Making better binary models and modeling distorted black holes using black hole perturbation theory

by

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Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2015

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Abstract

In this thesis, I discuss the application and development of black hole perturbation theory both from an observational standpoint via gravitational waves and also tidal distortions of black hole horizons.

The promise of gravitational wave astronomy depends on our ability to accurately model gravitational wave signals from astrophysical sources. This requires large numbers of accurate theoretical template waveforms spanning large regions of parameter space to be cross-correlated against the output of gravitational-wave detectors. Numerical simulations of binary black-hole evolution are now possible but remain CPU costly. They also have problems with small mass ratios where perturbative analyses are efficient. This high computational cost has motivated the development of the effective-one-body (EOB) formalism, a framework which models the three phases of binary black hole coalescence — inspiral, plunge/merger, and ringdown — by combining information from a variety of modeling techniques. In this thesis, we combine EOB with black hole perturbation theory to study the transition from inspiral to plunge-merger and ringdown. This allows us to tune and improve the accuracy of EOB.

In Newtonian gravity, tidal coupling between members of a binary system has an influence on that binary’s dynamics. There are also well-understood connections between the geometry of the binary’s distorted members and the impact of tides on the orbit’s evolution. In this thesis we develop tools for investigating the tidal distortion of black holes for tides arising from a body in a bound orbit. We also develop tools to visualize the horizon’s distortion for black hole spin $a/M \leq \sqrt{3}/2$. In analyzing how a Kerr black hole is distorted by a small body for a circular equatorial orbit, we find that Newtonian intuition is not applicable. We also apply these techniques to generic Kerr black hole orbits, which enables us to look at time-dependent phenomena on the horizon. In particular, we find significant offsets between the applied tide and the horizon’s response, as well as small amplitude coherent wiggles in the horizon’s shear response to the applied tide. These appear to arise from the teleological nature of the horizon’s response to tides.
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Acknowledgments

This dissertation is the culmination of many years of work which involved assistance by a great number of people. I would first like to thank my advisor, Prof. Scott Hughes, for guiding me through my time as a graduate student at MIT. I am a much better researcher due to his input. I had extensive discussions with Prof. Daniel Kennefick (University of Arkansas) and Dr. Kostas Glampedakis (University of Murcia) about black hole physics which eventually lead to my investigations of black hole tidal coupling. These works constitute a large part of this thesis.

I would also like to thank Prof. Niall O’Murchadha, my advisor during my time at UCC, for introducing me to research in general relativity. I am grateful to the many teachers who have guided my intellectual curiosity over the years, especially Eugene Riordan and Derry O’Donovan who helped foster my interest in math and physics during my secondary school education at Coláiste Treasa.

I have made some great friends in the MIT Kavli Institute and the MIT Physics Department at large. In particular, I would like to thank my fellow group members, past and present, Ryan Lang, Pranesh Sundararajan, Leo Stein, Sarah Vigeland and Uchupol Ruangsri for their help and encouragement. This work involved intensive computation power and could not have been completed without our trusty cluster, Antares, and our helpful systems administrator, Paul Hsi.

I thank my parents, Eugene and Teresa O’Sullivan, as well as my extended family, for instilling in me a great work ethic, supporting my interests and assisting me in whatever avenues I chose to go through in life. Finally I would like to thank my wife, Jialing, and daughter, Kiera, for their guidance, inspiration and laughter.
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Chapter 1

Introduction

1.1 Background

Einstein established his general theory of relativity (GR) 100 years ago and radically changed how we understand gravitational physics. Using the principle of equivalence, in which gravitation can be locally transformed away in a freely falling system, he was able to generalize the rigid spacetime structure of the special theory of relativity (SR). In GR, spacetime is described by a differentiable pseudo-Riemannian manifold where the metric field $g_{\mu\nu}$ describes not only measures of space and time but the gravitational field as well. This means that we have a dynamically curved metric field which influences and is influenced by all other physical processes. The governing equations can be written succinctly as

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \quad (1.1)$$

The stress-energy tensor $T_{\mu\nu}$ at a given event (point on the manifold) generates curvature $G_{\mu\nu}$ at the same event. Written out explicitly, these equations become a set of coupled, nonlinear differential equations involving the metric $g_{\mu\nu}$ and its first two derivatives on the left-hand side, and the stress-energy tensor (which typically depends on $g_{\mu\nu}$) on the right-hand side.

The static, spherically symmetric solution of the field equations was found by
Schwarzschild only two months after Einstein published his field equations. However, we know that all stars rotate to some degree and if a rotating star were to undergo gravitational collapse then the resulting object would be expected to retain some fraction of its initial angular momentum. As such, a perfect Schwarzschild solution is not likely to be astrophysically relevant. It would be nearly another 50 years before Kerr discovered the solution which bears his name [1], which eventually came to be understood as describing a rotating black hole. The solution was fully characterized by just two parameters, the mass and spin of the black hole. Remarkably, the so-called “no hair” theorems soon established that Kerr’s solution uniquely describes the exterior of black holes (neglecting the influence of charge, which is not relevant in macroscopically for astrophysical black holes) [2, 3].

The term “Black Hole” (BH) itself only came to prominence in the aftermath of this discovery. In the subsequent decade, the core properties of BHs were developed, including the global properties of BH spacetimes, the definition of event horizons, as well as the singularity theorems of Penrose and Hawking. These were followed by the emergence of black hole thermodynamics by Bekenstein [4] and Bardeen, Carter and Hawking [5] in 1973 and Hawking radiation [6] in 1974. Penrose formulated the cosmic censorship hypothesis (the singularity is always behind the horizon and therefore has no causal connection to an external observer) in 1979 [7]. Whether or not the cosmic censorship hypothesis holds for generic gravitational collapse scenarios remains a conjecture.

Central to this advancement was the construction of the event horizon which delineates a one-way surface through which energy and information can transfer to the interior but not vice-versa. More specifically it is defined as the boundary of a region from which null rays (e.g. photons) cannot escape to future null infinity. To know the future path of these null rays requires knowledge of the entire future of the spacetime. Event horizons are therefore determined by future causes – they are teleological

¹Teleological is a term that originally appeared in philosophical discussions of phenomena that arose in order to produce their desired effects. The general relativity community has adopted it to describe the manner in which a black hole’s event horizon apparently evolves in anticipation of
Because of the aforementioned history, black holes have typically been defined by their event horizons. An alternative approach is to use trapped surfaces as introduced by Penrose. These geometrical definitions allow observers to locate, at least approximately, a BH's surface at some moment in time. This is not possible for the teleological event horizon since, in principle at least, the entire future history must be known. Work is ongoing in this direction [8] but there is no established quasi-local alternative description of BHs as of today. A mortal observer would have no way of knowing that a gravitational collapse will occur sometime in the future and may actually be near an event horizon. Indeed, event horizons usually form in locally flat regions of spacetime.

These results from the 1970's on BH dynamics and thermodynamics modified the previous view of BHs as simple passive potential wells by endowing them with global dynamical and thermodynamic quantities such as mass, charge, entropy and temperature. Christodoulou and Ruffini [9] used a Penrose process (see Sec 1.4.1) to establish a fundamental irreversibility in BH dynamics which in turn led to a further thermodynamic quantity, irreducible mass. The difference between the mass and the irreducible mass gives the available free energy of the BH, the maximum extractible energy. This active participation of a BH's energy quotient has a visible astrophysical manifestation where supermassive BHs at the centers of galaxies power quasars, AGNs and relativistic jets via the Blandford-Znajek process [10]. The extraction of rotational energy from a BH will occur if the rotation and turbulence in an accretion disk around it generates sufficiently large magnetic fields [11].

The prospect of gravitational-wave detection has motivated a large body of research into the study of black hole mergers, using both perturbative and fully numerical techniques. One avenue of research, perhaps the most immediately relevant, involves calculating the gravitational waveforms emitted by such mergers. However a second branch is focused on studying the features of spacetime geometry in the strong-field regime such as the Blandford-Znajek process [12], BH kicks [13], and BH momentum flow [14, 15]. This thesis concerns both of these avenues. I will first elaborate on some stresses it will be feeling later.
of the relevant background material to these two programs and then give a synopsis of my work at the end of this chapter.

1.1.1 Waveform calculations and Numerical Relativity

Gravitational perturbations of a BH are of special interest because of the ongoing search for Gravitational Waves (GWs). If we manage to open the gravitational-wave window to the Universe it is important that we have an understanding of the physics involved in the generation of such waves. BHs with their extremely large gravitational fields are considered among the most promising sources of GWs in the Universe. Astronomers have already been successful in locating a large number of BH candidates ranging from the most massive at the centers of galaxies, on the order of billions of solar masses, to solar mass scale BHs resident in X-ray binaries. Many questions remain though. For instance, we do not know the evolutionary process of galactic nuclei. How did these BHs form: in situ or through the hierarchical assembly of smaller galaxies? Did they precede the formation of galaxies or develop afterwards? The recently-discovered tight correlation between central black hole mass and stellar bulge velocity dispersion strongly implies that black hole formation and growth is closely tied to the processes that form galaxies [16]. This result also suggests that super massive black holes are at the centers of most or all large galaxies. Besides the upcoming extremely large telescopes, gravitational wave observations would assist greatly in this endeavor. Instruments such as eLISA would be able to take a snapshot of supermassive BH binaries in the mass range $10^4 - 10^7 M_\odot$ out to high redshift, which in turn would provide complementary input to testing various galaxy formation scenarios.

To observe a gravitational wave, you need to know what you are looking for: noise is strong enough that an accurate model of the wave plays an important role in extracting the signal from the detector's datastream. One of the main hurdles, the long-standing problem of calculating the evolution of two BHs numerically, was overcome by Pretorius [17] in 2005. Even so, 10 years later, the calculation is no small feat and involves many CPU cycles. In particular, it is difficult (CPU costly)
to resolve different length scales simultaneously on a numerical grid and so there will always be a region of the parameter space where perturbation theory is the preferred method.

1.2 Perturbation Theory

Black hole perturbation theory is the natural tool to model compact binaries where the mass ratio \( q = m_1/m_2 \ll 1 \). This technique is valid even for strong-field orbits with \( r/M \sim a \) few or high velocities \( v \lesssim 1 \). To zeroth order, the small body moves along a timelike geodesic of the background Kerr spacetime. Note that the description of the geodesics for the Kerr solution is far more complicated than for the Schwarzschild metric. The problem, however, is completely integrable. The Kerr solution produces a stationary and axisymmetric metric so that \( E \) and \( L_z \) are constants of the motion. The Hamiltonian itself also produces a constant of motion, the rest mass of the test particle. In addition to these three, Carter [18] found a fourth constant of motion using Hamilton-Jacobi methods to separate the motions in \( r \) and \( \theta \). The ability to use this technology is obviously core to this thesis given that particles that follow these paths will generate the perturbations to the metric which we wish to calculate.

To first order, the body is affected by small metric perturbations of \( O(q) \) as sourced by its small mass. These perturbations accelerate the body with respect to the background metric under the effects of a gravitational self-force (GSF). This self-force can be decomposed into a conservative, time-symmetric component with secular effects like orbital precession and ISCO shifts [19] and a dissipative, time-asymmetric part whose average is directly related to gravitational-wave emission. The leading order gravitational emission is described by the Teukolsky equation [20]. This is a wave equation which describes the propagation of arbitrary spin fields on the Kerr background; for spin \( s = 2 \), it describes first order gravitational perturbations propagating on the black hole background. Going beyond leading order, there has been significant effort in recent years to compute the conservative GSF components in the case of Kerr [21], non-circular orbits [22] and second-order gravitational effects [23] and research
in these directions is still ongoing.

We use both frequency- and time-domain Teukolsky solvers in this thesis. The time-domain solutions are used where we need to calculate the full waveform through adiabatic inspiral, plunge and coalescence; they work well when the solution is not slowly evolving or nearly stationary. The frequency domain code is used to produce solutions where the trajectory is (almost) stationary and can therefore be represented by a set of (slowly changing) harmonics. This method is extensively used for the BH distortion chapters. The physical fields considered here are always assumed to be weak in the sense that the effect of their energy-momentum on the background metric of the BH can be neglected. In other words we neglect any changes to both mass and spin of the BH.

The typical usage of BH perturbation theory involves calculating the perturbations generated at spatial infinity by a specific worldline. One can then examine the waveform and use it in a multitude of applications. For instance, one could test its viability for experimental detection by second generation ground-based interferometers such as advanced LIGO, advanced Virgo, or KAGRA using the matched-filtering technique. This method takes the noisy output of an interferometer and correlates it with known templates. However, we can apply perturbation theory to calculate amplitudes anywhere on the spacetime grid. Another interesting place to analyze these perturbations is at the black hole surface, the event horizon. One can then look at how the black hole’s geometrical features like shape, size and angular velocity are affected by exterior disturbances. This is the course of action in chapters 3 and 4 of this thesis.

1.3 Inspirals and Black Holes

Before going on further I would like to elucidate on three ideas which are central to this thesis: (i) the trajectories of small bodies around a massive central black hole, (ii) null generators and (iii) the behavior of black holes as described by BH thermodynamics.
1.3.1 EMRIs

Extreme mass ratio inspirals (EMRIs) denote coalescences which have one of the masses much smaller than its companion. An astrophysical realization of this is where a stellar-mass body inspirals into a super-massive black hole (SMBH). The small body will follow nearly geodesic motion in the black hole geometry as the rate of loss of its orbital energy is small. Bound geodesics in the Kerr geometry have a far more complicated structure than orbits in Newtonian gravity. Highly eccentric orbits which have a close approach to the black hole can exhibit so-called zoom-whirl motion whereby the test particle can go through several near circular orbits near periastron and end up at a vastly different position at apastron. The radiated flux is much stronger at periastron than at apastron which leads to complicated emitted signals.

The full multipole structure of the BH spacetime will be imprinted in the observed signal’s phasing and it should be possible to produce a map of the Kerr geometry of the central object. An experimental detection of such a system would be a stringent test to check if the spacetime geometry is really described solely by their masses and spins or if BHs have hair [24].

1.3.2 Event horizons as null surfaces

A light cone is, by construction, a surface built from lines (individual null (light) rays). These lines are the null generators of the surface since each point on the light cone lies on one and only one light ray, a congruence of curves. Null surfaces are 3-dimensional surfaces orthogonal to a null vector field (or in other words, a surface whose normal vector is everywhere null and non-zero) and are a generalized notion of light cones whereby the restriction to a single source is relaxed. The event horizon is one example of a null surface. A normal vector to a null surface is simultaneously tangent to the null surface since $n^\alpha n_\alpha = 0$. It is this peculiar property which means that null surfaces are naturally endowed with a congruence of null lines, which are the flow lines of the normal vectors within the surface and the null generators of the
surface. These lines themselves are not arbitrary and are actually null geodesics.

1.3.3 Black Hole Hydrodynamics and Thermodynamics

The first law of BH thermodynamics was developed by Bekenstein [4] and Bardeen et al. [5] quickly developed BH thermodynamics in general. Of note, the zeroth law states that the surface gravity $\kappa$ is constant over the BH horizon and the second law states that the area of the event horizon can never decrease in time. The first law states

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J$$

(1.2)

where $M$ is the mass measured at spatial infinity, $\kappa$ is the surface gravity at the event horizon but using a vector field normalized at infinity, $A$ is the area of the event horizon, $\Omega_H$ is the angular velocity at the horizon, $J$ is the angular momentum at infinity. $\kappa$, $A$ and $\Omega_H$ are known functions of $M$ and $J$. These laws dictate the behavior of BHs as defined by their event horizons.

A classical black hole represents a perfect absorber at zero temperature. Bekenstein [4] and Hawking [6] demonstrated that an isolated black hole must possess finite temperature and entropy proportional to its surface gravity $\kappa$ and surface area $A$

$$T = \frac{\hbar \kappa}{2\pi k_B} = \frac{\hbar \sqrt{M^2 - a^2}}{4\pi Mr_+ k_B}$$

(1.3)

$$S = \frac{k_B A}{4\hbar G} = \frac{2\pi k_B Mr_+}{\hbar G}$$

(1.4)

The second form listed in these equations specializes to a Kerr black hole of mass $M$ and spin parameter $a$. This means that a black hole will emit thermal radiation and will have a finite lifespan. Classical general relativity treats spacetime as a continuum much like fluid dynamics treats a fluid as a continuum. In fact, BHs display hydrodynamic behavior and this has been known for many years. Hawking and Hartle [25, 26, 27] introduced the idea of BH viscosity when studying the response of the event horizon to external perturbations. This then led to a viscous-fluid analogy which
was developed by Thorne, Price and Macdonald in the “membrane paradigm” [11]. Therein, the physical properties of the BH are discussed in terms of mechanical and electromagnetic properties of a 2-dimensional viscous fluid. By projecting the Einstein equations on the horizon’s world tube one can construct energy and momentum balance equations for this fluid with viscosity being identified with the dissipative terms. A quick exercise will allow this relationship to be made explicit. Via classical gravity, we have

$$\frac{dA}{dt} = \sigma_{\mu\nu}\sigma^{\mu\nu}$$

(1.5)

where $\sigma^{\mu\nu}$ is the trace-free shear of the generators of the horizon (whose value is governed by the horizon’s deformations), and $\kappa$ is the horizon’s surface gravity [25]. By multiplying and dividing by $h/k_B$ the left-hand side can be written as $TdS/dt$. Then, when all the factors are written out properly, we find

$$T \frac{dS}{dt} = \frac{1}{8\pi G} \sigma_{\mu\nu}\sigma^{\mu\nu}$$

(1.6)

The equation governing entropy generation in a normal fluid with shear viscosity $\eta$ is

$$T \frac{dS}{dt} = 2\eta \sigma_{ij}\sigma^{ij}$$

(1.7)

These equations are identical provided we identify

$$\eta = \frac{1}{16\pi G} \frac{c^3}{G}$$

(1.8)

as the shear viscosity associated with the horizon. However, it is a slightly subtle identification: in the fluid case, it’s a shear of a 3-D flow, whereas for the horizon it is a 2-D flow.

Note that even though $T$ and $S$ are quantum mechanical constructs, their product is a classical quantity (see eqn(1.3)) as is $\sigma_{ab}$. Einstein’s field equations, in general, when projected onto any null surface in any spacetime, reduce to the form of Navier-
Stokes equations [28]. Therefore the results of chapters 3 and 4 which are focused directly on the event horizon may have further applicability given that event horizons are but one such null surface.

Boltzman emphasized that the thermal phenomena exhibited by a fluid implies the existence of substructure. We have just recalled both quantum and classical effects involved at BH horizons, therefore the same reasoning could be applied to spacetime. Does spacetime itself have a microstructure and classical gravity is just the thermodynamic limit of the statistical mechanics of some unknown sub-structure? This is just conjecture of course, but shows the importance of trying to understand physics at a “thermodynamic level” particular in those extreme regions where our intuition breaks down like at the horizon of a BH.

