
\left. \frac{1}{2\gamma} (1 + r_f)^{2k+2} \left( 1 - \vartheta_{T-k} + \lambda_{T-k} \, \sigma_{T-k}^2 \right) \, W_{T-k-1}^2 \right\}. \\
\gamma (1 + r_f)^{2k+1} \left( 1 - \vartheta_{T-k} + \lambda_{T-k} \, \sigma_{T-k}^2 \right) \, W_{T-k-1} \bar{u}_{T-k-1} - \\
\frac{1}{2\gamma} (1 + r_f)^{2k} \left( 1 - \vartheta_{T-k} + \lambda_{T-k} \, \sigma_{T-k}^2 \right) \, \bar{u}_{T-k-1}^2 \right\}. \\
\right. \\
\end{equation}

Using the fact that for a normally distributed random variable \( \epsilon \) with mean zero and variance one, \( E \{ \epsilon^3 \} = 0 \) and \( E \{ \epsilon^4 \} = 3 \), using the variance dynamics based on which

\[ E_{T-k-1} \left\{ \sigma_{T-k}^2 \right\} = \alpha_0 + (\alpha_1 + \beta) \, \sigma_{T-k-1}^2, \]

and letting

\[ \xi_{T-k-1} = \vartheta_{T-k} - \lambda_{T-k} \, \alpha_0 - \lambda_{T-k} \, (\alpha_1 + \beta) \, \sigma_{T-k-1}^2, \]
the value function simplifies to

\[
\max_{\tilde{u}_{T-k-1}} \left\{ \begin{aligned}
\frac{1}{2\gamma} \xi_{T-k-1} + (1 + r_f)^{k+1} (1 - \xi_{T-k-1}) \ W_{T-k-1} - \\
\frac{1}{2} \gamma (1 + r_f)^{2k+2} (1 - \xi_{T-k-1}) \ W_{T-k-1}^2 + \\
(1 + r_f)^k (1 - \xi_{T-k-1}) \ \tilde{u}_{T-k-1} \ \mu - \\
\gamma (1 + r_f)^{2k+1} (1 - \xi_{T-k-1}) \ W_{T-k-1} \ \tilde{u}_{T-k-1} \ \mu - \\
\frac{1}{2} \gamma (1 + r_f)^{2k} (1 - \xi_{T-k-1}) \ \tilde{u}_{T-k-1}^2 \ \mu^2 - \\
\frac{1}{2} \gamma (1 + r_f)^{2k} \left(1 - \vartheta_{T-k} + \lambda_{T-k} \ \alpha_0 + \lambda_{T-k} \ \beta \ \sigma_{T-k-1}^2 \right) \ \tilde{u}_{T-k-1}^3 \ \sigma_{T-k-1}^2 - \\
\frac{3}{2} \gamma (1 + r_f)^{2k} \ \lambda_{T-k} \ \alpha_1 \ \sigma_{T-k-1}^4 \ \tilde{u}_{T-k-1}^2
\end{aligned} \right\}.
\]

The solution to the above optimization problem is given by the first-order condition:

\[
(1 + r_f)^k (1 - \xi_{T-k-1}) \ \mu - \gamma (1 + r_f)^{2k+1} (1 - \xi_{T-k-1}) \ W_{T-k-1} \ \mu - \\
\gamma (1 + r_f)^{2k} (1 - \xi_{T-k-1}) \left(\mu^2 + \sigma_{T-k-1}^2\right) \ \tilde{u}_{T-k-1} - \\
2\gamma (1 + r_f)^{2k} \ \lambda_{T-k} \ \alpha_1 \ \sigma_{T-k-1}^4 \ \tilde{u}_{T-k-1} = 0.
\]

Let

\[
Q_{T-k-1} = (1 - \xi_{T-k-1}) \left(\mu^2 + \sigma_{T-k-1}^2\right) + 2\lambda_{T-k} \ \alpha_1 \ \sigma_{T-k-1}^4.
\]

Then

\[
\tilde{u}_{T-k-1} = \frac{(1 - \xi_{T-k-1}) \ \mu}{\gamma (1 + r_f)^k \ Q_{T-k-1}} - \frac{(1 + r_f) (1 - \xi_{T-k-1}) \ \mu}{Q_{T-k-1}} \ W_{T-k-1}.
\]

Substituting back in the expression of \(V_{T-k-1}\), we obtain

\[
V_{T-k-1} = \frac{1}{2\gamma} \xi_{T-k-1} + (1 + r_f)^{k+1} (1 - \xi_{T-k-1}) \ W_{T-k-1} - \\
\frac{1}{2} \gamma (1 + r_f)^{2k+2} (1 - \xi_{T-k-1}) \ W_{T-k-1}^2 + \frac{(1 - \xi_{T-k-1})^2 \ \mu^2}{\gamma \ Q_{T-k-1}} - \\
\frac{(1 + r_f)^{k+1} (1 - \xi_{T-k-1})^2 \ \mu^2}{Q_{T-k-1}} \ W_{T-k-1} - \frac{(1 + r_f)^k (1 - \xi_{T-k-1})^2 \ \mu^2}{Q_{T-k-1}} \ W_{T-k-1} + \\
\frac{1}{2} \gamma (1 + r_f)^{2k+1} (1 - \xi_{T-k-1}) \ W_{T-k-1}^2 - \frac{(1 - \xi_{T-k-1})^3 \left(\mu^2 + \sigma_{T-k-1}^2\right) \ \mu^2}{2\gamma \ Q_{T-k-1}^2} - \\
\frac{1}{2} \gamma (1 + r_f)^{2k+2} (1 - \xi_{T-k-1})^3 \left(\mu^2 + \sigma_{T-k-1}^2\right) \ \mu^2}{2Q_{T-k-1}^2} \ W_{T-k-1} + \\
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\]
\[ (1 + r_f)^{k+1} \frac{(1 - \xi_{T-k-1})^3 (\mu^2 + \sigma^2_{T-k-1}) \mu^2}{Q^2_{T-k-1}} W_{T-k-1} - \frac{\lambda_{T-k} \alpha_1 \sigma^4_{T-k-1} (1 - \xi_{T-k-1})^2 \mu^2}{\gamma Q^2_{T-k-1}} \]

\[ \frac{\gamma (1 + r_f)^{2k+2} \lambda_{T-k} \alpha_1 \sigma^4_{T-k-1} (1 - \xi_{T-k-1})^2 \mu^2}{Q^2_{T-k-1}} W^2_{T-k-1} + \frac{2 (1 + r_f)^{k+1} \lambda_{T-k} \alpha_1 \sigma^4_{T-k-1} (1 - \xi_{T-k-1})^2 \mu^2}{Q^2_{T-k-1}} W_{T-k-1}. \]

The term that is independent of \( W_{T-k-1} \) can be written as

\[ \frac{1}{2\gamma} \left[ \frac{\xi_{T-k-1} + \frac{2(1-\xi_{T-k-1})^2 \mu^2}{Q^2_{T-k-1}}}{(1-\xi_{T-k-1})^3 (\mu^2 + \sigma^2_{T-k-1}) \mu^2} - \frac{2\lambda_{T-k} \alpha_1 \sigma^4_{T-k-1} (1-\xi_{T-k-1})^2 \mu^2}{Q^2_{T-k-1}} \right] = \frac{1}{2\gamma} \left[ \frac{\xi_{T-k-1} + (1-\xi_{T-k-1})^2 \mu^2}{Q_{T-k-1}} \right]. \]

Performing similar algebraic calculations and letting

\[ S_{T-k-1} = \xi_{T-k-1} + \frac{(1 - \xi_{T-k-1})^2 \mu^2}{Q_{T-k-1}}, \]

we obtain that

\[ V_{T-k-1} = \frac{1}{2\gamma} S_{T-k-1} + (1 + r_f)^{k+1} \left( 1 - S_{T-k-1} \right) \left[ W_{T-k-1} - \frac{1}{2\gamma} (1 + r_f)^{k+1} W^2_{T-k-1} \right]. \]

The approximation step of the algorithm consists of approximating the function \( S_{T-k-1} \) with a linear function in the state variable \( \sigma^2_{T-k-1} \) by using the first-order Taylor's expansion around the unconditional expected value of the asset's variance, \( E_s \). Since \( \xi_{T-k-1} \) is already linear in \( \sigma^2_{T-k-1} \), we need only to approximate the term

\[ \frac{(1 - \xi_{T-k-1})^2 \mu^2}{\xi_{T-k-1}} = \frac{1}{1 - \vartheta_{T-k} + \lambda_{T-k} \alpha_0 + \lambda_{T-k} (\alpha_1 + \beta) \sigma^2_{T-k-1}} \left( \mu^2 + \sigma^2_{T-k-1} \right) + \frac{2\lambda_{T-k} \alpha_1 \sigma^4_{T-k-1}}{\lambda_{T-k} (\alpha_1 + \beta) \sigma^2_{T-k-1}} \left( \mu^2 + \sigma^2_{T-k-1} \right). \]
For convenience, let

$$\delta_{T-k-1} = 1 - \psi_{T-k} + \lambda_{T-k} \alpha_0 + \lambda_{T-k} (\alpha_1 + \beta) E_s.$$

Then,

$$\frac{(1 - \xi_{T-k-1})^2 \mu^2}{Q_{T-k-1}} \approx \frac{\mu^2 \delta_{T-k-1}^2}{(\mu^2 + E_s) \delta_{T-k-1}^2 + 2\lambda_{T-k} \alpha_1 E_s^2 + \frac{2\mu^2 \delta_{T-k-1} \lambda_{T-k} (\alpha_1 + \beta)}{((\mu^2 + E_s) \delta_{T-k-1}^2 + 2\lambda_{T-k} \alpha_1 E_s^2)} - \frac{\mu^2 \delta_{T-k-1}^2}{((\mu^2 + E_s) \delta_{T-k-1}^2 + 2\lambda_{T-k} \alpha_1 E_s^2)^2}} \left(\sigma_{T-k-1}^2 - E_s\right),$$

and by letting

$$\lambda_{T-k-1} = \lambda_{T-k} (\alpha_1 + \beta) + \frac{4\lambda_{T-k} \alpha_1 E_s + \delta_{T-k-1} + \frac{2\mu^2 \delta_{T-k-1} \lambda_{T-k} (\alpha_1 + \beta)}{(\mu^2 + E_s) \delta_{T-k-1}^2 + 2\lambda_{T-k} \alpha_1 E_s^2}}{((\mu^2 + E_s) \delta_{T-k-1}^2 + 2\lambda_{T-k} \alpha_1 E_s^2)^2},$$

$$\psi_{T-k-1} = \frac{\mu^2 \delta_{T-k-1}^2}{((\mu^2 + E_s) \delta_{T-k-1}^2 + 2\lambda_{T-k} \alpha_1 E_s^2)^2} + \lambda_{T-k-1} E_s,$$

we conclude that

$$\tilde{S}_{T-k-1} = \psi_{T-k-1} - \lambda_{T-k-1} \sigma_{T-k-1}^2.$$

Therefore,

$$\tilde{V}_{T-k-1} = \frac{1}{2\gamma} \tilde{S}_{T-k-1} + (1 + r_f)^{k+1} \left(1 - \tilde{S}_{T-k-1}\right) \left[W_{T-k-1} - \frac{1}{2\gamma} (1 + r_f)^{k+1} W_{T-k-1}^2\right].$$

The approximate dynamic policy given by Equation (5.7) is qualitatively quite different from the one given by Equations (5.5)-(5.6) because it depends on the level of wealth: the higher the wealth, the more risk averse the investor becomes and the smaller the investment in the risky asset. Under the exponential utility specification though, the investment decisions do not depend on the manager's wealth. Therefore, we expect the control policies and, as a result, the corresponding risky holdings to deviate significantly.
An alternative approximate policy that does not exhibit dependence on the level of wealth can be derived by simply ignoring the term in Equation (5.7) that depends on wealth, thus given by
\[ \tilde{u}_{T-k} \left( \sigma^2_{T-k} \right) = \frac{(1 - \xi_{T-k}) \mu}{\gamma (1 + r_f)^{k-1} Q_{T-k}}. \] (5.8)

In what follows, we analyze the performance of the proposed investment decision policies and we compare them with the “myopic” one, where investors follow a sequence of optimal single-period investment strategies. The single-period optimization problem can be solved in closed-form, as we have shown earlier in this section. Therefore, the investor that follows the myopic policy performs the following decisions at every time period, \( k = T, \ldots, 1 \) :
\[ u_{T-k}^{\text{static}} \left( \sigma^2_{T-k} \right) = \frac{\mu}{\gamma \sigma^2_{T-k}}, \] (5.9)
which are inversely proportional to the state variable \( \sigma^2_{T-k} \) and independent of the risk-free rate \( r_f \) and of the time to horizon \( k \). The static policy given by equation (5.9) significantly underperforms relative to the proposed approximate dynamic policies, since it is unable to capture the time diversification effect and the conservatism needed in the beginning of the investment horizon.

We also consider an alternative dynamic trading strategy that is a combination of the static policy given by Equation (5.9) and the proposed approximate policy given by Equation (5.7). We will refer to this policy as the expanded static policy. More specifically, we consider the policy given by
\[ u_{T-k}^{E\text{-static}} \left( \sigma^2_{T-k} \right) = \frac{\mu}{\gamma (1 + r_f)^{k-1} \sigma^2_{T-k}}. \] (5.10)
The difference between Equation (5.10) and Equation (5.9) is the term \((1 + r_f)^{k-1}\) in the denominator that causes time diversification and conservatism at the beginning of the investment period. This policy is the one followed by an investor that makes his investment decisions myopically, but faces different initial conditions at every point in time and regards as his time horizon the remaining time to expiration.

In the next section we compare the approximate dynamic policies considered and evaluate their time dependence.
5.2.3 A Numerical Illustration

We consider again the base example, where \( T = 24, \ r_f = 0.05, \ x_0^0 = 1, \ x_0 = 1, \ \gamma = 0.01, \mu = 0.1, \ \alpha_0 = 0.01875, \ \alpha_1 = 0.2 \) and \( \beta = 0.5 \). We simulate the following policies:

1. The approximate dynamic policy (Appr. DP) given by Equations (5.5)-(5.6).

2. The approximate quadratic policy (Quadratic 1) given by Equation (5.7).

3. The alternative quadratic policy (Quadratic 2) given by Equation (5.8).

4. The static policy (Static) described in Equation (5.9).

5. The expanded static policy (E-Static) described in Equation (5.10).

<table>
<thead>
<tr>
<th></th>
<th>Simulated ( V_0 )</th>
<th>Expected ( V_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appr. DP</td>
<td>-0.1475</td>
<td>-0.1611</td>
</tr>
<tr>
<td>Quadratic 1</td>
<td>-1.0864</td>
<td>-0.1356</td>
</tr>
<tr>
<td>Quadratic 2</td>
<td>-0.1518</td>
<td>-</td>
</tr>
<tr>
<td>Static</td>
<td>-1.6145</td>
<td>-</td>
</tr>
<tr>
<td>E-Static</td>
<td>-0.1500</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 5.1: The expected utility of final wealth as given by the simulation experiment.

As it is reported in Table 5.1, the policies with superior performance are the approximate dynamic policy given by (5.5)-(5.6), the alternative quadratic policy given by (5.8) and the expanded static strategy given by (5.10). The best policy found is the approximate dynamic policy that outperforms the expanded static approach by 1.67%.

In Figure 5-9, the expected risky holdings increase monotonically with time for all but the policy given by the approximate quadratic algorithm. As mentioned earlier, the policy given by Equation (5.7) depends linearly on the level of wealth. As time to expiration approaches, more wealth is accumulating and, therefore, the quadratic policy decreases the amount of risky investment. The static policy is the least conservative one with a corresponding expected risky amount that is almost constant over time. The policies given by Equations (5.5)-(5.6), (5.10) and (5.8) are very similar and outperform the static policy. Under these policies, the rate of change in the risky holdings increases over time and thus the investor follows a slightly more aggressive strategy as time progresses.
Figure 5-9: The expected risky holdings $\tilde{u}_t$, plotted as a function of time for the different policies considered. The expectations are taken over 20,000 simulated paths of the risky asset return.

In Figure 5-10, we plot the time dependence of the expected risky investment, $u_t$. As we have seen earlier, the transacted amount in the risky asset $u_t$ relates to the investment holdings $\tilde{u}_t$ by $u_t = \tilde{u}_t - x_t$, where $x_t$ is the holdings in the risky asset before transacting at time $t$. As wealth grows over time there is a tendency, all else equal, for the risky investment to decline, because the CARA utility function in use implies the absence of any wealth effect on the risky holdings. The behavior of all policies, but the static and the quadratic one, is very similar. As time progresses and wealth accumulates, the investment decision is to sell small amounts of the risky asset and lock more wealth to the riskless one. In contrast, the quadratic policy (Quadratic 1) suggests selling more stock in the beginning of the investment horizon and then gradually move towards minimal transaction at the end. On the other hand, the myopic policy is more conservative and suggests transacting heavily out of the stock.

Comparison with the Exact DP Policy
Figure 5-10: The expected risky investment $u_t$, plotted as a function of time for the different policies considered. The expectations are taken over 20,000 simulated paths of the risky asset return.

We compare the performance of the proposed approximate dynamic policy given by (5.5)-(5.6) relative to the optimal strategy (exact) in an instance where exact dynamic programming is feasible. We consider again the base example with a smaller investment horizon $T = 10$. We first present the case of a risky asset with low volatility with $\alpha_0 = 0.01875$ ($E_s = 0.25^2$) and then the case of a high volatility risky asset with $\alpha_0 = 0.147$ ($E_s = 0.70^2$).

In Tables 5.2 and 5.3 we present the resulted utility levels and initial investment decisions for the two cases considered. In both instances, the proposed approximate dynamic policy is almost identical to the optimal policy. As shown in Figure 5-11 the expected risky investment for the approximate and optimal policies are practically identical across time. This is evidence that the proposed dynamic strategy also produces near optimal policies in even larger instances of the problem.
<table>
<thead>
<tr>
<th>$\alpha_0 = 0.01875$</th>
<th>Simulated $V_0$</th>
<th>Expected $V_0$</th>
<th>$\tilde{u}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>-0.4485</td>
<td>-0.4242</td>
<td>96.4754</td>
</tr>
<tr>
<td>Appr. DP</td>
<td>-0.4486</td>
<td>-0.4555</td>
<td>96.9404</td>
</tr>
<tr>
<td>Static</td>
<td>-0.5360</td>
<td>-</td>
<td>160.0000</td>
</tr>
</tbody>
</table>

Table 5.2: Performance comparison between the exact DP and the proposed approximate policy for $E_s = 0.25^2$.

<table>
<thead>
<tr>
<th>$\alpha_0 = 0.147$</th>
<th>Simulated $V_0$</th>
<th>Expected $V_0$</th>
<th>$\tilde{u}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>-0.8596</td>
<td>-0.8669</td>
<td>13.0355</td>
</tr>
<tr>
<td>Appr. DP</td>
<td>-0.8597</td>
<td>-0.8747</td>
<td>13.0490</td>
</tr>
<tr>
<td>Static</td>
<td>-0.8720</td>
<td>-</td>
<td>20.4082</td>
</tr>
</tbody>
</table>

Table 5.3: Performance comparison between the exact DP and the proposed approximate policy for $E_s = 0.70^2$.

Figure 5-11: A comparison between the risky holdings given by the exact DP algorithm and the proposed approximate policy (ADP). We consider a risky asset with low volatility ($E_s = 0.25^2$) and an asset with high volatility ($E_s = 0.70^2$).
Chapter 6

Exponential Utility: Transaction Costs and Factor Models

In the previous chapter we investigated the effect of the asset return process to the investment behavior of an investment manager who maximizes his expected utility of terminal wealth under the assumption that transaction costs are ignored. In this chapter we examine the effect of quadratic transaction costs on dynamic portfolio strategies followed by an investor with CARA utility assuming models that account for lagged correlations in asset returns. We develop iterative solution algorithms that approximate the optimal dynamic trading strategy, and show that an in-depth investigation of the dynamic optimization problem at every point in time enables us to capture essential characteristics of the optimal investment policy and level of utility. We compare the performance of the proposed dynamic trading policies and investigate the impact of transaction costs, risk aversion, return autocorrelation and asset and/or factor volatilities to the investor’s portfolio composition.

In the presence of transaction costs, it does not suffice to fully characterize the state space with the state variable $W_t$, the wealth accumulated at time $t$. We need to differentiate between the holdings of the various assets that comprise the portfolio. We consider an investor who faces the problem of making sequential investment decisions at discrete times $t = 0, \ldots, T - 1$. The evolution of the wealth dynamics is described in Section 1.6. The asset return dynamics
are given by the multifactor pricing model described in Equations (1.1)-(1.2):

\[ r_t = c_t + A_t f_t + \epsilon_t, \]
\[ f_t = d_{t-1} + B_{t-1} f_{t-1} + \eta_t, \]

where \( K \) is the total number of factors, \( r_t \) is the \( N \times 1 \) vector of the rate of returns, \( f_t \) is the \( K \times 1 \) vector of the factor realizations at time \( t \), \( A_t \) is the \( N \times K \) matrix of the factor sensitivities, \( B_{t-1} \) is the \( K \times K \) symmetric matrix of the factor correlations, \( c_t \) and \( d_{t-1} \) are \( N \times 1 \) and \( K \times 1 \) vectors of constants respectively, and \( \epsilon_t, \eta_t \) are uncorrelated normally distributed random vectors with mean zero and covariance matrices \( \Sigma_\epsilon \) and \( \Sigma_\eta \) respectively. Finally, the investor has a CARA utility function with absolute risk aversion parameter equal to \( \gamma \). The state of the system at time \( t = 0, 1, \ldots, T - 1 \) consists of the asset holdings \( (x_t^0, x_t) \) before a transaction is made at time \( t \), and \( f_t \) the factor realizations at time \( t \). The portfolio manager faces the following optimization problem

\[
\max_{\{u_0, \ldots, u_{T-1}\}} E_0 \{ -\exp (-\gamma W_T) \}
\]

subject to

\[
W_t = x_t^0 + x_t' e
\]
\[
= (1 + r_f) \left[ x_{t-1}^0 - e' u_{t-1} - e' \Gamma u_{t-1}^2 \right] + (e + r_t) \left( x_{t-1} + u_{t-1} \right)
\]
\[
r_t = c_t + A_t f_t + \epsilon_t
\]
\[
f_t = d_{t-1} + B_{t-1} f_{t-1} + \eta_t
\]
\[
\epsilon_t \sim N(0, \Sigma_\epsilon) \text{ and } \eta_t \sim N(0, \Sigma_\eta).
\]

The remainder of this chapter is organized as follows. In Section 6.1, we present the single-period optimization problem and its closed-form solution. In Section 6.2, we present a structured approximation that uses characteristics of the optimal cost-to-go function at every point in time. In Section 6.3, we present a quadratic approximation and in Section 6.4, we evaluate the relative performance of the proposed approximation algorithms and obtain insight about the qualitative behavior of the optimal portfolio composition over time.
6.1 The Single Period Problem

In this section we show that there exists a closed-form solution for the single-period problem that is linear in the state variables. The state at time $t = 0, 1, \ldots, T - 1$ consists of the manager’s holdings at time $t$, $(x_t^0, x_t)$ and the factor realizations, $f_t$. We begin by characterizing the optimal value function $V_{T-1}$ by using the boundary condition:

$$V_{T-1}(x_{T-1}^0, x_{T-1}, f_{T-1}) = \max_{u_{T-1}} E_{T-1} \left\{ -\exp \left[ -\gamma \left(1 + r_f\right) \begin{bmatrix} x_{T-1}^0 - e' u_{T-1} - e' \Gamma u_{T-1}^2 \\ e' \Gamma u_{T-1}^2 \\ -\gamma (e + r_T)' (x_{T-1} + u_{T-1}) \end{bmatrix} \right] \right\} =$$

$$\max_{u_{T-1}} E_{T-1} \left\{ -\exp \left[ -\gamma \left(1 + r_f\right) \begin{bmatrix} x_{T-1}^0 - e' u_{T-1} - e' \Gamma u_{T-1}^2 \\ -\gamma (e + c_T + A_T d_{T-1} + A_T B_{T-1} f_{T-1} + A_T \eta_T + e_T)' \\ (x_{T-1} + u_{T-1}) \end{bmatrix} \right] \right\} =$$

$$\max_{u_{T-1}} -\exp \left\{ -\gamma \left(1 + r_f\right) \begin{bmatrix} x_{T-1}^0 - e' u_{T-1} - e' \Gamma u_{T-1}^2 \\ -\gamma (e + c_T + A_T d_{T-1} + A_T B_{T-1} f_{T-1})' (x_{T-1} + u_{T-1}) \\ +\frac{1}{2} \gamma^2 (x_{T-1} + u_{T-1})' A_T \Sigma_\eta A_T' (x_{T-1} + u_{T-1}) \\ +\frac{1}{2} \gamma^2 (x_{T-1} + u_{T-1})' \Sigma_e (x_{T-1} + u_{T-1}) \end{bmatrix} \right\}.$$

The maximization problem is equivalent to

$$\max_{u_{T-1}} \left\{ \gamma \left(1 + r_f\right) \begin{bmatrix} x_{T-1}^0 - e' u_{T-1} - e' \Gamma u_{T-1}^2 \\ e' \Gamma u_{T-1}^2 \\ -\gamma (e + c_T + A_T d_{T-1} + A_T B_{T-1} f_{T-1})' (x_{T-1} + u_{T-1}) \\ +\frac{1}{2} \gamma^2 (x_{T-1} + u_{T-1})' (A_T \Sigma_\eta A_T' + \Sigma_e) (x_{T-1} + u_{T-1}) \end{bmatrix} \right\}.$$

This is a convex optimization problem since the matrix $A_T \Sigma_\eta A_T' + \Sigma_e$ is symmetric positive semi-definite. The first order conditions are necessary and sufficient and given by

$$-\gamma (1 + r_f) e - 2 \gamma (1 + r_f) \Gamma u_{T-1} + \gamma (e + c_T + A_T d_{T-1} + A_T B_{T-1} f_{T-1}) -$$

$$\gamma^2 (A_T \Sigma_\eta A_T' + \Sigma_e) (x_{T-1} + u_{T-1}) = 0.$$
The optimal control, therefore, at time \( T - 1 \) is linear in the state variables \( f_{T-1} \) and \( x_{T-1} \) and independent of \( x_{T-1}^0 \). If we let

\[
Q_{T-1} = \left[ 2 \left( 1 + r_f \right) \Gamma + \gamma A_T \Sigma_\eta \ A_T' + \gamma \Sigma_e \right]^{-1},
\]

\[
m_{T-1} = Q_{T-1} \left( -r_f \ e + c_T + A_T \ d_{T-1} \right),
\]

\[
G_{T-1} = Q_{T-1} A_T \ B_{T-1},
\]

\[
L_{T-1} = Q_{T-1} \left( \gamma A_T \Sigma_\eta \ A_T' + \gamma \Sigma_e \right),
\]

we obtain that the optimal investment decision at time \( T - 1 \) is given by

\[
u_{T-1}^* = m_{T-1} + G_{T-1} f_{T-1} - L_{T-1} \ x_{T-1},
\] (6.1)

where \( Q_{T-1} \) is a \( N \times N \) symmetric matrix, and \( G_{T-1}, L_{T-1} \) are \( N \times K \) and \( N \times N \) matrices respectively. Substituting for the optimal control given by Equation (6.1), we obtain the value function at time \( T - 1 \):

\[
V_{T-1} \left( x^0_{T-1}, x_{T-1}, f_{T-1} \right) =

\begin{bmatrix}
-\gamma \left( 1 + r_f \right) x^0_{T-1} \\
+\gamma \left( 1 + r_f \right) e' \left( m_{T-1} + G_{T-1} f_{T-1} - L_{T-1} \ x_{T-1} \right) \\
+\gamma \left( 1 + r_f \right) e' \Gamma \left( m_{T-1} + G_{T-1} f_{T-1} - L_{T-1} \ x_{T-1} \right)^2 \\
-\exp \left( m_{T-1} + G_{T-1} f_{T-1} - L_{T-1} \ x_{T-1} \right) \\
-\gamma \left( e + c_T + A_T \ d_{T-1} + A_T \ B_{T-1} \ f_{T-1} \right)' \\
\left( m_{T-1} + G_{T-1} f_{T-1} + \left( I - L_{T-1} \right) x_{T-1} \right)' \\
\left( A_T \Sigma_\eta \ A_T' + \Sigma_e \right) \left( m_{T-1} + G_{T-1} f_{T-1} + \left( I - L_{T-1} \right) x_{T-1} \right)
\end{bmatrix}.
\] (6.2)

The function \( V_{T-1} \) is linear in the state variable \( x^0_{T-1} \) and quadratic in the variables \( x_{T-1}, f_{T-1} \).

Using Equation (3.4) we can simplify the above expression to the following.

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\[ V_{T-1} \left( x_{T-1}^0, x_{T-1}, f_{T-1} \right) = - \exp \begin{bmatrix}
- \gamma (1 + r_f) x_{T-1}^0 + x_{T-1} \\
+ p_{T-1}' f_{T-1} + f_{T-1}' H_{T-1} f_{T-1} \\
- b_{T-1}' x_{T-1} + x_{T-1}' C_{T-1} x_{T-1} \\
- f_{T-1}' S_{T-1} x_{T-1}
\end{bmatrix}, \quad (6.3) \]

where

\[
\begin{align*}
 z_{T-1} &= \gamma (1 + r_f) e' m_{T-1} + \gamma (1 + r_f) e' \Gamma m_{T-1}^2 - \gamma (e + c_T + A_T d_{T-1})' m_{T-1} + \\
&\quad \frac{1}{2} \gamma^2 m_{T-1}' (A_T \Sigma \eta A_T' + \Sigma \epsilon) m_{T-1}, \\
p_{T-1} &= \gamma (1 + r_f) G_{T-1} e + 2 \gamma (1 + r_f) G_{T-1}' \Gamma m_{T-1} - \gamma G_{T-1}' (e + c_T + A_T d_{T-1}) - \\
&\quad \gamma B_{T-1}' A_T m_{T-1} + \gamma^2 G_{T-1}' (A_T \Sigma \eta A_T' + \Sigma \epsilon) m_{T-1}, \\
b_{T-1} &= \gamma (1 + r_f) L_{T-1} e + 2 \gamma (1 + r_f) L_{T-1}' \Gamma m_{T-1} + \\
&\quad \gamma (I - L_{T-1})' (e + c_T + A_T d_{T-1}) - \gamma^2 (I - L_{T-1})' (A_T \Sigma \eta A_T' + \Sigma \epsilon) m_{T-1}, \\
H_{T-1} &= \gamma (1 + r_f) G_{T-1}' \Gamma G_{T-1} - \gamma B_{T-1}' A_T G_{T-1} + \\
&\quad \frac{1}{2} \gamma^2 G_{T-1}' (A_T \Sigma \eta A_T' + \Sigma \epsilon) G_{T-1}, \\
C_{T-1} &= \gamma (1 + r_f) L_{T-1} \Gamma L_{T-1} + \frac{1}{2} \gamma^2 (I - L_{T-1})' (A_T \Sigma \eta A_T' + \Sigma \epsilon) (I - L_{T-1}), \\
S_{T-1} &= 2 \gamma (1 + r_f) G_{T-1}' \Gamma L_{T-1} + \gamma B_{T-1}' A_T (I - L_{T-1}) - \\
&\quad \gamma^2 G_{T-1}' (A_T \Sigma \eta A_T' + \Sigma \epsilon) (I - L_{T-1}).
\end{align*}
\]

From the above relations it is evident that the matrices \( H_{T-1} \) and \( C_{T-1} \) are symmetric.

Even though we are able to solve the single-period problem in closed-form, we cannot proceed recursively and solve Bellman's equation for arbitrary times. Nevertheless, in the sections that follow we propose two different classes of approximation algorithms and investigate their behavior.

### 6.2 Approximation A: A Structured Approximation

In this section, we propose an approximation algorithm that utilizes characteristics of the optimal cost-to-go function at every point in time. In order to motivate the algorithm that
follows, consider the value function at time $T - 2$:

$$V_{T-2} \left( x_{T-2}^0, x_{T-2}, f_{T-2} \right) = \max_{u_{T-2}} E_{T-2} \left\{ -\exp \left[ \begin{array}{c}
-\gamma (1 + r_f) x_{T-1}^0 + z_{T-1} \\
+ p'_{T-1} f_{T-1} + f_{T-1}' H_{T-1} f_{T-1} \\
-b'_{T-1} x_{T-1} + x_{T-1}' C_{T-1} x_{T-1} \\
- f'_{T-1} S_{T-1} x_{T-1}
\end{array} \right] \right\}. $$

Substituting for the wealth and factor dynamics, the exponent of $V_{T-2}$ can be written as

$$-\gamma (1 + r_f)^2 \left[ x_{T-2}^0 - e' u_{T-2} - e' \Gamma u_{T-2}^2 \right] + z_{T-1}$$

$$+ p'_{T-1} \left( d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1} \right)$$

$$+ \left( d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1} \right)' H_{T-1} \left( d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1} \right)$$

$$- \left[ b_{T-1} \otimes (e + r_{T-1}) \right]' \left( x_{T-2} + u_{T-2} \right)$$

$$+ (e + r_{T-1})' \{ C_{T-1} \otimes \left[ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right] \} (e + r_{T-1})$$

$$- (d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1})' \{ S_{T-1} \otimes \left[ e \ (x_{T-2} + u_{T-2})' \right] \} (e + r_{T-1}).$$

The value function $V_{T-2}$ is thus given by

$$V_{T-2} \left( x_{T-2}^0, x_{T-2}, f_{T-2} \right) = \ldots$$
\begin{align*}
\max_{\mathbf{u}_{T-2}} E_{T-2} &= -\exp \left\{ -\gamma (1 + r_f)^2 \left[ x_{T-2}^0 - e' \mathbf{u}_{T-2} - e' \mathbf{u}_{T-2}^2 \right] + z_{T-1} \\
+ &p'_{T-1} \left( d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1} \right) \\
+ &\left( d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1} \right)' \mathbf{H}_{T-1} \\
+ &\left( \mathbf{x}_{T-2} + \mathbf{u}_{T-2} \right) \otimes \left( \mathbf{e} + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + \eta_{T-1} + \epsilon_{T-1} \right) \\
- &\left( \mathbf{x}_{T-2} + \mathbf{u}_{T-2} \right)' \mathbf{C}_{T-1} \otimes \left[ \mathbf{x}_{T-2} + \mathbf{u}_{T-2} \right]' \mathbf{S}_{T-1} \otimes \left[ \mathbf{e} \left( \mathbf{x}_{T-2} + \mathbf{u}_{T-2} \right)' \right] \\
+ &\left( \mathbf{e} + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + \eta_{T-1} + \epsilon_{T-1} \right) \\
- &\left( d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1} \right)' \left( \mathbf{S}_{T-1} \otimes \left[ \mathbf{e} \left( \mathbf{x}_{T-2} + \mathbf{u}_{T-2} \right)' \right] \right) \\
- &\left( \mathbf{e} + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + \eta_{T-1} + \epsilon_{T-1} \right) \\
- &\left( \mathbf{e} + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + \eta_{T-1} + \epsilon_{T-1} \right) \\
\right\}.
\end{align*}

By constructing the symmetric matrix \( \mathbf{B} \Theta_{T-1} \) of dimension \( N \times N \) similarly as before:

\[ \mathbf{B} \Theta_{T-1} = \text{diag} \left( \mathbf{b}_{T-1} \right), \]

the exponent in the above relation can be broken into two parts: \( \Phi_1 \), that is independent of the noise terms \( \epsilon_{T-1} \) and \( \eta_{T-1} \) and \( \Phi_2 \), that is quadratic in the noise at time \( T - 1 \), and thus influences the expectation of the exponential function. So, the above relation is equivalent to

\[ V_{T-2} \left( x_{T-2}^0, \mathbf{x}_{T-2}, f_{T-2} \right) = \max_{\mathbf{u}_{T-2}} \left\{ \exp \left[ \Phi_1 \right], E_{T-2} \left\{ \exp \left[ \Phi_2 \right] \right\} \right\}, \quad (6.4) \]

where

\[ \Phi_1 = -\gamma (1 + r_f)^2 \left[ x_{T-2}^0 - e' \mathbf{u}_{T-2} - e' \mathbf{u}_{T-2}^2 \right] + z_{T-1} \]

\[ + p'_{T-1} \left( d_{T-2} + B_{T-2} f_{T-2} \right) \]
\[ + (d_{T-2} + B_{T-2} f_{T-2})' H_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) \]
\[ - (x_{T-2} + u_{T-2})' B \Theta_{T-1} (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \]
\[ + (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \]
\[ \left\{ C_{T-1} \otimes \left[ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right] \right\} \]
\[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \]
\[ - (d_{T-2} + B_{T-2} f_{T-2})' \left\{ S_{T-1} \otimes \left[ e (x_{T-2} + u_{T-2})' \right] \right\} \]
\[ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \],

that can be written for notation convenience as

\[ \Phi_1(f, \tilde{x}) = a + b' f + c' \tilde{x} + f' A_1 f + f' A_2 \tilde{x} + \tilde{x}' A_3 \tilde{x} + L_1(f, \tilde{x}) + L_2(f, \tilde{x}), \]

for \( \tilde{x} = x + u \). Note that \( L_1, L_2 \) are third and fourth order polynomials with respect to the state variables. Moreover,

\[ \Phi_2 = \left( \begin{array}{c}
 p_{T-1} + 2 H_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - \\
 A'_{T-1} B \Theta_{T-1} (x_{T-2} + u_{T-2}) + \\
 2 A'_{T-1} \left\{ C_{T-1} \otimes \left[ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right] \right\} \\
 (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) - \\
 A'_{T-1} \left\{ S_{T-1} \otimes \left[ e (x_{T-2} + u_{T-2})' \right] \right\}' (d_{T-2} + B_{T-2} f_{T-2}) - \\
 \left\{ S_{T-1} \otimes \left[ e (x_{T-2} + u_{T-2})' \right] \right\}' (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \\
 \left( -B \Theta_{T-1} (x_{T-2} + u_{T-2}) + \\
 2 \left\{ C_{T-1} \otimes \left[ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right] \right\} \\
 (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) - \\
 \left\{ S_{T-1} \otimes \left[ e (x_{T-2} + u_{T-2})' \right] \right\}' (d_{T-2} + B_{T-2} f_{T-2}) \\
 \end{array} \right) \]

\[ \eta_{T-1} + \]

\[ \left( \begin{array}{c}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 \end{array} \right) \]

\[ \epsilon_{T-1} + \]

\[ \left( \begin{array}{c}
 H_{T-1} + A'_{T-1} \left\{ C_{T-1} \otimes \left[ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right] \right\} \\
 A_{T-1} - \left\{ S_{T-1} \otimes \left[ e (x_{T-2} + u_{T-2})' \right] \right\} A_{T-1} \\
 \left\{ C_{T-1} \otimes \left[ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right] \right\} \end{array} \right) \]

\[ \eta_{T-1} + \]

\[ \epsilon_{T-1} + \]

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\[
\eta'_{T-1} \left( \frac{2A'_{T-1} \{ C_{T-1} \otimes \{ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \} \} - \{ S_{T-1} \otimes \{ e (x_{T-2} + u_{T-2})' \} \}}{e_{T-1}} \right)
\]

For notation convenience we rewrite \( \Phi_2 \) as

\[
\Phi_2 = g' \eta + q' \epsilon + \eta' A \eta + \epsilon' B \epsilon + \eta' C \epsilon,
\]

(6.5)

where \( g, q, A, B, C \) are independent of \( \eta, \epsilon \), but depend on \( (f, x + u) \). Applying Proposition 5.1 we obtain the desired \( E \left[ e^{\Phi_2} \right] \) by the following:

**Lemma 6.1** For two normally distributed random vectors \( \eta \) and \( \epsilon \) the following result is true:

\[
E \left[ e^{\Phi_2} \right] = \frac{1}{|I - 2\Sigma_\epsilon B|^{1/2}} \frac{1}{|I - 2\Sigma_\eta (A + \frac{1}{2} C \Lambda_\epsilon C')|^{1/2}} \exp \left[ \frac{1}{2} q' \Lambda_\epsilon q + \frac{1}{2} (g + C \Lambda_\epsilon q)' \Lambda_\eta (g + C \Lambda_\epsilon q) \right],
\]

(6.6)

where

\[
\Lambda_\epsilon = [\Sigma_\epsilon^{-1} - 2B]^{-1}, \quad \Lambda_\eta = [\Sigma_\eta^{-1} - 2(A + \frac{1}{2} C \Lambda_\epsilon C')]^{-1}.
\]

**Proof.** By conditioning first on \( \epsilon \) and then on \( \eta \) and using Proposition 5.1 we obtain

\[
E \left[ e^{\Phi_2} \right] = E_\eta \left[ \exp \left( g' \eta + \eta' A \eta \right) E_\epsilon \left[ (q + C' \eta)' \epsilon + \epsilon' B \epsilon \right] \right]
\]

\[
= \sqrt{|\Lambda_\epsilon| \Sigma_\epsilon} E_\eta \left[ \exp \left( g' \eta + \eta' A \eta + \frac{1}{2} (q + C' \eta)' \Lambda_\epsilon (q + C' \eta) \right) \right]
\]

\[
= \sqrt{|\Lambda_\epsilon| \Sigma_\epsilon} \exp \left( \frac{1}{2} q' \Lambda_\epsilon q \right)
\]

\[
E_\eta \left[ \exp \left( (g + C \Lambda_\epsilon q)' \eta + \eta' \left( A + \frac{1}{2} C \Lambda_\epsilon C' \right) \eta \right) \right]
\]

\[
= \sqrt{|\Lambda_\epsilon| \Sigma_\epsilon} \sqrt{|\Lambda_\eta| \Sigma_\eta} \exp \left( \frac{1}{2} q' \Lambda_\epsilon q + \frac{1}{2} (g + C \Lambda_\epsilon q)' \Lambda_\eta (g + C \Lambda_\epsilon q) \right)
\]

and the result follows. \( \blacksquare \)

The DP recursion is solvable when the exponent in the value function is quadratic in the state variables (see Equation (6.3)). Unfortunately, when substituting for the wealth and return
dynamics in Equation (6.4) we obtain third and fourth order terms with respect to $f$ and $\tilde{x} = (x + u)$ of the type:

$$L_1 (f, \tilde{x}) = (d + B f)' \left( S \otimes e \tilde{x}' \right) (c + D f),$$

$$L_2 (f, \tilde{x}) = (c + D f)' \left( C \otimes \tilde{x} \tilde{x}' \right) (c + D f).$$

Our proposed approximation is motivated by the need to keep the quadratic nature of the exponent, i.e. we want to approximate $L_1 (f, \tilde{x}), L_2 (f, \tilde{x})$ by quadratic functions in $f, \tilde{x}:

- For $L_1$, we achieve this by performing a Taylor's series expansion around the conditional expectation of $f$ defined in Equation (3.9) as $E_f$, keeping up to second order terms.

- For $L_2$, we achieve this by replacing $f$ by $E_f$.

Regarding the terms in Equation (6.5), we perform the following operations:

- We approximate the matrices $A(\tilde{x}), B(\tilde{x}), C(\tilde{x})$ by $A(x_0), B(x_0), C(x_0)$ replacing $\tilde{x}$ by the initial risky holdings $x_0$.

- We approximate the vectors $g(f, \tilde{x})$ and $q(f, \tilde{x})$ with linear function in the state variables.

The details of the approximation procedure are presented below.

**Approximations Performed in $\Phi_1$**

The approximations performed in $\Phi_1$ are identical to the approximations performed in the case of a quadratic utility function. More specifically, consider the term $L_1$ in $\Phi_1$

$$L_1 = (d_{T-2} + B_{T-2} f_{T-2})' \left\{ S_{T-1} \otimes \left[ e \left( x_{T-2} + u_{T-2} \right)' \right] \right\} (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) +
\begin{align*}
&d_{T-2}' B'_{T-2} \left\{ S_{T-1} \otimes \left[ e \left( x_{T-2} + u_{T-2} \right)' \right] \right\} (e + c_{T-1} + A_{T-1} d_{T-2}) +
&f_{T-2}' B'_{T-2} \left\{ S_{T-1} \otimes \left[ e \left( x_{T-2} + u_{T-2} \right)' \right] \right\} A_{T-1} B_{T-2} f_{T-2},
\end{align*}$$

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which is identical to the corresponding term appearing in Section 3.2. Using the same analysis as in Section 3.2, we obtain the following approximation formula for $L_1$:

\[
L_1 \approx (x_{T-2} + u_{T-2})' \left\{ S_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \right\} + \\
(x_{T-2} + u_{T-2})' \left\{ S_{T-1} B_{T-2} f_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \right\} + \\
(x_{T-2} + u_{T-2})' \left\{ S_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} (f_{T-2} - E_{f,T-2}) \right\},
\]

(6.7)

thus observing that now the coefficient of $(x_{T-2} + u_{T-2})$ is linear in $f_{T-2}$.

The second approximating operation performed involves the term $L_2$:

\[
L_2 = (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \left\{ C_{T-1} \otimes \left[ \begin{array}{c}
(x_{T-2} + u_{T-2}) \\
(x_{T-2} + u_{T-2})'
\end{array} \right] \right\}
\]

\[
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})
\]

that is quadratic in $(x_{T-2} + u_{T-2})$. Again, we perform the same approximation as done in Section 3.2. We replace $f_{T-2}$ with its conditional expectation $E_{f,T-2}$ and thus approximate $L_2$ with

\[
L_2 \approx (x_{T-2} + u_{T-2})' \\
\left\{ C_{T-1} \otimes \left( \begin{array}{c}
e + c_{T-1} + A_{T-1} d_{T-2}+ \\
A_{T-1} B_{T-2} E_{f,T-2}
\end{array} \right) \right\} \\
(x_{T-2} + u_{T-2}).
\]

(6.8)

As a result, $\Phi_1$ can be approximated by

\[
\Phi_1 \approx -\gamma (1 + r_f)^2 \left[ x_{T-2}^0 - e' u_{T-2} - e' \Gamma u_{T-2}^2 \right] + z_{T-1} \\
+ p_{T-1}' (d_{T-2} + B_{T-2} f_{T-2}) \\
+ (d_{T-2} + B_{T-2} f_{T-2})' H_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) \\
- (x_{T-2} + u_{T-2})' B \Theta_{T-1} (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \\
+ L_2 - L_1.
\]

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Approximations Performed in $\Phi_2$

The second class of approximations performed concerns $\Phi_2$. We begin by approximating the coefficient of $\eta_{T-1}$ with a linear and separable function in $(x_{T-2} + u_{T-2})$ and $f_{T-2}$ by using the first-order Taylor’s expansion around $x_0$ and $E_{f_{T-2}}$. We concentrate on the term $L_3$

$$
L_3 = 2 \ A'_{T-1} \ \left\{ C_{T-1} \otimes \left[ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right] \right\} \\
(\ e + c_{T-1} + A_{T-1} \ d_{T-2} + A_{T-1} B_{T-2} \ f_{T-2}) - \\
A'_{T-1} \ \left\{ S_{T-1} \otimes \left[ e (x_{T-2} + u_{T-2})' \right] \right\} \ (d_{T-2} + B_{T-2} f_{T-2}) - \\
\left\{ S_{T-1} \otimes \left[ e (x_{T-2} + u_{T-2})' \right] \right\} \ (e + c_{T-1} + A_{T-1} \ d_{T-2} + A_{T-1} B_{T-2} \ f_{T-2}) - \\
2 \ A'_{T-1} \ \left\{ C_{T-1} \otimes (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right\} \ (e + c_{T-1} + A_{T-1} \ d_{T-2}) + \\
2 \ A'_{T-1} \ \left\{ C_{T-1} \otimes \left[ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right] \right\} A_{T-1} B_{T-2} f_{T-2} - \\
A'_{T-1} \ \left\{ (x_{T-2} + u_{T-2})' e' \right\} d_{T-2} - \\
A'_{T-1} \ \left\{ (x_{T-2} + u_{T-2})' e' \right\} B_{T-2} f_{T-2} - \\
\left\{ S_{T-1} \otimes \left[ e (x_{T-2} + u_{T-2})' \right] \right\} \ (e + c_{T-1} + A_{T-1} \ d_{T-2}) - \\
\left\{ S_{T-1} \otimes \left[ e (x_{T-2} + u_{T-2})' \right] \right\} A_{T-1} B_{T-2} f_{T-2},
$$

that is a function of $(x_{T-2} + u_{T-2})$ and $f_{T-2}$. Once again, the approximations performed are similar to the corresponding ones of Section 3.2. Next we prove a formula necessary for the approximation procedure.

**Lemma 6.2** For arbitrary matrices $S$, $B$ and vectors $x$, $f$ of dimensions $(K \times N)$, $(K \times K)$, $(N \times 1)$ and $(K \times 1)$ respectively, the first-order Taylor’s expansion of $\{S' \otimes x \ e'\}Bf$ around $x_0$ and $f_E$ can be written as

$$
\{S' \otimes x_0 \ e'\} \ B f_E + SE (x - x_0) + \{S' \otimes x_0 \ e'\} \ B (f - f_E),
$$

where the matrix $SE$ is constructed as follows for $i = 1, \ldots, N$

$$
SE = \text{diag} [S' \ B \ f_E].
$$
Proof. We proceed similarly as before. We write \(\{S' \otimes x \ e'\} Bf\) as

\[
\begin{bmatrix}
S_{11} x_1 (B_{11} f_1 + \ldots + B_{1K} f_K) + \ldots + S_{K1} x_1 (B_{K1} f_1 + \ldots + B_{KK} f_K) \\
\vdots \\
S_{1N} x_N (B_{11} f_1 + \ldots + B_{1K} f_K) + \ldots + S_{KN} x_N (B_{K1} f_1 + \ldots + B_{KK} f_K)
\end{bmatrix}.
\]

We must construct a matrix with rows the gradients with respect to \(x\) of every element of the above vector. Consider the matrix \(SE\). Its \(ii\)-th element is the \(i\)-th element of the vector

\[
\begin{bmatrix}
S_{11} (B_{11} f_{E,1} + \ldots + B_{1K} f_{E,K}) + \ldots + S_{K1} (B_{K1} f_{E,1} + \ldots + B_{KK} f_{E,K}) \\
\vdots \\
S_{1N} (B_{11} f_1 + \ldots + B_{1K} f_K) + \ldots + S_{KN} (B_{K1} f_1 + \ldots + B_{KK} f_K)
\end{bmatrix}
\]

and the result follows. ■

Using the above formula and Lemmas 3.3 and 3.4, we can now approximate \(L_3\) with

\[
L_3 \approx 2 A'_{T-1} \{C_{T-1} \otimes x_0 \ x'_0\} (e + c_{T-1} + A_{T-1} \ d_{T-2} + A_{T-1}B_{T-2} E_{f,T-2}) - \\
A'_{T-1} \{S'_{T-1} \otimes x_0 \ e'\} (d_{T-2} + B_{T-2} E_{f,T-2}) - \\
\{S_{T-1} \otimes e \ x'_0\} (e + c_{T-1} + A_{T-1} \ d_{T-2} + A_{T-1}B_{T-2} E_{f,T-2}) + \\
2 A'_{T-1} \left\{ \frac{C_{T-1} \otimes e \ (e + c_{T-1} + A_{T-1} \ d_{T-2})'}{CD_{T-2}} \otimes x_0 \ e' \right\} + (x_{T-2} + u_{T-2} - x_0) + \\
2 A'_{T-1} \left\{ \frac{C_{T-1} \otimes e \ (A_{T-1}B_{T-2} E_{f,T-2})'}{CE_{T-2}} \otimes x_0 \ e' \right\} + (x_{T-2} + u_{T-2} - x_0) + \\
2 A'_{T-1} \{C_{T-1} \otimes x_0 \ x'_0\} A_{T-1}B_{T-2} (f_{T-2} - E_{f,T-2}) - \\
A'_{T-1} SD_{T-2} (x_{T-2} + u_{T-2} - x_0) - \\
A'_{T-1} SE_{T-2} (x_{T-2} + u_{T-2} - x_0) - A'_{T-1} \left\{ \{S'_{T-1} \otimes x_0 \ e'\} B_{T-2} (f_{T-2} - E_{f,T-2}) - \\
\{S_{T-1} \otimes e \ (e + c_{T-1} + A_{T-1} \ d_{T-2})'\} (x_{T-2} + u_{T-2} - x_0) - \\
\{S_{T-1} \otimes e \ (A_{T-1}B_{T-2} E_{f,T-2})'\} (x_{T-2} + u_{T-2} - x_0) - \\
\{S_{T-1} \otimes e \ x'_0\} A_{T-1}B_{T-2} (f_{T-2} - E_{f,T-2}) ,
\]

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where the matrices $\mathbf{CD}_{T-2}$, $\mathbf{CE}_{T-2}$, $\mathbf{SD}_{T-2}$, and $\mathbf{SE}_{T-2}$ are diagonal with $ii$-th elements given by

\[
\begin{align*}
\mathbf{CD}_{T-2} &= \text{diag} \left[ \{C_{T-1} \otimes e \ x'_0\} \ (e + c_{T-1} + A_{T-1} \ d_{T-2}) \right], \\
\mathbf{CE}_{T-2} &= \text{diag} \left[ \{C_{T-1} \otimes e \ x'_0\} \ A_{T-1} \mathbf{B}_{T-2} \mathbf{E}_{f,T-2} \right], \\
\mathbf{SD}_{T-2} &= \text{diag} \left[ S'_{T-1} \ d_{T-2} \right], \\
\mathbf{SE}_{T-2} &= \text{diag} \left[ S'_{T-1} \ \mathbf{B}_{T-2} \mathbf{E}_{f,T-2} \right].
\end{align*}
\]

Let

\[
\begin{align*}
\mathbf{a}_{1T-2} &= 2 \left\{ C_{T-1} \otimes x_0 \ x'_0 \right\} \ (e + c_{T-1} + A_{T-1} \ d_{T-2}) - \left\{ S'_{T-1} \otimes x_0 \ e' \right\} \ d_{T-2} - \\
&\quad 2 \left[ \left\{ C_{T-1} \otimes e \ (e + c_{T-1} + A_{T-1} \ d_{T-2})' \otimes x_0 \ e' \right\} + \mathbf{CD}_{T-2} \right] x_0 - \\
&\quad 2 \left[ \left\{ C_{T-1} \otimes e \ (A_{T-1} \mathbf{B}_{T-2} \mathbf{E}_{f,T-2})' \otimes x_0 \ e' \right\} + \mathbf{CE}_{T-2} \right] x_0 + \\
&\quad \mathbf{SD}_{T-2} x_0 + \mathbf{SE}_{T-2} x_0, \\
\mathbf{a}_{2T-2} &= A'_{T-1} \ \mathbf{a}_{1T-2} - \left\{ S_{T-1} \otimes e \ x'_0 \right\} \ (e + c_{T-1} + A_{T-1} \ d_{T-2}) + \\
&\quad \left\{ S_{T-1} \otimes e \ (e + c_{T-1} + A_{T-1} \ d_{T-2})' \right\} x_0 + \\
&\quad \left\{ S_{T-1} \otimes e \ (A_{T-1} \mathbf{B}_{T-2} \mathbf{E}_{f,T-2})' \right\} x_0, \\
\Psi_{1T-2} &= 2 \left[ \left\{ C_{T-1} \otimes e \ (e + c_{T-1} + A_{T-1} \ d_{T-2})' \otimes x_0 \ e' \right\} + \mathbf{CD}_{T-2} \right] + \\
&\quad 2 \left[ \left\{ C_{T-1} \otimes e \ (A_{T-1} \mathbf{B}_{T-2} \mathbf{E}_{f,T-2})' \otimes x_0 \ e' \right\} + \mathbf{CE}_{T-2} \right] - \\
&\quad \mathbf{SD}_{T-2} - \mathbf{SE}_{T-2}, \\
\Psi_{2T-2} &= A'_{T-1} \ \Psi_{1T-2} - \left\{ S_{T-1} \otimes e \ (e + c_{T-1} + A_{T-1} \ d_{T-2})' \right\} - \\
&\quad \left\{ S_{T-1} \otimes e \ (A_{T-1} \mathbf{B}_{T-2} \mathbf{E}_{f,T-2})' \right\}, \\
\Omega_{1T-2} &= 2 \left\{ C_{T-1} \otimes x_0 \ x'_0 \right\} \ A_{T-1} \mathbf{B}_{T-2} - \left\{ S'_{T-1} \otimes x_0 \ e' \right\} \ \mathbf{B}_{T-2}, \\
\Omega_{2T-2} &= A'_{T-1} \ \Omega_{1T-2} - \left\{ S_{T-1} \otimes e \ x'_0 \right\} \ A_{T-1} \mathbf{B}_{T-2}.
\end{align*}
\]

Then,

\[
\mathcal{L}_3 \approx \mathbf{a}_{2T-2} + \Psi_{2T-2} \ (x_{T-2} + u_{T-2}) + \Omega_{2T-2} \ f_{T-2},
\]

and thus $\mathcal{L}_3$ is linear in $(x_{T-2} + u_{T-2})$ and $f_{T-2}$ as desired.