Relevance to AdS/CFT

In a broader context, BHs have been the subject of intense study in quantum gravity and the calculation of BH entropy has been seen as a key milestone. More specifically, it is known through the Holographic principle that the features of an interacting quantum field theory at finite temperature can be mapped via gauge/gravity duality to features of BH horizons. Any interacting quantum field theory at finite temperature can be described by hydrodynamics when viewed on long enough length scales and the physics therein is governed by the near-horizon portion of the dual geometry because of the UV/IR connection [29]. The most rigorous realization of the holographic principle is the AdS/CFT correspondence by Juan Maldacena [30]. Attempts have already been made to tie in this formalism with the membrane paradigm [31, 32]. Of course the connection between our work and BHs in anti-de Sitter space in d dimensions is somewhat tenuous at best given that we are analyzing three dimensional BHs in flat space, but the point is that understanding the behavior of the event horizon may have broader applications beyond BH physics.
1.4 Definition of $\Omega_H$

Throughout this section and for the remainder of this thesis we use geometric units $G = c = 1$.

We introduce a few definitions in Chapter 3 without much background. One especially important definition is that of $\Omega_H$, the angular velocity of rotation of the black hole (or horizon), since much of the narrative of chapter 3 involves comparing it against the orbital angular velocity of the orbiting body in determining the geometry of the horizon. The Kerr metric in Boyer-Lindquist coordinates (useful as it tends to the Lorentz frame at spatial infinity), describing a BH with mass $M$ and spin parameter $a = J/M$ where $J$ is the angular momentum of the BH, is given by

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mra\sin^2 \theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$+ \frac{(r^2 + a^2)^2 - a^2\Delta \sin^2 \theta}{\Sigma} \sin^2 \theta d\phi^2$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2Mr + a^2$. Let the angular velocity of a zero-angular-momentum observer (ZAMO)$^2$ at a fixed coordinate radius $r$ be given by $\omega$. Thanks to the dragging of inertial frames by the black hole’s spin, the ZAMO has some orbital velocity despite lacking angular momentum their angular velocity with respect to the time of the distant observer $t$ is given by:

$$\omega \equiv \frac{d\phi}{dt} = -\frac{g_{\phi t}}{g_{\phi \phi}} = \frac{2Mar}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}$$

The event horizon is defined as the surface at which $\Delta = 0$; this occurs at the coordinate $r = r_+ = M + \sqrt{M^2 - a^2}$. In that limit, the angular velocity $\omega$ becomes

$$\Omega_H = \frac{a}{2Mr_+} = \frac{a}{r_+^2 + a^2}$$

This limit is constant at the horizon, independent of $\theta$. It is called the angular velocity of rotation of the BH. The fact that $\Omega_H$ is constant over the event horizon

$^2$This ZAMO is accelerated, otherwise they would fall into the black hole.
is an important one as it allows us to treat the black hole as a rigid body and as such allows us to make comparisons to a rigid body in Newtonian physics. Even though this result was obtained from explicit formulas, one could argue, on physical grounds, that if the horizon did not rigidly rotate then one would expect gravitational radiation to be emitted from the region near the horizon, this in turn would violate the stationarity of the Kerr solution. There also exists a general mathematical proof by Carter [33], the “weak rigidity theorem” which establishes the Killing property of the horizon and the rigidity property of its rotation. Note that the surface gravity $\kappa$ is also constant on the horizon and is a central quantity when considering distortions on the horizon as it determines the damping timescale of oscillations.

1.4.1 Black Hole energy extraction

We can see that in the Kerr metric, eqn(1.9), the time Killing vector $\partial_t$ changes from being timelike to spacelike near the horizon. The region where $\partial_t$ is spacelike outside the event horizon is called the ergosphere. Its outer surface is situated where $\partial_t$ becomes null:

$$\partial_t \cdot \partial_t = g_{tt} = 0$$ \hspace{1cm} (1.13)

$$r = M + \sqrt{M^2 - a^2 \cos^2 \theta}$$ \hspace{1cm} (1.14)

The existence of the ergosphere makes it possible in principle to extract energy from a rotating BH. To understand the mechanism it is worthwhile to consider a concrete example. In the Penrose process, a particle falls freely into the ergosphere where it breaks up into two fragments, one of the particles then carries away a 4-momentum $\vec{p}$ and negative energy $E = -\vec{p} \cdot \partial_t$, and proceeds to fall into the hole. The other one goes out to infinity with more energy than the original particle and the BH loses some of its rotational energy.
1.5 Summary of Chapters

As we have seen the two-body problem plays a central role in gravitational physics and the orbital motion and gravitational emission of such systems can be modelled using a variety of approximation schemes and numerical methods. In this thesis we specifically concentrate on black hole perturbation theory and the effective-one-body model.

1.5.1 Tuning the EOB model

The central idea behind the effective-one-body model [34, 35] is to map the actual dynamics of a compact binary system of masses $m_1, m_2$, and mass ratio $q$ orbiting each other with an extreme mass ratio binary, where the more massive object is a deformed Kerr BH of mass $M = m_1 + m_2$ and has a particle of reduced mass $\mu = m_1 m_2 / M$ in its orbit. The deformation of the Kerr metric is controlled by the symmetric mass ratio $\nu = q / (1 + q)^2$. By construction the EOB Hamiltonian is able to reproduce the known post-Newtonian (PN) dynamics in the weak-field/small-velocity regime and also incorporates a description of the gravitational-wave emission and the related dissipative radiation-reaction force. The method relies heavily on resummation methods such as Padé approximants with the aim of improving the convergence of the PN series in the strong-field regime which makes it possible to analytically model the three main phases of binary black hole evolution, namely inspiral, plunge-merger and ringdown. Nonetheless the model is not self consistent, containing several free parameters which account for the undetermined relativistic corrections during the late inspiral and final plunge. These have been calibrated by comparison with fully nonlinear NR simulations in the comparable mass regime and with perturbative methods for EMRIs. Chapter 2 advances these works by considering the transition from inspiral through plunge-merger and ringdown using a time-domain computational framework based on the Teukolsky equation as developed in [36, 37, 38]. We concentrate on quasi-circular equatorial orbits in the Kerr spacetime and calibrate the leading EOB modes; the dominant (2,2) mode for a range of spins, plus
the three subleading modes in the Schwarzschild case. The resulting improved EOB model will allow it to cover a much larger region of parameter space including higher modes and extreme spins.

Work is still ongoing in improving the accuracy of the EOB model, for instance including gravitational self-force terms in the Hamiltonian [39] and spin effects [40].

1.5.2 Distortions of BHs

In this introduction we have discussed extensively the motivation behind looking at physics near BHs. Chapters 3 and 4 try to progress in this direction by considering tidal deformations of BHs under the influence of small orbiting bodies using perturbation theory. The formalism for describing these distortions of black holes with arbitrary spin is developed in Chapter 3 and contains some applications of the formalism in the restricted cases of circular orbits. In particular we analyze the relationship between the applied tidal field and the response of the black hole horizon with the result that simple Newtonian intuition breaks down in such cases. Chapter 4 uses the developed formalism and shows how it can be applied to eccentric inclined orbits which gives us insight into horizon dynamics. We find small amplitude coherent wiggles in the horizon’s shear response to the applied tide, which is explained via a teleological Green’s function relating the shear to the tide.

References


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Chapter 2

Modeling multipolar gravitational-wave emission from small mass-ratio mergers

Abstract

Using the effective-one-body (EOB) formalism and a time-domain Teukolsky code, we generate inspiral, merger, and ringdown waveforms in the small mass-ratio limit. We use EOB inspiral and plunge trajectories to build the Teukolsky equation source term, and compute full coalescence waveforms for a range of black hole spins. By comparing EOB waveforms that were recently developed for comparable mass binary black holes to these Teukolsky waveforms, we improve the EOB model for the (2,2), (2,1), (3,3), and (4,4) modes. Our results can be used to quickly and accurately extract useful information about merger waves for binaries with spin, and should be useful for improving analytic models of such binaries.

2.1 Introduction

Since the numerical relativity breakthrough of 2005 [41, 42, 43], there have been tremendous advances both in the computation of gravitational radiation from binary black-hole systems, and in analytical modeling of this radiation using approximate techniques. Despite rapid and ongoing advances, it remains a challenge for numerical relativity to quickly and accurately compute models that span large regions of
parameter space. Extreme conditions such as large spins and small mass ratios are particularly challenging, although there has been excellent recent progress on these issues [44, 45, 46].

The effective-one-body (EOB) formalism [47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57] makes it possible to analytically model the three main phases of binary black-hole evolution: inspiral, plunge-merger, ringdown. EOB has been used to model the dynamics and gravitational-wave emission from comparable-mass binaries [58, 59, 60, 61, 62, 63, 64, 65, 66, 67], extreme mass-ratio inspiraling binaries [68, 69, 70] (neglecting conservative self-force effects), and small mass-ratio non-spinning binaries [71, 72, 73, 74, 75]. In order to study the transition from inspiral to plunge-merger and ringdown, Refs. [71, 72] suggested combining EOB with black hole perturbation theory. Concretely, they used EOB in order to compute the trajectory followed by an object spiraling and plunging into a much larger black hole, and then used that trajectory to describe the source for the time-domain Regge-Wheeler-Zerilli (RWZ) equation [76, 77] describing metric perturbations to Schwarzschild black holes. They were then able to compute the full RWZ coalescence waveform and to compare with the EOB model. This was used in Ref. [73] to produce gravitational modes beyond the leading (2, 2) mode and compute the recoil velocity. More recently, Refs. [74, 75] have used information from the RWZ modes to improve the modeling of the subleading EOB modes. A particularly beautiful feature of Ref. [75] is the use, for the first time, of hyperboloidal slicings in such an analysis. This effectively compactifies the computational domain so that waveforms at future null infinity can be read out of the numerical calculation with great accuracy.

References [36, 37, 38] developed a time-domain computational framework based on the Teukolsky equation [78], which describes curvature perturbations of rotating (Kerr) black holes. The goal of these papers has been to understand gravitational waves produced by physically reasonable but otherwise arbitrary trajectories of small bodies bound to rotating black holes (such as slowly inspiraling orbits, or trajectories that plunge into the hole's event horizon). This has been used to understand the small-mass ratio limit of merging black holes, studying for example the dependence
of recoil velocity on black hole spin in this limit. This Teukolsky code has been optimized to make effective use of modern many-core processor architectures, such as Graphics Processing Units (GPUs) [79].

In this chapter, we combine EOB with the time-domain Teukolsky code developed in Refs. [36, 37, 38] to extend the ideas of Refs. [71, 72, 74, 75] in several directions. Our primary extension is, for the first time, producing full coalescence waveforms describing inspiral, merger, and ringdown for quasi-circular equatorial orbits in the Kerr spacetime. The energy flux we use in the EOB equations of motion comes from the factorized resummed waveforms of Refs. [53, 55]. For the Schwarzschild limit, we model analytically three subleading modes [(2, 1), (3, 3), and (4, 4)] plus the dominant (2, 2) mode, finding useful information about the plunge-merger which we use to improve the comparable mass EOB model described in Ref. [67]. For more general spins, we calibrate the leading EOB mode for spins $a/M = -0.9, -0.5, 0.5,$ and 0.7. We also extract some information regarding subleading modes, and regarding the high prograde spin $a/M = 0.9$. These results for spinning binaries provide valuable input for improving the spinning EOB model of Refs. [80, 54], as well as the spinning EOB waveforms of Refs. [62, 81]. This will in turn make it possible to develop models that can cover a much larger region of parameter space, including higher modes and extreme spins.

Several other groups have also been using perturbation theory tools recently to improve our understanding for comparable mass and intermediate mass ratio binaries. For example, in Refs. [82, 46, 83] the authors directly employ a moving-puncture trajectory (or a post-Newtonian–inspired fit to it) in the RWZ equation. They compare the resulting RWZ waveform with the results of full numerical relativity calculation for mass ratios $1/15$ and $1/10$, finding good agreement. Another recent suggestion is the hybrid approach of Ref. [84], in which inspiral-plunge intermediate-mass black-hole waveforms are computed by evolving the EOB equations of motion augmented by the perturbation-theory energy flux. An important issue in all attempts to model binary coalescence with perturbation theory is the computation of the so-called excitation coefficients, or more generally the question of which fundamental frequencies con-
tribute to the radiation. In this context, perturbation-theory calculations are offering new insights [85, 86, 87, 88].

The remainder of this chapter is organized as follows. We begin in Sec. 2.2 by reviewing the EOB formalism for a test particle moving along quasi-circular, equatorial orbits around a Kerr black hole. We then describe (Sec. 2.3) the time-domain Teukolsky equation calculation we use to compute the gravitational radiation emitted from a test particle that follows our EOB-generated trajectory. This section discusses in some detail numerical errors which arise from finite-difference discretization, and from the extrapolation procedure by which we estimate our waves at future null infinity. Since we began this analysis, the Teukolsky code we use has been upgraded to use the hyperboloidal layer method [89]. This upgrade came too late to be used throughout our analysis, but has been used to spot check our estimates of this extrapolation error.

Our results are presented in Sec. 2.4. We begin by comparing the leading and three subleading Teukolsky modes with $a/M = 0$ to the corresponding EOB modes calibrated to non-spinning comparable mass binaries [67]. We then improve the non-spinning EOB model by including some features we find in our test-particle-limit calculation. We next calibrate the leading $(2,2)$ EOB waveform with our Teukolsky-equation results for spins $a/M = -0.9, -0.5, 0.5,$ and $0.7$. We conclude our results by discussing the challenges of calibrating subleading modes and of modeling extreme spin configurations, such as ones with $a/M \geq 0.9$. Section 2.5 summarizes our main conclusions, and outlines some plans for future work. A particularly important goal for the future will be to move beyond equatorial configurations, modeling the important case of binaries with misaligned spins and orbits.
2.2 Dynamics and waveforms using the effective-one-body formalism

The Hamiltonian of a non-spinning test-particle of mass $\mu$ orbiting a Kerr black hole of mass $M$ and intrinsic angular momentum (or spin) per unit mass $a$ is

$$H = \beta^i p_i + \alpha \sqrt{\mu^2 + \gamma^{ij} p_i p_j}, \quad (2.1)$$

where the indices $i, j$ label spatial directions, and the functions introduced here are given by

$$\alpha = \frac{1}{\sqrt{-g^{tt}}}, \quad (2.2a)$$
$$\beta^i = \frac{g^{ti}}{g^{tt}}, \quad (2.2b)$$
$$\gamma^{ij} = g^{ij} - \frac{g^{ti} g^{tj}}{g^{tt}}; \quad (2.2c)$$

$t$ is the time index, and $g_{\mu\nu}$ is the Kerr metric. Working in Boyer-Lindquist coordinates $(t, r, \theta, \phi)$ and restricting ourselves to the equatorial plane $\theta = \pi/2$, the relevant metric components read

$$g^{tt} = -\frac{\Lambda}{r^2 \Delta}, \quad (2.3a)$$
$$g^{rr} = \frac{\Delta}{r^2}, \quad (2.3b)$$
$$g^{\phi\phi} = \frac{1}{\Lambda} \left( -\frac{4a^2 M^2}{\Delta} + r^2 \right), \quad (2.3c)$$
$$g^{t\phi} = -\frac{2a M}{r \Delta}, \quad (2.3d)$$

where we have introduced the metric potentials

$$\Delta = r^2 - 2Mr - a^2, \quad (2.4)$$
$$\Lambda = (r^2 + a^2)^2 - a^2 \Delta. \quad (2.5)$$
We replace the radial momentum \( p_r \) with \( p_{r^*} \), the momentum conjugate to the tortoise radial coordinate \( r^* \). The tortoise coordinate is related to the Boyer-Lindquist \( r \) by

\[
dr^* = \frac{r^2 + a^2}{\Delta} \, dr.
\]  

Since \( p_r \) diverges at the horizon while \( p_{r^*} \) does not, this replacement improves the numerical stability of the Hamilton equations

\[
\frac{dr}{dt} = \frac{\Delta}{r^2 + a^2} \frac{\partial H}{\partial p_{r^*}} (r, p_{r^*}, p_\phi),
\]
\[
\frac{d\phi}{dt} = M \Omega = \frac{\partial H}{\partial p_\phi} (r, p_{r^*}, p_\phi),
\]
\[
\frac{dp_{r^*}}{dt} = -\frac{\Delta}{r^2 + a^2} \frac{\partial H}{\partial r} (r, p_{r^*}, p_\phi) + nK \mathcal{F}_\phi \frac{p_{r^*}}{p_\phi},
\]
\[
\frac{dp_\phi}{dt} = nK \mathcal{F}_\phi.
\]

Our trajectory is produced by integrating these equations using initial conditions that specify a circular orbit. We typically find in our evolutions a small residual eccentricity on the order of \( 3 \times 10^{-4} \).