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In addition, the term $L_4$ in the coefficient of $\varepsilon_{T-1}$,

$$L_4 = 2 \left\{ C_{T-1} \otimes (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right\} \left( \begin{array}{c} e + c_{T-1} + A_{T-1} \ d_{T-2} + \\ A_{T-1}, B_{T-2} \ f_{T-2} \end{array} \right) - \left\{ S_{T-1} \otimes \left[ e \ (x_{T-2} + u_{T-2})' \right] \right\} ' \ (d_{T-2} + B_{T-2} \ f_{T-2}),$$

can be approximated by

$$L_4 \approx a_1 T_{-2} + \Psi_1 T_{-2} \ (x_{T-2} + u_{T-2}) + \Omega_1 T_{-2} \ f_{T-2}.$$

The final approximation step performed concerns the terms $\eta_{T-1} \ [\cdot] \ \eta_{T-1}, \ \epsilon_{T-1} \ [\cdot] \ \epsilon_{T-1}$ and $\eta_{T-1} \ [\cdot] \ \epsilon_{T-1}$ appearing in $\Phi_2$. We replace $(x_{T-2} + u_{T-2})$ with $x_0$ and thus make them independent of the state at time $T - 1$. More specifically, if we let

$$L_5 = A'_{T-1} \left\{ C_{T-1} \otimes \left[ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right] \right\} A_{T-1} - \left\{ S_{T-1} \otimes \left[ e \ (x_{T-2} + u_{T-2})' \right] \right\} A_{T-1},$$

$$L_6 = C_{T-1} \otimes \left[ (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right],$$

then the approximation yields

$$L_5 \approx A'_{T-1} \left\{ C_{T-1} \otimes x_0 \ x_0' \right\} A_{T-1} - \{ S_{T-1} \otimes e \ x_0 \} A_{T-1},$$

$$L_6 \approx C_{T-1} \otimes x_0 \ x_0'.$$

As a result, the value function at time $T - 2$ can be approximated with

$$V_{T-2} (x_{T-2}^0, x_{T-2}, f_{T-2}) = \max_{u_{T-2}} - \exp \left[ \tilde{\Phi}_1 \right] E_{T-2} \{ \exp \left[ \tilde{\Phi}_2 \right] \},$$

where

$$\tilde{\Phi}_1 = -\gamma (1 + r_f) \left[ x_{T-2}^0 - e' u_{T-2} - e' \ \Gamma \ u_{T-2}' \right] + z_{T-1} + p'_{T-1} \ (d_{T-2} + B_{T-2} \ f_{T-2})$$

and

$$\tilde{\Phi}_2 = \frac{e_{T-2}}{x_{T-2}^0}.$$
\[
+ (d_{T-2} + B_{T-2} f_{T-2})' H_{T-1} \ (d_{T-2} + B_{T-2} f_{T-2}) \\
- (x_{T-2} + u_{T-2})' B \Theta_{T-1} \ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \\
+ (x_{T-2} + u_{T-2})' \left\{ C_{T-1} \otimes \begin{pmatrix}
(e + c_{T-1}) + \\
A_{T-1} d_{T-2} + \\
A_{T-1} B_{T-2} E_{f,T-2}
\end{pmatrix}
\right\}' \\
(x_{T-2} + u_{T-2}) \\
- (x_{T-2} + u_{T-2})' \left\{ S_{T-1}' d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \right\} \\
- (x_{T-2} + u_{T-2})' \left\{ S_{T-1}' B_{T-2} f_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \right\} \\
- (x_{T-2} + u_{T-2})' \left\{ S_{T-1}' B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} (f_{T-2} - E_{f,T-2}) \right\},
\]

and using Lemma 6.1 we obtain that

\[
E_{T-2} \left\{ \exp \left[ \Phi_2 \right] \right\} = \sqrt{\frac{\Lambda \epsilon_{T-2} \Sigma_{\epsilon}}{\Sigma_{\epsilon} |\Sigma_{\epsilon}|}} \sqrt{\frac{\Lambda \eta_{T-2} \Sigma_{\eta}}{\Sigma_{\eta} |\Sigma_{\eta}|}} \\
\exp \left\{ \frac{1}{2} (a_1 T-2 + \Omega 1 T-2 \ f_{T-2} + [-B \Theta_{T-1} + \Psi 1 T-2] (x_{T-2} + u_{T-2}))' \Lambda \epsilon_{T-2} \right\} \\
\exp \left\{ \frac{1}{2} \begin{pmatrix}
[p_{T-1} + 2 H_{T-1} d_{T-2} + a_2 T-2 + \Theta_{T-2} \Lambda \epsilon_{T-2} \ a_1 T-2] + \\
[2 H_{T-1} B_{T-2} + \Omega 2 T-2 + \Theta_{T-2} \Lambda \epsilon_{T-2} \ \Omega 1 T-2] f_{T-2} + \\
-\Lambda T-1 B \Theta_{T-1} + \Psi 2 T-2 - \\
\Theta_{T-2} \Lambda \epsilon_{T-2} B \Theta_{T-1} + \\
\Theta_{T-2} \Lambda \epsilon_{T-2} \ \Psi 1 T-2
\end{pmatrix}' \Lambda \eta_{T-2} \\
(x_{T-2} + u_{T-2}) \right\} \\
\right\} \\
\exp \left\{ \frac{1}{2} \begin{pmatrix}
[p_{T-1} + 2 H_{T-1} d_{T-2} + a_2 T-2 + \Theta_{T-2} \Lambda \epsilon_{T-2} \ a_1 T-2] + \\
[2 H_{T-1} B_{T-2} + \Omega 2 T-2 + \Theta_{T-2} \Lambda \epsilon_{T-2} \ \Omega 1 T-2] f_{T-2} + \\
-\Lambda T-1 B \Theta_{T-1} + \Psi 2 T-2 - \\
\Theta_{T-2} \Lambda \epsilon_{T-2} B \Theta_{T-1} + \Theta_{T-2} \Lambda \epsilon_{T-2} \ \Psi 1 T-2
\end{pmatrix}' (x_{T-2} + u_{T-2}) \right\},
\]

where

\[
\Lambda \eta_{T-2} = \left[ \Sigma_{\eta}^{-1} - 2 \begin{pmatrix}
H_{T-1} + A'_{T-1} \ {C}_{T-1} \otimes x_0 \ x_0' \ A_{T-1} - \\
\{ S_{T-1} \otimes e \ x_0' \} A_{T-1} + \frac{1}{2} \Theta_{T-2} \Lambda \epsilon_{T-2} \ \Theta'_{T-2}
\end{pmatrix} \right]^{-1}
\]
\[ \Lambda \varepsilon_{T-2} = \left[ \Sigma^{-1}_c - 2 \left( C_{T-1} \otimes x_0 \ x'_0 \right) \right]^{-1}, \]
\[ \Theta_{T-2} = 2 \mathbf{A}'_{T-1} \left\{ C_{T-1} \otimes x_0 \ x'_0 \right\} - \left\{ S_{T-1} \otimes e \ x'_0 \right\}. \]

The solution to the optimization problem defined by \( V_{T-2} \) is given by

\[ -\gamma \left( 1 + r_f \right)^2 e - 2 \gamma \left( 1 + r_f \right)^2 \Gamma u_{T-2} + \]
\[ \mathbf{B} \Theta_{T-1} (e + c_{T-1} + A_{T-1} \ d_{T-2} + A_{T-1} \ B_{T-2} \ f_{T-2}) - \]
\[ 2 \left\{ C_{T-1} \otimes \left( \begin{array}{c} e + c_{T-1}+ \\ A_{T-1} \ d_{T-2}+ \\ A_{T-1} B_{T-2} E_{f,T-2} \end{array} \right) \right\}' \left( \begin{array}{c} e + c_{T-1}+ \\ A_{T-1} \ d_{T-2}+ \\ A_{T-1} B_{T-2} E_{f,T-2} \end{array} \right) \left\{ S'_{T-1} \ d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \right\} + \]
\[ \left\{ S'_{T-1} \ B_{T-2} f_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \right\} + \]
\[ \left\{ S'_{T-1} \ B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} \right\} \left( f_{T-2} - E_{f,T-2} \right) \right\} - \]
\[ \left[ -A'_{T-1} \ B \Theta_{T-1} + \Psi 2_{T-2} - \Theta_{T-2} \ \Lambda \varepsilon_{T-2} \ B \Theta_{T-1} + \Theta_{T-2} \ \Lambda \varepsilon_{T-2} \ \Psi 1_{T-2} \right]' \ \Lambda \eta_{T-2} \]
\[ \left( \begin{array}{c} p_{T-1} + 2 \ H_{T-1} \ d_{T-2} + a 2_{T-2} + \Theta_{T-2} \ \Lambda \varepsilon_{T-2} \ a 1_{T-2} \right) + \]
\[ 2 \ H_{T-1} \ B_{T-2} + \Omega 2_{T-2} + \Theta_{T-2} \ \Lambda \varepsilon_{T-2} \ \Omega 1_{T-2} \right] f_{T-2} + \]
\[ \left\{ \begin{array}{c} -A'_{T-1} \ B \Theta_{T-1} + \Psi 2_{T-2} - \\ \Theta_{T-2} \ \Lambda \varepsilon_{T-2} \ B \Theta_{T-1} + \Theta_{T-2} \ \Lambda \varepsilon_{T-2} \ \Psi 1_{T-2} \end{array} \right\}' \left( x_{T-2} + u_{T-2} \right) \]
\[ \left[ -B \Theta_{T-1} + \Psi 1_{T-2} \right]' \ \Lambda \varepsilon_{T-2} \left( a 1_{T-2} + \Omega 1_{T-2} \ f_{T-2} + \right) \]
\[ \left[ -B \Theta_{T-1} + \Psi 1_{T-2} \right] \left( x_{T-2} + u_{T-2} \right) = 0. \]
The approximate control, therefore, at time $T-2$ is linear in the state variables $f_{T-2}$ and $x_{T-2}$.

If we let

$$Q_{T-2} = \begin{bmatrix}
2\gamma \left(1 + r_f\right)^2 \Gamma + \\
\begin{bmatrix}
2 \begin{bmatrix}
C_{T-1} \otimes \begin{bmatrix} e + c_{T-1} + \\
A_{T-1} d_{T-2} + \\
A_{T-1} B_{T-2} E_{f,T-2}
\end{bmatrix}
\end{bmatrix}^\prime
\end{bmatrix} + \\
\begin{bmatrix}
-A_{T-1} \Theta_{T-1} + \Psi 2_{T-2} - \Theta_{T-2} \Lambda \epsilon_{T-2} B \Theta_{T-1} + \\
\Theta_{T-2} \Lambda \epsilon_{T-2} \Psi 1_{T-2}
\end{bmatrix}
\end{bmatrix}^{-1}
$$

$$m_{T-2} = Q_{T-2}$$

$$G_{T-2} = Q_{T-2}$$

$$p_{T-1} + 2 H_{T-1} d_{T-2} + a2_{T-2} + \Theta_{T-2} \Lambda \epsilon_{T-2} a1_{T-2}$$

$$B \Theta_{T-1} A_{T-1} B_{T-2} + \begin{bmatrix}
S_{T-1} d_{T-2} \otimes \begin{bmatrix} e' + A_{T-1} B_{T-2} \end{bmatrix}
\end{bmatrix}^\prime + \\
\begin{bmatrix}
S_{T-1} B_{T-2} \otimes \begin{bmatrix} e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2} \end{bmatrix}
\end{bmatrix}^\prime + \\
\begin{bmatrix}
S_{T-1} B_{T-2} E_{f,T-2} \otimes \begin{bmatrix} e' + A_{T-1} B_{T-2} \end{bmatrix}
\end{bmatrix}^\prime - \\
\begin{bmatrix}
-A_{T-1} \Theta_{T-1} + \Psi 2_{T-2} - \Theta_{T-2} \Lambda \epsilon_{T-2} B \Theta_{T-1} + \\
\Theta_{T-2} \Lambda \epsilon_{T-2} \Psi 1_{T-2}
\end{bmatrix}
\end{bmatrix}$$

$$2 H_{T-1} B_{T-2} + \Omega 2_{T-2} + \Theta_{T-2} \Lambda \epsilon_{T-2} \Omega 1_{T-2}$$
\[ L_{T-2} = Q_{T-2} \left[ \begin{array}{c}
2 \left\{ C_{T-1} \otimes \begin{pmatrix} e + c_{T-1}+ \\ A_{T-1}d_{T-2}+ \\ A_{T-1}B_{T-2}E_{f,T-2} \end{pmatrix} \right\} + \\
\begin{pmatrix} e + c_{T-1}+ \\ A_{T-1}d_{T-2}+ \\ A_{T-1}B_{T-2}E_{f,T-2} \end{pmatrix} \right\}' \\
\begin{pmatrix} -A_{T-1}'B\Theta_{T-1} + \Psi 2_{T-2} - \Theta_{T-2}A\epsilon_{T-2}B\Theta_{T-1} + \\
\Theta_{T-2}A\epsilon_{T-2}B1_{T-2} \\
-A_{T-1}'B\Theta_{T-1} + \Psi 2_{T-2} - \Theta_{T-2}A\epsilon_{T-2}B\Theta_{T-1} + \\
\Theta_{T-2}A\epsilon_{T-2}B1_{T-2} \\
[-B\Theta_{T-1} + \Psi 1_{T-2}]' \Lambda \epsilon_{T-2} [-B\Theta_{T-1} + \Psi 1_{T-2}] \end{pmatrix}' \end{array} \right] + \Lambda \eta_{T-2}, \]

we obtain that

\[ \hat{u}_{T-2} = m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2}. \]

The value function then becomes

\[ V_{T-2} = -\sqrt{\frac{|\Lambda \eta_{T-2}|}{|\Sigma_{\eta}|}} \sqrt{\frac{|\Lambda \epsilon_{T-2}|}{|\Sigma_{\epsilon}|}} \exp \left[ \hat{\Phi}_1 + \hat{\Phi}_2 \right], \]

where

\[ \hat{\Phi}_1 = -\gamma (1 + r_f)^2 x_{T-2}^0 + z_{T-1} + \gamma (1 + r_f)^2 e' \left[ m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2} \right] + \gamma (1 + r_f)^2 e' \Gamma \left[ m_{T-2} + G_{T-2} f_{T-2} - L_{T-2} x_{T-2} \right]^2 + p_{T-1}' (d_{T-2} + B_{T-2} f_{T-2}) + (d_{T-2} + B_{T-2} f_{T-2})' H_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' B \Theta_{T-1} [m_{T-2} + G_{T-2} f_{T-2} + (I - L_{T-2}) x_{T-2}] + \\
[m_{T-2} + G_{T-2} f_{T-2} + (I - L_{T-2}) x_{T-2}]' \]

\[ \begin{pmatrix} e + c_{T-1}+ \\ A_{T-1}d_{T-2}+ \\ A_{T-1}B_{T-2}E_{f,T-2} \end{pmatrix} \left\{ C_{T-1} \otimes \begin{pmatrix} e + c_{T-1}+ \\ A_{T-1}d_{T-2}+ \\ A_{T-1}B_{T-2}E_{f,T-2} \end{pmatrix} \right\}', \]

\[ \{ S_{T-1}' d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \}' \]

\[ [m_{T-2} + G_{T-2} f_{T-2} + (I - L_{T-2}) x_{T-2}] - \]

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\[ \{S_{T-1}' B_{T-2} f_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f, T-2})\}^\prime \]
\[ [m_{T-2} + G_{T-2} f_{T-2} + (I - L_{T-2}) x_{T-2}] - \]
\[ \{S_{T-1}' B_{T-2} E_{f, T-2} \otimes A_{T-1} B_{T-2} f_{T-2}\}^\prime [m_{T-2} + G_{T-2} f_{T-2} + (I - L_{T-2}) x_{T-2}] + \]
\[ \{S_{T-1}' B_{T-2} E_{f, T-2} \otimes A_{T-1} B_{T-2} E_{f, T-2}\}^\prime [m_{T-2} + G_{T-2} f_{T-2} + (I - L_{T-2}) x_{T-2}], \]

and

\[
\hat{\Phi}_2 = \frac{1}{2} \left( \begin{array}{c}
[p_{T-1} + 2 H_{T-1} d_{T-2} + a2_{T-2} + \Theta_{T-2} \Lambda \epsilon_{T-2} a1_{T-2}] + \\
[2 H_{T-1} B_{T-2} + \Omega_{T-2} + \Theta_{T-2} \Lambda \epsilon_{T-2} \Omega_{1T-2} f_{T-2}] + \\
[-A_{T-1}' B \Theta_{T-1} + \Psi_{2T-2}] + \\
[\Theta_{T-2} \Lambda \epsilon_{T-2} B \Theta_{T-1} + \Psi_{1T-2}] + \\
[\Theta_{T-2} \Lambda \epsilon_{T-2} \Psi_{1T-2}]
\end{array} \right)^\prime \Lambda \eta_{T-2}
\]

\[
\left( \begin{array}{c}
p_{T-1} + 2 H_{T-1} d_{T-2} + a2_{T-2} + \Theta_{T-2} \Lambda \epsilon_{T-2} a1_{T-2} + \\
[2 H_{T-1} B_{T-2} + \Omega_{T-2} + \Theta_{T-2} \Lambda \epsilon_{T-2} \Omega_{1T-2} f_{T-2}] + \\
[-A_{T-1}' B \Theta_{T-1} + \Psi_{2T-2}] + \\
[\Theta_{T-2} \Lambda \epsilon_{T-2} B \Theta_{T-1} + \Psi_{1T-2}] + \\
[\Theta_{T-2} \Lambda \epsilon_{T-2} \Psi_{1T-2}]
\end{array} \right)^\prime + \\
\frac{1}{2} \left( \begin{array}{c}
a1_{T-2} + \Omega_{1T-2} f_{T-2} + \\
(-B \Theta_{T-1} + \Psi_{1T-2}) [m_{T-2} + G_{T-2} f_{T-2} + (I - L_{T-2}) x_{T-2}]
\end{array} \right)^\prime \Lambda \epsilon_{T-2}
\]

As a result, the exponent in the value function is linear in \( x_{T-2}^0 \) and quadratic in the state variables \( x_{T-2} \) and \( f_{T-2} \). We now present the following theorem that yields the first proposed approximation algorithm:

**Theorem 6.1** Under Approximation A, the optimal investment decisions and the value function for \( k = 2, \ldots, T \) can be approximated by the following relations:

\[
\tilde{u}_{T-k}(x_{T-k}, f_{T-k}) = m_{T-k} + G_{T-k} f_{T-k} - L_{T-k} x_{T-k},
\]

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\[ \tilde{V}_{T-k} \left( x_{T-k}, x'_{T-k}, f_{T-k} \right) = -\prod_{m=2}^{k} \sqrt{\frac{|\Lambda \eta_{T-m}|}{|\Sigma_{\eta}|}} \sqrt{\frac{|\Lambda \epsilon_{T-m}|}{|\Sigma_{\epsilon}|}} \exp \left[ -\gamma \left( 1 + r_f \right)^k \frac{x_{T-k}^0}{x_{T-k}^0} + z_{T-k} + p_{T-k} \ f_{T-k} - b'_{T-k} x_{T-k} + f'_{T-k} H_{T-k} f_{T-k} - x'_{T-k} C_{T-k} x_{T-k} + f'_{T-k} S_{T-k} x_{T-k} \right] \right] , \]

where the following matrices are constructed recursively

\[ \mathbf{B} \mathbf{T}_{T-k} = \text{diag} [b_{T-k}] , \]
\[ \mathbf{C} \mathbf{D}_{T-k} = \text{diag} \left[ (C_{T-k+1} \otimes e) \ x_0 \ (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) \right] , \]
\[ \mathbf{C} \mathbf{E}_{T-k} = \text{diag} \left[ (C_{T-k+1} \otimes e) \ A_{T-k+1} B_{T-k} E_{f,T-k} \right] , \]
\[ \mathbf{S} \mathbf{D}_{T-k} = \text{diag} \left[ S_{T-k+1} d_{T-k} \right] , \]
\[ \mathbf{S} \mathbf{E}_{T-k} = \text{diag} \left[ S'_{T-k+1} B_{T-k} E_{f,T-k} \right] , \]

and

\[ a_{1T-k} = 2 \left\{ C_{T-k+1} \otimes x_0 \ x_0' \right\} \ (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) - \left\{ S'_{T-k+1} \otimes x_0 \ e' \right\} \ d_{T-k} - 2 \left[ \left\{ C_{T-k+1} \otimes e \ (e + c_{T-k+1} + A_{T-k+1} d_{T-k})' \otimes x_0 \ e' \right\} + \mathbf{C} \mathbf{D}_{T-k} \right] x_0 - 2 \left[ \left\{ C_{T-k+1} \otimes e \ (A_{T-k+1} B_{T-k} E_{f,T-k})' \otimes x_0 \ e' \right\} + \mathbf{C} \mathbf{E}_{T-k} \right] x_0 + \mathbf{S} \mathbf{D}_{T-k} x_0 + \mathbf{S} \mathbf{E}_{T-k} x_0 , \]
\[ a_{2T-k} = A'_{T-k+1} a_{1T-k} - \left\{ S_{T-k+1} \otimes e \ x_0' \right\} (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) + \left\{ S_{T-k+1} \otimes e \ (e + c_{T-k+1} + A_{T-k+1} d_{T-k})' \right\} x_0 + \left\{ S_{T-k+1} \otimes e \ (A_{T-k+1} B_{T-k} E_{f,T-k})' \right\} x_0 , \]
\[ \Psi_{1T-k} = 2 \left[ \left\{ C_{T-k+1} \otimes e \ (e + c_{T-k+1} + A_{T-k+1} d_{T-k})' \otimes x_0 \ e' \right\} + \mathbf{C} \mathbf{D}_{T-k} \right] + 2 \left[ \left\{ C_{T-k+1} \otimes e \ (A_{T-k+1} B_{T-k} E_{f,T-k})' \otimes x_0 \ e' \right\} + \mathbf{C} \mathbf{E}_{T-k} \right] - \mathbf{S} \mathbf{D}_{T-k} - \mathbf{S} \mathbf{E}_{T-k} , \]
\[ \Psi_{2T-k} = A'_{T-k+1} \Psi_{1T-k} - \left\{ S_{T-k+1} \otimes e \ (e + c_{T-k+1} + A_{T-k+1} d_{T-k})' \right\} - \]
\[
\left\{ S_{T-k+1} \otimes e \ (A_{T-k+1} B_{T-k} \ E_{f,T-k})' \right\},
\]

\[
\Omega_{1T-k} = 2 \left\{ C_{T-k+1} \otimes x_0 \ x_0' \right\} A_{T-k+1} B_{T-k} - \left\{ S'_{T-k+1} \otimes x_0 \ e' \right\} B_{T-k},
\]

\[
\Omega_{2T-k} = A'_{T-k+1} \Omega_{1T-k} - \left\{ S_{T-k+1} \otimes e \ x_0' \right\} A_{T-k+1} B_{T-k},
\]

\[
\Lambda_{\eta_{T-k}} = \left[ \Sigma_{\eta}^{-1} - 2 \left( H_{T-k+1} + A'_{T-k+1} \left\{ C_{T-k+1} \otimes x_0 \ x_0' \right\} A_{T-k+1} - \frac{1}{2} \Theta_{T-k} \Lambda \epsilon_{T-k} \Theta_{T-k}^T \right) \right]^{-1},
\]

\[
\Lambda_{\epsilon_{T-k}} = \left[ \Sigma_{\epsilon}^{-1} - 2 \left( C_{T-k+1} \otimes x_0 \ x_0' \right) \right]^{-1},
\]

\[
\Theta_{T-k} = 2 A'_{T-k+1} \left\{ C_{T-k+1} \otimes x_0 \ x_0' \right\} - \left\{ S_{T-k+1} \otimes e \ x_0' \right\}.
\]

In addition,

\[
Q_{T-k} = \left[ \begin{array}{c}
2 \gamma (1 + r_f)^k \Gamma + \\
2 \left\{ C_{T-k+1} \otimes \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) \right\} \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right)'
\end{array} \right]^{-1},
\]

\[
\left[ \begin{array}{c}
A'_{T-k+1} B_{T-k+1} \Psi_{2T-k} - \Theta_{T-k} \Lambda \epsilon_{T-k} B_{T-k+1} + \\
-A'_{T-k+1} B_{T-k+1} \Psi_{1T-k} + \\
A'_{T-k+1} B_{T-k+1} \Psi_{1T-k}
\end{array} \right] \Lambda_{\eta_{T-k}} \left[ -B_{T-k+1} + \Psi_{1T-k} \right]
\]

\[
m_{T-k} = Q_{T-k} \left[ \begin{array}{c}
-\gamma (1 + r_f)^k e + B_{T-k+1} \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) + \\
\left\{ S'_{T-k+1} d_{T-k} \otimes (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) \right\} - \\
\left\{ S'_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right\} - \\
-A'_{T-k+1} B_{T-k+1} \Psi_{2T-k} - \Theta_{T-k} \Lambda \epsilon_{T-k} B_{T-k+1} + \\
\Theta_{T-k} \Lambda \epsilon_{T-k} \Psi_{1T-k}
\end{array} \right]' \Lambda_{\eta_{T-k}},
\]

\[
\left[ \begin{array}{c}
p_{T-k+1} + 2 H_{T-k+1} d_{T-k} + a_{2T-k} + \Theta_{T-k} \Lambda \epsilon_{T-k} a_{1T-k} - \\
-B_{T-k+1} + \Psi_{1T-k}
\end{array} \right] \Lambda \epsilon_{T-k} a_{1T-k}
\]

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\[ G_{T-k} = Q_{T-k} \]

\[ L_{T-k} = Q_{T-k} \]

Also,

\[ z_{T-k} = z_{T-k+1} + \gamma (1 + r_f)^k e' m_{T-k} + \gamma (1 + r_f)^k e' \Gamma m_{T-k}^2 + p_{T-k+1} d_{T-k} + \]

\[ d'_{T-k} H_{T-k+1} d_{T-k} - (e + c_{T-k+1} + A_{T-k+1} d_{T-k})' B\Theta_{T-k+1} m_{T-k} + \]

\[ m'_{T-k} \left\{ C_{T-k+1} \otimes \left( \begin{array}{c} e + c_{T-k+1}^+ \\ A_{T-k+1} d_{T-k}^+ \\ A_{T-k+1} B_{T-k} E_{f,T-k} \\ A_{T-k+1} B_{T-k} E_{f,T-k}^+ \end{array} \right) \right\}' m_{T-k} - \]

\[ \left\{ S'_{T-k+1} d_{T-k} \otimes (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) \right\}' m_{T-k} + \]

\[ \left\{ S'_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right\}' m_{T-k} + \]

\[ \left[ \frac{1}{2} p_{T-k+2} H_{T-k+1} d_{T-k} + a2_{T-k} + \Theta_{T-k} \Lambda \epsilon_{T-k} a_{1,T-k} + \right. \]

\[ \left. \left( \begin{array}{c} -A'_{T-k+1} B\Theta_{T-k+1} + \Psi 2_{T-k} - \Theta_{T-k} \Lambda \epsilon_{T-k} B\Theta_{T-k+1} \end{array} \right) m_{T-k} \right\}' \Lambda \eta_{T-k} \]

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\[
\begin{align*}
\begin{bmatrix}
\frac{p_{T-k+1}}{2} + 2 H_{T-k+1} d_{T-k} + a2_{T-k} + \Theta_{T-k} \Lambda \epsilon_{T-k} a1_{T-k} + \\
\left(-A'_{T-k+1} B \Theta_{T-k+1} + \Psi_{2T-k} - \Theta_{T-k} \Lambda \epsilon_{T-k} B \Theta_{T-k+1} + \right) m_{T-k} \\
\Theta_{T-k} \Lambda \epsilon_{T-k} \Psi_{1T-k}
\end{bmatrix} + \\
\frac{1}{2} \left[ a1_{T-k} + [-B \Theta_{T-k+1} + \Psi_{1T-k}] m_{T-k} \right] ' \Lambda \epsilon_{T-k}
\end{align*}
\]

\[
p_{T-k} = \gamma (1 + r_f)^k G'_{T-k} e + 2 \gamma (1 + r_f)^k G'_{T-k} \Gamma m_{T-k} + B'_{T-k} p_{T-k+1} + \\
2 B'_{T-k} H_{T-k+1} d_{T-k} - G'_{T-k} B \Theta_{T-k+1} (e + c_{T-k+1} + A_{T-k+1} d_{T-k} - \\
B'_{T-k} A'_{T-k+1} B \Theta_{T-k+1} m_{T-k} + \\
2 G'_{T-k} \left\{ C_{T-k+1} \otimes \begin{bmatrix}
\frac{e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{j,T-k}}{\frac{e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{j,T-k}}{}}
\end{bmatrix} \right\}' m_{T-k} - \\
\left\{ B'_{T-k} A'_{T-k+1} \otimes e d'_{T-k} S_{T-k+1} \right\} m_{T-k} - \\
\left\{ B'_{T-k} S_{T-k+1} \otimes e (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{j,T-k} \right\} m_{T-k} - \\
\left\{ B'_{T-k} A'_{T-k+1} \otimes e E'_{j,T-k} B'_{T-k} S_{T-k+1} \right\} m_{T-k} + \\
G'_{T-k} \left\{ S'_{T-k+1} B_{T-k} E_{j,T-k} \otimes A_{T-k+1} B_{T-k} E_{j,T-k} \right\} + \\
\begin{bmatrix}
2 H_{T-k+1} B_{T-k} + \Omega 2_{T-k} + \Theta_{T-k} \Lambda \epsilon_{T-k} \Omega 1_{T-k} + \\
\left(-A'_{T-k+1} B \Theta_{T-k+1} + \Psi_{2T-k} - \Theta_{T-k} \Lambda \epsilon_{T-k} B \Theta_{T-k+1} + \right) G_{T-k} \\
\Theta_{T-k} \Lambda \epsilon_{T-k} \Psi_{1T-k}
\end{bmatrix} + \\
\left[ \Omega 1_{T-k} + (-B \Theta_{T-k+1} + \Psi_{1T-k}) G_{T-k} \right] ' \Lambda \epsilon_{T-k}
\end{align*}
\]

\[
b_{T-k} = \gamma (1 + r_f)^k L'_{T-k} e + 2 \gamma (1 + r_f)^k L'_{T-k} \Gamma m_{T-k} + \\
(I - L_{T-k})' B \Theta_{T-k+1} (e + c_{T-k+1} + A_{T-k+1} d_{T-k} - \\
p_{T-k+1} + 2 H_{T-k+1} d_{T-k} + a2_{T-k} + \Theta_{T-k} \Lambda \epsilon_{T-k} a1_{T-k} + \\
\left(-A'_{T-k+1} B \Theta_{T-k+1} + \Psi_{2T-k} - \Theta_{T-k} \Lambda \epsilon_{T-k} B \Theta_{T-k+1} + \right) m_{T-k} \\
\Theta_{T-k} \Lambda \epsilon_{T-k} \Psi_{1T-k}
\end{align*}
\]

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\[ 2 \left( I - L_{T-k} \right)' \left\{ C_{T-k+1} \otimes \begin{pmatrix} e + c_{T-k+1}^- \\ A_{T-k+1} d_{T-k}^- \\ A_{T-k+1} B_{T-k} E_{f,T-k} \end{pmatrix} \right\} \left( \begin{pmatrix} e + c_{T-k+1}^- \\ A_{T-k+1} d_{T-k}^- \\ A_{T-k+1} B_{T-k} E_{f,T-k} \end{pmatrix} \right)' \]

\[ m_{T-k} + \left( I - L_{T-k} \right)' \left\{ S_{T-k+1} d_{T-k} \otimes (e + c_{T-k+1}^- + A_{T-k+1} d_{T-k}) \right\} = \]

\[ \left( I - L_{T-k} \right)' \left\{ S_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right\} = \]

\[ \left( I - L_{T-k} \right)' \left[ \begin{array}{c} -A_{T-k+1} B_{T-k+1} + \psi_{2T-k}^- \\ \Theta_{T-k} \Lambda e_{T-k} B_{T-k+1} + \Theta_{T-k} \Lambda e_{T-k} \psi_{1T-k} \end{array} \right] \Lambda \eta_{T-k} \]

\[ \left[ p_{T-k+1} + 2 H_{T-k+1} d_{T-k} + a_{2T-k} + \Theta_{T-k} \Lambda e_{T-k} a_{1T-k} + \right. \]

\[ \left. \left( -A_{T-k+1} B_{T-k+1} + \psi_{2T-k} - \Theta_{T-k} \Lambda e_{T-k} B_{T-k+1} \right) m_{T-k} \right] = \]

\[ \left( I - L_{T-k} \right)' \left( -B_{T-k+1} + \psi_{1T-k} \right) \Lambda e_{T-k} \left[ a_{1T-k} + (-B_{T-k+1} + \psi_{1T-k}) m_{T-k} \right] . \]

and

\[ H_{T-k} = \gamma \left( 1 + r_f \right)^k G_{T-k} G_{T-k} + B_{T-k} H_{T-k+1} B_{T-k} = \]

\[ B_{T-k} A_{T-k+1} B_{T-k+1} G_{T-k} + \]

\[ G_{T-k} \left\{ C_{T-k+1} \otimes \begin{pmatrix} e + c_{T-k+1}^- \\ A_{T-k+1} d_{T-k}^- \\ A_{T-k+1} B_{T-k} E_{f,T-k} \end{pmatrix} \right\} \left( \begin{pmatrix} e + c_{T-k+1}^- \\ A_{T-k+1} d_{T-k}^- \\ A_{T-k+1} B_{T-k} E_{f,T-k} \end{pmatrix} \right)' \]

\[ \{ B_{T-k} S_{T-k+1} \otimes e d_{T-k} S_{T-k+1} \} G_{T-k} = \]

\[ \{ B_{T-k} S_{T-k+1} \otimes e (e + c_{T-k+1}^- + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})' \} G_{T-k} = \]

\[ \{ B_{T-k} A_{T-k+1} \otimes e E_{f,T-k} B_{T-k} S_{T-k+1} \} G_{T-k} + \]

\[ \left[ \begin{array}{c} 2 H_{T-k+1} B_{T-k} + \Omega_{2T-k} + \Theta_{T-k} \Lambda e_{T-k} \Omega_{1T-k} + \end{array} \right. \]

\[ \left. \begin{array}{c} -A_{T-k+1} B_{T-k+1} + \psi_{2T-k}^- \\ \Theta_{T-k} \Lambda e_{T-k} B_{T-k+1} + \Theta_{T-k} \Lambda e_{T-k} \psi_{1T-k} \end{array} \right] \Lambda \eta_{T-k} \]

\[ \frac{1}{2} \left[ \begin{array}{c} 2 H_{T-k+1} B_{T-k} + \Omega_{2T-k} + \Theta_{T-k} \Lambda e_{T-k} \Omega_{1T-k} + \end{array} \right. \]

\[ \left. \begin{array}{c} -A_{T-k+1} B_{T-k+1} + \psi_{2T-k}^- \\ \Theta_{T-k} \Lambda e_{T-k} B_{T-k+1} + \Theta_{T-k} \Lambda e_{T-k} \psi_{1T-k} \end{array} \right] G_{T-k} \]

\[ \frac{1}{2} \left[ \begin{array}{c} 2 H_{T-k+1} B_{T-k} + \Omega_{2T-k} + \Theta_{T-k} \Lambda e_{T-k} \Omega_{1T-k} + \end{array} \right. \]

\[ \left. \begin{array}{c} -A_{T-k+1} B_{T-k+1} + \psi_{2T-k}^- \\ \Theta_{T-k} \Lambda e_{T-k} B_{T-k+1} + \Theta_{T-k} \Lambda e_{T-k} \psi_{1T-k} \end{array} \right] G_{T-k} . \]
\[
\frac{1}{2} [\Omega_{T-k} + (-B \Theta_{T-k+1} + \Psi_{1T-k}) G_{T-k}]' \Lambda e_{T-k} \\
[\Omega_{T-k} + (-B \Theta_{T-k+1} + \Psi_{1T-k}) G_{T-k}],
\]

\[
C_{T-k} = \gamma (1 + \tau_f)^k L_{T-k} \Gamma L_{T-k} + \\
(I - L_{T-k})' \left\{ C_{T-k+1} \otimes \begin{pmatrix} e + c_{T-k+1} \\
A_{T-k+1} d_{T-k} + \\
A_{T-k+1} B_{T-k} e_{f,T-k} \end{pmatrix} \begin{pmatrix} e + c_{T-k+1} \\
A_{T-k+1} d_{T-k} + \\
A_{T-k+1} B_{T-k} e_{f,T-k} \end{pmatrix}' \right\} \\
(I - L_{T-k}) + \\
\frac{1}{2} (I - L_{T-k})' \left[ -A'_{T-k+1} B \Theta_{T-k+1} + \Psi_{2T-k} - \\
\Theta_{T-k} \Lambda e_{T-k} B \Theta_{T-k+1} + \Theta_{T-k} \Lambda e_{T-k} \Psi_{1T-k} \right] \Lambda \eta_{T-k} \\
\left[ -A'_{T-k+1} B \Theta_{T-k+1} + \Psi_{2T-k} - \\
\Theta_{T-k} \Lambda e_{T-k} B \Theta_{T-k+1} + \Theta_{T-k} \Lambda e_{T-k} \Psi_{1T-k} \right] (I - L_{T-k}) + \\
\frac{1}{2} (I - L_{T-k})' (-B \Theta_{T-k+1} + \Psi_{1T-k})' \Lambda e_{T-k} (-B \Theta_{T-k+1} + \Psi_{1T-k}) (I - L_{T-k}),
\]

\[
S_{T-k} = 2\gamma (1 + \tau_f)^k G'_{T-k} \Gamma L_{T-k} + B'_{T-k} A'_{T-k+1} B \Theta_{T-k+1} (I - L_{T-k}) - \\
2 G'_{T-k} \left\{ C_{T-k+1} \otimes \begin{pmatrix} e + c_{T-k+1} \\
A_{T-k+1} d_{T-k} + \\
A_{T-k+1} B_{T-k} e_{f,T-k} \end{pmatrix} \begin{pmatrix} e + c_{T-k+1} \\
A_{T-k+1} d_{T-k} + \\
A_{T-k+1} B_{T-k} e_{f,T-k} \end{pmatrix}' \right\} \\
(I - L_{T-k}) + \{ B'_{T-k} A'_{T-k+1} \otimes e d'_{T-k} s_{T-k+1} \} (I - L_{T-k}) + \\
\{ B'_{T-k} s_{T-k+1} \otimes e (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} e_{f,T-k})' \} \\
(I - L_{T-k}) + \{ B'_{T-k} A'_{T-k+1} \otimes e e'_{f,T-k} B'_{T-k} s_{T-k+1} \} (I - L_{T-k}) - \\
2 H_{T-k+1} B_{T-k} + \Omega_{T-k} \Lambda e_{T-k} \Omega_{1T-k} + \\
\left( -A'_{T-k+1} B \Theta_{T-k+1} + \Psi_{2T-k} - \\
\Theta_{T-k} \Lambda e_{T-k} B \Theta_{T-k+1} + \Theta_{T-k} \Lambda e_{T-k} \Psi_{1T-k} \right) G_{T-k} \Lambda \eta_{T-k} \\
\left[ -A'_{T-k+1} B \Theta_{T-k+1} + \Psi_{2T-k} - \\
\Theta_{T-k} \Lambda e_{T-k} B \Theta_{T-k+1} + \Theta_{T-k} \Lambda e_{T-k} \Psi_{1T-k} \right] (I - L_{T-k}) - \\
[\Omega_{1T-k} + (-B \Theta_{T-k+1} + \Psi_{1T-k}) G_{T-k}]' \Lambda e_{T-k} \\
(-B \Theta_{T-k+1} + \Psi_{1T-k}) (I - L_{T-k}),
\]

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with boundary conditions $Q_{T-1}$, $m_{T-1}$, $G_{T-1}$, $L_{T-1}$, $z_{T-1}$, $p_{T-1}$, $b_{T-1}$, $H_{T-1}$, $C_{T-1}$ and $S_{T-1}$ as calculated in section 6.1.

**Proof.** We have shown that the expressions are valid for $k = 2$. Since by following the same steps as for $k = 2$ and by changing the indexes considered we can prove that they are true for arbitrary $k$, we omit the details of the calculations. ■

The resulting approximate investment policy is again linear in the risky holdings and the factor realizations, and independent of the holdings in the riskless asset. The proposed algorithm provides an approximation of the value function at every point in time using characteristics of the optimal cost-to-go function. Thus the resulting policy, as we exhibit in Section 6.4 with numerical examples, outperforms other existing dynamic policies and provides an efficient way to measuring investors' utility levels as a function of time.

### 6.3 Approximation B: A Quadratic Approximation

In this section, we propose a different suboptimal control policy by approximating the optimal value function at time $T - 1$ with a quadratic using the second order Taylor's expansion around the initial holdings $x^0_T$, $x_0$ and the conditional expectation of $f_{T-1}$, $E_{f,T-1}$. More specifically, let

$$y = [x_{T-1} f_{T-1}]',$$

$$y_0 = [x_0 E_{f,T-1}]',$$

$$c = [-b_{T-1} p_{T-1}]',$$

$$Q = \begin{bmatrix} C_{T-1} & -\frac{1}{2} S'_{T-1} \\ -\frac{1}{2} S_{T-1} & H_{T-1} \end{bmatrix}.$$ 

Then the value function can be written as

$$V_{T-1}(x^0_{T-1}, y) = -\exp \left[ -\gamma (1 + r_f)x^0_{T-1} + z_{T-1} + c' y + y' Q y \right],$$

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and thus can be approximated with

\[ V_{T-1} \approx V_{T-1}(x_0^0, y_0) + \left[ \frac{\partial V_{T-1}}{\partial x_0^0} \right]_{(x_0^0, y_0)}(x_{T-1}^0 - x_0^0) + [\nabla V_{T-1}]_{(x_0^0, y_0)}(y - y_0) + \frac{1}{2}(y - y_0)'H_{(x_0^0, y_0)}(y - y_0), \]

where

\[ \nabla V_{T-1} = \left\{ -\exp\left[ -\gamma (1 + r_f) x_0^0 + x_{T-1} + c' y + y' Q y \right] \right\} (c + 2Q y) \]

\[ H = \left\{ -\exp\left[ -\gamma (1 + r_f) x_0^0 + x_{T-1} + c' y + y' Q y \right] \right\} [2Q + (c + 2Q y) (c + 2Q y)']. \]

In addition, define

\[ \mathcal{P} = \exp \left[ \frac{z_{T-1} - \gamma (1 + r_f) x_0^0 + p_{T-1} E_{f,T-1} + E_{f,T-1}'}{S_{T-1} x_0} \right], \quad (6.9) \]

\[ \tilde{z}_{T-1} = -1 - \gamma (1 + r_f) x_0^0 - \left( b_{T-1} - 2C_{T-1} x_0 + S_{T-1} E_{f,T-1} \right)' x_0 + \frac{1}{2} x_0 (-b_{T-1} + 2C_{T-1} x_0 - S_{T-1} E_{f,T-1})' x_0 - \frac{1}{2} E_{f,T-1}' \left( p_{T-1} + 2H_{T-1} E_{f,T-1} - S_{T-1} x_0 \right) + \left( p_{T-1} + 2H_{T-1} E_{f,T-1} - S_{T-1} x_0 \right)' E_{f,T-1} + \quad (6.10) \]

\[ E_{f,T-1}' \left( p_{T-1} + 2H_{T-1} E_{f,T-1} - S_{T-1} x_0 \right)' E_{f,T-1} - \frac{1}{2} E_{f,T-1}' \left( p_{T-1} + 2H_{T-1} E_{f,T-1} - S_{T-1} x_0 \right) \]

\[ \hat{p}_{T-1} = -p_{T-1} + \quad (6.11) \]

\[ \left( p_{T-1} + 2H_{T-1} E_{f,T-1} - S_{T-1} x_0 \right) (p_{T-1} + 2H_{T-1} E_{f,T-1} - S_{T-1} x_0)' E_{f,T-1} - \]

\[ \left( p_{T-1} + 2H_{T-1} E_{f,T-1} - S_{T-1} x_0 \right) (-b_{T-1} + 2C_{T-1} x_0 - S_{T-1} E_{f,T-1})' x_0, \]

\[ \hat{b}_{T-1} = b_{T-1} + \left( -b_{T-1} + 2C_{T-1} x_0 - S_{T-1} E_{f,T-1} \right)' x_0 \]
\[ (-b_{T-1} + 2C_{T-1}x_0 - S'_{T-1}E_{f,T-1}) (p_{T-1} + 2H_{T-1}E_{f,T-1} - S_{T-1}x_0)' E_{f,T-1}, \]

\[
\hat{H}_{T-1} = H_{T-1} + \frac{1}{2} (p_{T-1} + 2H_{T-1}E_{f,T-1} - S_{T-1}x_0)(p_{T-1} + 2H_{T-1}E_{f,T-1} - S_{T-1}x_0)',
\]

\[
\hat{C}_{T-1} = C_{T-1} + \frac{1}{2} (-b_{T-1} + 2C_{T-1}x_0 - S'_{T-1}E_{f,T-1}) (-b_{T-1} + 2C_{T-1}x_0 - S'_{T-1}E_{f,T-1})',
\]

\[
\hat{S}_{T-1} = S_{T-1} + (p_{T-1} + 2H_{T-1}E_{f,T-1} - S_{T-1}x_0)(-b_{T-1} + 2C_{T-1}x_0 - S'_{T-1}E_{f,T-1})'.
\]

Then, \( V_{T-1} \) is approximated by

\[
V_{T-1} \left( x^0_{T-1}, x_{T-1}, f_{T-1} \right) \approx \mathcal{P} \left\{ \begin{array}{l}
\tilde{e}_{T-1} + \gamma (1 + r_f) x^0_{T-1} + p'_{T-1} f_{T-1} - f'_{T-1} \hat{H}_{T-1} f_{T-1} \\
+ b_{T-1} x_{T-1} - x'_{T-1} \hat{C}_{T-1} x_{T-1} + f'_{T-1} \hat{S}_{T-1} x_{T-1}
\end{array} \right\}.
\]

The approximate optimization problem considered above is still not solvable in closed-form unless the asset returns are normally distributed. In order to motivate the approximation algorithm that follows, consider the cost-to-go function at time \( T - 2 \):

\[
V_{T-2} \left( x^0_{T-2}, x_{T-2}, f_{T-2} \right) =
\]

\[
\mathcal{P} \max_{u_{T-2}} E_{T-2} \left( \begin{array}{l}
\tilde{e}_{T-1} + \gamma (1 + r_f) x^0_{T-2} - e' u_{T-2} - e' \Gamma u^2_{T-2} \\
+ \tilde{p}'_{T-1} (d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1}) - \\
(d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1})' \hat{H}_{T-1} (d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1}) + \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1})' \\
\hat{B}_{T-1} (x_{T-2} + u_{T-2}) - \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1})' \\
\{ \tilde{C}_{T-1} \otimes (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \} \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1}) + \\
(d_{T-2} + B_{T-2} f_{T-2} + \eta_{T-1})' \{ \hat{S}_{T-1} \otimes e (x_{T-2} + u_{T-2})' \} \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} + A_{T-1} \eta_{T-1} + \epsilon_{T-1})
\end{array} \right).
\]

where the matrix \( \hat{B}_{T-1} \) is constructed similarly as before

\[
\hat{B}_{T-1} = \text{diag} \left( \hat{b}_{T-1} \right).
\] (6.12)
The maximization problem is equivalent to

\[
\mathcal{D}_{T-2} = \max_{u_{T-2}} \left\{ \varepsilon_{T-1} + \gamma (1 + r_f)^2 \left[ x_{T-2}^0 - e' u_{T-2} - e' \Gamma u_{T-2}^2 \right] + \right. \\
\left. \tilde{p}_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - \right. \\
\left. (d_{T-2} + B_{T-2} f_{T-2})' \tilde{H}_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - e' \left( \tilde{H}_{T-1} \otimes \Sigma_\eta \right) e + \right. \\
\left. (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \tilde{\Theta}_{T-1} (x_{T-2} + u_{T-2}) - \right. \\
\left. (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \right\} \\
\left. \left\{ \tilde{C}_{T-1} \otimes (x_{T-2} + u_{T-2}) \right\} (x_{T-2} + u_{T-2})' \right\} \\
\left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} \right) - \left. e' \right[ \left( A_{T-1}' \{ \tilde{C}_{T-1} \otimes (x_{T-2} + u_{T-2}) \} A_{T-1} \right) \otimes \Sigma_\eta \right] e + \right. \\
\left. e' \left\{ \tilde{C}_{T-1} \otimes (x_{T-2} + u_{T-2}) \right\} (x_{T-2} + u_{T-2})' \otimes \Sigma_\epsilon \right] e + \right. \\
\left. (d_{T-2} + B_{T-2} f_{T-2})' \{ \tilde{S}_{T-1} \otimes e \} (x_{T-2} + u_{T-2})' \right\} \\
\left( e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} \right) + \left. e' \right[ \left\{ \{ \tilde{S}_{T-1} \otimes e \} (x_{T-2} + u_{T-2})' \} A_{T-1} \right) \otimes \Sigma_\eta \right] e \right\}.
\]

The approximation performed for \( k = 2, \ldots, T \) is similar to the one described in the previous section for the term \( \Phi_1 \):

1. We linearize the vector coefficient of \((x_{T-k} + u_{T-k})\) with respect to the state variable \( f_{T-k} \).

2. We approximate the matrix coefficient \( P_1 \) in \((x_{T-2} + u_{T-2})' P_1 (x_{T-2} + u_{T-2})\) with a constant.

So, using the approximation formulae of (6.7) and (6.8) the optimization problem \( \mathcal{D}_{T-2} \)
reduces to

\[
\begin{align*}
\left( z_{T-1} + \gamma (1 + r_f)^2 \left[ x_{T-2}^0 - e' \Gamma' u_{T-2}^2 \right] + \tilde{\nu}_{T-1}' \left( d_{T-2} + B_{T-2} f_{T-2} \right) - \\
(d_{T-2} + B_{T-2} f_{T-2})' \tilde{H}_{T-1} (d_{T-2} + B_{T-2} f_{T-2}) - e' \left( \tilde{H}_{T-1} \otimes \Sigma_{e} \right) e + \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2})' \tilde{\Theta}_{T-1} (x_{T-2} + u_{T-2}) - \\
(x_{T-2} + u_{T-2})'ight) \\
\max_{u_{T-2}} \left( (x_{T-2} + u_{T-2}) - \\
e' \left( (A_{T-1}' \left( \tilde{C}_{T-1} \otimes (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \right) A_{T-1} ) \otimes \Sigma_{\epsilon} \right) e -
\right. \\
e' \left( (x_{T-2} + u_{T-2})' \otimes \Sigma_{e} \right) e + \\
e' \left( (S_{T-1}' e (x_{T-2} + u_{T-2})' A_{T-1} ) \otimes \Sigma_{\epsilon} \right) e + \\
(x_{T-2} + u_{T-2})' \left( S_{T-1}' e (x_{T-2} + u_{T-2})' \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \right) + \\
(x_{T-2} + u_{T-2})' \left( S_{T-1}' e (x_{T-2} + u_{T-2})' \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \right)
\right). 
\end{align*}
\]

In order to write the first order conditions of the above optimization problem in an efficient way, we use the following lemma:

**Lemma 6.3** For arbitrary matrices $S$, $A$, $\Sigma$ and vector $x$ of dimensions $(K \times N)$, $(N \times K)$, $(K \times K)$ and $(N \times 1)$ respectively, the following relation is valid

\[
e' \left[ \left( S \otimes e' x' \right) \otimes \Sigma \right] e = sa' x,
\]

where the $i$-th element of vector $sa$ is the $ii$-th element of the matrix $S' \Sigma A'$:

\[
[sa]_i = [S' \Sigma A']_ii \quad \text{for} \quad i = 1, \ldots, N.
\]
Proof. We write \([\{(S \otimes e \mathbf{x}') A\} \otimes \Sigma]\) as

\[
\begin{bmatrix}
S_{11} x_1 & \cdots & S_{1N} x_N \\
\vdots & \ddots & \vdots \\
S_{K1} x_1 & \cdots & S_{KN} x_N \\
\end{bmatrix}
\begin{bmatrix}
A_{11} & \cdots & A_{1K} \\
\vdots & \ddots & \vdots \\
A_{N1} & \cdots & A_{NK} \\
\end{bmatrix}
\otimes
\begin{bmatrix}
\Sigma_{11} & \cdots & \Sigma_{1K} \\
\vdots & \ddots & \vdots \\
\Sigma_{N1} & \cdots & \Sigma_{NK} \\
\end{bmatrix}
= \\
\begin{bmatrix}
\Sigma_{11} \sum_{m=1}^{N} (A_{m1} S_{1m} x_i) & \cdots & \Sigma_{1K} \sum_{m=1}^{N} (A_{mK} S_{1m} x_i) \\
\vdots & \ddots & \vdots \\
\Sigma_{K1} \sum_{m=1}^{N} (A_{m1} S_{Km} x_i) & \cdots & \Sigma_{KK} \sum_{m=1}^{N} (A_{mK} S_{Km} x_i) \\
\end{bmatrix}
= \\
\begin{bmatrix}
\sum_{i=1}^{K} S_{i1} \sum_{j=1}^{K} (\Sigma_{ij} A_{1j}) & \cdots & \sum_{i=1}^{K} S_{iN} \sum_{j=1}^{K} (\Sigma_{ij} A_{Nj}) \\
\vdots & \ddots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_N \\
\end{bmatrix}
\]

The matrix \(S' \Sigma A'\) is

\[
\begin{bmatrix}
S_{11} & \cdots & S_{K1} \\
\vdots & \ddots & \vdots \\
S_{1N} & \cdots & S_{KN} \\
\end{bmatrix}
\begin{bmatrix}
\sum_{j=1}^{K} (\Sigma_{ij} A_{1j}) & \cdots & \sum_{j=1}^{K} (\Sigma_{ij} A_{Nj}) \\
\vdots & \ddots & \vdots \\
\sum_{j=1}^{K} (\Sigma_{Kj} A_{1j}) & \cdots & \sum_{j=1}^{K} (\Sigma_{Kj} A_{Nj}) \\
\end{bmatrix}
= \\
\begin{bmatrix}
\sum_{i=1}^{K} S_{i1} \sum_{j=1}^{K} (\Sigma_{ij} A_{1j}) & \cdots & \sum_{i=1}^{K} S_{i1} \sum_{j=1}^{K} (\Sigma_{ij} A_{Nj}) \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{K} S_{iN} \sum_{j=1}^{K} (\Sigma_{ij} A_{1j}) & \cdots & \sum_{i=1}^{K} S_{iN} \sum_{j=1}^{K} (\Sigma_{ij} A_{Nj}) \\
\end{bmatrix}
\]

and thus the result follows. \(\blacksquare\)

Using the above formula and the fact that

\[
e' \left[(A_{T-1}' \{ \tilde{C}_{T-1} \otimes (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})'\} A_{T-1}) \otimes \Sigma_{\eta} \right] e = \\
(x_{T-2} + u_{T-2})' \left\{ A_{T-1} \Sigma_{\eta} A_{T-1}' \otimes \tilde{C}_{T-1} \right\} (x_{T-2} + u_{T-2}),
\]

\[
e' \left\{ \tilde{C}_{T-1} \otimes (x_{T-2} + u_{T-2}) (x_{T-2} + u_{T-2})' \otimes \Sigma_{\epsilon} \right\} e = \\
(x_{T-2} + u_{T-2})' \left\{ \tilde{C}_{T-1} \otimes \Sigma_{\epsilon} \right\} (x_{T-2} + u_{T-2}),
\]

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the first order conditions are given by

\[
-\gamma (1 + r_f)^2 e - 2\gamma (1 + r_f)^2 \Gamma u_{T-2} + \\
\widehat{B}\Theta_{T-1} \begin{pmatrix} e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2} \end{pmatrix} - \\
2 \left\{ \partial_{T-1} \otimes \begin{pmatrix} (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \end{pmatrix} \right\} (x_{T-2} + u_{T-2}) - \\
2 \left\{ \Sigma_{\eta} A'_{T-1} \otimes \partial_{T-1} \right\} (x_{T-2} + u_{T-2}) + \\
\tilde{a}_{T-1} + \left\{ \tilde{S}'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} f_{T-2}) \right\} + \\
\tilde{S}'_{T-1} B_{T-2} f_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) + \\
\tilde{S}'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} (f_{T-2} - E_{f,T-2}) \right\} = 0,
\]

with

\[
[\tilde{a}_{T-1}]_i = \left[ \tilde{S}'_{T-1} \Sigma_{\eta} A'_{T-1} \right]_{ii}.
\] (6.13)

The approximate control, therefore, is linear in the state variables \( f_{T-2}, x_{T-2} \) and given by

\[
\hat{u}_{T-2} = \hat{m}_{T-2} + \hat{G}_{T-2} f_{T-2} - \hat{L}_{T-2} x_{T-2},
\]

where

\[
\hat{Q}_{T-2} = \begin{bmatrix} 2\gamma (1 + r_f)^2 \Gamma + \\
2 \left\{ \partial_{T-1} \otimes \begin{pmatrix} (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \end{pmatrix} \right\} + \\
2 \left\{ \Sigma_{\eta} A'_{T-1} \otimes \partial_{T-1} \right\} + 2 \left\{ \partial_{T-1} \otimes \Sigma_{\eta} \right\} \end{bmatrix}^{-1}
\]

\[
\hat{m}_{T-2} = \hat{Q}_{T-2} \begin{bmatrix} -\gamma (1 + r_f)^2 e + \widehat{B}\Theta_{T-1} (e + c_{T-1} + A_{T-1} d_{T-2}) + \tilde{a}_{T-1} + \\
\tilde{S}'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2}) - \\
\tilde{S}'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} E_{f,T-2} \end{bmatrix},
\]

\[
\hat{G}_{T-2} = \hat{Q}_{T-2} \begin{bmatrix} \widehat{B}\Theta_{T-1} A_{T-1} B_{T-2} + \tilde{S}'_{T-1} d_{T-2} e' \otimes A_{T-1} B_{T-2} + \\
\tilde{S}'_{T-1} B_{T-2} E_{f,T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) e' + \\
\tilde{S}'_{T-1} B_{T-2} E_{f,T-2} e' \otimes A_{T-1} B_{T-2} \end{bmatrix},
\]

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\[ \hat{L}_{T-2} = \hat{Q}_{T-2} \left\{ \begin{array}{c}
2 \left\{ \hat{C}_{T-1} \otimes \left[ \begin{array}{c}
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})'
\end{array} \right] \\
+ 2 \left\{ A_{T-1} \Sigma \eta \ A'_{T-1} \otimes \hat{C}_{T-1} \right\} + 2 \left\{ \hat{C}_{T-1} \otimes \Sigma \epsilon \right\}
\end{array} \right\} \right\}.
\]

The value function then becomes

\[ \hat{V}_{T-2} = \mathcal{P} \mathcal{D}_{T-2} \]

where

\[ \mathcal{D}_{T-2} = \hat{\xi}_{T-1} + \gamma (1 + \tau_f)^2 x_{T-2}^0 - \gamma (1 + \tau_f)^2 e' \left[ \hat{m}_{T-2} + \hat{G}_{T-2} \ f_{T-2} - \hat{L}_{T-2} \ x_{T-2} \right] - \gamma (1 + \tau_f)^2 e' \Gamma \left[ \hat{m}_{T-2} + \hat{G}_{T-2} \ f_{T-2} - \hat{L}_{T-2} \ x_{T-2} \right]' + \hat{p}_{T-1} (\hat{d}_{T-2} + \hat{B}_{T-2} \ f_{T-2} - \hat{H}_{T-1} (\hat{d}_{T-2} + \hat{B}_{T-2} \ f_{T-2} - \hat{L}_{T-2} \ x_{T-2} - \hat{C}_{T-1} \otimes \left[ \begin{array}{c}
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})'
\end{array} \right] \right) \right] \]

\[ \hat{m}_{T-2} + \hat{G}_{T-2} \ f_{T-2} + (\mathcal{I} - \hat{L}_{T-2}) \ x_{T-2} \]

\[ \left\{ \begin{array}{c}
\hat{C}_{T-1} \otimes \left[ \begin{array}{c}
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})'
\end{array} \right] \right\} \]

\[ \hat{m}_{T-2} + \hat{G}_{T-2} \ f_{T-2} + (\mathcal{I} - \hat{L}_{T-2}) \ x_{T-2} \]

\[ \left\{ \begin{array}{c}
\hat{C}_{T-1} \otimes \left[ \begin{array}{c}
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})'
\end{array} \right] \right\} \]

\[ \hat{m}_{T-2} + \hat{G}_{T-2} \ f_{T-2} + (\mathcal{I} - \hat{L}_{T-2}) \ x_{T-2} \]

\[ \tilde{s}_{T-1} \left[ \hat{m}_{T-2} + \hat{G}_{T-2} \ f_{T-2} + (\mathcal{I} - \hat{L}_{T-2}) \ x_{T-2} \right] + \left\{ \begin{array}{c}
\hat{S}_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2}) \}
\end{array} \right\} \left[ \hat{m}_{T-2} + \hat{G}_{T-2} \ f_{T-2} + (\mathcal{I} - \hat{L}_{T-2}) \ x_{T-2} \right] + \left\{ \begin{array}{c}
\hat{B}_{T-2} A'_{T-1} \otimes e \ d_{T-2} \hat{S}_{T-1} \}
\end{array} \right\} \left[ \hat{m}_{T-2} + \hat{G}_{T-2} \ f_{T-2} + (\mathcal{I} - \hat{L}_{T-2}) \ x_{T-2} \right] + \left\{ \begin{array}{c}
\hat{B}_{T-2} \hat{S}_{T-1} \otimes (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})'
\end{array} \right\} \left[ \hat{m}_{T-2} + \hat{G}_{T-2} \ f_{T-2} + (\mathcal{I} - \hat{L}_{T-2}) \ x_{T-2} \right] + \]}
\[ f_{T-2} \left\{ B'_{T-2} A'_{T-1} \otimes e \mathbf{E}'_{T-2} \right\} \left[ \bar{m}_{T-2} + \hat{G}_{T-2} f_{T-2} + \left( I - \hat{L}_{T-2} \right) x_{T-2} \right] - \left\{ \hat{S}'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} E_{f,T-2} \right\}' \left[ \bar{m}_{T-2} + \hat{G}_{T-2} f_{T-2} + \left( I - \hat{L}_{T-2} \right) x_{T-2} \right]. \]