In Eqs. (2.7a)-(2.7d), radiation-reaction effects are included following the EOB formalism. For the \( \phi \) component of the radiation-reaction force we use the non-Keplerian (nK) force

\[
nK \mathcal{F}_\phi = -\frac{1}{\nu v_\Omega^3} \frac{dE}{dt},
\]

where \( v_\Omega \equiv (M \Omega)^{1/3} \), and \( dE/dt \) is the energy flux for quasi-circular orbits obtained by summing over gravitational-wave modes \((l, m)\). We use

\[
\frac{dE}{dt} = \frac{1}{16\pi} \sum_{\ell=2}^{8} \sum_{m=-\ell}^{\ell} \ell^2 v_\Omega^6 |h_{\ell m}|^2.
\]

The non-Keplerian behavior of the radiation-reaction force is implicitly introduced through the definition of \( h_{\ell m} \). To describe the inspiral and plunge dynamics, we use the modes

\[
h_{\ell m}^{\text{insp-plunge}} = h_{\ell m}^F N_{\ell m}.
\]
The coefficients $N_{\ell m}$ describe effects that go beyond the quasi-circular assumption and will be defined below [see Eq. (2.17)]. The factors $h_{\ell m}^F$ are the factorized resummed modes, and are given by [53]

$$h_{\ell m}^F = h_{\ell m}^{(N,e)} \tilde{S}(\epsilon) T_{\ell m} e^{i\delta_{\ell m}} (\rho_{\ell m})^\epsilon. \quad (2.11)$$

Here, $\epsilon = \pi(\ell + m)$ is the parity of the multipolar waveform. The leading term in Eq. (2.11), $h_{\ell m}^{(N,e)}$, is the Newtonian contribution

$$h_{\ell m}^{(N,e)} = \frac{M\nu}{R} n_{\ell m}^{(e)} c_{\ell+\epsilon}(\nu) V_\phi^\ell Y_{\ell-\epsilon,-m} \left(\frac{\pi}{2}, \phi\right), \quad (2.12)$$

where $R$ is distance from the source, $Y_{\ell m}(\theta, \phi)$ are the scalar spherical harmonics, and the functions $n_{\ell m}^{(e)}$ and $c_{\ell+\epsilon}(\nu)$ are given in Eqs. (4a), (4b) and (5) of Ref. [90] with $\nu = \mu/M$. For reasons that we will explain in Sec. 2.4.1, we choose

$$V_\phi^\ell = v_{\phi}^{(\ell+\epsilon)} \quad (\ell, m) \neq (2,1), (4,4), \quad (2.13a)$$

$$V_\phi^\ell = \frac{1}{r_\Omega} v_{\phi}^{(\ell+\epsilon-2)} \quad (\ell, m) = (2,1), (4,4). \quad (2.13b)$$

The quantities $v_{\phi}$ and $r_\Omega$ introduced here are defined by

$$v_{\phi} \equiv M\Omega r_\Omega \equiv M\Omega \left[(r/M)^{3/2} + a/M\right]^{2/3}. \quad (2.14)$$

The function $\tilde{S}(\epsilon)$ in Eq. (2.11) is given by

$$\tilde{S}(\epsilon)(r, p_r, p_\phi) = \begin{cases} H(r, p_r, p_\phi), & \epsilon = 0, \\ L = p_\phi v_\Omega, & \epsilon = 1. \end{cases} \quad (2.15)$$

The factor $T_{\ell m}$ in Eq. (2.11) resums the leading order logarithms of tail effects, and
is given by

\[ T_{\ell m} = \frac{\Gamma(\ell + 1 - 2imM\Omega)}{\Gamma(\ell + 1)} \times e^{\pi mM\Omega} e^{2imM\Omega \log(2m\Omega r_0)}, \]  

(2.16)

where \( r_0 = 2M/\sqrt{e} \) [90] and \( \Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t} dt \) is the complex gamma function. The factor \( e^{i\delta_{\ell m}} \) in Eq. (2.11) is a phase correction due to subleading order logarithms; \( \delta_{\ell m} \) is computed using Eqs. (27a)-(27i) of Ref. [90]. The factor \( (\rho_{\ell m})^\ell \) in Eq. (2.11) collects the remaining post-Newtonian terms, and is computed using Eqs. (29a)-(29i) and Eqs. (D1a)-(D1m) of Ref. [90].

Finally, the function \( N_{\ell m} \) entering Eq. (2.10) is given by

\[ N_{\ell m} = \left[ 1 + a_{1h_{\ell m}} \frac{p_{r*}^2}{(r\Omega)^2} + a_{2h_{\ell m}} \frac{p_{r*}^2 M}{(r\Omega)^2 r} \right. \\
+ a_{3h_{\ell m}} \frac{p_{r*}^2}{(r\Omega)^2} \left( \frac{M}{r} \right)^{3/2} + a_{4h_{\ell m}} \frac{p_{r*}^2 M}{(r\Omega)^2 r} \left( \frac{M}{r} \right) \\
+ a_{5h_{\ell m}} \frac{p_{r*}^2}{(r\Omega)^2} \left( \frac{M}{r} \right)^{5/2} \left[ \exp \left[ i \left( b_{1h_{\ell m}} \frac{p_{r*}}{r\Omega} + b_{2h_{\ell m}} \frac{p_{r*}}{r\Omega} \right. \\
+ b_{3h_{\ell m}} \sqrt{\frac{M}{r} \frac{p_{r*}^3}{r\Omega} + b_{4h_{\ell m}} \frac{r^3}{r\Omega}} \right) \right] \]  

(2.17)

where the quantities \( a_{i}^{h_{\ell m}} \) and \( b_{i}^{h_{\ell m}} \) are non-quasicircular (NQC) orbit coefficients. We will explain in detail how these coefficients are fixed in Sec. 2.4.

We conclude this section by describing how we build the final merger-ringdown portion of the EOB waveform. For each mode \( (\ell, m) \) we have

\[ h_{\text{merger-RD}}^{\ell m}(t) = \sum_{n=0}^{N-1} A_{\ell mn} e^{-i \sigma_{\ell mn}(t-t_{\text{match}})}, \]  

(2.18)

where \( n \) labels the overtone of the Kerr quasinormal mode (QNM), \( N \) is the number of overtones included in our model, and \( A_{\ell mn} \) are complex amplitudes to be determined.
by a matching procedure described below. The complex frequencies \( \sigma_{\ell mn} = \omega_{\ell mn} - i/\tau_{\ell mn} \), where the quantities \( \omega_{\ell mn} > 0 \) are the oscillation frequencies and \( \tau_{\ell mn} > 0 \) are the decay times, are known functions of the final black-hole mass and spin and can be found in Ref. [91]. In this chapter, we model the ringdown modes as a linear combination of eight QNMs (i.e., \( N = 8 \)).

The complex amplitudes \( A_{\ell mn} \) in Eq. (2.18) are determined by matching the merger-ringdown waveform (2.18) with the inspiral-plunge waveform (2.10). In order to do this, \( N \) independent complex equations needs to be specified throughout the comb of width \( \Delta t_{\text{match}} \). Details on the procedure are given in Ref. [67]. The full inspiral(-plunge)-merger-ringdown waveform is then given by

\[
h_{\ell m} = h_{\ell m}^{\text{insp-plunge}} \theta(t_{\text{match}} - t) + h_{\ell m}^{\text{merger-RD}} \theta(t - t_{\text{match}}). \tag{2.19}
\]

In this analysis, we focus on waveforms emitted by a test-particle of mass \( \mu \) orbiting a Kerr black hole. Thus, we shall set to zero terms proportional to \( \nu = \mu/M \) in Eq. (2.11), excepting the leading \( \nu \) term in Eq. (2.12). Throughout this chapter we restrict ourselves to the case \( \nu = 10^{-3} \).

### 2.3 The time-domain Teukolsky code

#### 2.3.1 Overview: The Teukolsky equation and its solution

The evolution of scalar, vector, and tensor perturbations of a Kerr black hole is described by the Teukolsky master equation [78], which takes the following form in
Boyer-Lindquist coordinates:

\[
- \left[ \frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \partial_t \Psi - \frac{4Mr}{\Delta} \partial_{t\phi} \Psi \\
- 2s \left[ r - \frac{M(r^2 - a^2)}{\Delta} + ia \cos \theta \right] \partial_t \Psi \\
+ \Delta^{-s} \partial_r \left( \Delta^{s+1} \partial_r \Psi \right) + \frac{1}{\sin \theta} \partial_{\theta} \left( \sin \theta \partial_{\theta} \Psi \right) + \\
\left[ \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right] \partial_{\phi\phi} \Psi + 2s \left[ \frac{a(r - M)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right] \partial_{\phi} \Psi \\
- (s^2 \cot^2 \theta - s) \Psi = -4\pi (r^2 + a^2 \cos^2 \theta) T .
\] (2.20)

The coordinates, the mass \( M \), the spin parameter \( a \), and the function \( \Delta \) are as defined in the previous section. The number \( s \) is the “spin weight” of the field. When \( s = \pm 2 \), this equation describes radiative degrees of freedom for gravity. We focus on the case \( s = -2 \), for which \( \Psi = (r - ia \cos \theta)^4 \psi_4 \), where \( \psi_4 \) is the Weyl curvature scalar that characterizes outgoing gravitational waves.

To solve Eq. (2.20), we use an approach introduced by Krivan et al. [92]. First, we change from radial coordinate \( r \) to tortoise coordinate \( r^* \) [Eq. (2.6)], and from axial coordinate \( \phi \) to \( \tilde{\phi} \), defined by

\[
d\tilde{\phi} = d\phi + \frac{a}{\Delta} dr .
\] (2.21)

These coordinates are much better suited to numerical evolutions, as detailed in Ref. [92]. Next, we exploit axisymmetry to expand \( \Psi \) in azimuthal modes:

\[
\Psi(t, r, \theta, \tilde{\phi}) = \sum_m e^{im\tilde{\phi}} r^3 \phi_m(t, r, \theta) .
\] (2.22)

This reduces Eq. (2.20) to a set of decoupled (2+1)-dimensional hyperbolic partial differential equations (PDEs). We rewrite this system in first-order form by introducing a momentum-like field,

\[
\Pi_m \equiv \partial_t \phi_m + \frac{(r^2 + a^2)}{\Sigma} \partial_r \phi_m ,
\] (2.23)
where $\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta$. We then integrate this system using a two-step, 2nd-order Lax-Wendroff finite-difference method. Details are presented in Refs. [36, 37]. Following Ref. [92], we set $\phi_m$ and $\Pi_m$ to zero on the inner and outer radial boundaries. Symmetries of the spheroidal harmonics are used to determine the angular boundary conditions: For $m$ even, we have $\theta \phi_m = 0$ at $\theta = 0, \pi$; for $m$ odd, $\phi_m = 0$ at $\theta = 0, \pi$.

The right-hand side (RHS) of Eq. (2.20) is a source term constructed from the energy-momentum tensor describing a point-like object moving in the Kerr spacetime. The expression for $T$ is lengthy and not particularly illuminating. For this chapter, it is suffices to point out that $T$ is constructed from Dirac-delta functions in $r$ and $\theta$, as well as first and second derivatives of the delta function in these variables. These terms have coefficients that are complex functions of the black hole’s parameters and the location of the point-like object. Details and discussion of how we model the deltas and their derivatives on a numerical grid are given in Ref. [36]. The delta functions are sourced at the location of the point-like object; the source $T$ thus depends on the trajectory that this body follows in the Kerr spacetime. In this analysis, we use a trajectory constructed using the EOB formalism to specify the small body’s location.

One point worth emphasizing is that the source term is scaled by a factor of $1/t$ [see Eq. (2.39) of Ref. [36]]; i.e., the source is inversely weighted by the rate of change of coordinate time per unit proper time experienced by the orbiting object. This means that the source term “redshifts away” as the object approaches the horizon. As a consequence, when describing a body that falls into a black hole, the Teukolsky equation (2.20) smoothly transitions into its homogeneous form, connecting the gravitational radiation from the last few orbital cycles to the Kerr hole’s quasinormal modes in a very natural way. The same behavior is seen in other analyses which model plunging trajectories using black hole perturbation theory (e.g., Refs. [93, 73, 85]).

We implement this numerical scheme with a Fortran code, parallelized using a standard domain decomposition (on the radial coordinate grid), and with OpenMPI\(^1\).

\(^1\)http://openmpi.org
enabled message passing. Good scaling has been observed for several hundred processor cores. In this analysis, we used 128 processor cores for computing each \( m \)-mode for all cases we studied.

### 2.3.2 Waveforms and multipole decomposition

Far from the black hole, \( \psi_4 \) is directly related to \( h_+ \) and \( h_\times \) via

\[
\psi_4 = \frac{1}{2} \left( \frac{\partial^2 h_+}{\partial t^2} - i \frac{\partial^2 h_\times}{\partial t^2} \right) = \frac{1}{2} \frac{\partial^2 h}{\partial t^2} .
\]

The waveform \( h \equiv h_+ - ih_\times \) is then found by integrating \( \psi_4 \) twice, choosing constants of integration so that \( h \to 0 \) at very late times (long after the system’s waves have decayed to zero).

As detailed in Sec. 2.3.1, our computation naturally decomposes the field \( \Psi \) (and hence \( \psi_4 \) and the waveform \( h \)) into axial modes with index \( m \). For comparison with EOB waveforms, it is necessary to further decompose into modes of spin-weighted spherical harmonics. Following standard practice, we define

\[
\psi_4 = \frac{1}{\mathcal{R}} \sum_{\ell,m} C_{\ell m}(t, r) - 2 Y_{\ell m}(\theta, \phi) ,
\]

\[
h = \frac{1}{\mathcal{R}} \sum_{\ell,m} h_{\ell m}(t, r) - 2 Y_{\ell m}(\theta, \phi) .
\]

In these equations, \( -2 Y_{\ell m} \) is a spherical harmonic of spin-weight \(-2\). Defining the inner product

\[
\langle Y_{\ell m} | f \rangle = \int d\Omega \, -2 Y_{\ell m}^*(\theta, \phi) f ,
\]

(where \( * \) denotes complex conjugation), extracting \( C_{\ell m} \) and \( h_{\ell m} \) is simple:

\[
C_{\ell m}(t, r) = \mathcal{R} \langle Y_{\ell m} | \psi_4 \rangle ,
\]

\[
h_{\ell m}(t, r) = \mathcal{R} \langle Y_{\ell m} | h \rangle .
\]

The complex wave mode \( h_{\ell m} \) can also be obtained from \( C_{\ell m} \) by integrating twice,
again choosing the constants of integration so that \( h_{\ell m} \to 0 \) at very late times.

### 2.3.3 Numerical errors

Our numerical solutions are contaminated by two dominant sources of error: Discretization error due to our finite-difference grid, and extraction error due to computing \( \Psi \) and associated quantities at finite spatial location rather than at null infinity.

**Discretization error**

As discussed in Ref. [37], our time-domain Teukolsky solver is intrinsically second-order accurate. Since we compute our solutions on a two-dimensional grid in tortoise radius \( r^* \) and angle \( \theta \), we expect our raw numerical output to have errors of order \((dr^*)^2\), \((d\theta)^2\), and \((dr^*d\theta)\). We mitigate this error with a variant of Richardson extrapolation, which we now describe.

Consider waveforms generated at three resolutions: \( h_1^{(2)} \) at \((dr^*, d\theta) = (0.064M, 0.2)\); \( h_2^{(2)} \) at \((0.032M, 0.1)\); and \( h_3^{(2)} \) at \((0.016M, 0.05)\). Superscript "(i)" means the solution is \( i \)th-order accurate. We convert from second-order to third-order accuracy using [94]

\[
\begin{align*}
  h_{1.5}^{(3)} &= h_1^{(2)} - \frac{h_1^{(2)} - h_2^{(2)}}{1 - 1/n^2} , \\
  h_{2.5}^{(3)} &= h_2^{(2)} - \frac{h_2^{(2)} - h_3^{(2)}}{1 - 1/n^2} .
\end{align*}
\]

Here, \( n = 2 \) is the ratio of grid spacing between the two resolutions.

To estimate the remaining error in this extrapolated solution, we compare \( h_{2.5}^{(3)} \) and \( h_{1.5}^{(3)} \). Let us define

\[
\begin{align*}
  \Delta h &= h_{2.5}^{(3)} - h_{1.5}^{(3)} , \\
  h^{(4)} &= h_{2.5}^{(3)} - \frac{h_{2.5}^{(3)} - h_2^{(3)}}{1 - 1/n^3} .
\end{align*}
\]

\( h^{(4)} \) is a fourth-order estimate of the Teukolsky solution \( h \), assuming that errors in
$h_{2.5,1.5}^{(3)}$ are third order. Defining the amplitude $|h|$ and phase $\phi$ as

$$h = |h|e^{i\phi},$$  \hspace{1cm} (2.33)

the amplitude error $\delta|h|/|h|$ and phase error $\delta\phi$ are

$$\frac{\delta|h|}{|h|} = \Re\left(\frac{\Delta h}{h^{(4)}}\right),$$  \hspace{1cm} (2.34)
$$\delta\phi = \Im\left(\frac{\Delta h}{h^{(4)}}\right).$$  \hspace{1cm} (2.35)

**Figure 2-1:** Errors in amplitude (left panel) and phase (right panel) due to grid discretization for $a = 0$ at mass ratio $\mu/M = 10^{-3}$. These errors are the residual we find following the Richardson extrapolation procedure described in the text.

Figure 2-1 shows discretization errors for several gravitational modes $h_{\ell m}$ extracted at $r^* = 950M$. For this case, the large black hole is non spinning ($a = 0$). Amplitude discretization errors are steady over almost the entire waveform, until very late times. In all cases, $\delta|h|/|h| \lesssim \text{a few} \times 10^{-3}$. Similar behavior is observed for phase errors. For most modes, $\delta\phi \lesssim \text{a few} \times 10^{-3}$ radians over the coalescence. The highest $(\ell, m)$ modes we consider approach $10^{-2}$ radian error at the latest times. Because higher $(\ell, m)$ modes require higher grid densities to be resolved, they tend to have larger discretization errors.
2.3. THE TIME-DOMAIN TEUKOLSKY CODE

Extraction error

The code used in the bulk of this analysis extracts $\Psi$ (and derived quantities such as $\psi_4$ and $h$) at large but finite radius. These quantities are more properly extracted at future null infinity. Although it has very recently become possible to extract waveforms at future null infinity (see Refs. [75, 89]), we did not have this capability when we began this analysis. Instead, following Ref. [95], we extract waveforms at multiple radii, and fit to a polynomial in $1/r$. Again defining amplitude $|h|$ and phase $\phi$ using Eq. (2.33), we put

$$|h|(t-r^*, r) = |h|(0)(t-r^*) + \sum_{k=1}^{N} \frac{|h|(k)(t-r^*)}{r^k}, \quad (2.36)$$
$$\phi(t-r^*, r) = \phi(0)(t-r^*) + \sum_{k=1}^{N} \frac{\phi(k)(t-r^*)}{r^k}. \quad (2.37)$$

The time $t-r^*$ is retarded time, taking into account the finite speed of propagation to tortoise radius $r^*$; $N$ is the order of the polynomial fit we choose. The functions $|h|(0)(t-r^*)$ and $\phi(0)(t-r^*)$ are the asymptotic amplitudes and phases describing the waves at future null infinity.

We extract waveforms at radii $r = 150M, 350M, 550M, 750M$ and $950M$. We then perform non-linear, least-squares fits for $|h|(k)$ and $\phi(k)$ using the Levenberg-Marquardt method [96] to find the asymptotic waveform amplitudes and phases. Following Ref. [95], we use $N = 3$ for the order of our fit, and estimate errors by comparing the fits for $N = 3$ and $N = 2$.

Figure 2-2 shows the extrapolation errors we find for the same case shown in Fig. 2-1. For most of the evolution, extrapolation errors are smaller than discretization errors. In particular, the amplitude errors are at or below $10^{-3}$ for most of the coalescence; phase errors are at or below $10^{-3}$ radians. Both phase and amplitude errors grow to roughly $10^{-2}$ very late in the evolution. Note that the largest errors in $\psi_4$ come at the latest times, when the waves have largely decayed away. In other words, the largest errors occur when the waves are weakest. Because we compute $\psi_4$ and then infer amplitude and phase, both amplitude and phase are affected in roughly
Figure 2-2: Errors in amplitude (left panel) and phase (right panel) following extrapolation to infinity for the non-spinning case at mass ratio $\mu/M = 10^{-3}$.

Our numerical errors appear to be of similar size to error estimates seen in related analyses (e.g., Ref. [75]).