If we let

\[ \hat{\tilde{z}}_{T-2} = \bar{z}_{T-1} - \gamma (1 + r_f)^2 e' \bar{m}_{T-2} - \gamma (1 + r_f)^2 e' \mathbf{I} \bar{m}_{T-2} + \tilde{p}_{T-1} \mathbf{d}_{T-2} - \]

\[ d_{T-2} \hat{H}_{T-1} d_{T-2} - e' \left( \hat{H}_{T-1} \otimes \Sigma_{\eta} \right) e + (e + c_{T-1} + A_{T-1} d_{T-2})' \hat{B} \hat{\Theta}_{T-1} \bar{m}_{T-2} - \]

\[ \bar{m}'_{T-2} = \left[ \hat{C}_{T-1} \otimes \left( \begin{array}{c} (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \end{array} \right) \right] \bar{m}_{T-2} + \]

\[ \left\{ A_{T-1} \Sigma_{\eta} A_{T-1}' \otimes \hat{C}_{T-1} \right\} + \{ \hat{C}_{T-1} \otimes \Sigma_{\epsilon} \} \]

\[ \bar{s}a_{T-1} \bar{m}_{T-2} + \left\{ \hat{S}'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2}) \right\}' \bar{m}_{T-2} - \]

\[ \left\{ \hat{S}'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} E_{f,T-2} \right\}' \bar{m}_{T-2}, \]

\[ \hat{p}_{T-2} = -\gamma (1 + r_f)^2 \hat{G}'_{T-2} e - 2\gamma (1 + r_f)^2 \hat{G}'_{T-2} \mathbf{I} \bar{m}_{T-2} + B'_{T-2} \hat{p}_{T-1} - 2 B'_{T-2} \hat{H}_{T-1} d_{T-2} + B'_{T-2} A'_{T-1} \hat{B} \hat{\Theta}_{T-1} \bar{m}_{T-2} + \]

\[ \hat{G}'_{T-2} \hat{B} \hat{\Theta}_{T-1} (e + c_{T-1} + A_{T-1} d_{T-2}) - \]

\[ 2 \hat{G}'_{T-2} \left[ \hat{C}_{T-1} \otimes \left( \begin{array}{c} (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2}) \\ (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \end{array} \right) \right] \bar{m}_{T-2} + \]

\[ \left\{ A_{T-1} \Sigma_{\eta} A_{T-1}' \otimes \hat{C}_{T-1} \right\} + \{ \hat{C}_{T-1} \otimes \Sigma_{\epsilon} \} \]

\[ \hat{G}'_{T-2} \bar{s}a_{T-1} + \hat{G}'_{T-2} \left\{ \hat{S}'_{T-1} d_{T-2} \otimes (e + c_{T-1} + A_{T-1} d_{T-2}) \right\} + \]

\[ \left\{ B'_{T-2} A'_{T-1} \otimes e d_{T-2} \hat{S}_{T-1} \right\} \bar{m}_{T-2} + \]

\[ \left\{ B'_{T-2} \hat{S}_{T-1} \otimes e \left( (e + c_{T-1} + A_{T-1} d_{T-2} + A_{T-1} B_{T-2} E_{f,T-2})' \right) \right\} \bar{m}_{T-2} + \]

\[ \left\{ B'_{T-2} A'_{T-1} \otimes e' E'_{f,T-2} B'_{T-2} \hat{S}_{T-1} \right\} \bar{m}_{T-2} - \]

\[ \hat{G}'_{T-2} \left\{ \hat{S}'_{T-1} B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} E_{f,T-2} \right\}, \]

\[ \hat{b}_{T-2} = \gamma (1 + r_f)^2 \hat{L}_{T-2} e + 2\gamma (1 + r_f)^2 \hat{L}_{T-2} \mathbf{I} \bar{m}_{T-2} + \]

\[ \left( I - \hat{L}_{T-2} \right)' \hat{B} \hat{\Theta}_{T-1} (e + c_{T-1} + A_{T-1} d_{T-2}) - \]

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\[2 \left( I - \hat{L}_{T-2} \right) \begin{bmatrix} \hat{C}_{T-1} \otimes \begin{bmatrix} (e + c_{T-1} + A_{T-1}d_{T-2} + A_{T-1}B_{T-2}E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1}d_{T-2} + A_{T-1}B_{T-2}E_{f,T-2})' \\
+ \{ A_{T-1} \Sigma \eta A'_{T-1} \otimes \hat{C}_{T-1} \} + \{ \hat{C}_{T-1} \otimes \Sigma \} 
\end{bmatrix} \right) \]

\[\hat{m}_{T-2} + \left( I - \hat{L}_{T-2} \right)' s \Phi_{T-1} + \]

\[\left( I - \hat{L}_{T-2} \right)' \{ \hat{S}_{T-1}d_{T-2} \otimes (e + c_{T-1} + A_{T-1}d_{T-2}) \} - \]

\[\left( I - \hat{L}_{T-2} \right)' \{ \hat{S}_{T-1}B_{T-2} E_{f,T-2} \otimes A_{T-1} B_{T-2} E_{f,T-2} \}, \]

and

\[\hat{H}_{T-2} = \gamma (1 + r_f)^2 \hat{G}'_{T-2} \Gamma \hat{G}_{T-2} + B'_{T-2} \hat{H}_{T-2} - B'_{T-2} A'_{T-1} B\Theta_{T-1} \hat{G}_{T-2} + \]

\[\hat{G}'_{T-2} \begin{bmatrix} \hat{C}_{T-1} \otimes \begin{bmatrix} (e + c_{T-1} + A_{T-1}d_{T-2} + A_{T-1}B_{T-2}E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1}d_{T-2} + A_{T-1}B_{T-2}E_{f,T-2})' \\
+ \{ A_{T-1} \Sigma \eta A'_{T-1} \otimes \hat{C}_{T-1} \} + \{ \hat{C}_{T-1} \otimes \Sigma \} 
\end{bmatrix} \right) \hat{G}_{T-2} - \]

\[\{ B'_{T-2} A'_{T-1} \otimes e d'_{T-2} \hat{S}_{T-1} \} \hat{G}_{T-2} - \]

\[\{ B'_{T-2} \hat{S}_{T-1} \otimes e (e + c_{T-1} + A_{T-1}d_{T-2} + A_{T-1}B_{T-2}E_{f,T-2})' \} \hat{G}_{T-2} - \]

\[\{ B'_{T-2} A'_{T-1} \otimes e E'_{f,T-2} B'_{T-2} \hat{S}_{T-1} \} \hat{G}_{T-2}, \]

\[\hat{C}_{T-2} = \gamma (1 + r_f)^2 \hat{L}'_{T-2} \Gamma \hat{L}_{T-2} + \]

\[\left( I - \hat{L}_{T-2} \right)' \begin{bmatrix} \hat{C}_{T-1} \otimes \begin{bmatrix} (e + c_{T-1} + A_{T-1}d_{T-2} + A_{T-1}B_{T-2}E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1}d_{T-2} + A_{T-1}B_{T-2}E_{f,T-2})' \\
+ \{ A_{T-1} \Sigma \eta A'_{T-1} \otimes \hat{C}_{T-1} \} + \{ \hat{C}_{T-1} \otimes \Sigma \} 
\end{bmatrix} \right) \]

\[\left( I - \hat{L}_{T-2} \right), \]

\[\hat{S}_{T-2} = 2\gamma (1 + r_f)^2 \hat{G}'_{T-2} \Gamma \hat{L}_{T-2} + B'_{T-2} A'_{T-1} B\Theta_{T-1} \left( I - \hat{L}_{T-2} \right) - \]

\[2 \hat{G}'_{T-2} \begin{bmatrix} \hat{C}_{T-1} \otimes \begin{bmatrix} (e + c_{T-1} + A_{T-1}d_{T-2} + A_{T-1}B_{T-2}E_{f,T-2}) \\
(e + c_{T-1} + A_{T-1}d_{T-2} + A_{T-1}B_{T-2}E_{f,T-2})' \\
+ \{ A_{T-1} \Sigma \eta A'_{T-1} \otimes \hat{C}_{T-1} \} + \{ \hat{C}_{T-1} \otimes \Sigma \} 
\end{bmatrix} \right) \]

\[\left( I - \hat{L}_{T-2} \right) + \{ B'_{T-2} A'_{T-1} \otimes e d'_{T-2} \hat{S}_{T-1} \} \left( I - \hat{L}_{T-2} \right) + \]

\[\{ B'_{T-2} \hat{S}_{T-1} \otimes e (e + c_{T-1} + A_{T-1}d_{T-2} + A_{T-1}B_{T-2}E_{f,T-2})' \} \left( I - \hat{L}_{T-2} \right) + \]

\[\{ B'_{T-2} A'_{T-1} \otimes e E'_{f,T-2} B'_{T-2} \hat{S}_{T-1} \} \left( I - \hat{L}_{T-2} \right), \]

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we finally obtain that the approximate value function at time $T-2$ is just

$$\hat{V}_{T-2} = \mathcal{P} \left\{ \hat{z}_{T-2} + \gamma (1 + r_f)^2 x_{T-2}^0 + \hat{p}'_{T-2} f_{T-2} - f'_{T-2} \hat{H}_{T-2} f_{T-2} - b_{T-2} x_{T-2} - x'_{T-2} \hat{C}_{T-2} x_{T-2} + f'_{T-2} \hat{S}_{T-2} x_{T-2} \right\},$$

with the matrices $\hat{H}_{T-2}$, $\hat{C}_{T-2}$ being symmetric. We prove the following theorem that yields the approximation algorithm.

**Theorem 6.2** Under Approximation B, the optimal investment decisions and the value function for $k = 2, \ldots, T$ can be approximated by the following relations:

$$\hat{u}_{T-k}(x_{T-k}, f_{T-k}) = \tilde{m}_{T-k} + \hat{G}_{T-k} f_{T-k} - \hat{L}_{T-k} x_{T-k},$$

$$\hat{v}_{T-k}(x_{T-k}^0, x_{T-k}, f_{T-k}) = \mathcal{P} \left\{ \hat{z}_{T-k} + \gamma (1 + r_f)^k x_{T-k}^0 + \hat{p}'_{T-k} f_{T-k} - f'_{T-k} \hat{H}_{T-k} f_{T-k} \right\},$$

where $\mathcal{P}$ and the initial conditions are given by (6.9), (6.10)-(6.13) respectively. The matrix $\hat{b}_{T-k}$ and the vector $\hat{s}_{T-k}$ are constructed recursively by

$$\hat{b}_{T-k} = \text{diag} \left[ \hat{b}_{T-k} \right],$$

$$[\hat{s}_{T-k}]_i = \left[ \hat{S}_{T-k} \Sigma_{\eta} A'_{T-k} \right]_{ii} \quad \text{for } i = 1, \ldots, N.$$

In addition, the parameters involved are given by

$$\hat{Q}_{T-k} = \begin{bmatrix} 2\gamma (1 + r_f)^k \Gamma + \\ 2 \left\{ \hat{C}_{T-k+1} \otimes \left[ (e + c_{T-k+1} A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}) \\
(e + c_{T-k+1} A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})' \right] \right\} \\ + 2 \left\{ A_{T-k+1} \Sigma_{\eta} A'_{T-k+1} \otimes \hat{C}_{T-k+1} \right\} + 2 \left\{ \hat{C}_{T-k+1} \otimes \Sigma_{\epsilon} \right\} \end{bmatrix}^{-1},$$

$$\hat{m}_{T-k} = \hat{Q}_{T-k} \begin{bmatrix} -\gamma (1 + r_f)^k e + \hat{b}_{T-k+1} (e + c_{T-k+1} A_{T-k+1} d_{T-k}) + \hat{s}_{T-k+1} - \\ \left\{ \hat{S}_{T-k+1} A_{T-k+1} B_{T-k} E_{f,T-k} \otimes \left( e + c_{T-k+1} A_{T-k+1} d_{T-k} \right) \right\} - \\ \left\{ \hat{S}_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right\} \end{bmatrix}.$$
\[ \hat{G}_{T-k} = \hat{Q}_{T-k} \left[ \hat{B}_T \Theta_{T-k+1} A_{T-k+1} B_{T-k} + \left\{ \hat{S}_{T-k+1} d_{T-k} e' \otimes A_{T-k+1} B_{T-k} \right\} + \left\{ \hat{S}_{T-k+1} B_{T-k} \otimes \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) e' \right\} + \left\{ \hat{S}'_{T-k+1} B_{T-k} E_{f,T-k} e' \otimes A_{T-k+1} B_{T-k} \right\} \right] \]

\[ \hat{L}_{T-k} = \hat{Q}_{T-k} \left[ 2 \left\{ \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right) \right\} \right] \]

Also,

\[ \hat{z}_{T-k} = \hat{z}_{T-k+1} - \gamma (1 + r_f)^k e' \hat{m}_{T-k} + \gamma (1 + r_f)^k e' \Gamma \hat{m}_{T-k}^2 + \hat{p}_{T-k+1} d_{T-k} - \]

\[ (e + c_{T-k+1} + A_{T-k+1} d_{T-k})' \hat{B}_T \Theta_{T-k+1} \hat{m}_{T-k} - \]

\[ \hat{m}'_{T-k} \left[ \hat{C}_{T-k+1} \otimes \left( \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right) \right) \right] \]

\[ + \left\{ A_{T-k+1} \Sigma_\eta A_{T-k+1} \otimes \hat{C}_{T-k+1} \right\} \]

\[ \hat{m}_{T-k} + \hat{s}_T \cdot T_{k+1} \hat{m}_{T-k} + \left\{ \hat{S}'_{T-k+1} B_{T-k} \otimes (e + c_{T-k+1} + A_{T-k+1} d_{T-k})' \right\} \hat{m}_{T-k} - \]

\[ \left\{ \hat{S}'_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k} \right\} \hat{m}_{T-k}, \]

\[ \hat{p}_{T-k} = -\gamma (1 + r_f)^k \hat{G}'_{T-k} e - 2 \gamma (1 + r_f)^k \hat{G}'_{T-k} \Gamma \hat{m}_{T-k} + B_{T-k} \hat{p}_{T-k+1} - \]

\[ 2 B_{T-k} \hat{H}_{T-k+1} d_{T-k} + B_{T-k} A_{T-k+1} \hat{B}_T \Theta_{T-k+1} \hat{m}_{T-k} + \]

\[ \hat{G}'_{T-k} \hat{B}_T \Theta_{T-k+1} \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} \right) - \]

\[ 2 \hat{G}'_{T-k} \left[ \hat{C}_{T-k+1} \otimes \left( \left( e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right) \right) \right] \]

\[ + \left\{ A_{T-k+1} \Sigma_\eta A_{T-k+1} \otimes \hat{C}_{T-k+1} \right\} \]

\[ \hat{m}_{T-k} + \hat{G}'_{T-k} \hat{s}_T \cdot T_{k+1} + \hat{G}'_{T-k} \left\{ \hat{S}'_{T-k+1} d_{T-k} \otimes (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) \right\} + \]

\[ \left\{ B_{T-k} A_{T-k+1} \otimes e d_{T-k} \hat{S}_{T-k+1} \right\} \hat{m}_{T-k} + \]

\[ \left\{ B_{T-k} \hat{S}_{T-k+1} \otimes e (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})' \right\} \hat{m}_{T-k} + \]

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\[
\begin{align*}
\{B_{T-k} A'_{T-k+1} \otimes e E_{f,T-k} B_{T-k} \hat{S}_{T-k+1}\} \quad \widehat{m}_{T-k} - \\
\hat{G}_{T-k} \quad \{\hat{S}_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k}\}, \\
\hat{b}_{T-k} &= \gamma (1 + r_f)^k \hat{L}_{T-k} e + 2\gamma (1 + r_f)^k \hat{L}_{T-k} \Gamma \widehat{m}_{T-k} + \\
&(I - \hat{L}_{T-k})' \quad \widehat{B\Theta}_{T-k+1} (e + c_{T-k+1} + A_{T-k+1} d_{T-k}) - \\
&2 (I - \hat{L}_{T-k})' \quad \left[ \begin{array}{c} \hat{C}_{T-k+1} \\
\{e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}\} \\
\{e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}\}' \\
+ \{A_{T-k+1} \Sigma \eta A'_{T-k+1} \otimes \hat{C}_{T-k+1}\} + \{\hat{C}_{T-k+1} \otimes \Sigma\} \end{array} \right] \\
&\widehat{m}_{T-k} + (I - \hat{L}_{T-k})' \quad \widehat{s}_{\hat{a}}_{T-k+1} + \\
(I - \hat{L}_{T-k})' \quad \{\hat{S}_{T-k+1} d_{T-k} \otimes (e + c_{T-k+1} + A_{T-k+1} d_{T-k})\} - \\
(I - \hat{L}_{T-k})' \quad \{\hat{S}_{T-k+1} B_{T-k} E_{f,T-k} \otimes A_{T-k+1} B_{T-k} E_{f,T-k}\}, \\
\end{align*}
\]
and

\[
\begin{align*}
\hat{H}_{T-k} &= \gamma (1 + r_f)^k \hat{G}_{T-k} \Gamma \hat{G}_{T-k} + B'_{T-k} \hat{H}_{T-k+1} B_{T-k} - \\
B'_{T-k} A'_{T-k+1} \quad \widehat{B\Theta}_{T-k+1} \quad \hat{G}_{T-k} + \\
\hat{G}_{T-k} \quad \left[ \begin{array}{c} \hat{C}_{T-k+1} \otimes \\
\{e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}\} \\
\{e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}\}' \\
+ \{A_{T-k+1} \Sigma \eta A'_{T-k+1} \otimes \hat{C}_{T-k+1}\} + \{\hat{C}_{T-k+1} \otimes \Sigma\} \end{array} \right] \\
\hat{G}_{T-k} - \{B'_{T-k} A'_{T-k+1} \otimes e d_{T-k} \hat{S}_{T-k+1}\} \quad \hat{G}_{T-k} - \\
\{B'_{T-k} \hat{S}_{T-k+1} \otimes e (e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})'\} \quad \hat{G}_{T-k} - \\
\{B'_{T-k} A'_{T-k+1} \otimes e E_{f,T-k} B'_{T-k} \hat{S}_{T-k+1}\} \quad \hat{G}_{T-k}, \\
\hat{C}_{T-k} &= \gamma (1 + r_f)^k \hat{L}_{T-k} \Gamma \hat{L}_{T-k} + \\
&(I - \hat{L}_{T-k})' \quad \left[ \begin{array}{c} \hat{C}_{T-k+1} \otimes \\
\{e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}\} \\
\{e + c_{T-k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}\}' \\
+ \{A_{T-k+1} \Sigma \eta A'_{T-k+1} \otimes \hat{C}_{T-k+1}\} + \{\hat{C}_{T-k+1} \otimes \Sigma\} \end{array} \right] \\
(I - \hat{L}_{T-k}), \\
\end{align*}
\]
\[ \mathcal{S}_{T-k} = 2\gamma (1+r_f)^k \bar{G}'_{T-k} \Gamma \bar{L}_{T-k} + B_{T-k} A'_{T-k+1} \bar{H} \Theta_{T-k+1} (I - \bar{L}_{T-k}) - 2 \bar{G}'_{T-k} \mathcal{C}_{T,k+1} \otimes \begin{bmatrix} (e + c_{T,k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k}) \\ (e + c_{T,k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k})' \end{bmatrix} + \begin{bmatrix} \Sigma \eta A'_{T-k+1} \otimes \bar{C}_{T,k+1} \\ \bar{C}_{T,k+1} \otimes \Sigma \epsilon \end{bmatrix} \left( I - \bar{L}_{T-k} \right) + \{ B'_{T-k} A'_{T-k+1} \otimes e \} d'_{T-k} \mathcal{S}_{T-k+1} \left( I - \bar{L}_{T-k} \right) + \{ B'_{T-k} \mathcal{S}_{T-k+1} \otimes e \} \left( e + c_{T,k+1} + A_{T-k+1} d_{T-k} + A_{T-k+1} B_{T-k} E_{f,T-k} \right)' \left( I - \bar{L}_{T-k} \right) + \{ B'_{T-k} A'_{T-k+1} \otimes e \} E'_{f,T-k} B'_{T-k} \mathcal{S}_{T-k+1} \left( I - \bar{L}_{T-k} \right) \right). \]

**Proof.** We prove the theorem by induction. We have shown that the relations are true for \( k = 2 \). Assume that they hold for arbitrary \( k \); we will prove that they are valid for \( k + 1 \). The value function at time \( T - k - 1 \) is

\[ V_{T-k-1} (x^0_{T-k-1}, x_{T-k-1}, f_{T-k-1}) = \max_{u_{T-k-1}} E_{T-k-1} \left\{ \tilde{V}_{T-k} \left(x^0_{T-k}, x_{T-k}, f_{T-k} \right) \right\} = \]

\[ \mathcal{P} \max_{u_{T-k-1}} E_{T-k-1} \left\{ \begin{aligned} &\tilde{S}_{T-k} + \gamma (1+r_f)^k x^0_{T-k} + \tilde{P}'_{T-k} f_{T-k} - f'_{T-k} \tilde{H}_{T-k} f_{T-k} \\ + &\tilde{G}_{T-k} x_{T-k} - x^0_{T-k} \tilde{C}_{T-k} x_{T-k} + f'_{T-k} \tilde{S}_{T-k} x_{T-k} \end{aligned} \right\}. \]

Substituting for the wealth and return dynamics, the above maximization problem is equivalent to

\[
\max_{u_{T-k-1}} \left\{ \begin{aligned}
\tilde{S}_{T-k} + \gamma (1+r_f)^k &\left[ x^0_{T-k-1} - e' u_{T-k-1} - e' \Gamma u^2_{T-k-1} \right] + \\
\tilde{P}'_{T-k} &\left( d_{T-k-1} + B_{T-k-1} f_{T-k-1} \right) - \\
(d_{T-k-1} + B_{T-k-1} f_{T-k-1})' &\tilde{H}_{T-k} \left( d_{T-k-1} + B_{T-k-1} f_{T-k-1} \right) + \\
(e + c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1})' &\tilde{B} \Theta_{T-k} \\
(x_{T-k-1} + u_{T-k-1}) &- \\
(e + c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1}) &\\
\left\{ \tilde{C}_{T-k} \otimes (x_{T-k-1} + u_{T-k-1}) \right\} (x_{T-k-1} + u_{T-k-1})' \\
(e + c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1}) &+ \\
(d_{T-k-1} + B_{T-k-1} f_{T-k-1})' &\left\{ \tilde{S}_{T-k} \otimes e \right\} (x_{T-k-1} + u_{T-k-1})' \\
(e + c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1}) &
\end{aligned} \right\}
\]

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\[
E_{T-k-1} = \begin{cases}
\tilde{p}_{T-k} \eta_{T-k} - 2(d_{T-k-1} + B_{T-k-1} f_{T-k-1})' \hat{H}_{T-k} \eta_{T-k} - \\
\eta'_{T-k} \hat{H}_{T-k} \eta_{T-k} + \\
(x_{T-k-1} + u_{T-k-1})' B\Theta_{T-k} (A_{T-k} \eta_{T-k} + \epsilon_{T-k}) - \\
2(e + c_{T-k} + A_{T-k} \sigma_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1})' \\
\{\tilde{C}_{T-k} \otimes (x_{T-k-1} + u_{T-k-1}) (x_{T-k-1} + u_{T-k-1})'\} (A_{T-k} \eta_{T-k} + \epsilon_{T-k}) - \\
\eta'_{T-k} A_{T-k}' \{\tilde{C}_{T-k} \otimes (x_{T-k-1} + u_{T-k-1}) (x_{T-k-1} + u_{T-k-1})'\} A_{T-k} \eta_{T-k} - \\
\epsilon'_{T-k} \{\tilde{C}_{T-k} \otimes (x_{T-k-1} + u_{T-k-1}) (x_{T-k-1} + u_{T-k-1})'\} \epsilon_{T-k} + \\
(d_{T-k-1} + B_{T-k-1} f_{T-k-1})' \{\tilde{S}_{T-k} \otimes e (x_{T-k-1} + u_{T-k-1})'\} + \\
(A_{T-k} \eta_{T-k} + \epsilon_{T-k}) + \eta_{T-k}' \{\tilde{S}_{T-k} \otimes e (x_{T-k-1} + u_{T-k-1})'\} A_{T-k} \eta_{T-k}
\end{cases}
\]

where the \(ii\)-th element of the matrix \(B\Theta_{T-k}\) is the \(i\)-th element of the vector \(\tilde{b}_{T-k}\). Using Lemma 6.3 the expectation in the above equation is

\[
E_{T-k-1} = \begin{cases}
-\eta'_{T-k} \hat{H}_{T-k} \eta_{T-k} - \\
\eta_{T-k}' A_{T-k}' \{\tilde{C}_{T-k} \otimes (x_{T-k-1} + u_{T-k-1}) (x_{T-k-1} + u_{T-k-1})'\} A_{T-k} \eta_{T-k} - \\
-\epsilon'_{T-k} \{\tilde{C}_{T-k} \otimes (x_{T-k-1} + u_{T-k-1}) (x_{T-k-1} + u_{T-k-1})'\} \epsilon_{T-k} + \\
+\eta_{T-k}' \{\tilde{S}_{T-k} \otimes e (x_{T-k-1} + u_{T-k-1})'\} A_{T-k} \eta_{T-k} - \\
-\epsilon' \{\tilde{H}_{T-k} \otimes \Sigma_{\eta}\} e - (x_{T-k-1} + u_{T-k-1})' \{A_{T-k} \Sigma_{\eta} A_{T-k}' \otimes \tilde{C}_{T-k}\} (x_{T-k-1} + u_{T-k-1}) - \\
-(x_{T-k-1} + u_{T-k-1})' \{\tilde{C}_{T-k} \otimes \Sigma_{\epsilon}\} (x_{T-k-1} + u_{T-k-1}) + \tilde{s}_{T-k}' (x_{T-k-1} + u_{T-k-1})
\end{cases}
\]

where

\[
[s_{T-k}]_{ii} = [\tilde{S}_{T-k} \Sigma_{\eta} A_{T-k}']_{ii}.
\]

We perform the following two approximations using the formulae of (6.7) and (6.8):

1. Make the term that is quadratic in \((x_{T-k-1} + u_{T-k-1})\) independent of the state variable \(f_{T-k-1}\) by replacing it with its expectation \(E_{f,T-k-1}\).