Finally, it is worth noting that, thanks to the hyperboloidal layer method introduced to time-domain black hole perturbation theory in Refs. [75, 89], it will not be necessary to perform this extrapolation in future work. The codes will, to very good accuracy, compute the waveform directly at future null infinity. Although this advance did not come in time for the bulk of our present analysis, we have used it to check our error estimates in several cases. We find that our total numerical error estimates (discretization plus extrapolation error, combined in quadrature) is similar to the errors we compute using the hyperboloidal layer method\(^2\). This gives us confidence that our error estimates are reliable.

\(^2\)It is worth emphasizing that codes which use the hyperboloidal layer method are much faster than those which use the extrapolation described here; we find a speedup of roughly ten (for the scale of the evolutions performed in the context of this work). Although it is gratifying that these extrapolations reliably improve our numerical accuracy, the substantial speed-up means that upgrading our method is worthwhile for future work.
2.3.4 Comparing time-domain and frequency-domain Teukolsky codes

As a further check on the accuracy of our numerical Teukolsky-based waveforms, we compare time-domain (TD) waveforms computed using the techniques described here with frequency-domain (FD) waveforms [97, 98]. Since we only calibrate the higher-order modes in the EOB model for \(a = 0\), we focus on that case here. We expect our conclusions to be similar for spinning cases since we use the same procedure to estimate errors in that case.

As described in Secs. 4.1 and 2.3.1, in our analysis the source term for the TD waveforms [cf. Eq. (2.20)] depends on the EOB inspiral and plunge trajectory. For FD waveforms, by contrast, the source is built from a purely geodesic trajectory. This is because the FD code uses the existence of discrete orbital frequencies. For this analysis, we specialize further to circular-orbit equatorial geodesics, but allow these geodesics to evolve adiabatically using FD Teukolsky fluxes, as described in Ref. [99]. Previous work has shown that a self-consistent adiabatic evolution implemented with our FD code is in excellent agreement with the EOB model during the inspiral [68], and so it makes sense to compare TD and FD waveforms during this phase of the coalescence. It is also worth noting that FD waveforms can generally be computed to near machine accuracy using spectral techniques [100]. The only limitation on their accuracy is truncation of the (formally infinite) sums over multipoles and frequency harmonics. We can thus safely assume that the difference between TD and FD waveforms is only due to errors in the TD waves.

To perform this comparison, we align the \(\ell = m = 2\) TD and FD waveforms by introducing time and phase shifts \(\Delta t\) and \(\Delta \phi\) which minimize the gravitational phase difference at low frequencies. More specifically, we choose \(\Delta t\) and \(\Delta \phi\) in order to minimize

\[
\int_{t_1}^{t_2} \left[ \phi_{22}^{FD}(t) - \phi_{22}^{TD}(t + \Delta t) + \Delta \phi \right]^2 dt,
\]

(2.38)

where \(t_1\) and \(t_2\) are separated by 1000\(M\) and correspond to \(M\omega_{22} \approx 0.108\) and a \(M\omega_{22} \approx 0.111\), respectively. This low-frequency alignment is necessary for three
reasons. First, the time coordinate of the TD waveform includes the effect of the extraction radii of the data used for the extrapolation; the FD waveforms are truly extracted at future null infinity. Second, the initial phases of the TD and FD trajectories are not necessarily the same, which introduces a phase offset between the two models. Third and last, as discussed in detail in Ref. [36], TD waveforms include an initial burst of "junk" radiation, which must be discarded. During that burst, the TD and FD trajectories may accumulate a small phase difference. We have found that small changes to $t_1$ and $t_2$ do not significantly affect the alignment.

Once $\Delta t$ and $\Delta \phi$ are fixed, we have no freedom to introduce further time or phase shifts for the other modes. For instance, the difference between the FD and TD phases for the mode $(\ell, m)$ is

$$
\delta \phi_{\ell m}^{\text{FD-TD}} = \left| \phi_{\ell m}^{\text{FD}}(t) - \phi_{\ell m}^{\text{TD}}(t + \Delta t) + m \frac{\Delta \phi}{2} \right|. 
$$

(2.39)

The fractional amplitude difference is

$$
\frac{\delta |h|_{\ell m}^{\text{FD-TD}}}{|h|_{\ell m}} = \left| \frac{|h|_{\ell m}^{\text{FD}}(t)}{|h|_{\ell m}^{\text{TD}}(t + \Delta t)} - 1 \right|.
$$

(2.40)

The $\Delta t$ and $\Delta \phi$ used here are the ones which minimize (2.38).

Table 2.1 compares $\delta \phi_{\ell m}^{\text{FD-TD}}$ and $\delta |h|_{\ell m}^{\text{FD-TD}}/|h|_{\ell m}$ with the errors computed using the techniques described in Sec. 2.3.3. In particular, we examine the averages of $\delta \phi_{\ell m}^{\text{FD-TD}}$ and $\delta |h|_{\ell m}^{\text{FD-TD}}/|h|_{\ell m}$ over the alignment interval $(t_1, t_2)$, and compare them to the averages over the same interval of the TD numerical errors discussed in the previous section. For this comparison, we average the sum (in quadrature) of discretization and extrapolation errors. We see that the difference between TD and FD is always within the TD numerical errors, except for the $\ell = 3, m = 2$ mode. This mode is among the weakest of those that we show in Table 2.1, which makes the extraction procedure described in the previous section considerably more difficult.
Table 2.1: The phase difference and fractional amplitude difference for various modes, averaged over the time interval $t_2 - t_1$ [see Eq. (2.38)]. We compare the amplitude and phase error found by comparing TD and FD waveforms (columns 2 and 4) with the TD errors we estimate using the techniques discussed in Sec. 2.3.3. In all cases but one [phase error for the $(3,2)$ mode], our numerical error estimates are larger than those we find comparing the two calculations; in that single discrepant case, the errors themselves are particularly small. This is further evidence that our numerical error estimates are reliable.

| $(\ell, m)$ | $\delta \phi_{\ell m}^{\text{FD-TD}}$ | $\delta \phi_{\ell m}^{\text{TD}}$ | $\frac{\delta |h|^\text{FD-TD}_{\ell m}}{|h|_{\ell m}}$ | $\frac{\delta |h|^\text{TD}_{\ell m}}{|h|_{\ell m}}$ |
|------------|-----------------|-----------------|-----------------|-----------------|
| (2,2)      | $7.71 \times 10^{-4}$ | $1.27 \times 10^{-3}$ | $1.22 \times 10^{-4}$ | $8.20 \times 10^{-4}$ |
| (3,3)      | $1.18 \times 10^{-3}$ | $2.22 \times 10^{-3}$ | $1.95 \times 10^{-4}$ | $1.64 \times 10^{-3}$ |
| (2,1)      | $4.05 \times 10^{-4}$ | $1.28 \times 10^{-3}$ | $2.55 \times 10^{-4}$ | $1.03 \times 10^{-3}$ |
| (4,4)      | $1.58 \times 10^{-3}$ | $2.94 \times 10^{-3}$ | $2.80 \times 10^{-4}$ | $2.80 \times 10^{-3}$ |
| (3,2)      | $7.71 \times 10^{-4}$ | $3.92 \times 10^{-4}$ | $4.09 \times 10^{-4}$ | $3.13 \times 10^{-3}$ |

**2.3.5 Characteristics of time-domain Teukolsky merger waveforms**

We now turn to the waveforms produced by the TD Teukolsky analysis and their general characteristics. Figure 2-3 examines the behavior of the dominant TD modes $[(\ell, m) = (2,2), (3,3), (4,4), (2,1), (3,2)]$ during plunge, merger and ringdown. We also show the orbital frequency of the EOB trajectory used to produce the TD data.

For the non-spinning case (left panel), the peak of the $(2,2)$ amplitude comes slightly earlier than the peak in orbital frequency, while higher-order modes peak later. A summary of the time difference $t_{\text{peak}}^{\ell m} - t_{\text{peak}}^\Omega$ between the peak of the Teukolsky mode amplitude and that of the EOB orbital frequency is shown in Table 2.2. This difference can be as large as $6.25M$ for the $(3,2)$ mode, and even $8.82M$ for the $(2,1)$ mode.

The situation is different in the spinning case. A trend we see is that as the spin $a$ grows positive, higher-order modes become progressively more important. In the right-hand panel of Fig. 2-3, we show the amplitude of the eight strongest modes for $a/M = 0.9$. As noticed in Ref. [90], modes with $\ell = m$ tend to increase more during the plunge. For example, the $(5,5)$ mode is smaller than the $(3,2)$ mode.
Figure 2-3: Amplitude of the dominant modes during plunge, merger and ringdown for $a = 0$ (left panel) and $a/M = 0.9$ (right panel). We also show orbital frequency, scaled to fit on the plot. As expected, the orbital frequency asymptotes to the horizon's angular velocity at late times, because the frame dragging locks the particle's motion to that of the horizon. The vertical dashed line in the two panels marks the position of the peak of the orbital frequency.

<table>
<thead>
<tr>
<th>$(\ell, m)$</th>
<th>$(t_{\ell m}^{\text{peak}} - t_{\Omega}^{\text{peak}})/M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2,2)</td>
<td>-2.99</td>
</tr>
<tr>
<td>(3,3)</td>
<td>0.52</td>
</tr>
<tr>
<td>(4,4)</td>
<td>2.26</td>
</tr>
<tr>
<td>(2,1)</td>
<td>8.82</td>
</tr>
<tr>
<td>(3,2)</td>
<td>6.25</td>
</tr>
</tbody>
</table>

Table 2.2: The time difference $(t_{\ell m}^{\text{peak}} - t_{\Omega}^{\text{peak}})/M$ between the peak of the Teukolsky modes’ amplitude and of the orbital frequency, for $a/M = 0$. 
Table 2.3: The time difference \((t_{\text{peak}}^{22} - t_{\text{peak}}^{\Omega})/M\) between the peak of the Teukolsky (2, 2) amplitude of the orbital frequency, for various values of the spin \(a/M\). Also shown is the value of the orbital frequency at that peak, \(M\Omega_{\text{max}}\).

during inspiral, but becomes larger during the plunge. The (6,6) mode also grows quickly during the plunge (which roughly ends at the time near the peak of the orbital frequency). This behavior is not surprising, in fact should become more and more pronounced as spin \(a/M \to 1\); modes modes with large multipole moments become as important as low-\(\ell\) modes in that limit, at least for quasi-circular orbits [101, 102].

Table 2.3 shows the time difference \((t_{\text{peak}}^{22} - t_{\text{peak}}^{\Omega})/M\) between the peak of the (2, 2) amplitude and of the orbital frequency for the values of spin that we consider in this chapter. As the spin grows, the orbital frequency peaks later and later relative to the peak of the (2, 2) amplitude. This has important implications for modeling the EOB merger-ringdown waveform, as we discuss in detail later in the chapter.

Another interesting feature of the Teukolsky waveforms we find is shown in Fig. 2-4. This figure shows the gravitational-wave frequency (defined as the time derivative of the phase) for the (2, 2) mode during plunge, merger, and ringdown, for several spin values. Notice the strong oscillations seen at late times (during the final ringdown) for spins \(a \lesssim 0\). These oscillations grow as the spin decreases, and become very large for \(a/M = -0.9\). We have verified that these oscillations (even in the \(a/M = -0.9\) case) are \textit{not} numerical artifacts. We have found that they are insensitive to numerical resolution and floating-point precision, that they also appear in the context of other plunging retrograde trajectories. We find that they are due to a superposition of the dominant (2, 2) QNM with the (2, -2) QNM, which is also excited during the plunge.
CHAPTER 2. MODELING MULTIPOLAR GRAVITATIONAL-WAVE EMISSION FROM SMALL MASS-RATIO MERGERS

$\omega_2 \pm 20 = \omega_{2 \pm 20} - i/\tau_{2 \pm 20}$ are the complex QNM frequency [with overtone $n = 0$, as introduced in Eq. (2.18)]. The complex parameter $h_0$ and real parameters $\bar{\alpha}$, $\bar{\beta}$ are left unspecified. From Eq. (2.41) one can then calculate the frequency as $\Re[-i h(t)/h(t)]$. Because $h_0$ cancels in this expression, we are left with the parameters $\bar{\alpha}$ and $\bar{\beta}$, which can be determined by numerical fitting [73]. We find that the relative excitation $\bar{\alpha}$ of the $(2, -2)$ modes goes from $\bar{\alpha} \approx 0.005$ for $a/M = 0$, to $\bar{\alpha} \approx 0.03$ for $a/M = -0.5$ and $\bar{\alpha} \approx 0.46$ for $a/M = -0.9$. This is nicely in accord with the growing strength of the oscillations as $a/M \to -1$ that is seen in Fig. 2-4.

A possible reason why the $(2, -2)$ QNM is strongly excited for large negative
spins can be understood by examining the particle’s trajectory. When \( a < 0 \), the spin angular momentum is oppositely directed from the orbital angular momentum. During the inspiral, when the orbit is very wide, the orbit’s angular velocity is opposite to the sense in which the horizon rotates. At late times (during the final plunge), the particle’s motion becomes locked to the horizon by frame dragging. The particle’s angular velocity thus flips sign at some point during the plunge when \( a < 0 \). This change in angular velocity is most pronounced for large negative spins, since the difference between the frequency at the innermost stable circular orbit (ISCO) and at the event horizon is largest for large negative \( a \).

Figure 2-5 shows, as an example, the EOB trajectory we used to produce the \( a/M = -0.9 \) Teukolsky waveforms. As viewed here, the horizon rotates in the clock-
wise sense. After the anti-clockwise inspiral, the particle plunges and its angular velocity flips sign before the particles settles on a quasi-circular orbit with \( r \rightarrow r_+ \) and \( \Omega \rightarrow \Omega_+ = a/2Mr_+ \) as \( t \rightarrow +\infty \) (where \( r_+ = M + \sqrt{M^2 - a^2} \) is the coordinate radius of the event horizon). This behavior leads us to conjecture that the \((2, -2)\) QNM is excited by the last part of the plunge, when the particle is corotating with the black hole. The \((2, 2)\) QNM is excited by the final inspiral and initial plunge, when the particle is counter-rotating relative to the black hole. When \( a > 0 \) the particle’s motion is always co-rotating with the black hole, both during inspiral and through the plunge. This conjecture thus explains why oscillations in the ringdown frequency are much less significant for \( a > 0 \) and seem to disappear when \( a/M \approx 1 \) (see Fig. 2-4).

### 2.4 Comparison of the EOB model with the Teukolsky time-domain waveforms

In this Section, we present the main results of this chapter, comparing the EOB waveforms for binary coalescence with waveforms calculated using the time-domain Teukolsky equation tools described in the previous section. We begin by comparing Teukolsky waveforms (for \( a = 0 \)) with an EOB model that has been calibrated for the comparable-mass case (Sec. 2.4.1). The agreement is good for some modes \([(2, 2) \text{ and } (3, 3)]\), but is much less good for others \([(2, 1) \text{ and } (4, 4)]\). We nail down the reason for this disagreement, recalibrate the EOB model, and show much better agreement in the comparison in Sec. 2.4.2. We then consider \( a \neq 0 \). Focusing on the \((2, 2)\) mode, we compare Teukolsky and EOB waveforms for a range of spins in Sec. 2.4.3.

We stress that all the comparisons that we present in this chapter have been performed between EOB and Teukolsky waveforms produced with the same EOB trajectory. For example, in order to recalibrate the EOB model, we start with a reasonable EOB trajectory, we feed that to the Teukolsky code, and compare the resulting Teukolsky waveforms to the EOB waveforms. If the waveforms do not
2.4. COMPARISON OF THE EOB MODEL WITH THE TEUKOLSKY TIME-DOMAIN WAVEFORMS

Figure 2-6: The amplitude $h$ (top panels) and gravitational-wave frequency $\omega$ (bottom panels) when the (2, 2) (left) and (4, 4) (right) modes reach their peak. Circles at $\nu \geq 0.12$ denote data points extracted from the numerical-relativity simulations; the left-most points at $\nu = 10^{-3}$ are data extracted with the Teukolsky code. The solid lines are quadratic fits to the data points.

agree, we modify the EOB trajectory so that the EOB waveforms agree with the Teukolsky waveforms produced with the old EOB trajectory, and then feed the new EOB trajectory to the Teukolsky code. We then iterate until this procedure has converged.

2.4.1 Comparison of comparable-mass EOB waveforms and Teukolsky waveforms for $a = 0$

Reference [67] presents an EOB model calibrated to numerical-relativity simulations of non-spinning black-hole binaries with mass ratios $m_2/m_1 = 1, 1/2, 1/3, 1/4$ and 1/6. This model achieves very good agreement between the phase and amplitude of the EOB and numerical-relativity waveforms; see Secs. II and III of Ref. [67] for details. As background for the comparison we will make to the Teukolsky waveform, we briefly discuss how the EOB inspiral-plunge waveform was built, and how the merger-ringdown waveform was attached to build the full waveform.

For each mode, Ref. [67] set $a_4^{h_{\ell m}} = a_5^{h_{\ell m}} = b_3^{h_{\ell m}} = 0$ in Eq. (2.17), and fixed the
remaining coefficients $a_{i}^{\ell m}$ [$i \in (1, 2, 3)]$ and $b_{i}^{\ell m}$ [$i \in (1, 2)]$ by imposing the following five conditions:

1. The time at which the EOB $h_{22}$ reaches its peak should coincide with the time at which the EOB orbital frequency $\Omega$ reaches its peak. We denote this time with $t_{\text{peak}}^{\Omega}$. The peaks of higher-order numerical modes differ from the peak of the numerical $h_{22}$; we define this time difference as

$$
\Delta t_{\text{peak}}^{\ell m} = t_{\text{peak}}^{\ell m} - t_{\text{peak}}^{22} = t_{d} |h_{\text{NR}}^{\ell m}|/dt = 0 - t_{d} |h_{22}^{\text{NR}}|/dt = 0.
$$

(2.42)

We require that the peaks of the EOB $h_{\ell m}$ occur at the time $t_{\text{peak}}^{\Omega} + \Delta t_{\text{peak}}^{\ell m}$:

$$
\frac{d |h_{\ell m}^{\text{EOB}}|}{dt} \bigg|_{t_{\text{peak}}^{\Omega} + \Delta t_{\text{peak}}^{\ell m}} = 0.
$$

(2.43)

2. The peak of the EOB $h_{\ell m}$ should have the same amplitude as the peak of the
2.4. COMPARISON OF THE EOB MODEL WITH THE TEUKOLSKY TIME-DOMAIN WAVEFORMS

numerical $h_{\ell m}$:

$$|h_{\ell m}^{\text{EOB}}(t_{\text{peak}}^\Omega + \Delta t_{\text{peak}}^{\ell m})| = |h_{\ell m}^{\text{NR}}(t_{\text{peak}}^{\ell m})|.$$  \hfill (2.44)

3. The peak of the EOB $h_{\ell m}$ should have the same second time derivative as the peak of the numerical $h_{\ell m}$:

$$\frac{d^2}{dt^2} |h_{\ell m}^{\text{EOB}}|_{t_{\text{peak}}^\Omega + \Delta t_{\text{peak}}^{\ell m}} = \frac{d^2}{dt^2} |h_{\ell m}^{\text{NR}}|_{t_{\text{peak}}^{\ell m}}.$$  \hfill (2.45)

4. The frequency of the numerical and EOB $h_{\ell m}$ waveforms should coincide at their peaks:

$$\omega_{\ell m}^{\text{EOB}}(t_{\text{peak}}^\Omega + \Delta t_{\text{peak}}^{\ell m}) = \omega_{\ell m}^{\text{NR}}(t_{\text{peak}}^{\ell m}).$$  \hfill (2.46)

5. The time derivative of the frequency of the numerical and EOB $h_{\ell m}$ waveforms should coincide at their peaks:

$$\dot{\omega}_{\ell m}^{\text{EOB}}(t_{\text{peak}}^\Omega + \Delta t_{\text{peak}}^{\ell m}) = \dot{\omega}_{\ell m}^{\text{NR}}(t_{\text{peak}}^{\ell m}).$$  \hfill (2.47)

[Note that the quantities $h_{\ell m}^{\text{EOB}}$ referenced in the above equations are the same as the quantities $h_{\ell m}^{\text{ins} \rightarrow \text{plunge}}$ defined in Eq. (2.10).]

The functions $\Delta t_{\text{peak}}^{\ell m}$, $|h_{\ell m}^{\text{NR}}(t_{\text{peak}}^{\ell m})|$, $d^2|h_{\ell m}^{\text{NR}}|/dt^2|_{t_{\text{peak}}^{\ell m}}$, $\omega_{\ell m}^{\text{NR}}(t_{\text{peak}}^{\ell m})$, and $\dot{\omega}_{\ell m}^{\text{NR}}(t_{\text{peak}}^{\ell m})$ described in Ref. [67] were extracted from numerical-relativity and Teukolsky data, and approximated by smooth functions of the symmetric mass ratio $\nu$. Least-square fits for these quantities were given in Table III of Ref. [67]. These fits included information about the $\nu = 10^{-3}$ case from the analysis of this chapter (which was in preparation as Ref. [67] was completed).

Since $|h_{\ell m}(t_{\text{peak}}^{\ell m})|$ and $d^2|h_{\ell m}|/dt^2|_{t_{\text{peak}}^{\ell m}}$ approach zero in the test-particle limit, their input values at $\nu = 10^{-3}$ do not affect the least-square fits very much. For the $(2, 2)$ and $(3, 3)$ modes, the data points are very regular. This is illustrated for the $(2, 2)$ case in Fig. 2-6. The residues of the fit at $\nu = 10^{-3}$ are very small, and the $\nu$-fits agree well with the values from the Teukolsky code at $\nu = 10^{-3}$. Unfortunately,
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Figure 2-8: The same as Fig. 2-7, but for the (2,1) and (4,4) modes.

this is not the case for the (2,1) and (4,4) modes. Figure 2-6 shows this for the (4,4) mode. The $|h_{44}(t^4_{\text{peak}})|$ data points do not lie on a smooth curve, and so the fit is intrinsically unstable. By minimizing the relative residue instead of the absolute residue in the least-square fit, we increase the weight on the $\nu = 10^{-3}$ data point and get a much better fit at low mass ratio, but at the cost of a much poorer fit in the comparable-mass regime. The situation is even worse for $\omega_{44}(t^4_{\text{peak}})$, for which the data points for comparable masses have a rather irregular trend. These results emphasize the need for more accurate numerical-relativity data describing the higher-order modes in order to smoothly connect these quantities from the test-particle limit to the equal-mass case.

Once the coefficients $a_i^{h\ell m}$ and $b_i^{h\ell m}$ are known, we calculate $h_{\ell m}^{\text{insp-plunge}}$ using Eq. (2.10), and attach the QNMs using Eq. (2.19). We assume the following comb widths [67],

$$
\Delta t_{\text{match}}^{22} = 5M, \quad \Delta t_{\text{match}}^{33} = 12M,
$$

$$
\Delta t_{\text{match}}^{44} = 9M, \quad \Delta t_{\text{match}}^{21} = 8M,
$$

and choose $t_{\text{match}}^{\ell m} = t^\Omega_{\text{peak}} + \Delta t_{\text{peak}}^{\ell m}$ with $\Delta t_{\text{peak}}^{\ell m}$ given in Eq. (2.42). It was found in Ref. [67] that, after calibrating the EOB adjustable parameters and aligning the EOB
and numerical waveforms at low frequency, the difference between $t_{\text{peak}}^{22}$ and $t_{\text{peak}}^\Omega$ is typically $\sim 1M$. Thus, Ref. [67] assumed that $t_{\text{peak}}^\Omega = t_{\text{peak}}^{22}$ and consequently set $\Delta t_{\text{peak}}^{22} = 0$. Following these findings, we attach the QNMs at $t_{\text{peak}}^\Omega$ for the $(2,2)$ mode, and at $t_{\text{peak}}^\Omega + \Delta t_{\text{peak}}^\text{em}$ for all the other modes.

Figures 2-7 and 2-8 compare the leading modes generated by this EOB model with the modes generated by the time-domain Teukolsky code. We adopt the waveform alignment procedure used in Refs. [60, 61, 90, 67] and Sec. 2.3.4, aligning the waveforms at low frequency by minimizing

$$\int_{t_1}^{t_2} [\phi_1(t) - \phi_2(t - \Delta t) - \Delta \phi]^2 \, dt,$$

over a time shift $\Delta t$ and a phase shift $\Delta \phi$. Here, $\phi_1(t)$ and $\phi_2(t)$ are the gravitational phases of the EOB and Teukolsky $h_{22}$. We chose $t_2 - t_1 = 1000M$, and center these times when the orbital frequencies are low. We have verified that our results are insensitive to the precise location of this integration interval, provided that it is chosen during the inspiral phase.

As expected from the discussion above, Figure 2-7 shows that there is quite good agreement between the EOB and Teukolsky models for the $(2,2)$ and $(3,3)$ modes. In particular, the difference in both the amplitude and the phase is quite small until the inspiral reaches the ISCO. This excellent agreement is due to the resummed-factorized energy flux [53] employed in the EOB equations of motion and waveforms (see previous studies [71, 72, 55, 103, 73, 74, 104, 75]). The amplitude disagreement during merger and ringdown is due to our procedure of attaching QNMs to the EOB waveform [see Fig. 3 and discussion around in Ref. [67]]. The accumulation of some phase difference during plunge will be discussed at the end of this section.

Figure 2-8 shows that the agreement between the EOB and Teukolsky $(2,1)$ and $(4,4)$ modes remains excellent during the long inspiral, but is not very satisfactory during the merger and ringdown. For the $(4,4)$ mode, the EOB amplitude becomes too large toward merger. This is a consequence of the excessively large residue of the $\nu$-fit for $|h_{44}(t_{\text{peak}}^\text{em})|$ at $\nu = 10^{-3}$. For the $(2,1)$ mode, the EOB model of Ref. [67] fails
to reproduce a reasonable merger waveform. This problem is related to the fact that the value of $\Delta t^{21}_{\text{peak}}$ at $\nu = 10^{-3}$ used in Ref. [67] to determine the $\nu$-fit is too large (see Table 2.2). The problem is deeper than this, however. In particular, the value is uncertain due to the (unusual) broadness of the Teukolsky (2, 1) mode's peak (see Fig. 2-3). We shall see in the next section that to improve the agreement of the (2, 1) mode, we need a smaller value for $\Delta t^{21}_{\text{peak}}$.

An additional source of error arises from the procedure that was used to compute the NQC coefficients $a^{h22}_1$, $a^{h22}_2$, and $a^{h22}_3$ used in $N_{22}$ [see Eq. (2.17)]. In Ref. [67], these coefficients were calculated by an iterative procedure using the five conditions discussed at the beginning of this section. These coefficients have small but non-negligible effects on the EOB dynamics: through the amplitude $|h_{22}|$, they enter the energy flux [see Eq. (2.9)] and thereby influence the rate at which the small body spirals in. This iterative procedure increases by a factor of a few the computational cost of generating $h_{22}$. To mitigate this cost increase, Ref. [67] suggested replacing the iterative procedure with $\nu$-fits for $a^{h22}_1$, $a^{h22}_2$, and $a^{h22}_3$. These fits were obtained using data for mass ratios $1, 1/2, 1/3, 1/4$ and $1/6$. The EOB waveforms shown in

\begin{itemize}
  \item Note that the NQC coefficients $a^{h_{\nu m}}_i$ of higher-order modes contribute much less to the energy flux and can be safely ignored in the dynamics of comparable-mass binaries [67].
\end{itemize}
Figs. 2-7, 2-8 are then generated using these $\nu$-fits extrapolated to $\nu = 10^{-3}$. When comparing the fit and the true values of $a_1^{h2a}$, $a_2^{h2a}$ and $a_3^{h2a}$ at $\nu = 10^{-3}$, we find a non-negligible difference which is responsible for $\sim 0.4$ rad difference between the EOB and Teukolsky (2,2) modes, and for $\sim 0.6$ rad difference for the (3,3) modes. In the next section, we shall show that by returning to the iterative procedure, rather than using the fits, we can do much better.

Lastly, we comment on why for the (2,1) and (4,4) modes in Eq. (2.13b) we replaced $v_\phi^{(t+e)}$ with $v_\phi^{(t+e-2)}/r_\Omega$. As discussed above, the amplitude of the numerical (2,1) and (4,4) modes reaches a peak a fairly long time after the peak of the (2,2) mode. Thus, in order to impose the first condition (in our list of five) given above, the peak of the EOB mode should be moved to $t_\text{peak}^\Omega + \Delta t_\text{peak}^{\ell m}$. However, the leading EOB amplitude is proportional to a power of the orbital frequency. This frequency decreases to zero at the horizon, and so the EOB amplitude drops to an extremely small value at $t_\text{peak}^\Omega + \Delta t_\text{peak}^{\ell m}$. By replacing $v_\phi^2 = (M r_\Omega \Omega)^2$ with $1/r_\Omega$, we slow the decay of these modes after $t_\text{peak}^\Omega$, and can successfully move the peak of the mode to $t_\text{peak}^\Omega + \Delta t_\text{peak}^{\ell m}$. This modification was also adopted in Ref. [67] to successfully model the (2,1) and (4,4) modes in the comparable-mass case.

2.4.2 Comparison of calibrated EOB waveforms and Teukolsky waveforms for $a = 0$

We now improve on the EOB model of Ref. [67] to more accurately reproduce Teukolsky waveforms. We focus on comparisons for the $a = 0$ limit, although we discuss how we build our model for general spins. We start from the five conditions discussed
at the beginning of Sec. 2.4.1 which allow us to compute the NQC coefficients:

\[
\frac{d}{dt} h^{\text{EOB}}_{\ell m} |_{t_{\text{peak}} + \Delta t_{\text{peak}}^{\ell m}} = 0 \tag{2.50}
\]

\[
| h^{\text{EOB}}_{\ell m} (t_{\text{peak}} + \Delta t_{\text{peak}}^{\ell m}) | = | h^{\text{Teuk}}_{\ell m} (t_{\text{peak}}) | \tag{2.51}
\]

\[
\frac{d^2}{dt^2} h^{\text{EOB}}_{\ell m} |_{t_{\text{peak}} + \Delta t_{\text{peak}}^{\ell m}} = \frac{d^2}{dt^2} h^{\text{Teuk}}_{\ell m} |_{t_{\text{peak}}^{\ell m}} \tag{2.52}
\]

\[
\omega^{\text{EOB}}_{\ell m} (t_{\text{peak}} + \Delta t_{\text{peak}}^{\ell m}) = \omega^{\text{Teuk}}_{\ell m} (t_{\text{peak}}) \tag{2.53}
\]

\[
\dot{\omega}^{\text{EOB}}_{\ell m} (t_{\text{peak}} + \Delta t_{\text{peak}}^{\ell m}) = \dot{\omega}^{\text{Teuk}}_{\ell m} (t_{\text{peak}}) \tag{2.54}
\]

where

\[
\Delta t_{\text{peak}}^{\ell m} = t_{\text{peak}}^{\ell m} - t_{\text{peak}} \tag{2.55}
\]

As before, the quantities \( h^{\text{EOB}}_{\ell m} \) given above are equivalent to the quantities \( h^{\text{insp-plunge}}_{\ell m} \) in Eq. (2.10). The quantities on the RHSs of Eqs. (2.50)–(2.54) are now computed from the Teukolsky waveforms, rather than using the \( \nu \)-fits of Ref. [67], as in Sec. 2.4.1. Notice that \( \Delta t_{\text{peak}}^{\ell m} \) in Eq. (2.55) differs from the one in Eq. (2.42). In fact, since the
2.4. COMPARISON OF THE EOB MODEL WITH THE TEUKOLSKY TIME-DOMAIN WAVEFORMS

Teukolsky code uses the EOB trajectory, the time difference between the peak of any \((\ell, m)\) mode and the peak of the orbital frequency is unambiguous. This was not the case in Ref. [67] where numerical relativity waveforms are used. In that case, the time difference between the peak of the numerical relativity \((\ell, m)\) modes and the peak of the EOB orbital frequency depends on the alignment procedure between the numerical and EOB waveforms, and on the calibration of the EOB adjustable parameters. This is why Ref. [67] adopted the \(\Delta t_{\text{peak}}\) described in Eq. (2.42).

As discussed in Sec. 2.4.1, Ref. [67] assumed that \(t_{\text{peak}}^{\Omega} = t_{\text{peak}}^{22}\) and consequently set \(\Delta t_{\text{peak}}^{22} = 0\). However, as seen in Tables 2.2, 2.3, the Teukolsky data show that \(t_{\text{peak}}^{\Omega} - t_{\text{peak}}^{22}\) differs from zero when \(\nu = 10^{-3}\); this effect is particularly pronounced for large positive Kerr spin parameters. The modified prescription given by Eq. (2.55) is thus quite natural. By contrast, the prescription described in Ref. [67] is bound to fail for all the modes in the test-particle limit.

In the non-spinning case, we solve Eqs. (2.50), (2.51) for \(a_i h_{tm}\) with \(i = 1, 2, 3\), and set \(a_4 h_{tm} = a_5 h_{tm} = 0\). In the spinning case, in order to not introduce spin-dependence at leading order, we fix \(a_i h_{tm}\) and \(a_j h_{tm}\) to the values calculated for \(a = 0\), and solve for \(a_i h_{tm}\) (with \(i = 3, 4, 5\)). As for the phase NQC coefficients, in the non-spinning case we solve Eqs. (2.52) and (2.53) for \(b_i h_{tm}\) (with \(i = 1, 2\)) and set \(b_3 h_{tm} = b_4 h_{tm} = 0\). In the spinning case, we fix \(b_1 h_{tm}\) and \(b_2 h_{tm}\) to their \(a = 0\) values (again, in order to not introduce any spin-dependence at leading order), and solve for \(b_3 h_{tm}\) and \(b_4 h_{tm}\).

Once the coefficients \(a_i h_{tm}\) and \(b_i h_{tm}\) are known, we calculate \(h_{tm}^{\text{inap--plunge}}\) using Eq. (2.10) and attach the QNMs using Eq. (2.19), assuming the comb’s width as in Eq. (2.48). Furthermore, we choose \(t_{\text{match}}^{\Omega} = t_{\text{peak}}^{\Omega} + \Delta t_{\text{peak}}^{\text{m}}\) with \(\Delta t_{\text{peak}}^{\text{m}}\) given in Eq. (2.55). The merger-ringdown \((2, 2)\) mode is now attached at the time where the Teukolsky \((2, 2)\) amplitude peaks, in contrast to the approach used in Ref. [67]. All the other merger-ringdown \((\ell, m)\) modes are attached at the time the corresponding Teukolsky \((\ell, m)\) amplitudes peaks, which is the same procedure followed in Ref. [67].

Figures 2-9 and 2-10 compare these calibrated EOB models to the Teukolsky amplitudes. Table 2.4 lists the input parameters used on the RHSs of Eqs. (2.50)--(2.54). We emphasize that the value of \(\Delta t_{\text{peak}}^{21}\) we reported in this table and that we


**Table 2.4:** Input values for the RHSs of Eqs. (2.50)–(2.54) for the EOB model for non-spinning black-hole used in Figs. 2-9 and 2-10.

| $(\ell, m)$ | $\Delta t_{\ell m, \text{peak}}$ | $|h_{\ell m, \text{peak}}|_{\text{Teuk}}$ | $d^2|h_{\ell m, \text{peak}}|_{\text{Teuk}}/dt^2$ | $\omega_{\ell m, \text{peak}}_{\text{Teuk}}$ | $\dot{\omega}_{\ell m, \text{peak}}_{\text{Teuk}}$ |
|-----------|-----------------|------------------|-----------------|-----------------|-----------------|
| (2,2)     | -2.99           | 0.001450         | -3.171 x 10^{-6} | 0.2732          | 0.005831        |
| (2,1)     | 6.32            | 0.0005199        | -7.622 x 10^{-7} | 0.2756          | 0.01096         |
| (3,3)     | 0.52            | 0.0005662        | -1.983 x 10^{-6} | 0.4546          | 0.01092         |
| (4,4)     | 2.26            | 0.0002767        | -1.213 x 10^{-6} | 0.6347          | 0.01547         |

**Figure 2-11:** Comparison of Teukolsky-calibrated EOB and Teukolsky (2,2) modes for $a/M = -0.5$ (left panel) or $a/M = 0.5$ (right panel). Upper panels show the real part of the modes; lower panels show phase and fractional amplitude differences.

Use in our model is $2.5M$ smaller than the difference $t_{\ell m, \text{peak}}^{(21)} - t_{\ell m, \text{peak}}^{(2)}$. We do this because the peak of the Teukolsky (2,1) amplitude is quite broad, and at the time $t_{\ell m, \text{peak}}^{(21)}$ where the peak occurs the Teukolsky mode’s frequency oscillates due to superposition of the $\ell = 2$, $m \pm 1$ modes, as discussed in Sec. 2.3.5. (See also Ref. [73] and Fig. 3 of Ref. Ref. [74], which shows similar oscillations.) Although these oscillations are physical, we do not attempt to reproduce them in our EOB waveform, as their effect on the phase agreement between the EOB and Teukolsky waveforms is negligible. We therefore simply choose a slightly smaller value of $\Delta t_{\ell m, \text{peak}}^{(21)}$, ensuring that $t_{\ell m, \text{peak}}^{(2)} + \Delta t_{\ell m, \text{peak}}^{(21)}$ is within the broad peak of $h_{\ell m, \text{peak}}^{(21)}$. This in turn ensures that the oscillations in frequency do not impact our results.
2.4. COMPARISON OF THE EOB MODEL WITH THE TEUKOLSKY TIME-DOMAIN WAVEFORMS

Figure 2-10 demonstrates that, by calibrating against the Teukolsky waveforms using the input values shown in Table 2.4, the agreement between the EOB and Teukolsky waveforms is considerably improved during merger and ringdown for the (2, 1) and (4, 4) modes. Improvements to the (2, 2) and (3, 3) modes (Fig. 2-9) are less significant, since the model of Ref. [67] works quite well for these modes. As we discussed in the previous section, the input parameters listed in Table 2.4 are well predicted by the fitting formulas of Ref. [67] (see also Fig. 2-6). There is, however, noticeable improvement in phase agreement between Figs. 2-7 and 2-9. This is due to the fact that in the latter case we use the iterative procedure to compute the NQC coefficients \( a_i \), as discussed at the end of Sec. 2.4.1.