2. Make the term that is linear in \((x_{T-k-1} + u_{T-k-1})\) linear in the state variable \(f_{T-k-1}\) by using the first-order Taylor's expansion around its expectation \(E_{f,T-k-1}\).
The objective function in the maximization problem is now approximated by

\[\tilde{v}_{T-k} + \gamma (1 + r_f)^{k+1} \left[ x^0_{T-k-1} - e' u_{T-k-1} - e' \Gamma u^2_{T-k-1} \right] + \]

\[\tilde{p}'_{T-k} \left( d_{T-k-1} + B_{T-k-1} f_{T-k-1} \right) - \]

\[(d_{T-k-1} + B_{T-k-1} f_{T-k-1})' \tilde{H}_{T-k} \left( d_{T-k-1} + B_{T-k-1} f_{T-k-1} \right) + \]

\[(e + c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1})' \tilde{B}_{T-k} \left( x_{T-k-1} + u_{T-k-1} \right) - \]

\[(e + c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} E_{f,T-k-1})' \]

\[\{ C_{T-k} \otimes \left( x_{T-k-1} + u_{T-k-1} \right) (x_{T-k-1} + u_{T-k-1})' \} \]

\[(e + c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} E_{f,T-k-1}) + \]

\[(x_{T-k-1} + u_{T-k-1})' \left\{ \tilde{S}_{T-k} d_{T-k-1} \otimes \begin{pmatrix} e + c_{T-k} + A_{T-k} d_{T-k-1} + \\ A_{T-k} B_{T-k-1} f_{T-k-1} \end{pmatrix} \right\} + \]

\[(x_{T-k-1} + u_{T-k-1})' \left\{ \tilde{S}_{T-k} B_{T-k-1} f_{T-k-1} \otimes \begin{pmatrix} e + c_{T-k} + A_{T-k} d_{T-k-1} + \\ A_{T-k} B_{T-k-1} E_{f,T-k-1} \end{pmatrix} \right\} + \]

\[(x_{T-k-1} + u_{T-k-1})' \left\{ \tilde{S}_{T-k} B_{T-k-1} E_{f,T-k-1} \otimes A_{T-k} B_{T-k-1} (f_{T-k-1} - E_{f,T-k-1}) \right\} - \]

\[\tilde{e}' \left\{ \tilde{H}_{T-k} \otimes \Sigma_\eta \right\}_e \left( x_{T-k-1} + u_{T-k-1} \right)' \left\{ A_{T-k} \Sigma_\eta A_{T-k}' \otimes \tilde{C}_{T-k} \right\} \left( x_{T-k-1} + u_{T-k-1} \right) - \]

\[(x_{T-k-1} + u_{T-k-1})' \left\{ \tilde{C}_{T-k} \otimes \Sigma_\epsilon \right\}_\epsilon \left( x_{T-k-1} + u_{T-k-1} \right) + \tilde{s}_{T-k} \left( x_{T-k-1} + u_{T-k-1} \right), \]

and thus the approximate control policy satisfies the first-order optimality conditions

\[-\gamma (1 + r_f)^{k+1} e - 2\gamma (1 + r_f)^{k+1} \Gamma u_{T-k-1} + \]

\[\tilde{B}_{T-k} \left( e + c_{T-k} + A_{T-k} d_{T-k-1} + A_{T-k} B_{T-k-1} f_{T-k-1} \right) - \]

\[\begin{array}{c}
2 \left\{ C_{T-k} \otimes \left( x_{T-k-1} + u_{T-k-1} \right) \right\} \\
\{ \tilde{S}_{T-k} d_{T-k-1} \otimes \begin{pmatrix} e + c_{T-k} + A_{T-k} d_{T-k-1} + \\ A_{T-k} B_{T-k-1} f_{T-k-1} \end{pmatrix} \right\} + \\
\{ \tilde{S}_{T-k} B_{T-k-1} f_{T-k-1} \otimes \begin{pmatrix} e + c_{T-k} + A_{T-k} d_{T-k-1} + \\ A_{T-k} B_{T-k-1} E_{f,T-k-1} \end{pmatrix} \right\} + \\
\{ \tilde{S}_{T-k} B_{T-k-1} E_{f,T-k-1} \otimes A_{T-k} B_{T-k-1} (f_{T-k-1} - E_{f,T-k-1}) \right\} - \\
2 \left\{ A_{T-k} \Sigma_\eta A_{T-k}' \otimes \tilde{C}_{T-k} \right\} \left( x_{T-k-1} + u_{T-k-1} \right) - 2 \left\{ \tilde{C}_{T-k} \otimes \Sigma_\epsilon \right\}_\epsilon \left( x_{T-k-1} + u_{T-k-1} \right) + \\
\end{array} \]

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\[ \tilde{\mathbf{a}}_{T-k} = 0. \]

Let

\[
\hat{\mathbf{Q}}_{T-k-1} = \left[ \begin{array}{c}
2\gamma (1 + r_f)^{k+1} \mathbf{I} + \\
2 \left\{ \tilde{\mathbf{C}}_{T-k} \otimes \left[ (\mathbf{e} + c_{T-k} + A_{T-k} \mathbf{d}_{T-k-1} + A_{T-k} B_{T-k-1} E_{f,T-k-1}) \right] \right. \\
+ 2 \left\{ A_{T-k} \Sigma_{\eta} A'_{T-k} \otimes \tilde{\mathbf{C}}_{T-k} \right\} + 2 \left\{ \tilde{\mathbf{C}}_{T-k} \otimes \Sigma_{\epsilon} \right\}
\end{array} \right]^{-1},
\]

\[
\hat{\mathbf{m}}_{T-k-1} = \hat{\mathbf{Q}}_{T-k-1} \left[ \begin{array}{c}
-\gamma (1 + r_f)^{k+1} \mathbf{e} + \tilde{\mathbf{B}} \tilde{\mathbf{Q}}_{T-k} \left( \mathbf{e} + c_{T-k} + A_{T-k} \mathbf{d}_{T-k-1} \right) + \tilde{\mathbf{a}}_{T-k} + \\
\left\{ \tilde{\mathbf{S}}_{T-k} \mathbf{d}_{T-k-1} \otimes (\mathbf{e} + c_{T-k} + A_{T-k} \mathbf{d}_{T-k-1}) \right\} - \\
\left\{ \tilde{\mathbf{S}}_{T-k} \mathbf{B}_{T-k-1} \otimes A_{T-k} \mathbf{d}_{T-k-1} \right\} E_{f,T-k-1} E_{f,T-k-1}
\end{array} \right],
\]

\[
\hat{\mathbf{G}}_{T-k-1} = \hat{\mathbf{Q}}_{T-k-1} \left[ \begin{array}{c}
\tilde{\mathbf{B}} \tilde{\mathbf{Q}}_{T-k} \mathbf{B}_{T-k-1} + \left\{ \tilde{\mathbf{S}}_{T-k} \mathbf{d}_{T-k-1} \otimes \mathbf{e}' \otimes A_{T-k} \mathbf{B}_{T-k-1} \right\} + \\
\left\{ \tilde{\mathbf{S}}_{T-k} \mathbf{B}_{T-k-1} \otimes \left( \mathbf{e} + c_{T-k} + A_{T-k} \mathbf{d}_{T-k-1} \right) \right\} \left( \mathbf{e}' \otimes A_{T-k} \mathbf{B}_{T-k-1} \right) + \\
\left\{ \tilde{\mathbf{S}}_{T-k} \mathbf{B}_{T-k-1} \otimes \mathbf{e}' \otimes A_{T-k} \mathbf{B}_{T-k-1} \right\}
\end{array} \right],
\]

\[
\hat{\mathbf{L}}_{T-k-1} = \hat{\mathbf{Q}}_{T-k-1} \left[ \begin{array}{c}
2 \left\{ \tilde{\mathbf{C}}_{T-k} \otimes \\
(\mathbf{e} + c_{T-k} + A_{T-k} \mathbf{d}_{T-k-1} + A_{T-k} B_{T-k-1} E_{f,T-k-1}) \right. \\
+ 2 \left\{ A_{T-k} \Sigma_{\eta} A'_{T-k} \otimes \tilde{\mathbf{C}}_{T-k} \right\} + 2 \left\{ \tilde{\mathbf{C}}_{T-k} \otimes \Sigma_{\epsilon} \right\}
\end{array} \right].
\]

The approximate dynamic policy is then linear in the state variables \( x_{T-k-1}, f_{T-k-1} \):

\[ \tilde{\mathbf{u}}_{T-k-1} = \hat{\mathbf{m}}_{T-k-1} + \hat{\mathbf{G}}_{T-k-1} f_{T-k-1} - \hat{\mathbf{L}}_{T-k-1} x_{T-k-1}. \]

Substituting back into the value function \( \tilde{V}_{T-k-1} \), we can show the rest of the equations. ■

The resulting approximate investment policy has the same qualitative characteristics as the previous approximate policy and we investigate its performance in Section 6.4. In the following section, we propose the simplification of the above described quadratic approximation algorithm under the assumption of independent and identically distributed asset returns.
6.3.1 Special case: IID returns

The approximate optimization problem described in the previous section can be solved in closed-form when asset returns are normally distributed. As a result, the proposed quadratic algorithm decomposes into two parts:

1. Approximation of the optimal value function at time $T - 1$ with a quadratic using the second-order Taylor’s expansion.

2. Exact solution of the resulting approximate optimization problem.

More specifically, consider the asset return generating process

$$r_t = \mu + \epsilon_t,$$

where $\mu$ is the $(N \times 1)$ vector of the mean returns. This is a special case of the return dynamics given by (1.1)-(1.2) for $A_t = 0$ and $c_t = \mu$. The resulting optimal investment decisions and value function at $T - 1$ are given by

$$u^*_T(x_{T-1}) = m_{T-1} - L_{T-1} x_{T-1},$$

$$V_{T-1}(x^0_{T-1}, x_{T-1}) = -\exp \left[ z_{T-1} - \gamma (1 + r_f) x^0_{T-1} - b'_{T-1} x_{T-1} + x'_{T-1} C_{T-1} x_{T-1} \right],$$

where

$$Q_{T-1} = [2 (1 + r_f) \Gamma + \gamma \Sigma_\epsilon]^{-1},$$

$$m_{T-1} = Q_{T-1} (-r_f e + \mu),$$

$$L_{T-1} = \gamma Q_{T-1} \Sigma_\epsilon,$$

$$z_{T-1} = \gamma (1 + r_f) e' m_{T-1} + \gamma (1 + r_f) e' \Gamma m^2_{T-1} - \gamma (e + \mu)' m_{T-1} + \frac{1}{2} \gamma^2 m^2_{T-1} \Sigma_\epsilon m_{T-1},$$

$$b_{T-1} = \gamma (1 + r_f) L'_{T-1} e + 2 \gamma (1 + r_f) L'_{T-1} \Gamma m_{T-1} + \gamma (I - L_{T-1})' (e + \mu) - \gamma^2 (I - L_{T-1})' \Sigma_\epsilon m_{T-1},$$

$$C_{T-1} = \gamma (1 + r_f) L'_{T-1} \Gamma L_{T-1} + \frac{1}{2} \gamma^2 (I - L_{T-1})' \Sigma_\epsilon (I - L_{T-1}).$$
The optimal investment depends only on the risky holdings: as the investor's position in the risky assets increases, the change in the portfolio composition decreases. The proposed quadratic approximation algorithm is provided below.

**Corollary 6.1** The optimal investment decisions and the value function for \( k = 2, \ldots, T \) can be approximated by the following relations:

\[
\hat{u}_{T-k}(x_{T-k}) = \hat{m}_{T-k} - \hat{L}_{T-k} x_{T-k},
\]

\[
\hat{V}_{T-k}(x^0_{T-k}, x_{T-k}) = \mathcal{P}\{\hat{z}_{T-k} + \gamma (1 + r_f)^k x^0_{T-k} + \hat{b}_{T-k} x_{T-k} - x'_{T-k} \hat{C}_{T-k} x_{T-k}\},
\]

where

\[
\mathcal{P} = \exp\left[\hat{z}_{T-1} - \gamma (1 + r_f) x^0_0 - b'_{T-1} x_0 + x'_0 C_{T-1} x_0\right],
\]

the initial conditions are given by

\[
\hat{z}_{T-1} = -1 - \gamma (1 + r_f) x^0_0 - b'_{T-1} x_0 + x'_0 C_{T-1} x_0 - \frac{1}{2} x'_0 (-b_{T-1} + 2C_{T-1} x_0)(-b_{T-1} + 2C_{T-1} x_0)' x_0,
\]

\[
\hat{b}_{T-1} = b_{T-1} + (-b_{T-1} + 2C_{T-1} x_0)(-b_{T-1} + 2C_{T-1} x_0)' x_0,
\]

\[
\hat{C}_{T-1} = C_{T-1} + \frac{1}{2} (-b_{T-1} + 2C_{T-1} x_0)(-b_{T-1} + 2C_{T-1} x_0)',
\]

and the parameters influencing the solution are provided by

\[
\hat{Q}_{T-k} = \left[2 \gamma (1 + r_f)^k \Gamma + 2 \left\{\hat{C}_{T-k+1} \otimes (e + \mu)(e + \mu)' + \hat{C}_{T-k+1} \otimes \Sigma_\epsilon\right\}\right]^{-1},
\]

\[
\hat{m}_{T-k} = \hat{Q}_{T-k} \left[-\gamma (1 + r_f)^k e + \hat{B}_{T-k+1} (e + \mu)\right],
\]

\[
\hat{L}_{T-k} = \hat{Q}_{T-k} \left[2 \left\{\hat{C}_{T-k+1} \otimes (e + \mu)(e + \mu)\right\}' + 2 \left\{\hat{C}_{T-k+1} \otimes \Sigma_\epsilon\right\}\right],
\]

\[
\hat{z}_{T-k} = \hat{z}_{T-k+1} - \gamma (1 + r_f)^k e' \hat{m}_{T-k} - \gamma (1 + r_f)^k e' \Gamma \hat{m}^2_{T-k} + (e + \mu)' \hat{B}_{T-k+1} \hat{m}_{T-k} - \frac{1}{2} x'_0 \hat{m}_{T-k} \left\{\hat{C}_{T-k+1} \otimes (e + \mu)(e + \mu)\right\} + \left\{\hat{C}_{T-k+1} \otimes \Sigma_\epsilon\right\}\right\} \hat{m}_{T-k},
\]

\[
\hat{b}_{T-k} = \gamma (1 + r_f)^k \hat{L}_{T-k} e + 2 \gamma (1 + r_f)^k \hat{L}_{T-k} e + 2 \gamma (1 + r_f)^k \hat{L}_{T-k} e + 2 \gamma (1 + r_f)^k \hat{L}_{T-k} e,
\]

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\[(I - L_{T-k})' \tilde{B} \tilde{\Theta}_{T-k+1} (e + \mu) - \]
\[2 (I - L_{T-k})' \left[ \left\{ \tilde{C}_{T-k+1} \otimes (e + \mu) (e + \mu)' \right\} + \left\{ \tilde{C}_{T-k+1} \otimes \Sigma_{e} \right\} \right] \tilde{m}_{T-k}, \]
\[\tilde{C}_{T-k} = \gamma (1 + r_f)^k \tilde{L}_{T-k} \Gamma L_{T-k}, \]
\[\left( I - L_{T-k} \right)' \left[ \left\{ \tilde{C}_{T-k+1} \otimes (e + \mu) (e + \mu)' \right\} + \left\{ \tilde{C}_{T-k+1} \otimes \Sigma_{e} \right\} \right] (I - L_{T-k}). \]

6.4 A Numerical Illustration

In this section, we examine the impact of transaction costs, risk aversion, return autocorrelation and volatility on the investor’s portfolio composition.

A host of papers show that in the presence of proportional transaction costs, the optimal investment policy for an investor who maximizes a power utility function of terminal wealth assuming that prices follow a geometric Brownian motion is described in terms of a no transaction region. The optimal policy is to refrain from trading if the ratio of stock to bond holdings lies within the region, and to transact to the nearest boundary of the region if portfolio holdings lie outside the region.

For infinite horizon models with intermediate consumption Constantinides [19] has numerically derived that transaction costs shift the region of no transactions towards the riskless asset. Dumas and Luciano [25], however, for no intermediate consumption show that the region of no transaction is wider than the one obtained by Constantinides and that there is no tendency for increased transaction costs to bias the portfolio one way or the other. The differences lie in the fact that, when a steady flow of consumption expenditures must be met out of the existing cash on hand, there is both less room for fluctuations in the amount of cash available and more need to bias the portfolio in favor of cash than when consumption is postponed to infinity.

In contrast, for portfolio managers with a finite horizon, the boundaries of the region of no transaction changes as the maturity date approaches. The derivation of the optimal strategy of an investor who maximizes his expected power utility of terminal wealth under a multiplicative binomial stock process is numerically presented in Gennette and Jung [35]. They showed that the no transaction region (1) narrows to a constant width as the time horizon lengthens, (2) converges faster to a constant width as transaction costs decrease, stock volatility increases, or risk aversion decreases, (3) narrows as transaction costs decrease and is of width zero when costs
are zero, (4) narrows and shifts towards the riskless asset as the investor becomes more risk averse or the volatility of stock returns increases. In addition, they obtained that transaction costs reduce investors’ utility levels relative to the no transaction cost case, but most of the reduction in utility is due to the costs of establishing and closing a position. Comparing with the myopic strategy, where investors follow a sequence of optimal 1-year policies, it was found that the utility loss incurred by following the myopic policy is small.

In what follows, we illustrate the performance of the proposed policies described in Sections 6.2 (App. A) and 6.3 (App. B) for changing values of the parameters involved, and compare them with the myopic strategy, where investors follow a sequence of optimal single-period policies, and the optimize-and-hold policy described in Section 3.3. We also demonstrate the effect of transaction costs on the utility level and the approximate optimal portfolio composition.

We consider the same example as given in Section 5.1.1. The return dynamics are given by

\[
    r_t = c + A f_t + \epsilon_t \\
    f_t = d + B f_{t-1} + \eta_t
\]

where it is assumed that \( c = [0.04, 0.14]' \), \( A = [0.2, 0.2]' \), \( d = 0.24 \), \( B = 0.2 \), the correlation coefficient is \( \rho_{12} = 0.4 \), and the volatilities are \( \sigma_{\epsilon,11} = 0.1 \), \( \sigma_{\epsilon,22} = 0.3 \) and \( \sigma_{\eta} = 0.25 \). We assume that changing the portfolio position in the less risky Asset 1 is less costly than Asset 2: \( \tau_1 = 0.01 \) and \( \tau_2 = 0.02 \). In addition, we consider that \( r_f = 5\% \), \( \gamma = 0.01 \), \( x_0^0 = 1 \), \( x_0 = [1, 1]' \) and \( T = 10 \). In our comparative analysis we present:

1. A comparison of the performance of the following dynamic trading strategies as a function of the factor correlation:
   - The approximate policy described in Section 6.2 (App. A).
   - The quadratic policy described in Section 6.3 (App. B).
   - The optimize-and-hold policy described in Section 3.3 (Opt-and-Hold).
   - The static policy that derives as the solution to a series of single-period optimization problems (static).
2. An investigation of the following parameters to the investment behavior of an investment manager who trades according to policy B:

- Transaction Costs
- Risk Aversion
- Time Horizon
- Asset Volatilities

For all policies considered, 1,000 independent sample paths of the asset returns are simulated and for each path the approximate dynamic policies are implemented. We denote as \( \hat{V}_0 \) the expected utility of terminal wealth derived analytically from the theorems presented in the previous sections, and as \( \overline{V}_0 \) the simulated expected utility of terminal wealth defined as the average (over 1,000 sample paths) of the investment manager's utility. We also plot the average risky investments (over the 1,000 sample paths) as a function of time in order to explore the evolution of the portfolio composition as time to expiration decreases.

Relative Performance of Dynamic Policies (Positive Factor Correlation \( B = 0.2 \))

In Table 6.1 we present the resulted utility levels from the application of the different dynamic policies considered. The approximate policy A outperforms the static strategy by almost 20% for the chosen parameters in our simulation experiment and the optimize-and-hold by 10%. Still though the myopic policy achieves a higher utility level than the one obtained by the quadratic policy B. We also notice a significant increase in the utility level in the absence of transaction costs. The certainty equivalent (defined as \(-1/\gamma \ln \left(-\overline{V}_0\right)\) and expressed in account units) with no transaction costs is 159.013, while the one in the presence of transaction costs is just 75.929 for the approximate policy A and 65.508 for the optimize-and-hold strategy. Indeed, when trading costs are ignored risky holdings are significantly higher and costless portfolio rebalancing allows investors a more effective asset allocation resulting in higher utility levels.

In Table 6.2 we report the initial risky investments for the different policies considered. All policies suggest buying both assets in the beginning of the investment horizon and then gradually reduce the risky share in the portfolio composition, as it is expected due to the CARA utility specification. The approximate policy A and the optimize-and-hold strategy
<table>
<thead>
<tr>
<th>Policies</th>
<th>$\bar{V}_0$</th>
<th>$\tilde{V}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>App. A</td>
<td>-0.4680</td>
<td>-0.4879</td>
</tr>
<tr>
<td>App. B</td>
<td>-0.6575</td>
<td>-0.5450</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>-0.5194</td>
<td>-</td>
</tr>
<tr>
<td>Static</td>
<td>-0.5911</td>
<td>-</td>
</tr>
<tr>
<td>No Costs</td>
<td>-0.2039</td>
<td>-0.2092</td>
</tr>
</tbody>
</table>

Table 6.1: Monte Carlo simulation of the investment policies under investigation for positive factor correlation $B = 0.2$. 1,000 independent sample paths were simulated, each path containing 10 periods, and $f_0 = 0.3$.

favor the less risky Asset 1 in the beginning of the investment horizon, but policy A suggests a smaller risky investment. Once again, in the absence of transaction costs the change in the initial portfolio composition is significantly higher.

<table>
<thead>
<tr>
<th>Policies</th>
<th>$\tilde{u}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>App. B</td>
<td>[2.2474, 12.8153]''</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>[18.4537, 10.8945]''</td>
</tr>
<tr>
<td>Static</td>
<td>[2.3303, 3.4617]''</td>
</tr>
<tr>
<td>No Costs</td>
<td>[154.4932, 77.6552]''</td>
</tr>
</tbody>
</table>

Table 6.2: The initial investment decision for positive factor correlation $B = 0.2$.

In order to investigate the investment behavior and the evolution of the risky holdings, we plot the expected risky investment (Figure 6-1) and expected risky holdings (Figure 6-2) as a function of time.

For positive factor correlation $B$, the proposed risky investments do not exhibit the same behavior over time. According to policy A and the optimize-and-hold strategy, the investor buys both assets for the entire duration of his investment horizon favoring the less risky Asset 1. The risky investment in Asset 2 decreases early on and then increases slightly towards the end of the horizon. Expected holdings in both assets increase over time, always being above the holdings when the static policy is applied and below the holdings in the case when transaction costs are ignored.

The quadratic approximate policy B exhibits a jump in the risky investment at the end of the investment horizon; this is due to the boundary conditions imposed at time $T - 1$ when the
approximation procedure commences. This policy is qualitatively quite different. It suggests a continuous decrease in the risky holdings and, thus, treats the risky assets as inferior goods because of the increasing absolute risk aversion characteristic inherent in the quadratic utility. The investor, as a result, tends to sell more of Asset 1 than of Asset 2, since he can expect a greater return by holding more of the riskier asset.

Finally, the myopic policy remains quite flat for this short horizon problem and suggests investing slightly more on the riskier asset 2. For longer horizons, though, the policy drastically changes to selling both assets and thus trusting the risk-free investment instead.

**Relative Performance of Dynamic Policies (Negative Factor Correlation $B = -0.2$)**

In Table 6.3 we present the resulted utility levels from the application of the different dynamic policies considered in the case of a negative factor correlation. We observe that, all else equal,
Figure 6-2: The expected risky holdings plotted as a function of time for the different policies considered. In Panel (a) we present the holdings for Asset 1 and in Panel (b) for Asset 2, for positive factor correlation $B = 0.2$.

the expected utility is lower in the presence of negative serial correlation. This is due to the fact that the magnitude of the change in both risky positions is smaller resulting in lower asset holdings, as it is also shown in Table 6.4. In the presence of transaction costs, the investor does not react as much to anticipated movements in future returns, as it is shown in Figure 6-3.

Similarly with the case of a positive factor autocorrelation, both the approximate policy B and the static policy exhibit similar qualitative characteristics with the ones described earlier.

On the contrary, policy A and the optimize-and-hold strategy introduce an interesting and intuitive behavior. In the presence of negative autocorrelation, the investor is suggested to follow an investment strategy that favors the riskier Asset 2 in the beginning of the investment horizon. Indeed, negative autocorrelation makes it more likely that bad returns will be offset by good future returns, more so when many future periods remain, making risky assets more
<table>
<thead>
<tr>
<th>Policies</th>
<th>$\bar{V}_0$</th>
<th>$V_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>App. A</td>
<td>-0.5743</td>
<td>-0.6067</td>
</tr>
<tr>
<td>App. B</td>
<td>-0.6957</td>
<td>-0.6025</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>-0.6220</td>
<td>-</td>
</tr>
<tr>
<td>Static</td>
<td>-0.6686</td>
<td>-</td>
</tr>
<tr>
<td>No Costs</td>
<td>-0.3540</td>
<td>-0.3456</td>
</tr>
</tbody>
</table>

Table 6.3: Monte Carlo simulation of the investment policies under investigation for negative factor correlation $B = -0.2$. 1,000 independent sample paths were simulated, each path containing 10 periods, and $f_0 = 0.3$.

<table>
<thead>
<tr>
<th>Policies</th>
<th>$\hat{u}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>App. A</td>
<td>6.7800, 9.7078</td>
</tr>
<tr>
<td>App. B</td>
<td>-1.6620, 12.3601</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>10.0267, 11.6613</td>
</tr>
<tr>
<td>Static</td>
<td>1.3868, 2.9989</td>
</tr>
<tr>
<td>No Costs</td>
<td>64.8609, 79.9534</td>
</tr>
</tbody>
</table>

Table 6.4: The initial investment decision for negative factor correlation $B = -0.2$.

attractive farther from the horizon. Thus, the riskier Asset 2 provides a hedging effect: if the current risky return is low, the opportunity set improves canceling some of its risk and making it more attractive compared to Asset 1.

Comparison with the Exact DP Policy

We compare the performance of the proposed approximate dynamic policy $A$ relative to the optimal strategy (exact) in an instance where exact dynamic programming is feasible. We consider a portfolio consisting of one risky and one riskless asset and we assume that $\varepsilon = 0.04$, $A = 0.2$, $d = 0.24$, and the volatilities are $\sigma_\varepsilon = 0.25$ and $\sigma_\eta = 0.30$. In addition, we assume that $r_f = 5\%$, $\gamma = 0.01$, $\tau = 0.01$, $x_0^0 = 1$, $x_0 = 1$, and $T = 3$. In Table 6.5 we present the resulted utility levels and initial investment decisions, and in Figure 6-5 we plot the expected risky investment as a function of time. It is evident that the approximate policy $A$ results in a near optimal performance.

Relative Performance of Policy $A$ with respect to Static policy

We illustrate the performance of the approximation policy $A$ relative to the myopic strategy by varying one parameter at a time, while keeping the values of the other parameters fixed. In Figure 6-6, we provide evidence for the superior performance of the proposed policy relative
Figure 6-3: The expected risky investment plotted as a function of time for the different policies considered. In Panel (a) we present the investment decision for Asset 1 and in Panel (b) for Asset 2 for positive factor correlation $B = -0.2$.

to the myopic strategy. As the time horizon increases, the myopic policy significantly deviates from the optimal sequence of investment decisions, since it suggests selling both assets with no lower bound on the amount transacted. Therefore, it suggests an unnatural increase in the investment activity near expiration when the risky holdings are higher. On the other hand, according to the approximate policy A transactions are limited at the end of the time horizon, since the expected return earned over the remaining time period decreases. Therefore, near maturity the investor practically does not trade.

In Figure 6-7, we depict the relative performance of the two policies for changing values of (1) the risk aversion parameter $\gamma$, (2) the transaction cost coefficient of the less risky Asset 1, $\tau_1$, (3) the factor volatility, $\sigma_\eta$ and (4) the volatility of Asset 2, $\sigma_{\epsilon,2}$. The performance is unaffected by changes in the transaction cost coefficients. On the other hand, as the investor
Figure 6-4: The expected risky holdings plotted as a function of time for the different policies considered. In Panel (a) we present the holdings for Asset 1 and in Panel (b) for Asset 2, for negative factor correlation $B = -0.2$.

becomes more risk averse and the volatilities involved increase, the myopic policy performs relatively better. The increase in the uncertainty of future payoffs reduces the holdings in the risky assets and, as a result, the investment decisions according to the myopic policy; thus near maturity, we do not experience big stock sellouts and the trading activity is not as intense.

**Effect of Transaction Costs on Policy A**  
Not surprisingly, as illustrated in Figure 6-8, increasing the size of transaction costs reduces the size of the trades. Policy A suggests a gradual decrease in the investment position for both assets as time to expiration decreases, as is also shown in Figures 6-1 and 6-3. In the absence of transaction costs though, the changes in the manager's investment positions are significant from period to period. In addition, the trading activity does not cease at the end of the investment horizon, as it is the case when transaction costs are introduced.
<table>
<thead>
<tr>
<th></th>
<th>Expected $V_0$</th>
<th>$\tilde{u}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>-0.9628</td>
<td>4.1867</td>
</tr>
<tr>
<td>App. A</td>
<td>-0.9677</td>
<td>6.2171</td>
</tr>
<tr>
<td>Opt-and-Hold</td>
<td>-</td>
<td>6.5724</td>
</tr>
<tr>
<td>Static</td>
<td>-</td>
<td>2.2778</td>
</tr>
</tbody>
</table>

Table 6.5: Performance comparison between the exact DP and the proposed approximate policy.

**Effect of Risk Aversion on Policy A**

Figure 6-8 shows that increasing the investor's risk aversion parameter decreases the magnitude of the risky investments. As the investor's aversion to risk increases, he becomes less tolerant of fluctuations in the value of his portfolio. Therefore, he moves out of the risky stocks and into the riskless bond.

**Effect of Time to Maturity on Policy A**

In Figure 6-9, we plot the expected risky investment as a function of time for different investment periods. For all time horizons, the transacted amount reduces as time to maturity decreases, because the expected return earned over the remaining time period decreases. Therefore, near maturity, portfolio rebalancing is minimal. In addition, as time to maturity increases, the reduction in the risky holdings decreases, since stock selling is limited, and for longer investment horizons the initial risky investment is smaller.

**Effect of Volatilities on Policy A**

Finally, the effect of varying the factor volatility, $\sigma_\eta$, and the volatility of Asset 2, $\sigma_{4,22}$, on the risky investment under the proposed approximate policy A is shown in Figure 6-10. Because the investor is risk averse, as $\sigma_\eta$ increases the investment in Asset 1 decreases, but the investment in Asset 2, that is riskier and positively correlated with Asset 1, practically remains unaffected. In addition, as Asset 2 becomes more risky the investor moves out of the riskier asset and into both the riskless bond and Asset 1.
Figure 6-5: A comparison between the risky holdings given by the exact DP algorithm and the proposed approximate policy A (ADP).
Figure 6-6: In Panel (a) we plot the percentage improvement over the myopic policy for $T = 5$, $T = 10$, $T = 20$ and $T = 30$ and positive factor correlation $B = 0.2$. In Panel (b) we plot the proposed expected risky investment of the two policies considered for $T = 30$. 
Figure 6-7: The percentage improvement over the myopic policy for changing values of the parameters $\gamma$, $\tau_1$, $\sigma_\eta$ and $\sigma_{\epsilon,22}$. 1,000 independent sample paths of the asset returns are simulated, each sample path containing 10 observations ($T = 10$.)
Figure 6-8: The expected risky investment plotted as a function of time for $T = 10$. In Panels (a)-(b) we show the dependence on the risk aversion parameter $\gamma$ and in Panels (c)-(d) the dependence on the transaction cost coefficient $\tau_1$. The solid line corresponds to Asset 1 and the dotted line to Asset 2.
Figure 6-9: The expected risky investment plotted as a function of time for different time horizons $T = 5, T = 10, T = 30$ and $T = 50$. 
Figure 6-10: The expected risky investment, plotted as a function of time for $T = 10$. In Panels (a)-(b) we show the dependence on the factor volatility $\sigma_\eta$ and in Panels (c)-(d) the dependence on the volatility of Asset 2 $\sigma_{t,22}$. The solid line corresponds to Asset 1 and the dotted one to Asset 2.
Chapter 7

Exponential Utility: Transaction Costs and Stochastic Volatility Models

In the previous chapter we examined the effect of transaction costs on dynamic portfolio strategies in the presence of models that account for lagged correlations in asset returns and a CARA utility specification. In this chapter we explore the application of models that are nonlinear in the variance, concentrating on univariate Generalized Autoregressive Conditionally Heteroskedastic (GARCH) models.