Comparing to the discussion of numerical error in Sec. 2.3.3, we note that the differences in phase and amplitude between the EOB and Teukolsky modes shown in Figs. 2-9 and 2-10 are within the numerical errors essentially through the plunge; the differences grow larger than these errors during merger and ringdown. As this analysis was being completed, we acquired the capability to produce Teukolsky waveforms using the hyperboloidal layer method (Ref. [89]; see also Ref. [75]). We have compared phase and amplitude differences for the (2, 2) mode between the Teukolsky code used for the bulk of this analysis, and the hyperboloidal variant. We have found that these differences are within the errors discussed in Sec. 2.3.3.

2.4.3 Comparisons for general spin

We conclude our discussion of results by comparing, for the first time, EOB and Teukolsky coalescence waveforms with \( a \neq 0 \). These waveforms are produced using the trajectory of the spinning EOB model described in Sec. 2.2. As in the non-spinning case, understanding the transition from inspiral to ringdown in the test-particle limit when the central black hole carries spin can help modeling the plunge-merger waveforms from comparable-mass spinning black holes.

As in the non-spinning case [71, 72, 55, 103, 73, 74, 104, 75], we expect that the resummed-factorized energy-flux and mode amplitudes agree quite well with the Teukolsky data at least up to the ISCO, provided that the spin is not too high. In fact,
CHAPTER 2. MODELING MULTIPOLAR GRAVITATIONAL-WAVE EMISSION FROM SMALL MASS-RATIO MERGERS

Figure 2-12: The same as Fig. 2-11, but for \( a/M = -0.9 \) (left panel) and \( a/M = 0.7 \) (right panel).

| \( a/M \) | \( \Delta t_{\text{peak}}^{22} \) | \( |h^{\text{Teuk}}_{22,\text{peak}}| \) | \( d^2|h^{\text{Teuk}}_{22,\text{peak}}|/dt^2 \) | \( \omega_{22,\text{peak}}^{\text{Teuk}} \) | \( \dot{\omega}_{22,\text{peak}}^{\text{Teuk}} \) |
|---|---|---|---|---|---|
| -0.9 | 1.60 | 0.001341 | -3.532 \times 10^{-6} | 0.2195 | 0.005676 |
| -0.5 | -0.08 | 0.001382 | -2.536 \times 10^{-6} | 0.2376 | 0.006112 |
| 0.5 | -7.22 | 0.001542 | -1.334 \times 10^{-6} | 0.3396 | 0.005095 |
| 0.7 | -12.77 | 0.001582 | -1.212 \times 10^{-6} | 0.3883 | 0.004068 |
| 0.9 | -39.09 | 0.001576 | -8.102 \times 10^{-8} | 0.4790 | 0.001779 |

Table 2.5: Input values for the RHSs of Eqs. (2.50)-(2.54) for the EOB (2,2) mode for spinning black-holes used in Figs. 2-11 \( (a/M = \pm 0.5) \) and 2-12 \( (a/M = 0.7 \) and \( a/M = -0.9) \). We also include data for the case \( a/M = 0.9 \), although we do not compare waveforms for this example.

Ref. [55] showed that, in the adiabatic limit, the resummed-factorized (2,2) modes agree very well with frequency-domain Teukolsky modes up to the ISCO, at least over the range \(-1 \leq a/M \lesssim 0.7\). The relative difference between amplitudes in the two models is less than 0.5% when \( a/M \leq 0.5 \), but grows to 3.5% when \( a/M \simeq 0.75 \).

In this work, we focus on the (2,2) mode comparison, leaving to a future publication a thorough study of the higher modes. We have already seen in Sec. 2.3.5 that as \( a/M \to 1 \), many more modes become excited during the plunge and merger. For this limit, the resummed-factorized waveforms will need to be improved in order to match higher-order Teukolsky modes with good precision.
Figures 2-11 and 2-12 compare (2,2) modes for the EOB and Teukolsky waves for spin values \(a/M = \pm 0.5\), \(a/M = 0.7\) and \(a/M = -0.9\). We build the full EOB waveform following the prescription described in Sec. 2.4.2, using the input parameters shown in Table 2.5. For the cases \(a/M = 0.5\) and \(a/M = 0.7\), we also use a pseudo QNM (pQNM) (in addition to the standard QNMs) as suggested in Refs. [61, 67]. A possible physical motivation of these pQNMs follows. The peak of the orbital frequency comes from orbits that are very close to the light-ring position [54], which in turn corresponds nearly to the peak of the effective potential for gravitational perturbations [105, 106, 107, 108, 109]. Therefore, before the orbital frequency peaks, the gravitational-wave emission is dominated by the source of the Teukolsky equations (i.e. by the particle); afterwards, the emission is dominated by the black-hole’s QNMs. In the standard EOB approach, the waveform is a superposition of QNMs already after the peak of the numerical amplitude \(t_{\text{peak}}^\text{em}\). However, we have seen that this precedes the peak of the orbital frequency by a considerable time interval: \(-\Delta t_{\text{em}} \approx 12-40M\) for \(a/M = 0.5\) and \(a/M = 0.7\) (see Table 2.5). To account for the effect of the particle emission before the peak of the orbital frequency, we therefore introduce a pQNM having frequency \(\omega_{22}^{\text{pQNM}} = 2\Omega_{\text{max}}\) (cf. Table 2.3) and decay time \(\tau_{22}^{\text{pQNM}} = -\Delta t_{\text{peak}}^{22}/2\). We included this pQNM only for \(a/M = 0.5\) and \(a/M = 0.7\). For smaller spins, since \(\Delta t_{\text{peak}}^{22}\) is small, the pQNM would be short lived and would not alter our results significantly.

As in the non-spinning case, phase and amplitude agreement are excellent until the ISCO. The phase differences remain small during the plunge, until merger, and grow up to \(\sim 0.1\) rad during the ringdown. The amplitude difference grows to larger values, \(\sim 20-30\%\) through merger and ringdown, because of the limitations of our procedure to attach the QNMs in the EOB waveforms (see Ref. [67], Fig. 3 and associated discussion). The phase difference during the merger-ringdown for the case \(a/M = 0.7\) is larger, because for larger and larger spins the resummed-factorized waveforms [55] perform less and less accurately around and beyond the ISCO. In the case \(a/M = -0.9\), the disagreement between the EOB and Teukolsky (2, 2) becomes large and oscillatory during ringdown. This is a consequence of the fact that the
oscillatory frequency behavior discussed in Sec. 2.3.5 is particularly strong in this case, but we are not including the associated (2, −2) QNMs in our EOB model.

Finally, although in this chapter we do not attempt to calibrate higher-order modes for \( a \neq 0 \), it is useful for ongoing work on the comparable-mass case to extract relevant information, such as the time delay between the peaks of the \((\ell, m)\) modes, and the input parameters entering the RHS of Eqs. (2.50)–(2.54). In Table 2.6, we show the time delays \( \Delta t^{\ell m}_{\text{peak}} \) and \( h^{\ell m, \text{peak}} \) as functions of \( a/M \). Quadratic fits to these functions are as follows:

\[
\begin{align*}
|h^{22, \text{peak}}_{, \text{Teuk}}| & = 0.001 \left[ 1.46 + 0.144 a/M + 0.00704 (a/M)^2 \right], \\
|h^{21, \text{peak}}_{, \text{Teuk}}| & = 0.001 \left[ 0.527 - 0.445 a/M + 0.016 (a/M)^2 \right], \\
|h^{33, \text{peak}}_{, \text{Teuk}}| & = 0.001 \left[ 0.566 + 0.133 a/M + 0.0486 (a/M)^2 \right], \\
|h^{44, \text{peak}}_{, \text{Teuk}}| & = 0.001 \left[ 0.276 + 0.0773 a/M + 0.0405 (a/M)^2 \right], \\
\omega^{\text{peak}}_{22, \text{Teuk}} & = 0.266 + 0.129 a/M + 0.0968 (a/M)^2, \\
\omega^{\text{peak}}_{21, \text{Teuk}} & = 0.291 + 0.0454 a/M - 0.0857 (a/M)^2, \\
\omega^{\text{peak}}_{33, \text{Teuk}} & = 0.441 + 0.224 a/M + 0.163 (a/M)^2, \\
\omega^{\text{peak}}_{44, \text{Teuk}} & = 0.616 + 0.315 a/M + 0.227 (a/M)^2. \quad (2.56)
\end{align*}
\]

We postpone to future work the study of \( \frac{d^2}{dt^2} |h^{\ell m, \text{peak}}_{, \text{Teuk}}|/dt^2 \) and \( \omega^{\ell m, \text{peak}}_{, \text{Teuk}} \) for higher-order modes.

<table>
<thead>
<tr>
<th>( a/M )</th>
<th>( \Delta t^{21}_{\text{peak}} )</th>
<th>( \Delta t^{33}_{\text{peak}} )</th>
<th>( \Delta t^{44}_{\text{peak}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9</td>
<td>14.26</td>
<td>3.81</td>
<td>5.41</td>
</tr>
<tr>
<td>-0.5</td>
<td>12.63</td>
<td>2.71</td>
<td>4.36</td>
</tr>
<tr>
<td>0.5</td>
<td>3.87</td>
<td>-1.99</td>
<td>0.28</td>
</tr>
<tr>
<td>0.7</td>
<td>0.10</td>
<td>-5.02</td>
<td>-2.16</td>
</tr>
<tr>
<td>0.9</td>
<td>-34.48</td>
<td>-17.57</td>
<td>-11.80</td>
</tr>
</tbody>
</table>

**Table 2.6:** The time delay \( \Delta t^{21}_{\text{peak}} \), \( \Delta t^{33}_{\text{peak}} \) and \( \Delta t^{44}_{\text{peak}} \) defined in Eq. (2.55) for \( a/M = -0.9, -0.5, 0.5, 0.7 \) and 0.9. Time delay information for the non-spinning case \( a/M = 0 \) and the dominant (2, 2) mode are given in Table 2.2 and Table 2.3.
2.5 Conclusions

The similarity of the transition from inspiral to merger to ringdown over all mass ratios studied in Refs. [48, 51] suggested the possibility of using the test-particle limit as a laboratory to investigate quickly and accurately the main features of the merger signal. The authors of Refs. [71, 72] were the first to exploit this possibility. They proposed using the EOB inspiral-plunge trajectory to build the source for the time-domain Regge-Wheeler-Zerilli equations. They also improved the EOB modeling, notably the energy flux and the non-quasi-circular orbit effects, by requiring that the EOB and RWZ leading (2, 2) mode agreed during plunge, merger and ringdown.

Here, we have employed the time-domain Teukolsky code developed in Refs. [36, 37, 38] and extended previous works [71, 72, 74, 75] in several directions. In the Schwarzschild case, we first discussed how the EOB model developed in Ref. [67] for comparable-mass non-spinning black holes performs when $\nu = 10^{-3}$ for the leading (2, 2) mode, as well as for three subleading modes, (2, 1), (3, 3) and (4, 4). Confirm-
Figure 2-14: Comparison between Teukolsky-calibrated EOB and Teukolsky $h_+$ and $h_x$ polarizations for $a/M = 0$. The four dominant modes $(2,2)$, $(2,1)$, $(3,3)$ and $(4,4)$ are included.

Previous results [71, 72, 55, 103, 73, 74, 104, 75], we found that the agreement between the Teukolsky and EOB modes is excellent during the long inspiral. During the merger, whereas the agreement of the $(2,2)$ and $(3,3)$ modes is still good, that of the $(4,4)$ and $(2,1)$ is not very satisfactory. We find that this is due to the irregular behavior of the numerical-relativity input values for the peak of the mode amplitude and the gravitational frequency at that peak. This motivates the need for more accurate numerical-relativity data for these higher-order modes, which will presumably be available in the future. By calibrating the EOB model using input values directly extracted from the Teukolsky modes (Tables 2.2 and 2.3), we found very good agreement for the four largest modes. In Fig. 2-14, we compare $h_+$ and $h_x$ constructed for these four modes, using

$$h_+(\theta, \phi; t) - i h_x(\theta, \phi) = \sum_{\ell,m} Y_{\ell m}(\theta, \phi) h_{\ell m}(t). \quad (2.57)$$

The sum here is over $(\ell, m) = (2, \pm 2), (2, \pm 1), (3, \pm 3)$ and $(4, \pm 4)$. The agreement between EOB and Teukolsky polarizations is very good as expected. There are some minor differences during the ringdown, which are mainly due to the underestimated ringdown amplitudes of the $(2,2)$ and $(3,3)$ modes in the EOB model.
Moreover, for the first time, we employed the EOB inspiral-plunge trajectory to produce merger waveforms for quasi-circular, equatorial inspiral in the Kerr spacetime. The energy flux in the EOB equations of motion uses the factorized resummed waveforms of Refs. [53, 55]. We calibrated the leading EOB (2, 2) mode for spins $a/M = -0.9, -0.5, 0.5, 0.7$, and extracted information on the subleading modes. We also investigated the high spin case $a/M = 0.9$. We found that several modes which are subleading during the inspiral become relevant during plunge and merger. The major new feature of the EOB calibration (based on Teukolsky data) is that we relaxed the assumption used in previous works [57, 58, 59, 61, 62, 63, 64, 65, 66, 67] that the matching of the QNMs for the leading (2, 2) mode occurs at the peak of the orbital frequency. In fact, we found that the peak of the orbital frequency does not occur at the same time as the peak of the Teukolsky (2, 2) mode, and that the time difference grows as the spin parameter increases. Our work represents a first step in exploring and taking advantage of test-particle limit results to build a better spin EOB model in the comparable-mass case [61].

In the future, we plan to extend this work in at least two directions. First, we want to calibrate the EOB model in the test-particle limit for higher spins and for higher-order modes, and to connect it to the spin EOB model in the comparable-mass case [62, 81]. To achieve this goal, we would need to introduce adjustable parameters in the functions $\rho_{tm}$ in Eq. (2.11) to improve the resummed-factorized energy flux and amplitude modes for large spin values.

In our future analyses, we will use a Teukolsky code which uses hyperboloidal slicing [75, 89]. Although we were able to achieve similar accuracy by extrapolating our results from finite radius to future null infinity, hyperboloidal slicing is far faster, and has proven to be very robust. Second, we would like to extend this model to inclined orbits. To tackle this case, we need to generalize the resummed-factorized waveforms to generic spin orientations. If we were only interested in extracting the input values, as in Tables 2.4, 2.5, it might be sufficient to use the hybrid method suggested in Ref. [84]. In this case, we could use in the EOB equations of motion the energy flux computed with a frequency-domain Teukolsky code [97], but extend it to
plunging trajectories.

Finally, besides improving the EOB model, the possibility of generating quickly and accurately merger waveforms in the test-particle limit will allow us to investigate several interesting phenomena, such as the distribution of kick velocities for spinning black-hole mergers [38], the energy and angular-momentum released when a test particle plunges into a Kerr black hole [110, 111], and the generic ringdown frequencies suggested in Refs. [85, 86].

References


Chapter 3

Strong-field tidal distortions of rotating black holes: Formalism and results for circular, equatorial orbits

Abstract

Tidal coupling between members of a compact binary system can have an interesting and important influence on that binary's dynamical inspiral. Tidal coupling also distorts the binary's members, changing them (at lowest order) from spheres to ellipsoids. At least in the limit of fluid bodies and Newtonian gravity, there are simple connections between the geometry of the distorted ellipsoid and the impact of tides on the orbit's evolution. In this chapter, we develop tools for investigating tidal distortions of rapidly rotating black holes using techniques that are good for strong-field, fast-motion binary orbits. We use black hole perturbation theory, so our results assume extreme mass ratios. We develop tools to compute the distortion to a black hole's curvature for any spin parameter, and for tidal fields arising from any bound orbit, in the frequency domain. We also develop tools to visualize the horizon's distortion for black hole spin $a/M \leq \sqrt{3}/2$ (leaving the more complicated $a/M > \sqrt{3}/2$ case to a future analysis). We then study how a Kerr black hole's event horizon is distorted by a small body in a circular, equatorial orbit. We find that the connection between the geometry of tidal distortion and the orbit's evolution is not as simple as in the Newtonian limit.
3.1 Introduction

3.1.1 Tidal coupling and binary inspiral

Tidal coupling in binary inspiral has been a topic of much recent interest. A great deal of attention has focused in particular on systems which contain neutron stars, where tides and their backreaction on the binary’s evolution may allow a new probe of the equation of state of neutron star matter [112, 113, 114]. A great deal of work has been done to rigorously define the distortion of fluid stars [115, 116], the coupling of the tidal distortion to the binary’s orbital energy and angular momentum [117], and most recently the importance of nonlinear fluid modes which can be sourced by tidal fields [118, 119].

Tidal coupling also plays a role in the evolution of binary black holes. Indeed, the influence of tidal coupling on binary black holes has been studied in some detail over the past two decades, but using rather different language: instead of “tidal coupling,” past literature typically discusses gravitational radiation “down the horizon.” This down-horizon radiation has a dual description in the tidal deformation of the black hole’s event horizon. A major purpose of this chapter is to explore this dual description, examining quantitatively how a black hole is deformed by an orbiting companion.