Using the asset return dynamics for the stochastic volatility model of Section 1.1.2, the portfolio manager faces the following dynamic optimization problem:

$$\text{maximize}_{\{u_0, \ldots, u_{\tau - 1}\}} \quad E_0 \left\{ - \exp \left[ -\gamma \left( x_T^0 + x_T \right) \right] \right\}$$

subject to

$$x_t^0 = (1 + r_f) \left[ x_{t-1}^0 - u_{t-1} - \tau u_{t-1}^2 \right]$$

$$x_t = (1 + r_t) \left[ x_{t-1} + u_{t-1} \right]$$

$$r_t = \mu + \sigma_{t-1} \varepsilon_t$$

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\[ \sigma_t^2 = \alpha_0 + \beta \sigma_{t-1}^2 + \alpha_1 \sigma_{t-1}^2 \epsilon_t^2 \]
\[ \epsilon_t \sim N(0,1) \, . \]

Once again, \( x_t^0 \) denotes the dollar amount holdings in cash at time \( t \) that exhibit a constant rate of return \( r_f \), and \( x_t \) the holdings in the risky asset with rate of return \( r_t \). The transaction cost coefficient is \( \tau \), the conditional mean and variance of the return process at time \( t \) are \( \mu \) and \( \sigma_{t-1}^2 \) respectively. We also assume that \( \alpha_1 + \beta < 1 \); thus the unconditional expectation of \( \sigma_t^2 \) is \( E_\sigma = \alpha_0 / (1 - \alpha_1 - \beta) \).

The remainder of this chapter is organized as follows. In Section 7.1, we present the solution to the single-period optimization problem and in Section 7.2, we propose a structured approximation algorithm that uses characteristics of the optimal investment policy and in-depth analysis of the DP recursion. Finally, in Section 7.3, we present numerical examples that compare the performance of the proposed dynamic policies and illustrate the impact of transaction costs, risk aversion and return characteristics to the investment behavior over time.

### 7.1 The Single Period Problem

In this section we show that there exists a closed-form solution for the single period problem that results in an investment policy inversely proportional to the variance realization.

The state at time \( t = 0, 1, \ldots, T - 1 \) consists of the agent’s holdings at time \( t \), \((x_t^0, x_t)\) and the variance realizations at time \( t \), \( \sigma_t^2 \). The control at time \( t \) is the risky investment, \( u_t \). We begin by characterizing the optimal value function \( V_{T-1} \) by using the boundary condition:

\[
V_{T-1} \left( x_{T-1}^0, x_{T-1}, \sigma_{T-1}^2 \right) = \max_{u_{T-1}} \left\{ -\exp \left[ -\gamma \left( 1 + r_f \right) \left( x_{T-1}^0 - u_{T-1} - \tau u_{T-1}^2 \right) \right] -\gamma \left( 1 + r_T \right) \left( x_{T-1} - u_{T-1} \right) -\gamma \left( 1 + \mu + \sigma_{T-1} \epsilon_T \right) \left( x_{T-1} + u_{T-1} \right) \right\}
\]

\[
= \max_{u_{T-1}} \left\{ -\exp \left[ -\gamma \left( 1 + r_f \right) \left( x_{T-1}^0 - u_{T-1} - \tau u_{T-1}^2 \right) \right] -\gamma \left( 1 + \mu \right) \left( x_{T-1} + u_{T-1} \right) + \frac{1}{2} \gamma^2 \sigma_{T-1}^2 \left( x_{T-1} + u_{T-1} \right)^2 \right\}
\]

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The maximization problem is equivalent to

$$\max_{u_{T-1}} \left\{ \gamma (1 + r_f) \left( x_{T-1}^0 - u_{T-1} - \tau \ u_{T-1}^2 \right) + \gamma (1 + \mu) \ (x_{T-1} + u_{T-1}) - \frac{1}{2} \gamma^2 \sigma_{T-1}^2 \ (x_{T-1} + u_{T-1})^2 \right\},$$

and its solution is given by

$$-\gamma (1 + r_f) - 2\tau \gamma (1 + r_f) \ u_{T-1} + \gamma (1 + \mu) - \gamma^2 \sigma_{T-1}^2 \ (x_{T-1} + u_{T-1}) = 0.$$ 

Therefore, the optimal control at time $T - 1$ is linear in the risky holdings and inversely proportional to the conditional variance, and is given by

$$u_{T-1}^* = \frac{(\mu - r_f) - \gamma \sigma_{T-1}^2 \ x_{T-1}}{2\tau (1 + r_f) + \gamma \sigma_{T-1}^2}.$$  \hspace{1cm} (7.1)

Notice that the optimal investment decision given by Equation (7.1) is identical to the one derived in Section 4.1 for a quadratic utility function with $\gamma = 2\lambda$. This observation constitutes the main justification for using quadratic approximation models in one of the recursive algorithms analyzed later in this chapter. Substituting for the optimal control given by Equation (7.1) and letting

$$m_{T-1} \left( \sigma_{T-1}^2 \right) = \frac{\mu - r_f}{2\tau (1 + r_f) + \gamma \sigma_{T-1}^2},$$

$$L_{T-1} \left( \sigma_{T-1}^2 \right) = \frac{\gamma \sigma_{T-1}^2}{2\tau (1 + r_f) + \gamma \sigma_{T-1}^2},$$

the value function at time $T - 1$ becomes

$$V_{T-1} \left( x_{T-1}^0, x_{T-1}, \sigma_{T-1}^2 \right) = -\exp \left[ -\gamma (1 + r_f) x_{T-1}^0 + \mathcal{F}_{T-1} \right],$$  \hspace{1cm} (7.2)

where

$$\mathcal{F}_{T-1} \left( x_{T-1}, \sigma_{T-1}^2 \right) = \gamma (1 + r_f) [m_{T-1} - L_{T-1} \ x_{T-1}] + \tau \gamma (1 + r_f) [m_{T-1} - L_{T-1} \ x_{T-1}]^2 - \gamma (1 + \mu) [m_{T-1} + (1 - L_{T-1}) \ x_{T-1}] + \frac{1}{2} \gamma^2 \sigma_{T-1}^2 \ [m_{T-1} + (1 - L_{T-1}) \ x_{T-1}]^2.$$
The exponent in Equation (7.2) is linear in the holdings in the riskless asset, \( x^0_{T-1} \), and quadratic in the risky holdings, \( x_{T-1} \). However, it is a highly nonlinear function with respect to the state variable \( \sigma^2_{T-1} \), the variance of the single risky asset conditioned on the information available at time \( T - 1 \). As a result, a closed-form solution to the dynamic optimization problem is unattainable, since the DP recursion is not solvable. In response, we propose two classes of recursive algorithms that approximate the optimal investment policy and level of utility over time. The first approximation algorithm described in Section 7.2 investigates in-depth the encountered dynamic optimization problem at every point in time and captures essential characteristics of the optimal cost-to-function. The second approximation algorithm arises from the solution of a series of stochastic optimization problems with different initial conditions and successively smaller investment horizon, and was described in Section 3.3.

### 7.2 A Structured Approximation

In this section, we propose an approximation algorithm that utilizes characteristics of the optimal cost-to-go function at every time period and is a generalization of the algorithm presented in Section 5.2.1. The exponent in the value function, as given by Equation (7.2), is quadratic in the risky portfolio holdings and nonlinear in the conditional variance of the risky asset. Since the dynamic optimization problem is solvable when the exponent in the value function is linear in the asset’s conditional variance, as we have seen in the previous section, we propose the following operations for \( k = 1, \ldots, T \):

1. Approximation of the function \( \mathcal{F}_{T-k} \) in the exponent of \( V_{T-k} \) with a function \( \hat{\mathcal{F}}_{T-k} \) that is quadratic in the risky holdings and linear in the conditional variance of the single asset, and is of the following form

\[
\hat{\mathcal{F}}_{T-k} = z_{T-k} - b_{T-k} x_{T-k} + p_{T-k} \sigma^2_{T-k} + C_{T-k} x^2_{T-k}.
\]

The approximations performed use Taylor’s expansion around the initial risky holdings \( x_0 \) and the unconditional expectation of the asset’s variance \( E_\sigma \).
2. Approximation of the coefficient of $\epsilon_{T-k+1}$ in the exponent of $V_{T-k}$ with a function linear in $(x_{T-k} + u_{T-k})$, using the first-order Taylor's expansion around the initial risky holdings $x_0$.

3. Approximation of the coefficient of $\sigma_{T-k}^2$ in the exponent of $V_{T-k}$ with a constant, replacing $\sigma_{T-k}^2$ with its unconditional expectation $E_s$ and $(x_{T-k} + u_{T-k})^2$ with the initial risky holdings $x_0$.

More specifically, the first step of the procedure involves approximating the function $F_{T-1}$ in the exponent of $V_{T-1}$ with

$$
\hat{F}_{T-1} = F_{T-1}(x_0, E_s) + \left[ \frac{\partial F_{T-1}}{\partial x_{T-1}} \right]_{(x_0, E_s)} (x_{T-1} - x_0) + \left[ \frac{\partial^2 F_{T-1}}{\partial \sigma_{T-1}^2} \right]_{(x_0, E_s)} (\sigma_{T-1}^2 - E_s) + \frac{1}{2} \left[ \frac{\partial^2 F_{T-1}}{\partial x_{T-1}^2} \right]_{(x_0, E_s)} (x_{T-1} - x_0)^2.
$$

Thus, if we let

$$
\hat{Q}_{T-1} = 2\tau (1 + r_f) + \gamma E_s,
\hat{m}_{T-1} = \frac{\mu - r_f}{\hat{Q}_{T-1}}, \quad \hat{m}_{T-1} = \frac{\gamma (\mu - r_f)}{(\hat{Q}_{T-1})^2},
\hat{L}_{T-1} = \frac{\tau E_s}{\hat{Q}_{T-1}}, \quad \hat{L}_{T-1} = \frac{2\tau \gamma (1 + r_f)}{(\hat{Q}_{T-1})^3},
$$

we can evaluate the partial derivatives of $F_{T-1}$ appearing in the desired approximate operation.

As a result, by denoting

$$
\begin{align*}
0_{T-1} & = -\gamma (\mu - r_f) \hat{m}_{T-1} + \gamma \left[ \tau (1 + r_f) + \frac{1}{2} \gamma E_s \right] (\hat{m}_{T-1})^2 - \\
& \quad \gamma \left\{ 1 + \mu - (\mu - r_f) \hat{L}_{T-1} - \gamma E_s \hat{m}_{T-1} + [2\tau (1 + r_f) + \gamma E_s] \hat{m}_{T-1} \hat{L}_{T-1} \right\} x_0 + \\
& \quad \left\{ \left[ \tau (1 + r_f) + \frac{1}{2} \gamma E_s \right] (\hat{L}_{T-1})^2 + \frac{1}{2} \gamma^2 E_s - \gamma^2 E_s \hat{L}_{T-1} \right\} x_0^2,
\end{align*}
$$

$$
\begin{align*}
f_{1, T-1} & = -\gamma (1 + \mu) + \gamma (\mu - r_f) \hat{L}_{T-1} + \gamma^2 E_s \hat{m}_{T-1} - \gamma [2\tau (1 + r_f) + \gamma E_s] \hat{m}_{T-1} \hat{L}_{T-1} + \\
& \quad 2\gamma \left\{ \left[ \tau (1 + r_f) + \frac{1}{2} \gamma E_s \right] (\hat{L}_{T-1})^2 + \frac{1}{2} \gamma E_s - \gamma E_s \hat{L}_{T-1} \right\} x_0,
\end{align*}
$$

$$
\begin{align*}
f_{11, T-1} & = 2\gamma \left\{ \left[ \tau (1 + r_f) + \frac{1}{2} \gamma E_s \right] (\hat{L}_{T-1})^2 + \frac{1}{2} \gamma E_s - \gamma E_s \hat{L}_{T-1} \right\},
\end{align*}
$$

$$
\begin{align*}
f_{2, T-1} & = -\gamma (\mu - r_f) \hat{m}_{T-1} + 2\gamma \tau (1 + r_f) \hat{m}_{T-1} \hat{m}_{T-1} + \frac{1}{2} \gamma^2 (\hat{m}_{T-1})^2 + \\
& \quad \gamma^2 E_s \hat{m}_{T-1} \hat{m}_{T-1} -
\end{align*}
$$

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we approximate $\tilde{F}_{T-1}$ with $\hat{F}_{T-1}$ using the following relation

$$\hat{F}_{T-1} = f_{0,T-1} + f_{1,T-1} (x_{T-1} - x_0) + f_{2,T-1} (\sigma_{T-1}^2 - E_s) + \frac{1}{2} f_{11,T-1} (x_{T-1} - x_0)^2.$$ 

Therefore, the value function at time $T - 1$ is approximated by

$$\hat{V}_{T-1} (x_{T-1}^0, x_{T-1}, \sigma_{T-1}^2) = -\exp \left[ -\gamma (1 + r_f) x_{T-1}^0 + \hat{F}_{T-1} \right] =$$

$$-\exp \left[ x_{T-1} - \gamma (1 + r_f) x_{T-1}^0 - b_{T-1} x_{T-1} + p_{T-1} \sigma_{T-1}^2 + C_{T-1} x_{T-1}^2 \right],$$

where

$$x_{T-1} = f_{0,T-1} - f_{1,T-1} x_0 - f_{2,T-1} E_s + \frac{1}{2} f_{11,T-1} x_0^2,$$

$$b_{T-1} = -f_{1,T-1} + f_{11,T-1} x_0,$$

$$p_{T-1} = f_{2,T-1},$$

$$C_{T-1} = \frac{1}{2} f_{11,T-1}.$$ 

Consequently, the value function at time $T - 2$ is given by the following optimization problem

$$V_{T-2} (x_{T-2}^0, x_{T-2}, \sigma_{T-2}^2) = \max_{u_{T-2}} E_{T-2} \{ \hat{V}_{T-1} \} =$$

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\[
\max_{u_{T-2}} E_{T-2} \left\{ - \exp \left[ \begin{array}{c}
  z_{T-1} - \gamma (1 + r_f)^2 \left( x_{T-2}^0 - u_{T-2} - \tau u_{T-2}^2 \right) - \\
  b_{T-1} (1 + \mu) (x_{T-2} + u_{T-2}) + \\
  p_{T-1} \left( \alpha_0 + \beta \sigma_{T-2}^2 \right) + C_{T-1} (1 + \mu)^2 (x_{T-2} + u_{T-2})^2 + \\
  \epsilon_{T-1} \left[ 2C_{T-1} (1 + \mu) \sigma_{T-2} (x_{T-2} + u_{T-2})^2 \right] + \\
  \epsilon_{T-1}^2 \left[ p_{T-1} \alpha_1 \sigma_{T-2}^2 + C_{T-1} \sigma_{T-2}^2 (x_{T-2} + u_{T-2})^2 \right]
\end{array} \right] \right\}.
\]

The second step of the approximation procedure involves approximating the coefficient of \( \epsilon_{T-1} \) in the exponent of \( V_{T-2} \), \( \mathcal{L} \), with a function linear in \( (x_{T-2} + u_{T-2}) \) using the first-order Taylor's expansion:

\[
\mathcal{L} \equiv 2C_{T-1} (1 + \mu) \sigma_{T-2} (x_{T-2} + u_{T-2})^2 - b_{T-1} \sigma_{T-2} (x_{T-2} + u_{T-2}) + [4C_{T-1} (1 + \mu) \sigma_{T-2} x_0] (x_{T-2} + u_{T-2} - x_0)
\]

\[
= -b_{T-1} \sigma_{T-2} (x_{T-2} + u_{T-2}) - 2C_{T-1} (1 + \mu) \sigma_{T-2} x_0^2 + 4C_{T-1} (1 + \mu) \sigma_{T-2} x_0 (x_{T-2} + u_{T-2}).
\]

The third step of the approximation procedure involves approximating the coefficient of \( \epsilon_{T-1}^2 \) in the exponent of \( V_{T-2} \) with a constant, replacing \( \sigma_{T-2}^2 \) with \( E_s \) and \( (x_{T-2} + u_{T-2}) \) with \( x_0 \). Therefore, the value function can be approximated with

\[
V_{T-2} \left( x_{T-2}^0, x_{T-2}, \sigma_{T-2}^2 \right) =
\max_{u_{T-2}} E_{T-2} \left\{ - \exp \left[ \begin{array}{c}
  z_{T-1} - \gamma (1 + r_f)^2 \left( x_{T-2}^0 - u_{T-2} - \tau u_{T-2}^2 \right) - \\
  b_{T-1} (1 + \mu) (x_{T-2} + u_{T-2}) + \\
  p_{T-1} \left( \alpha_0 + \beta \sigma_{T-2}^2 \right) + C_{T-1} (1 + \mu)^2 (x_{T-2} + u_{T-2})^2 + \\
  \epsilon_{T-1} \left[ -2C_{T-1} (1 + \mu) \sigma_{T-2} x_0^2 \right] + \\
  \epsilon_{T-1}^2 \left[ p_{T-1} \alpha_1 E_s + C_{T-1} E_s x_0^2 \right]
\end{array} \right] \right\}.
\]

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and by applying Proposition 5.1 the above optimization problem reduces to

\[
V_{T-2}(x_{T-2}^0, x_{T-2}, \sigma_{T-2}^2) = \max_{u_{T-2}} \sqrt{\Lambda_{T-2}} \exp \left\{ \frac{x_{T-1} - \gamma (1 + r_f)^2 (x_{T-2}^0 - u_{T-2} - \tau u_{T-2}^2)}{b_{T-1} (1 + \mu) (x_{T-2} + u_{T-2}) + p_{T-1} (\alpha_0 + \beta \sigma_{T-2}^2)} + C_{T-1} (1 + \mu)^2 (x_{T-2} + u_{T-2})^2 \right\} - \frac{1}{2} \Lambda_{T-2} \sigma_{T-2}^2 \left[ \frac{[4C_{T-1} (1 + \mu) x_0 - b_{T-1}] (x_{T-2} + u_{T-2}) - 2C_{T-1} (1 + \mu) x_0^2}{2C_{T-1} (1 + \mu) x_0^2} \right]^2,
\]

where

\[
\Lambda_{T-2} = \left[ 1 - 2 \left( p_{T-1} \alpha_1 E_s + C_{T-1} E_s x_0^2 \right) \right]^{-1}.
\]

The optimal solution of the above optimization problem is given by

\[
-\gamma (1 + r_f)^2 - 2r \gamma (1 + r_f)^2 u_{T-2} + b_{T-1} (1 + \mu) - 2C_{T-1} (1 + \mu)^2 (x_{T-2} + u_{T-2}) - \Lambda_{T-2} \sigma_{T-2}^2 \left\{ [4C_{T-1} (1 + \mu) x_0 - b_{T-1}] (x_{T-2} + u_{T-2}) - 2C_{T-1} (1 + \mu) x_0^2 \right\} \\
\left[ 4C_{T-1} (1 + \mu) x_0 - b_{T-1} \right] = 0.
\]

As a result, the approximate optimal investment in the risky asset at time \( T - 2 \) is

\[
\bar{u}_{T-2}(x_{T-2}, \sigma_{T-2}^2) = m_{T-2} (\sigma_{T-2}^2) - L_{T-2} (\sigma_{T-2}^2) x_{T-2},
\]

where

\[
\begin{align*}
\lambda_{T-2} &= 2C_{T-1} (1 + \mu)^2, \\
\delta_{T-2} &= \Lambda_{T-2} \left[ 4C_{T-1} (1 + \mu) x_0 - b_{T-1} \right]^2, \\
\pi_{T-2} &= -\gamma (1 + r_f)^2 + b_{T-1} (1 + \mu), \\
\varphi_{T-2} &= 2A_{T-2} C_{T-1} \left[ 4C_{T-1} (1 + \mu) x_0 - b_{T-1} \right] (1 + \mu) x_0^2,
\end{align*}
\]

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and

\[ Q_{T-2} (\sigma^2_{T-2}) = 2\tau \gamma (1 + r_f)^2 + \lambda_{T-2} + \delta_{T-2} \sigma^2_{T-2}, \]

\[ m_{T-2} (\sigma^2_{T-2}) = \frac{\pi_{T-2} + \phi_{T-2} \sigma^2_{T-2}}{Q_{T-2}}, \]

\[ L_{T-2} (\sigma^2_{T-2}) = \frac{\lambda_{T-2} + \delta_{T-2} \sigma^2_{T-2}}{Q_{T-2}}. \]

Substituting the approximate control policy in the value function yields the approximated cost-to-go function \( \hat{V}_{T-2} \)

\[ \hat{V}_{T-2} \left( x^0_{T-2}, x_{T-2}, \sigma^2_{T-2} \right) = \]

\[-\sqrt{\Lambda_{T-2}} \exp \left[ \begin{array}{c}
    z_{T-1} - \gamma (1 + r_f)^2 x^0_{T-2} + \gamma (1 + r_f)^2 [m_{T-2} - L_{T-2} x_{T-2}] + \\
    \tau \gamma (1 + r_f)^2 [m_{T-2} - L_{T-2} x_{T-2}]^2 - \\
    b_{T-1} (1 + \mu) [m_{T-2} - (1 - L_{T-2}) x_{T-2}] + \\
    p_{T-1} \left( \alpha_0 + \beta \sigma^2_{T-2} \right) + C_{T-1} (1 + \mu)^2 [m_{T-2} + (1 - L_{T-2}) x_{T-2}]^2 + \\
    \frac{1}{2} \Lambda_{T-2} \sigma^2_{T-2} \left\{ \begin{array}{c}
        4C_{T-1} (1 + \mu) x_0 - b_{T-1} [m_{T-2} + (1 - L_{T-2}) x_{T-2}] \\
        -2C_{T-1} (1 + \mu) x_0^2
    \end{array} \right\}^2
\end{array} \right]. \]

We prove the following theorem that yields the proposed approximation algorithm:

**Theorem 7.1** Under Approximation A, the optimal investment decisions \( u^*_x \) and the value function \( V_{T-k} \) are approximated for \( k = 1, \ldots, T \) by the following relations:

\[ \hat{u}_{T-k} \left( x_{T-k}, \sigma^2_{T-k} \right) = m_{T-k} \left( \sigma^2_{T-k} \right) - L_{T-k} \left( \sigma^2_{T-k} \right) x_{T-k}, \]

\[ \hat{V}_{T-k} \left( x^0_{T-k}, x_{T-k}, \sigma^2_{T-k} \right) = -\prod_{m=1}^{k} \sqrt{\Lambda_{T-m}} \exp \left[ -\gamma (1 + r_f)^k x^0_{T-k} + \mathcal{F}_{T-k} \right], \]

where

\[ m_{T-k} \left( \sigma^2_{T-k} \right) = \frac{\pi_{T-k} + \phi_{T-k} \sigma^2_{T-k}}{2\tau \gamma (1 + r_f)^k + \lambda_{T-k} + \delta_{T-k} \sigma^2_{T-k}}, \]

\[ L_{T-k} \left( \sigma^2_{T-k} \right) = \frac{\lambda_{T-k} + \delta_{T-k} \sigma^2_{T-k}}{2\tau \gamma (1 + r_f)^k + \lambda_{T-k} + \delta_{T-k} \sigma^2_{T-k}}. \]
\[ \lambda_{T-k} = 2C_{T-k+1} (1 + \mu)^2, \]
\[ \delta_{T-k} = \Lambda_{T-k} \left[ 4C_{T-k+1} (1 + \mu) x_0 - b_{T-k+1} \right]^2, \]
\[ \pi_{T-k} = -\gamma (1 + \tau_f)^k + b_{T-k+1} (1 + \mu), \]
\[ \varphi_{T-k} = 2\Lambda_{T-k} C_{T-k+1} \left[ 4C_{T-k+1} (1 + \mu) x_0 - b_{T-k+1} \right] (1 + \mu) x_0^2, \]

and

\[ \mathcal{F}_{T-k} = z_{T-k+1} + \gamma (1 + \tau_f)^k [m_{T-k} - L_{T-k} x_{T-k}] + \]
\[ p_{T-k+1} \left( \alpha_0 + \beta \frac{\sigma_{T-k}^2}{\gamma_{T-k}} \right) + C_{T-k+1} (1 + \mu)^2 [m_{T-k} + (1 - L_{T-k}) x_{T-k}]^2 + \]
\[ \frac{1}{2} \Lambda_{T-k} \sigma_{T-k}^2 \left[ 4C_{T-k+1} (1 + \mu) x_0 - b_{T-k+1} \right] [m_{T-k} + (1 - L_{T-k}) x_{T-k}]^2. \]

The parameters appearing are given by

\[ \Lambda_{T-k} = \left[ 1 - 2 \left( p_{T-k+1} \alpha_1 E_s + C_{T-k+1} E_s x_0^2 \right) \right]^{-1}, \]
\[ z_{T-k} = f_{0,T-k} - f_{1,T-k} x_0 - f_{2,T-k} E_s + \frac{1}{2} f_{11,T-k} x_0^2, \]
\[ b_{T-k} = -f_{21,T-k} + f_{11,T-k} x_0, \]
\[ p_{T-k} = f_{2,T-k}, \]
\[ C_{T-k} = \frac{1}{2} f_{11,T-k}, \]

where \( f_{0,T-k} \) is the value of \( \mathcal{F}_{T-k} \) at the point \((x_0, E_s)\) and

\[ f_{1,T-k} = \left[ \frac{\partial \mathcal{F}_{T-k}}{\partial x_{T-k}} \right]_{(x_0, E_s)}, \quad f_{2,T-k} = \left[ \frac{\partial \mathcal{F}_{T-k}}{\partial E_s} \right]_{(x_0, E_s)}, \quad f_{11,T-k} = \left[ \frac{\partial^2 \mathcal{F}_{T-k}}{\partial x_{T-k}^2} \right]_{(x_0, E_s)}.
\]

Finally, the boundary conditions are given by

\[ z_T = p_T = C_T = 0, \]
\[ b_T = \gamma. \]
Proof. We prove the theorem by induction. We have shown the relations for \( k = 1 \). Assuming that they are valid for arbitrary \( k \), we show that they are true for \( k + 1 \). The proof is similar to the one presented in Section 4.2 for a quadratic utility function. By construction \( F_{T-k} \) is a function that is quadratic in the risky holdings and nonlinear in the conditional variance at time \( T - k \). Therefore, the first step of the approximation procedure involves approximating \( F_{T-k} \left( x_{T-k}, \sigma^2_{T-k} \right) \) with a function \( \tilde{F}_{T-k} \left( x_{T-k}, \sigma^2_{T-k} \right) \) that is quadratic in the risky holdings and linear in the conditional variance of the single asset, and is of the following form

\[
\tilde{F}_{T-k} \left( x_{T-k}, \sigma^2_{T-k} \right) = F_{T-k} \left( x_0, E_s \right) + \left[ \frac{\partial F_{T-k}}{\partial x_{T-k}} \right]_{(x_0, E_s)} (x_{T-k} - x_0) + \frac{1}{2} \left[ \frac{\partial^2 F_{T-k}}{\partial x^2_{T-k}} \right]_{(x_0, E_s)} (x_{T-k} - x_0)^2.
\]

Let

\[
\tilde{Q}_{T-k} = 2\tau \gamma (1 + r_f)^k + \lambda_{T-k} + \delta_{T-k} E_s,
\]

\[
\tilde{m}_{T-k} = \frac{\tilde{Q}_{T-k}}{\tilde{Q}_{T-k}} \pi_{T-k} + \varphi_{T-k} E_s,
\]

\[
\tilde{L}_{T-k} = \frac{\lambda_{T-k} + \delta_{T-k} E_s}{\tilde{Q}_{T-k}},
\]

\[
\tilde{m1}_{T-k} = \frac{\varphi_{T-k} \tilde{Q}_{T-k} - \left( \pi_{T-k} + \varphi_{T-k} E_s \right) \delta_{T-k}}{\left( \tilde{Q}_{T-k} \right)^2},
\]

\[
\tilde{L1}_{T-k} = \frac{2\tau \gamma (1 + r_f)^k \delta_{T-k}}{\left( \tilde{Q}_{T-k} \right)^2},
\]

where \( \tilde{m}_{T-k} \) is the value of \( m_{T-k} \left( \sigma^2_{T-k} \right) \) at the point \( E_s \) and \( \tilde{m1}_{T-k} \) is the first derivative of \( m_{T-k} \) with respect to \( \sigma^2_{T-k} \) evaluated at \( E_s \). Similar notation prevails for the derivatives of \( L_{T-k} \). As a result,

\[
\tilde{F}_{T-k} = f_{0,T-k} + f_{1,T-k} (x_{T-k} - x_0) + f_{2,T-k} \left( \sigma^2_{T-k} - E_s \right) + \frac{1}{2} f_{11,T-k} (x_{T-k} - x_0)^2,
\]

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where $f_{0,T-k}$ is the value of $F_{T-k}$ at the point $(x_0, E_s)$ given by

$$f_{0,T-k} = x_{T-k+1} + \gamma (1 + \tau \rho T_{k})^{k} \left[ \hat{m}_{T-k} - \hat{L}_{T-k} \right] x_{0} + \gamma (1 + \tau \rho T_{k})^{k} \left[ \hat{m}_{T-k} - \hat{L}_{T-k} \right] x_{0}^{2} -$$

$$b_{T-k+1} (1 + \mu) \left[ \hat{m}_{T-k} + (1 - \hat{L}_{T-k}) x_{0} \right] + p_{T-k+1} (\alpha_{0} + \beta E_{s}) +$$

$$C_{T-k+1} (1 + \mu)^{2} \left[ \hat{m}_{T-k} + (1 - \hat{L}_{T-k}) x_{0} \right]^{2} +$$

$$\frac{1}{2} \Lambda_{T-k} E_{s} \left\{ \left[ 4 C_{T-k+1} (1 + \mu) x_{0} - b_{T-k+1} \right] \left[ \hat{m}_{T-k} + (1 - \hat{L}_{T-k}) x_{0} \right] - \right\}^{2},$$

and the desired partial derivatives of $F_{T-k}$ are

$$f_{1,T-k} = -\gamma (1 + \tau \rho T_{k})^{k} \hat{L}_{T-k} - 2 \gamma (1 + \tau \rho T_{k})^{k} \hat{m}_{T-k} \hat{L}_{T-k} - b_{T-k+1} (1 + \mu) \left( 1 - \hat{L}_{T-k} \right) +$$

$$2 C_{T-k+1} (1 + \mu)^{2} \hat{m}_{T-k} \left( 1 - \hat{L}_{T-k} \right) +$$

$$\Lambda_{T-k} E_{s} \left[ 4 C_{T-k+1} (1 + \mu) x_{0} \hat{m}_{T-k} - b_{T-k+1} \hat{m}_{T-k} - 2 C_{T-k+1} (1 + \mu) x_{0}^{2} \right]$$

$$+ \left\{ \frac{1}{2} \Lambda_{T-k} E_{s} \left[ 4 C_{T-k+1} (1 + \mu) x_{0} - b_{T-k+1} \right] \left( 1 - \hat{L}_{T-k} \right)^{2} \right\} x_{0},$$

$$f_{11,T-k} = 2 \gamma (1 + \tau \rho T_{k})^{k} \left( \hat{L}_{T-k} \right)^{2} + 2 C_{T-k+1} (1 + \mu)^{2} \left( 1 - \hat{L}_{T-k} \right)^{2} +$$

$$\Lambda_{T-k} E_{s} \left[ 4 C_{T-k+1} (1 + \mu) x_{0} - b_{T-k+1} \right] \left( 1 - \hat{L}_{T-k} \right)^{2},$$

$$f_{2,T-k} = \gamma (1 + \tau \rho T_{k})^{k} \hat{m}_{T-k} \hat{m}_{T-k} -$$

$$b_{T-k+1} (1 + \mu) \hat{m}_{T-k} + p_{T-k+1} \beta + 2 C_{T-k+1} (1 + \mu)^{2} \hat{m}_{T-k} \hat{m}_{T-k} +$$

$$\frac{1}{2} \Lambda_{T-k} \left\{ \left[ 4 C_{T-k+1} (1 + \mu) x_{0} - b_{T-k+1} \right] \hat{m}_{T-k} - 2 C_{T-k+1} (1 + \mu) x_{0}^{2} \right\}^{2} +$$

$$\Lambda_{T-k} E_{s} \left\{ \left[ 4 C_{T-k+1} (1 + \mu) x_{0} - b_{T-k+1} \right] \hat{m}_{T-k} - 2 C_{T-k+1} (1 + \mu) x_{0}^{2} \right\}$$

$$\left[ 4 C_{T-k+1} (1 + \mu) x_{0} - b_{T-k+1} \right] \hat{m}_{T-k} -$$
\[
\begin{align*}
\left\{ \begin{array}{l}
\gamma (1 + r_f)^k \tilde{\mu}_{T-k} + 2\tau \gamma (1 + r_f)^k \left[ \tilde{m}_{T-k} \tilde{L}_{T-k} + \tilde{m}_{T-k} \tilde{L}_{T-k} \right] - \\
\beta_{T-k+1} (1 + \mu) \tilde{L}_{T-k} - \\
2\gamma (1 + \mu)^2 \left[ \tilde{m}_{T-k} - \tilde{m}_{T-k} \tilde{L}_{T-k} - \tilde{m}_{T-k} \tilde{L}_{T-k} \right] - \\
\Lambda_{T-k} [4\gamma (1 + \mu) x_0 - b_{T-k+1}] (1 - \tilde{L}_{T-k}) \\
\{(4\gamma (1 + \mu) x_0 - b_{T-k+1}) \tilde{m}_{T-k} - 2\gamma (1 + \mu) x_0^2\} - \\
\Lambda_{T-k} [4\gamma (1 + \mu) x_0 - b_{T-k+1}]^2 E_s \tilde{m}_{T-k} (1 - \tilde{L}_{T-k}) + \\
\Lambda_{T-k} [4\gamma (1 + \mu) x_0 - b_{T-k+1}] E_s \tilde{L}_{T-k} \\
\{(4\gamma (1 + \mu) x_0 - b_{T-k+1}) \tilde{m}_{T-k} - 2\gamma (1 + \mu) x_0^2\}
\end{array} \right\}
\end{align*}
\]

x_0 +

\[
\begin{align*}
\left\{ \begin{array}{l}
2\gamma (1 + r_f)^k \tilde{L}_{T-k} \tilde{L}_{T-k} - 2\gamma (1 + \mu)^2 (1 - \tilde{L}_{T-k}) \tilde{L}_{T-k} + \\
\frac{1}{2} \Lambda_{T-k} [4\gamma (1 + \mu) x_0 - b_{T-k+1}]^2 (1 - \tilde{L}_{T-k})^2 - \\
\Lambda_{T-k} [4\gamma (1 + \mu) x_0 - b_{T-k+1}]^2 E_s (1 - \tilde{L}_{T-k}) \tilde{L}_{T-k}
\end{array} \right\}
\end{align*}
\]

x_0^2.

So, now by letting

\[
x_{T-k} = f_{0,T-k} - f_{1,T-k} x_0 - f_{2,T-k} E_s + \frac{1}{2} f_{11,T-k} x_0^2,
\]

\[
b_{T-k} = -f_{1,T-k} + f_{11,T-k} x_0,
\]

\[
p_{T-k} = f_{2,T-k},
\]

\[
C_{T-k} = \frac{1}{2} f_{11,T-k},
\]

we obtain that

\[
\tilde{F}_{T-k} (x_{T-k}, \sigma_{T-k}^2) = x_{T-k} - b_{T-k} x_{T-k} + p_{T-k} \sigma_{T-k}^2 + C_{T-k} x_{T-k}^2,
\]

and we express the value function at time \( T - k - 1 \), and consequently the optimization problem, as

\[
\begin{align*}
V_{T-k-1} (x_{T-k-1}, x_{T-k-1}, \sigma_{T-k-1}^2) &\equiv \max_{u_{T-k-1}} \mathcal{E}_{T-k-1} \{ \tilde{V}_{T-k} \} = \\
\max_{u_{T-k-1}} - \prod_{m=1}^{k} \sqrt{\Lambda_{T-m}} E_{T-k-1} \left\{ \exp \left[ -\gamma (1 + r_f)^k x_{T-k}^0 + z_{T-k} - b_{T-k} x_{T-k} + p_{T-k} \sigma_{T-k}^2 + C_{T-k} x_{T-k}^2 \right] \right\}.
\end{align*}
\]

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Substituting for the wealth and asset return dynamics, \( V_{T-k-1} \) is given by

\[
V_{T-k-1} \left( x_{T-k-1}^0, x_{T-k-1}, \sigma_{T-k-1}^2 \right) = - \prod_{m=1}^{k} \sqrt{\Lambda_{T-m}} \\
\mathop{\max}_{u_{T-k-1}} E_{T-k-1} \left\{ \exp \left[ \begin{array}{l} -\gamma (1 + r_f)^{k+1} \left[ x_{T-k-1}^0 - u_{T-k-1} - \tau u_{T-k-1}^2 \right] + z_{T-k}^- \\ b_{T-k} (1 + \mu + \sigma_{T-k-1} \ v_{T-k}) (x_{T-k-1} + u_{T-k-1}) + \\ p_{T-k} \left( \alpha_0 + \beta \sigma_{T-k-1}^2 + \alpha_1 \sigma_{T-k-1}^2 \ v_{T-k}^2 \right) + \\ C_{T-k} (1 + \mu + \sigma_{T-k-1} \ v_{T-k})^2 (x_{T-k-1} + u_{T-k-1})^2 \\ \end{array} \right] \right\}.
\]

We now perform the following operations in the above optimization problem:

1. We approximate the coefficient of \( \varepsilon_{T-k} \) with a linear function in \( (x_{T-k-1} + u_{T-k-1}) \).

2. We approximate the coefficient of \( \varepsilon_{T-k}^2 \) with a constant.

Therefore, the value function \( \hat{V}_{T-k-1} \) is approximated by

\[
\hat{V}_{T-k-1} \left( x_{T-k-1}^0, x_{T-k-1}, \sigma_{T-k-1}^2 \right) = - \prod_{m=1}^{k} \sqrt{\Lambda_{T-m}} \\
\mathop{\max}_{u_{T-k-1}} E_{T-k-1} \left\{ \exp \left[ \begin{array}{l} -\gamma (1 + r_f)^{k+1} \left[ x_{T-k-1}^0 - u_{T-k-1} - \tau u_{T-k-1}^2 \right] + z_{T-k}^- \\ b_{T-k} (1 + \mu) (x_{T-k-1} + u_{T-k-1}) + p_{T-k} \left( \alpha_0 + \beta \sigma_{T-k-1}^2 \right) + \\ C_{T-k} (1 + \mu)^2 (x_{T-k-1} + u_{T-k-1})^2 + \\ \varepsilon_{T-k} \left\{ \begin{array}{l} 4C_{T-k} (1 + \mu) \sigma_{T-k-1} x_0 \\ \sigma_{T-k-1} \\ -2C_{T-k} (1 + \mu) \sigma_{T-k-1} x_0^2 \\ \varepsilon_{T-k}^2 \left\{ p_{T-k} \alpha_1 E_s + C_{T-k} E_s x_0^2 \right\} \\ \end{array} \right\} (x_{T-k-1} + u_{T-k-1}) \right] \right\}.
\]

and by applying Proposition 5.1 the above optimization problem reduces to

\[
\hat{V}_{T-k-1} \left( x_{T-k-1}^0, x_{T-k-1}, \sigma_{T-k-1}^2 \right) = - \prod_{m=1}^{k+1} \sqrt{\Lambda_{T-m}}
\]
\[
\max_{u_{T-k-1}} \exp \left\{ -\gamma (1 + r_f)^{k+1} \left[ x^0_{T-k-1} - u_{T-k-1} - \tau u_x^2_{T-k-1} \right] + z_{T-k-1} \right. \\
\left. b_{T-k} \ (1 + \mu) \ (x_{T-k-1} + u_{T-k-1}) + p_{T-k} \ \left( a_0 + \beta \sigma^2_{T-k-1} \right) + \\
C_{T-k} \ (1 + \mu)^2 \ (x_{T-k-1} + u_{T-k-1})^2 + \\
\frac{1}{2} \Lambda_{T-k-1} \sigma^2_{T-k-1} \left[ 4C_{T-k} (1 + \mu) x_0 - b_{T-k} \right] (x_{T-k-1} + u_{T-k-1})^2 \right\}^{\gamma},
\]

where

\[
\Lambda_{T-k-1} = \left[ 1 - 2 \left( p_{T-k} \ a_1 E_s + C_{T-k} E_s x_0^2 \right) \right]^{-1}.
\]

As a result, the optimal investment decision at time \( T - k - 1 \) is approximated by

\[
\tilde{u}_{T-k-1} (x_{T-k}, \sigma^2_{T-k}) = m_{T-k-1} (\sigma^2_{T-k}) - L_{T-k-1} (\sigma^2_{T-k}) \ x_{T-k-1},
\]

where

\[
m_{T-k-1} (\sigma^2_{T-k}) = \frac{\pi_{T-k-1} + \varphi_{T-k-1} \ \sigma^2_{T-k-1}}{2\gamma (1 + r_f)^{k+1} + \lambda_{T-k-1} + \delta_{T-k-1} \ \sigma^2_{T-k}},
\]

\[
L_{T-k-1} (\sigma^2_{T-k}) = \frac{\lambda_{T-k-1} + \delta_{T-k-1} \ \sigma^2_{T-k}}{2\gamma (1 + r_f)^{k+1} + \lambda_{T-k-1} + \delta_{T-k-1} \ \sigma^2_{T-k}},
\]

\[
\lambda_{T-k-1} = 2C_{T-k} \ (1 + \mu)^2,
\]

\[
\delta_{T-k-1} = \Lambda_{T-k-1} \left[ \frac{4C_{T-k} (1 + \mu) x_0 - b_{T-k}}{2} \right]^2,
\]

\[
\pi_{T-k-1} = -\gamma (1 + r_f)^{k+1} + b_{T-k} \ (1 + \mu),
\]

\[
\varphi_{T-k-1} = 2\Lambda_{T-k-1} C_{T-k} (1 + \mu) x_0^2 \left[ \frac{4C_{T-k} (1 + \mu) x_0 - b_{T-k}}{2} \right].
\]

Finally, substituting back into the expression of \( \tilde{V}_{T-k-1} \) we obtain that

\[
\tilde{V}_{T-k-1} (x^0_{T-k-1}, x_{T-k-1}, \sigma^2_{T-k-1}) = -\prod_{m=1}^{k+1} \sqrt{\Lambda_{T-m}} \exp \left\{ -\gamma (1 + r_f)^{k+1} x^0_{T-k-1} + F_{T-k-1} \right\},
\]

where

\[
F_{T-k-1} = z_{T-k} + \gamma (1 + r_f)^{k+1} \left[ m_{T-k-1} - L_{T-k-1} x_{T-k-1} \right] + \\
\tau \gamma (1 + r_f)^{k+1} \left[ m_{T-k-1} - L_{T-k-1} x_{T-k-1} \right]^2 -
\]

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\[ b_{T-k} (1 + \mu) [m_{T-k-1} + (1 - L_{T-k-1}) x_{T-k-1}] + \]
\[ p_{T-k} \left( \alpha_0 + \beta \sigma^2_{T-k-1} \right) + C_{T-k} (1 + \mu)^2 [m_{T-k-1} + (1 - L_{T-k-1}) x_{T-k-1}]^2 + \]
\[ \frac{1}{2} \lambda_{T-k-1}\sigma^2_{T-k-1} \left[ \begin{array}{c}
4C_{T-k} (1 + \mu) x_0 - b_{T-k} [m_{T-k-1} + (1 - L_{T-k-1}) x_{T-k-1}] \\
-2C_{T-k} (1 + \mu) x_0^2
\end{array} \right]^2. \]

The approximation algorithm proposed in this section utilizes characteristics of the optimal investment policy over time and analyzes in-depth the DP recursion. In Section 7.3 we identify the parameters that influence the investment behavior and evaluate the performance of the proposed policy relative to alternative dynamic trading strategies.

### 7.3 Computational Experiments

In this section, we analyze the effect of stochastic volatility models on long-horizon asset allocation and quantify the impact of having some degree of predictability in the volatility of the time-series asset return dynamics to the manager's investment behavior over time. In our comparative analysis we present a comparison of the performance of the following dynamic trading strategies:

- The approximate structured policy described in Section 7.2 (Structured).
- The approximate optimize-and-hold policy described in Section 3.3 (Opt-and-Hold).
- The static policy that derives as the solution to a series of single-period optimization problems (Static).

We consider the case where \( \gamma = 0.1, \mu = 0.15, \alpha_1 = 0.15, \beta = 0.4, r_f = 0.05, \tau = 0.01 \) and \( x_0^0 = 1, x_0 = 1 \). Thus, the single risky asset is assumed to have a constant expected return of 15%. In what follows, we analyze the effect of the following parameters to the investment behavior over time:

1. The unconditional expected variance of the risky asset \( E_r \), through the parameter \( \alpha_0 \).
2. The time horizon \( T \).
Relative Performance of Dynamic Policies

In Table 7.1 we report the expected utility of final wealth for the different policies considered, for \( T = 10 \) and various values of \( \alpha_0 \). The best policy found for the ranges of the asset’s volatility considered is the optimize-and-hold strategy, since it is advantageous to invest not just inversely proportional to the realizations of the asset’s variance, but according to a way that is “more” nonlinear. Moreover, the structured policy always outperforms the myopic one, but its relative improvement decreases as the risky asset becomes more volatile.

<table>
<thead>
<tr>
<th>( T = 10 )</th>
<th>Structured</th>
<th>Opt-and-Hold</th>
<th>Static</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_0 = 0.0045 ) (( E_s = 0.1^2 ))</td>
<td>(-1.152 \times 10^{-6})</td>
<td>(-1.539 \times 10^{-10})</td>
<td>(-0.0144)</td>
</tr>
<tr>
<td>( \alpha_0 = 0.0405 ) (( E_s = 0.3^2 ))</td>
<td>(-0.1301)</td>
<td>(-0.0264)</td>
<td>(-0.2125)</td>
</tr>
<tr>
<td>( \alpha_0 = 0.1125 ) (( E_s = 0.5^2 ))</td>
<td>(-0.4684)</td>
<td>(-0.4287)</td>
<td>(-0.4772)</td>
</tr>
</tbody>
</table>

Table 7.1: The expected utility of final wealth for \( T = 10 \) as given by the simulation experiment. The expectations are taken over 1,000 simulated paths of the risky asset return.

For longer investment horizons, the levels of utility are increased as shown in Table 7.2. The investor is able to better allocate his asset holdings, due to the improved opportunity set, and thus to reduce his portfolio variance. For small asset volatilities the best policy found is once again the optimize-and-hold strategy, but as the asset becomes more volatile its performance deteriorates. Indeed, since this policy is appropriate for investors with quadratic instead of exponential utilities, it produces investment decisions that increase with age and thus results in a more risky investment behavior. The best policy found for high asset volatilities and big investment horizons is the myopic policy.

<table>
<thead>
<tr>
<th>( T = 20 )</th>
<th>Structured</th>
<th>Opt-and-Hold</th>
<th>Static</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_0 = 0.0045 ) (( E_s = 0.1^2 ))</td>
<td>(-4.373 \times 10^{-11})</td>
<td>(-2.126 \times 10^{-9})</td>
<td>(-3.043 \times 10^{-11})</td>
</tr>
<tr>
<td>( \alpha_0 = 0.0405 ) (( E_s = 0.3^2 ))</td>
<td>(-0.0145)</td>
<td>(-0.0024)</td>
<td>(-0.0153)</td>
</tr>
<tr>
<td>( \alpha_0 = 0.1125 ) (( E_s = 0.5^2 ))</td>
<td>(-0.2451)</td>
<td>(-0.2987)</td>
<td>(-0.2027)</td>
</tr>
</tbody>
</table>

Table 7.2: The expected utility of final wealth for \( T = 20 \) as given by the simulation experiment. The expectations are taken over 1,000 simulated paths of the risky asset return.

In Figures 7-1, 7-2 and 7-3 we plot the expected risky investment as a function of time for different time horizons. When the asset’s volatility is small an aggressive policy, given by the optimize-and-hold strategy, is suggested with the portfolio manager always buying the risky asset but with a decreasing rate. On the contrary, when the asset’s volatility is high, the
manager buys the risky asset in the beginning of the investment horizon, thus taking advantage of the possibility of high returns early on, and continues by selling it throughout the remaining time to expiration.

![Graphs showing expected investment over time](image)

Figure 7-1: The expected investment decision, plotted as a function of time for $\alpha_0 = 0.0045$. In Panel (a) we consider an investment horizon of $T = 10$ and in Panel (b) of $T = 20$. 
Figure 7-2: The expected investment decision, plotted as a function of time for $\alpha_0 = 0.0405$. In Panel (a) we consider an investment horizon of $T = 10$ and in Panel (b) of $T = 20$. 
Figure 7-3: The expected investment decision, plotted as a function of time for $\alpha_0 = 0.1125$. In Panel (a) we consider an investment horizon of $T = 10$ and in Panel (b) of $T = 20$. 
Chapter 8

A Comparative Study for Large Scale Portfolios

In this chapter, we present the results from the application of the proposed approximate dynamic policies to a large scale portfolio optimization problem. The purpose of this computational exercise is to illustrate the feasibility and value of solving dynamic portfolio optimization problems by the approximate dynamic programming algorithm suggested in this thesis, and to contrast the ADP policies found for the two utilities we considered in this thesis.

We consider an investment manager holding a portfolio of \( N = 10 \) assets and having an investment horizon of \( T = 50 \) time periods. The investor identifies \( K = 3 \) serially correlated factors that influence the asset return dynamics. Thus, the return generating process is given by

\[
\begin{align*}
    r_t &= c + A f_t + \epsilon_t, \\
    f_t &= d + B f_{t-1} + \eta_t.
\end{align*}
\]

We assume that \( r_f = 5\% \), \( \gamma = 0.2 \), \( \lambda = 0.1 \), and that the initial holdings in all assets are one million dollars (account unit=1M), \( x_0^0 = 1 \), \( x_0 = [1, \ldots, 1]' \). The transaction cost coefficients are considered to be

\[
\tau_1 = \tau_2 = \tau_3 = \tau_6 = \tau_8 = 0.01,
\]
\[ \tau_4 = \tau_5 = \tau_7 = \tau_9 = \tau_{10} = 0.008. \]

The symmetric correlation matrices are assumed to be given by

\[
\rho_\varepsilon = \begin{bmatrix}
1 \\
0.2 & 1 \\
0.3 & 0.1 & 1 \\
0.1 & 0.3 & 0.3 & 1 \\
0.4 & 0.2 & 0.2 & 0.1 & 1 \\
0.1 & 0.2 & 0.1 & 0.2 & 0.2 & 1 \\
0.3 & 0.1 & 0.4 & 0.2 & 0.1 & 0.2 & 1 \\
0.6 & 0.3 & 0.2 & 0.3 & 0.1 & 0.3 & 0.5 & 1 \\
0.2 & 0.1 & 0.3 & 0.4 & 0.2 & 0.1 & 0.2 & 0.5 & 1 \\
0.2 & 0.3 & 0.2 & 0.1 & 0.1 & 0.2 & 0.6 & 0.1 & 0.2 & 1
\end{bmatrix}
\]

and

\[
\rho_\eta = \begin{bmatrix}
1 \\
0.2 & 1 \\
-0.1 & 0.1 & 1
\end{bmatrix}
\]

It is also assumed that

\[
c = \begin{bmatrix}
-0.30 \\
-0.05 \\
-0.42 \\
0.31 \\
0 \\
-0.33 \\
0.10 \\
0.16 \\
0.13 \\
-0.10
\end{bmatrix},
A = \begin{bmatrix}
0.35 & 0.20 & 0.10 \\
0.25 & -0.15 & 0.35 \\
0.50 & 0.10 & 0.25 \\
-0.25 & -0.15 & 0.35 \\
-0.05 & 0.50 & -0.25 \\
0.55 & 0.05 & -0.05 \\
0.05 & 0.05 & -0.05 \\
0.13 & -0.21 & 0.07 \\
-0.08 & 0.14 & -0.12 \\
0.31 & 0.08 & -0.28
\end{bmatrix},
\sigma_\varepsilon = \begin{bmatrix}
0.25 \\
0.32 \\
0.28 \\
0.15 \\
0.15 \\
0.25 \\
0.21 \\
0.26 \\
0.10 \\
0.12
\end{bmatrix}
\]
\[
\mathbf{d} = \begin{bmatrix}
0.40 \\
0.45 \\
0.30
\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}
0.50 & 0.20 & -0.10 \\
0.20 & -0.30 & 0.10 \\
-0.10 & 0.10 & 0.30
\end{bmatrix}, \quad \sigma_\eta = \begin{bmatrix}
0.25 \\
0.35 \\
0.30
\end{bmatrix}.
\]

In our comparative analysis we present a comparison of the performance of the following dynamic trading strategies for a quadratic and an exponential utility function:

- The structured approximate policy described in Sections 3.2 and 6.2 (Structured).
- The myopic policy (Static).
- The optimal policy when transaction costs are ignored (No Costs).

For all policies considered, 2,000 independent sample paths of the asset returns are simulated and for each path the approximate dynamic policies are implemented. We denote as \( \hat{V}_0 \) the expected utility of terminal wealth derived analytically from the theorems presented in the previous chapters, and as \( \overline{V}_0 \) the simulated expected utility of terminal wealth defined as the averages (over 2,000 sample paths) of the manager's utility. We also plot the average risky investment in some of the assets (over the 2,000 sample paths) as a function of time in order to explore the evolution of the portfolio composition as time to expiration decreases.

As was illustrated in the previous chapters, the myopic policy under the exponential and quadratic utility specifications is the same when \( \gamma = 2\lambda \). In Table 8.1 we present the resulted utility levels for a quadratic utility, and in Table 8.2 the certainty equivalent, that is defined as \(-1/\gamma \ln(-\overline{V}_0)\) and is expressed in account units (millions), for an exponential utility. The proposed structured policy clearly outperforms the myopic policy in both cases. The proposed approximate policy for an exponential utility yields a certainty equivalent of 303.0231M, while the one when trading costs are ignored is 336.9477M. Thus, we notice a significant increase in the utility level in the absence of transaction costs. When trading costs are ignored risky holdings are significantly higher and costless portfolio rebalancing allows investors a more effective asset allocation resulting in higher utility levels.
<table>
<thead>
<tr>
<th>Policies</th>
<th>$\bar{V}_0$</th>
<th>$\tilde{V}_0$</th>
<th>$\bar{K}_0$</th>
<th>$\tilde{K}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structured</td>
<td>230.2981</td>
<td>229.8956</td>
<td>380.4234</td>
<td>379.7158</td>
</tr>
<tr>
<td>Static</td>
<td>$-1.0066 \times 10^4$</td>
<td>$-4.529 \times 10^3$</td>
<td>$-4.529 \times 10^3$</td>
<td>$-4.529 \times 10^3$</td>
</tr>
</tbody>
</table>

Table 8.1: Monte Carlo simulation of the investment policies under investigation for quadratic utility. 2,000 independent sample paths were simulated, each path containing 50 periods.

<table>
<thead>
<tr>
<th>Policies</th>
<th>$\bar{V}_0$</th>
<th>$\tilde{V}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structured</td>
<td>303.0231</td>
<td>255.8364</td>
</tr>
<tr>
<td>No Costs</td>
<td>336.9477</td>
<td>287.4699</td>
</tr>
</tbody>
</table>

Table 8.2: Monte Carlo simulation of the investment policies under investigation for exponential utility. 2,000 independent sample paths were simulated, each path containing 50 periods.

In Figures 8-1 to 8-6 we plot the expected risky investment as a function of time for the different policies considered. We observe that the structured policies under the quadratic and exponential utility specifications exhibit similar behavior. This is due to the fact that $\gamma = 2\lambda$, and, thus, the single-period optimal solution is the same for both utilities. The expected risky investment is negative most of the time under all policies considered; due to the long investment horizon the portfolio manager sells the risky assets. Moreover, Assets 7 and 10 have negatively autocorrelated returns across time, and thus their corresponding expected risky investment under the no-costs policy oscillates substantially as illustrated in Figures 8-5 and 8-6.
Figure 8-1: The expected risky investment of Asset 1, plotted as a function of time for the different policies considered.
Figure 8-2: The expected risky investment of Asset 2, plotted as a function of time for the different policies considered.
Figure 8-3: The expected risky investment of Asset 3, plotted as a function of time for the different policies considered.
Figure 8-4: The expected risky investment of Asset 4, plotted as a function of time for the different policies considered.
Figure 8-5: The expected risky investment of Asset 7, plotted as a function of time for the different policies considered.
Figure 8-6: The expected risky investment of Asset 10, plotted as a function of time for the different policies considered.
Chapter 9

Conclusions and Future Research Directions

In this thesis we study multiperiod, discrete time portfolio optimization problems under (a) quadratic transaction costs that model the price impact effects, (b) quadratic and exponential utility functions, and (c) multifactor autocorrelated pricing and stochastic volatility models. We investigate the impact of investor's preferences, payoff functions and asset returns' predictability to the optimal investment decisions. We also answer the question of how the investment horizon influences the optimal portfolio composition.

When transaction costs can be ignored, we find the optimal investment policy over time in closed form using stochastic dynamic programming for the case of multiple assets, exponential utility and multifactor autocorrelated pricing models. For the case of a quadratic utility and general return dynamics we show that the optimal risky holdings are independent of the level of wealth, but depend on the information available at every point in time in a highly nonlinear fashion. The optimal value function is proved to be separable in the corresponding state variables (accumulated wealth and available information). A closed-form exists only when asset returns are assumed to be independent and identically distributed.

In the presence of transaction costs and multiple assets closed form solutions are not achievable. We develop a new approximate dynamic programming (ADP) methodology to find near optimal policies for such high-dimensional problems. In problems of small dimension, where ex-
act dynamic programming is feasible, our approximation produces near optimal solutions. We show that in-depth investigation of the underlying dynamic optimization problem at every iteration of the algorithm enables us to capture essential characteristics of the optimal investment policy and level of utility. We also propose and evaluate alternative dynamic trading strategies that arise from the solution of a series of stochastic optimization problems with different initial conditions and successively smaller investment horizon. In the case of complicated return dynamics, we extend this idea and present it as a systematic approach to deriving approximate trading policies. One of the clear advantages of this policy-approximation procedure is that it can easily accommodate nonnegativity and any desirable budget constraints.

We also use our ADP algorithms to understand the qualitative behavior of the optimal investment policy: we examine the effect of transaction costs, time horizon, asset correlations and volatilities on the portfolio composition over time. We show that transaction costs significantly impact both the portfolio manager's utility levels, as well as, the dynamic investment policy. As time to maturity decreases, investment decisions decrease resulting in minimal trading at the end of the investment horizon. As transaction costs, risk aversion, risk-free rate of return and asset volatilities increase, the portfolio manager trades out of the risky assets favoring the riskless investment opportunities. Moreover, as the investment horizon increases we show that initial risky investment decreases. We also report on the effect of return autocorrelations to the investment behavior over time. Positive return autocorrelation increases both the achieved utility levels, as well as, the performed change in the risky holdings relative to the negative and no autocorrelation case. Negative autocorrelation causes higher risky investment in the beginning of the investment horizon, since risky assets provide a hedging effect and appear more attractive farther from the horizon.

A continuation of the present work will address the dynamic portfolio optimization problem under proportional transaction costs and multifactor pricing and stochastic volatility models. The main focus of the literature on optimization models for portfolio management has been the effect of proportional transaction costs to the investment strategies of an investor who maximizes the expected value of a power utility function in the presence of one risky and one riskless asset under the assumption that asset prices follow a geometric Brownian motion. Even under this particular setting, numerical methods have to be performed in order to evaluate the
boundaries of the no transactions region. Furthermore, due to dimensionality problems existing analyses cannot be extended to the case of multiple risky assets and/or alternative models for the asset return dynamics. The development of approximate dynamic policies will provide interesting intuition about the investment behavior over time and the effect of having different pricing and transaction costs models to the manager's portfolio composition.

Another interesting future research direction is the application of the methodology presented in this thesis to other financial applications. Options pricing with trading restrictions and other market imperfections is an area that has attracted significant interest in recent years. Since perfect replication of the option's payoff is not possible, approximate dynamic replication strategies can be developed and prices of complex securities can be potentially derived. We plan to explore implications of transaction costs and trading restrictions to general option pricing settings in future research.
Bibliography


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