Consider the down-horizon radiation picture first. The wave equation governing radiation produced in a black hole spacetime admits two solutions [120, 121], one describing outgoing radiation very far from the hole, and another describing radiation ingoing on the event horizon. Both solutions carry energy and angular momentum away from the binary, and drive (on average) a secular inspiral of the orbit. After suitable averaging, we require (for example) the orbital energy $E_{\text{orb}}$ to evolve according to

$$\frac{dE_{\text{orb}}}{dt} = - \left( \frac{dE}{dt} \right)_{\infty} - \left( \frac{dE}{dt} \right)^{H},$$

where $(dE/dt)_{\infty}$ describes energy carried far away by the waves, and $(dE/dt)^{H}$ describes energy carried into the event horizon.
The down-horizon flux has an interesting property. When it is computed for a small body that is in a circular, equatorial orbit of a Kerr black hole with mass $M$ and spin parameter $a$, we find that

$$\left(\frac{dE}{dt}\right)^{H} \propto (\Omega_{\text{orb}} - \Omega_{H}),$$

(3.2)

where $\Omega_{\text{orb}} = M^{1/2}/(r^{3/2} + aM^{1/2})$ is the orbital frequency, and $\Omega_{H} = a/2Mr_{+}$ is the hole's spin frequency (Ref. [11], Sec VIID; see also synopsis in Sec. 3.2.5). The radius $r_{+} = M + \sqrt{M^{2} - a^{2}}$ gives the location of the event horizon in Boyer-Lindquist coordinates. We assume that the orbit is prograde, so that the orbital angular momentum is parallel to the hole’s spin angular momentum.

When $\Omega_{\text{orb}} > \Omega_{H}$ (i.e., when the orbit rotates faster than the black hole spins), we have $(dE/dt)^{H} > 0$ — radiation carries energy into the horizon, taking it from the orbital energy. This is intuitively sensible, given that an event horizon generally acts as a sink for energy and matter. However, when $\Omega_{\text{orb}} < \Omega_{H}$ (the hole spins faster than the orbit’s rotation), we have $(dE/dt)^{H} < 0$. This means that the down-horizon component of the radiation augments the orbital energy — energy is transferred from the hole to the orbit. This is far more difficult to reconcile with the behavior of an event horizon.

One clue to understanding this behavior is that, when $\Omega_{H} > \Omega_{\text{orb}}$, the modes which contribute to the radiation are superradiant [122, 123]. Consider a plane wave which propagates toward the black hole. A portion of the wave is absorbed by the black hole (changing its mass and spin), and a portion is scattered back out to large radius. A superradiant mode (see, for example, Sec. 98 of Ref. [123]) is one in which the scattered wave has higher amplitude than the original ingoing wave. Some of the black hole’s spin angular momentum and rotational energy has been transferred to the radiation.
3.1.2 Tidally distorted strong gravity objects

Although the condition for superradiance is the same as the condition under which an orbit gains energy from the black hole, superradiance does not explain how energy is transferred from the hole to the orbit. A more satisfying picture of this can be built by invoking the dual picture of a tidal distortion. As originally shown by Hartle [26, 27], an event horizon’s intrinsic curvature is distorted by a tidal perturbation. In analogy with tidal coupling in fluid systems, the tidally distorted horizon can gravitationally couple to the orbiting body, transferring energy and angular momentum from the black hole to the orbit.

Let us examine the fluid analogy in more detail for a moment. Consider in particular a moon that raises a tide on a fluid body, distorting its shape from spherical to a prolate ellipsoid. The tidal response will produce a bulge that tends to point at the moon. Due to the fluid’s viscosity, the bulging response will lag the driving tidal force. As a consequence, if the moon’s orbit is faster than the body’s spin, then the bulge will lag behind. The bulge will exert a torque on the orbit that tends to slow down the orbit; the orbit exerts a torque that tends to speed up the body’s spin. Conversely, if the spin is faster than the orbit, the bulge will lead the moon’s position, and the torque upon the orbit will tend to speed it up (and torque from the orbit tends to slow down the spin). In both cases, the bulge and moon exert torques on one another in such a way that the spin and orbit frequencies tend to be equalized.1 The action of this torque is such that energy is taken out of the moon’s orbit if the orbit frequency is larger than the spin frequency, and vice versa.

Since a black hole’s shape is changed by tidal forces in a manner similar to the change in shape of a fluid body, one can imagine that the horizon’s tidal bulge likewise exerts a torque on an orbit. Examining Eq. (3.2), we see that the sign of the “horizon flux” energy loss is exactly in accord with the tidal fluid analogy — energy is lost from the orbit if the orbital frequency exceeds the black hole’s spin frequency, and

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1This is why our Moon keeps the same face to the Earth: Tidal coupling has spun down the Moon’s “day” to match its “year.” Tidal forces from the Moon likewise slow down the Earth’s spin, lengthening the day at a rate of a few milliseconds per century [124]. Given enough time, this effect would drive the Earth to keep the same face to the Moon.
vice versa. Using the membrane paradigm [11], one can assign a viscosity to the horizon, making the fluid analogy even more compelling.

However, as was first noted by Hartle [26], the geometry of a black hole's tidal bulge behaves in a rather counterintuitive manner. At least using a weak-field, slow spin analysis, the bulge leads the orbit when $\Omega_{\text{orb}} > \Omega_H$, and lags when $\Omega_{\text{orb}} < \Omega_H$. This is opposite to the geometry which the fluid analogy would lead us to expect. This is because an event horizon is a teleological object: Whether an event in spacetime is inside or outside a horizon depends on that event's null future. At some moment in a given time slicing, an event horizon arranges itself in anticipation of the gravitational stresses it will be feeling in the future. This is closely related to the manner in which the event horizon of a spherical black hole expands outward when a spherical shell falls into it. See Ref. [11], Sec. VI C6 for further discussion.

Much of this background has been extensively discussed in past literature [26, 27, 11, 125, 126, 116, 127, 128]. Recent work on this problem has examined in detail how one can quantify the tidal distortion of a black hole, demonstrating that the "gravitational Love numbers" which characterize the distortion of fluid bodies vanish for non-rotating black holes [116], but that the geometry's distortion can nonetheless be quantified assuming particularly useful coordinate systems [126, 127] and in a fully covariant manner [128]. Indeed, one can define "surficial Love numbers," which quantify the distortion of a body's surface, for Schwarzschild black holes [129]. These techniques have been used to study horizon distortion in the Schwarzschild and slow spin limits, and for slow orbital velocities [130, 126, 128].

### 3.1.3 Our analysis: Strong-field, rapid spin tidal distortions

The primary goal of this chapter is to develop tools to explore the distorted geometry of a black hole in a binary which are good for fast motion, strong field orbits. We use techniques originally developed by Hartle [27] to compute the Ricci scalar curvature $R_H$ associated with the 2-surface of the distorted horizon; this is closely related to the intrinsic horizon metric developed in Ref. [128]. We will restrict our binaries to large mass ratios in order to use the tools of black hole perturbation theory. We
also develop tools to embed the horizon in a 3-dimensional space in order to visualize the tidal distortions. In this chapter, we restrict our embeddings to black hole spins \( a/M \leq \sqrt{3}/2 \). This is the largest spin at which the horizon can be embedded in a global Euclidean space; black holes with spins in the range \( \sqrt{3}/2 < a/M \leq 1 \) must be embedded in a space that is partially Euclidean, partially Lorentzian \([131]\). Although no issue of principle prevents us for examining larger spins, the mixed Euclidean-Lorentzian case is technically rather complicated. Since it does not add very much to the physics we wish to study here, we defer embeddings for \( a/M > \sqrt{3}/2 \) to future work.

A secondary goal of this chapter is to investigate whether there is a simple connection between the geometry of the tidal bulge and the orbit's evolution. In particular, we wish to see if the sign of \( dE^H/dt \), which is determined by \( Q_{\text{orb}} - Q_H \), is connected to the bulge's geometry relative to the orbit. This turns out to be somewhat tricky to investigate. The orbit and the horizon are at different locations, so we must map the orbit's position onto the horizon. There is no unique way to do this, so the results depend at least in part on how we make the map. We present two maps from orbit to horizon. One, based on ingoing zero-angular momentum light rays, is useful for comparing with past literature. The other, based on the geometry of the horizon's embedding and the orbit at an instant of constant ingoing time, is useful for describing our numerical data (at least for small spin). Another way to characterize the bulge geometry is to examine the relative phase of the bulge's curvature to the tidal field which distorts the black hole. Both of these quantities are defined at \( r = r_+ \), so no mapping is necessary.

We find that, at the extremes, the response of a black hole to a perturbing tide follows Newtonian logic (modulo a swap of "lag" and "lead," thanks to the horizon's teleological nature). In particular, when \( \Omega_{\text{orb}} \gg \Omega_H \) (so that \( dE^H/dt > 0 \)), the bulge leads the orbit, no matter how we compare the bulge to the orbit. When \( \Omega_{\text{orb}} \ll \Omega_H \) (\( dE^H/dt < 0 \)), the bulge lags the orbit. However, relations between lag, lead, and \( dE^H/dt \) are not so clear cut when \( \Omega_{\text{orb}} \sim \Omega_H \). Consider, in particular the case \( \Omega_{\text{orb}} = \Omega_H \), with \( dE^H/dt = 0 \). For Newtonian, fluid bodies, the tidal bulge
points directly at the orbiting body in this case, with no exchange of torque between the body and the orbit. For black holes, we find no particular relation between the horizon’s bulge and the orbit’s position. The relation between tidal coupling and tidal distortion is far more complicated in black hole systems than it is for fluid bodies in Newtonian gravity — which is not especially surprising.

3.1.4 Outline of this chapter, units, and conventions

The remainder of this chapter is organized as follows. Our formalism for computing the geometry of distorted Kerr black holes is given in Sec. 3.2. We show how to compute the curvature of a tidally distorted black hole, and how to quantify the relation of the geometry of this distortion to the geometry of the orbit which produces the tidal field. We also discuss how to compute $dE^H/dt$, demonstrating that the information which determines this down-horizon flux is identical to the information which determines the geometry of the distorted event horizon.

Sections 3.3 and 3.4 present results for Schwarzschild and Kerr, respectively. In both sections, we first look at the black hole’s curvature in a slow motion, slow spin expansion (slow motion only for Schwarzschild). This allows us to develop analytic expressions for the curvature, which are useful for comparing to the fast motion, rapid spin numerical results that we then compute. We visualize tidally distorted black holes by embedding their horizons in a 3-dimensional space. This provides a useful way to see how tides change the shape of a black hole. In Sec. 3.5, we examine in some detail whether there is a simple connection between a black hole’s tidally distorted geometry and the coupling between the hole and the orbit. In short, the answer we find is “no” — Newtonian, fluid intuition breaks down for black holes and strong-field orbits.

Concluding discussion is given in Sec. 3.6, followed by certain lengthy technical details which we relegate to appendices. Appendix 3.A describes in detail how to compute $\bar{\delta}$, a Newman-Penrose operator which lowers the spin-weight of quantities needed for our analysis. Appendix 3.B describes how to embed a distorted black hole’s event horizon in a 3-dimensional Euclidean space. As mentioned above, one cannot
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embed black holes with $a/M > \sqrt{3}/2$ in Euclidean space, but must use a space that is partly Euclidean, partly Lorentzian [131]. This exercise may be carried out in the future. Appendix 3.C computes, to leading order in spin, the spheroidal harmonics which are used as basis functions in black hole perturbation theory. This is needed for the slow-spin expansions we present in Sec. 3.4. Finally, Appendix 3.D summarizes certain changes in notation that we have introduced versus previous papers that use black hole perturbation theory. These changes synchronize our notation with that used in the literature from which we have recently adopted our core numerical method [132, 100].

Throughout this chapter, we work in “relativist’s units,” with $G = c = 1$. All of our calculations are done in the background of a Kerr black hole. Two coordinate systems, described in detail in Ref. [133], are particularly useful for us. The Boyer-Lindquist coordinates $(t, r, \theta, \phi)$ yield the line element

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma}dt \ d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\Sigma} \sin^2 \theta \ d\phi^2,$$  

where

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta.$$  

The function $\Delta$ has two roots, $r_{\pm} = M \pm \sqrt{M^2 - a^2}$; $r_+$ is the location of the event horizon. We will also often find it useful to use ingoing coordinates $(v, r', \theta, \psi)$, related to the Boyer-Lindquist coordinates by [133]

$$dv = dt + \frac{(r^2 + a^2)}{\Delta} dr, \quad (3.5)$$

$$d\psi = d\phi + \frac{a}{\Delta} dr. \quad (3.6)$$

$$dr' = dr, \quad (3.7)$$

These coordinates are well-behaved on the event horizon, and so are useful tools for describing fields that fall into the hole. Although the relation between $r$ and
$r'$ is trivial, it can be useful to distinguish the two as a bookkeeping device when transforming between the two coordinate systems. When there is no ambiguity, we will drop the prime on the ingoing radial coordinate. The Kerr metric in ingoing coordinates is given by

$$ds^2 = -\left(1 - \frac{2Mr'}{\Sigma}\right)dv^2 + 2dv dr' - 2a \sin^2 \theta \, dr' \, d\psi - \frac{4Mar' \sin^2 \theta}{\Sigma} \, dv \, d\psi + \Sigma d\theta^2 + \frac{[(r')^2 + a^2]^2 - a^2 \Delta \sin^2 \theta}{\Sigma} \sin^2 \theta \, d\psi^2 .$$

(3.8)

The quantities $\Sigma$ and $\Delta$ here are exactly as in Eq. (3.4), but with $r \to r'$.

It is not difficult to integrate up Eqs. (3.5) and (3.6) to find

$$v = t + r^*, \quad \psi = \phi + \bar{r},$$

(3.9)

where [133]

$$r^* = r + \frac{Mr_+}{\sqrt{M^2 - a^2}} \ln \left(\frac{r}{r_+} - 1\right) - \frac{Mr_-}{\sqrt{M^2 - a^2}} \ln \left(\frac{r}{r_-} - 1\right),$$

(3.10)

$$\bar{r} = \frac{a}{2\sqrt{M^2 - a^2}} \ln \left(\frac{r - r_+}{r - r_-}\right).$$

(3.11)

Notice that $\psi = \phi$ when $a = 0$.

For $r = r_+ + \delta r, \delta r \ll M$,

$$\bar{r} - \Omega_H r^* = K(a) + O(\delta r),$$

(3.12)
where

\[ K(a) = \frac{a}{2M(Mr_+ - a^2)} \left\{ a^2 - Mr_+ + 2M^2 \arctanh \left( \sqrt{1 - \frac{a^2}{M^2}} \right) + \frac{M \sqrt{M^2 - a^2} \ln \left( \frac{a^2}{4(M^2 - a^2)} \right)}{2M} \right\} \]

\[ = -\frac{a}{2M} + \left[ 1 - 2\ln \left( \frac{a}{2M} \right) \right] \left( \frac{a}{2M} \right)^3 + O(a^5). \] (3.13)

This means that, near the horizon, the combination \( r^* - \Omega_H \bar{r} \) cancels out the logarithms in both \( r^* \) and \( \bar{r} \), trending to a constant \( K(a) \) that depends only on spin. The quantity \( K(a) \) plays an important role in setting the phase of tidal fields on the event horizon.

### 3.2 Formalism

In this section, we develop the formalism we use to study the geometry of deformed event horizons. The details of this calculation are presented in Sec. 3.2.1. Two pieces of this calculation are sufficiently involved that we present them separately. First, in Sec. 3.2.2, we give an overview of how one solves the radial perturbation equation to find the amplitude that sets the magnitude of the tidal distortion. This material has been discussed at great length in many other papers, so we present just enough detail to illustrate what is needed for our analysis. We include in our discussion the static limit, mode frequency \( \omega = 0 \). Since static modes do not carry energy or angular momentum, they have been neglected in almost all previous analyses. However, these modes affect the shape of a black hole, so they must be included here. Second, in Sec. 3.2.3 we provide detailed discussion of the angular operator \( \bar{\Delta} \) and its action upon the spin-weighted spheroidal harmonic.

Section 3.2.4 describes how we characterize the bulge in the event horizon which is raised by the orbiting body’s tide. The bulge is a simple consequence of the geometry, but this discussion deserves separate treatment in order to properly discuss certain
choices and conventions we must make. We conclude this section by briefly reviewing
down-horizon fluxes in Sec. 3.2.5. Although this discussion is tangential to our main
focus in this chapter, we do this to explicitly show that the deformed geometry and
the down-horizon flux are just different ways of presenting the same information about
the orbiting body’s perturbation to the black hole.

### 3.2.1 The geometry of an event horizon

We will characterize the geometry of distorted black holes using the Ricci scalar
curvature $R_H$ associated with their event horizon’s 2-surface. The scalar curvature of
an undistorted Kerr black hole is given by\(^2\) [131]

$$R_H = R_H^{(0)} = \frac{2}{r_+^2} \frac{(1 + a^2/r_+^2)(1 - 3a^2 \cos^2 \theta/r_+^2)}{(1 + a^2 \cos^2 \theta/r_+^2)^3}. \quad (3.15)$$

For $a = 0$, $R_H^{(0)} = 2/r_+^2$, the standard result for a sphere of radius $r_+$. For $a/M \geq \sqrt{3}/2$, $R_H^{(0)}$ changes sign near the poles. This introduces important and interesting
complications to how we represent the tidal distortions of a rapidly rotating black
hole’s horizon.

To first order in the mass ratio, tidal distortions leave the horizon at the coordinate
$r = r_+$, but change the scalar curvature on that surface. Using the Newman-Penrose
formalism [134], Hartle [27] shows that the perturbation $R_H^{(1)}$ to the curvature is
simply related to the perturbing tidal field $\psi_0$:

$$R_H^{(1)} = -4 \text{Im} \sum_{lmkn} \frac{\delta \psi_{0,lmkn}^{HH}}{p_{mkn}(ip_{mkn} + 2\epsilon)} \equiv \sum_{lmkn} R_{H,lmkn}^{(1)}, \quad (3.16)$$

with all quantities evaluated at $r = r_+$. The quantity $\psi_{0,lmkn}^{HH}$ is a term in a multipolar
and harmonic expansion of the Newman-Penrose curvature scalar $\psi_0$, computed using.

\(^2\)Reference [131] actually computes the horizon’s Gaussian curvature $R_H$. The Gaussian curvature
$R$ of any 2-surface is exactly half that surface’s scalar curvature $R$, so $R_H = 2R_H$. 
the Hawking-Hartle tetrad [135]:

$$\psi_0^\text{HH} \equiv -C_{\alpha \beta \gamma \delta} (l^\alpha)^\text{HH} (m^\beta)^\text{HH} (r^\gamma)^\text{HH} (s^\delta)^\text{HH}$$

$$= \sum_{l,m,k,n} \psi_{0,l,m,k,n}^\text{HH} \cdot$$ (3.17)

The tensor $C_{\alpha \beta \gamma \delta}$ is the Weyl curvature, and the vectors $(l^\alpha)^\text{HH}$ and $(m^\alpha)^\text{HH}$ are Newman-Penrose tetrad legs in the Hawking-Hartle representation. See Appendix 3.A for detailed discussion of this tetrad and related quantities.

We assume that $\psi_0$ arises from an object in a bound orbit of the Kerr black hole. This object’s motion can be described using the three fundamental frequencies associated with such orbits: an axial frequency $\Omega_\phi$, a polar frequency $\Omega_\theta$, and a radial frequency $\Omega_r$. The indices $m$, $k$, and $n$ label harmonics of these frequencies:

$$\omega_{mkn} = m\Omega_\phi + k\Omega_\theta + n\Omega_r .$$ (3.18)

The index $l$ labels a spheroidal harmonic mode, and is discussed in more detail below. The remaining quantities appearing in Eq. (3.16) are the wavenumber for ingoing radiation$^3$

$$p_{mkn} = \omega_{mkn} - m\Omega_H ,$$ (3.19)

and

$$\epsilon = \frac{\sqrt{M^2 - a^2}}{4Mr_+} \equiv \frac{\kappa}{2} .$$ (3.20)

The quantity $\kappa$ is the Kerr surface gravity. We will find this interpretation of $\epsilon$ to be useful when discussing the geometry of the horizon’s tidal distortion. We discuss the operator $\delta \bar{\delta}$ in detail in Sec. 3.2.3. For now, note that it involves derivatives with respect to $\theta$.

The calculation of $R_H^{(1)}$ involves several computations that use the Newman-Penrose derivative operator $D \equiv l^\alpha \partial_\alpha$. Using the Hawking-Hartle form of $l^\alpha$ and

---

$^3$This wavenumber is often written $k$ in the literature; we use $p$ to avoid confusion with harmonics of the $\theta$ frequency.
ingoing Kerr coordinates (see Appendix 3.A), we find that

\[ D \rightarrow \frac{\partial}{\partial v} + \Omega_H \frac{\partial}{\partial \psi} \]  

(3.21)

as \( r \rightarrow r_+ \). The fields to which we apply this operator have the form \( e^{i(m\psi - \omega_{mkn}v)} \) near the horizon, so

\[ DF = i(m\Omega_H - \omega_{mkn}) F = -ip_{mkn} F \]  

(3.22)

for all relevant fields \( F \). Hartle chooses a time coordinate \( t \) such that \( D \equiv \partial/\partial t \) near the horizon, effectively working in a frame that corotates with the black hole. As a consequence, his Eq. (2.21) has \( \omega \) in place of \( p \). Hartle’s (2.21) also corresponds to a single Fourier mode, and so is not summed over indices.

The Hawking-Hartle tetrad is used in Eq. (3.17) because it is well behaved on the black hole’s event horizon [135]. In many discussions of black hole perturbation theory based on the Teukolsky equation, we instead use the Kinnersley tetrad, which is well designed to describe distant radiation [120, 136]. The Kinnersley tetrad is described explicitly in Appendix 3.A. The relation between \( \psi_0 \) in these two tetrads is [cf. Ref. [121], Eq. (4.43)]

\[ \psi_{0,HH}^H = \frac{\Delta^2}{4(r^2 + a^2)^2} \psi_{0,K}^K. \]  

(3.23)

Further, we know that \( \psi_{0,K}^K \) on the horizon can be written [121]

\[ \psi_{0,lmkn}^K = \frac{W_{lmkn}^H + 2S_{lm}(\theta; a\omega_{mkn})}{\Delta^2} e^{i(m\phi - \omega_{mkn}t) - p_{mkn}r^*}. \]  

(3.24)

We have introduced \( W_{lmkn}^H \), a complex amplitude\(^4\) which we will discuss in more detail below, as well as the spheroidal harmonic of spin-weight +2, \( +2S_{lm}(\theta; a\omega_{mkn}) \). Spheroidal harmonics are often used in black hole perturbation theory, since the equations governing a field of spin-weight \( s \) in a black hole spacetime separate when these harmonics are used as a basis for the \( \theta \) dependence. In the limit \( a\omega_{mkn} \rightarrow 0 \),

---

\(^4\)This amplitude is written \( Y \) rather than \( W \) in Ref. [121]; we have changed notation to avoid confusion with the spherical harmonic.
they reduce to the spin-weighted spherical harmonics:

\[ s_{S_{lm}}(\theta; a\omega_{mkn}) \rightarrow s_{Y_{lm}}(\theta) \quad \text{as} \quad a\omega_{mkn} \rightarrow 0. \]  

(3.25)

\( s_{Y_{lm}}(\theta) \) denotes the spherical harmonic without the axial dependence: \( s_{Y_{lm}}(\theta, \phi) = s_{Y_{lm}}(\theta)e^{im\phi} \). In what follows, we will abbreviate:

\[ +2s_{S_{lm}}(\theta; a\omega_{mkn}) \equiv S_{lmkn}^+(\theta). \]  

(3.26)

We will likewise write the spin-weight -2 spheroidal harmonic as \( S_{lmkn}^{-}(\theta) \).

Combining Eqs. (3.23) and (3.24), we find

\[ \psi_{0,lmkn}^{HH} = \frac{W_{lmkn}^{H} s_{lmkn}^+(\theta)}{4(r^2 + a^2)^2} e^{i(m\phi - a\omega_{mkn}t - p_{mkn}r^*)}. \]  

(3.27)

Using Eqs. (3.9) and (3.19), we can rewrite the phase factor using coordinates that are well-behaved on the horizon:

\[ m\phi - a\omega_{mkn}t - p_{mkn}r^* = m(\psi - \bar{r}) - a\omega_{mkn}(v - r^*) \]

\[ = m(\psi - \bar{r}) - a\Omega_H r^* \]

\[ = m\psi - a\omega_{mkn}v - m(\bar{r} - \Omega_H r^*). \]  

(3.28)

Taking the limit \( r \rightarrow r^+ \) and using Eq. (3.12), we find

\[ \psi_{0,lmkn}^{HH} = \frac{W_{lmkn}^{H} s_{lmkn}^+(\theta)}{16M^2 r_+^2} e^{i\Phi_{mkn}(v, \psi)}, \]  

(3.29)

where

\[ \Phi_{mkn}(v, \psi) = m\psi - a\omega_{mkn}v - mK(a), \]  

(3.30)

with \( K(a) \) defined in Eq. (4.27). We finally find

\[ R_{H,lmkn}^{(1)} = -\text{Im} \left[ \frac{W_{lmkn}^{H} e^{i\Phi_{mkn}(v, \psi)} \partial \bar{\partial} S_{lmkn}^+(\theta)}{4M^2 r_+^2 p_{mkn}(ip_{mkn} + 2\epsilon)} \right]. \]  

(3.31)
We will use a Teukolsky equation solver [137, 138, 139] which computes the curvature scalar $\psi_4$ rather than $\psi_0$. Although $\psi_4$ is usually used to study radiation far from the black hole, one can construct $\psi_0$ from it using the Starobinsky-Churilov identities [121, 140]. In the limit $r \to r_+$,

$$\psi_4 = \frac{\Delta^2}{(r - ia \cos \theta)^4} \sum_{lmkn} Z_{lmkn}^H S_{lmkn}^-(\theta) \times e^{i(m\phi - \omega_{mkn}t - \phi_{mkn} r^*)} . \tag{3.32}$$

We briefly summarize how we compute $Z_{lmkn}^H$ in Sec. 3.2.2. Using the Starobinsky-Churilov identities, we find that $Z_{lmkn}^H$ and $W_{lmkn}^H$ are related by

$$W_{lmkn}^H = \beta_{lmkn} Z_{lmkn}^H , \tag{3.33}$$

where

$$\beta_{lmkn} = \frac{64(2Mr_+)^4 p_{mkn}(p_{mkn}^2 + 4\epsilon^2)(p_{mkn} + 4i\epsilon)}{c_{lmkn}} , \tag{3.34}$$

and where the complex number $c_{lmkn}$ is given by

$$|c_{lmkn}|^2 = \left\{ \left[ (\lambda + 2)^2 + 4ma \omega_{mkn} - 4a^2 \omega_{mkn}^2 \right] \times \left( \lambda^2 + 36ma \omega_{mkn} - 36a^2 \omega_{mkn}^2 \right) + (2\lambda + 3)(96a^2 \omega_{mkn}^2 - 48ma \omega_{mkn}) \right\} + 144a^2 \omega_{mkn}^2 (M^2 - a^2) , \tag{3.35}$$

$$\text{Im } c_{lmkn} = 12M \omega_{mkn} , \tag{3.36}$$

$$\text{Re } c_{lmkn} = +\sqrt{|c_{lmkn}|^2 - 144M^2 \omega_{mkn}^2} . \tag{3.37}$$

The real number $\lambda$ appearing here is

$$\lambda = \mathcal{E}_{lmkn} - 2am \omega_{mkn} + a^2 \omega_{mkn}^2 - 2 , \tag{3.38}$$

with $\mathcal{E}_{lmkn}$ the eigenvalue of $S_{lmkn}^- (\theta)$. In the limit $a \omega_{mkn} \to 0$, $\mathcal{E}_{lmkn} \to l(l + 1)$. For our later weak-field expansion, it will be useful to have $\lambda$ as an expansion in $a \omega_{mkn}$.
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See Appendix 3.C for discussion of this.

Using these results, we can write the tidal distortion of the horizon’s curvature as

\[ R^{(1)}_{H,lmkn} = -\text{Im} \left[ \frac{\beta_{lmkn} Z^H_{lmkn} e^{i\Phi_{lmkn}(\nu,\psi)} \delta\delta S^+_{lmkn}(\theta)}{4M^2 r^2_p(p_{lmkn} + 2\epsilon)} \right] \]

\[ = \text{Im} \left[ C_{lmkn} Z^H_{lmkn} e^{i\Phi_{lmkn}(\nu,\psi)} \delta\delta S^+_{lmkn}(\theta) \right], \tag{3.39} \]

where

\[ C_{lmkn} = 256 M^2 r^2_p c^{-1} (p_{lmkn} + 4i\epsilon)(ip_{lmkn} - 2\epsilon). \tag{3.40} \]

Equation (3.39) is the workhorse of our analysis. We use a slightly modified version of the code described in Refs. [137, 138, 139] to compute the complex numbers \( Z^H_{lmkn} \) and the angular function \( \delta\delta S^+_{lmkn} \). We briefly describe these calculations in the next two subsections.

### 3.2.2 Computing \( Z^H_{lmkn} \)

Techniques for computing the amplitude \( Z^H_{lmkn} \) have been discussed in great detail in other papers, so our discussion here will be very brief; our analysis follows that given in Ref. [138]. The major change versus previous works is that we need the solution for static modes (\( \omega = 0 \)). Our goal here is to present enough detail to see how earlier studies can be modified fairly simply to include these modes. It is worth noting that we have changed notation from that used in previous papers by our group in order to more closely follow the notation of Fujita and Tagoshi [132, 100]. Appendix 3.D summarizes these changes.

The complex number \( Z^H_{lmkn} \) is the amplitude of solutions to the Teukolsky equation for spin-weight \( s = -2 \), so we begin there:

\[ \Delta^2 \frac{d}{dr} \left( \frac{dR_{lm\omega}}{dr} \right) - V_{lm}(r) R_{lm\omega} = \mathcal{T}_{lm\omega}(r). \tag{3.41} \]

This is the frequency-domain version of this equation, following the introduction of a modal and harmonic decomposition which separates the original time-domain equation; see [120] for further details. The potential \( V_{lm} \) is discussed in Sec. IIIA of
Equation (3.41) has two homogeneous solutions relevant to our analysis: The "in" solution is purely ingoing on the horizon, but is a mixture of ingoing and outgoing at future null infinity; the "up" solution is purely outgoing at future null infinity, but is a mixture of ingoing and outgoing on the horizon. We discuss these solutions in more detail below. For now, it is enough that these solutions allow us to build a Green’s function \cite{141},

\[
G(r|r') = \begin{cases} 
\frac{1}{W} R_{\text{in}}^{\text{up}}(r) R_{\text{in}}^{\text{up}}(r'), & r' < r, \\
\frac{1}{W} R_{\text{in}}^{\text{in}}(r) R_{\text{up}}^{\text{up}}(r'), & r' > r, 
\end{cases} 
\] (3.42)

where

\[ W = \frac{1}{\Delta} \left[ R_{\text{in}}^{\text{in}} \frac{d R_{\text{in}}^{\text{up}}}{dr} - R_{\text{up}}^{\text{up}} \frac{d R_{\text{in}}^{\text{in}}}{dr} \right] \] (3.43)

is the equation’s Wronskian. This is then integrated against the source to build the general inhomogeneous solution:

\[
R_{\text{in}}(r) = \int_{r_*}^{\infty} G(r|r') \mathcal{T}(r')dr' \\
\equiv \left[ Z_{\text{in}}^{\text{in}}(r) R_{\text{in}}^{\text{up}}(r) + Z_{\text{in}}^{\text{up}}(r) R_{\text{in}}^{\text{in}}(r) \right]. 
\] (3.44)

We have defined

\[
Z_{\text{in}}^{\text{in}}(r) = \frac{1}{W} \int_{r_*}^{r} R_{\text{in}}^{\text{in}}(r') \mathcal{T}_{\text{in}}(r') dr', \\
Z_{\text{in}}^{\text{up}}(r) = \frac{1}{W} \int_{r}^{\infty} R_{\text{in}}^{\text{up}}(r') \mathcal{T}_{\text{in}}(r') dr'. 
\] (3.45)

A key property of \( \mathcal{T}_{\text{in}} \) is that it is the sum of three terms, one proportional to \( \delta(r - r_{\text{orb}}(t)) \), one proportional to \( \delta'(r - r_{\text{orb}}(t)) \), and one proportional to \( \delta''(r - r_{\text{orb}}(t)) \).
(where ' denotes \(d/dr\)). Putting this into Eqs. (3.45) and (3.46), we find that

\[
Z^*_t(r) = \frac{1}{\mathcal{W}} \left\{ \mathcal{T}^0_{t\omega} [R^*_t(r)] + \mathcal{T}^1_{t\omega} \left( \frac{dR^*_t(r)}{dr} \right) \right\}
\]

\[
+ \mathcal{T}^2_{t\omega} \left( \frac{d^2R^*_t(r)}{dr^2} \right)
\]  \hspace{1cm} (3.47)

(where * can stand for "up" or "in"). The factors \(\mathcal{T}_{t\omega}^{0,1,2}\) are operators which act on \(R^*_t\) and its derivatives. These operators integrate over the \(r\) and \(\theta\) motion of the orbiting body.

In this analysis, we are concerned with the solution of the perturbation equation on the event horizon, so we want \(R^*_{t\omega}\) as \(r \to r_+\). In this limit, \(Z^*_{t\omega} = 0\). We define

\[
Z_{t\omega}^H \equiv Z^*_{t\omega}(r_+).
\]  \hspace{1cm} (3.48)

For a source term corresponding to a small body in a bound Kerr orbit, we find that Eq. (3.47) has the form

\[
Z^H_{t\omega} = \sum_{kn} Z^H_{t\omega kn} \delta(\omega - \omega_{mkn}).
\]  \hspace{1cm} (3.49)

It is then not difficult to read off \(Z^H_{t\omega kn}\). See Ref. [138] for detailed discussion of how to evaluate Eq. (3.47) and read off these amplitudes.

Key to computing \(Z^H_{t\omega kn}\) is computing the homogeneous solutions \(R^*_{t\omega kn}(r), R^*_{t\omega kn}(r)\), and their derivatives. Our methods for doing this depend on whether \(\omega_{mkn}\) is zero or not.

**Case:** \(\omega_{mkn} \neq 0\)

The homogeneous solutions for \(\omega_{mkn} \neq 0\) have been amply discussed in the literature; our analysis is based on that of Ref. [138]. In brief, the two homogeneous solutions
of Eq. (3.41) have the following asymptotic behavior:

\[
\begin{align*}
R_{\text{in}}^{\text{in}}(r \to r_+) &= B_{\text{in}} \Delta^2 e^{-ipr^*}, \\
R_{\text{in}}^{\text{in}}(r \to \infty) &= B_{\text{in}} e^{i\omega r^*} + B_{\text{inc}} e^{-i\omega r^*}; \\
R_{\text{up}}^{\text{up}}(r \to r_+) &= C_{\text{up}} e^{ipr^*} + C_{\text{ref}} \Delta e^{-ipr^*}, \\
R_{\text{up}}^{\text{up}}(r \to \infty) &= C_{\text{trans}} e^{i\omega r^*}.
\end{align*}
\]

These asymptotic solutions yield the Wronskian:

\[
\mathcal{W} = 2i\omega B_{\text{in}}^{\text{inc}} C_{\text{in}}^{\text{trans}}.
\]

An effective algorithm for computing all of the quantities which we need is described by Fujita and Tagoshi [114, 132, 104]. It is based on expanding the solution in a basis of hypergeometric and Coulomb wave functions, with the coefficients of the expansion determined by solving a recurrence relation; see Secs. 4.2 - 4.4 of Ref. [142] for detailed discussion. We use a code based on these methods [139] for all of our \(\omega_{mkn} \neq 0\) calculations; the analytic limits we present in Secs. 3.3.1 and 3.4.1 are also based on these methods.

**Case:** \(\omega_{mkn} = 0\)

Static modes have been neglected in much past work. They do not carry any energy or angular momentum, and so are not important for many applications. These modes do play a role in setting the shape of the distorted event horizon, however, and must be included here.

It turns out that homogeneous solutions for \(\omega_{mkn} = 0\) are available as surprisingly simple closed form expressions. Teukolsky’s Ph.D. thesis [143] presents two solutions that satisfy appropriate boundary conditions. Defining

\[
x = \frac{r - r_+}{r_+ - r_-}, \quad \gamma = \frac{iam}{r_+ - r_-},
\]

(3.55)
the two solutions of the radial Teukolsky equation for \( s = -2 \) are

\[
R_{lm0}^{ln}(r) = (r_+ - r_-)^4 x^2 (1 + x)^2 \left( \frac{x}{1 + x} \right)^\gamma \times 
\]

\[
2F_1(2 - l, l + 3; 3 + 2\gamma, -x),
\]

\[
R_{lm0}^{up}(r) = (r_+ - r_-)^{(1-l)} x^{(1-l)} (1 + 1/x)^{(2-\gamma)} \times 
\]

\[
2F_1(l + 3, l + 1 - 2\gamma; 2l + 2, -1/x).
\]

In these equations, \( 2F_1(a, b; c, x) \) is the hypergeometric function. These solutions satisfy regularity conditions at infinity and on the horizon: \( R_{lm0}^{ln}(r \to r_+) \propto \Delta^2 \), and \( R_{lm0}^{up}(r \to \infty) \propto 1/r^{l+1} \) [143]. We have introduced powers of \( r_+ - r_- \) to insure that we have the correct asymptotic behavior in \( r \), rather than in the dimensionless variable \( x \). The Wronskian corresponding to these solutions is

\[
W = \frac{(2l + 1)!}{(l + 2)!} \frac{\Gamma(3 + 2\gamma)}{\Gamma(l + 1 + 2\gamma)} (r_+ - r_-)^{(2-l)}. 
\]

Using Eqs. (3.56), (3.57), and (3.58), it is simple to adapt existing codes to compute \( Z_{lmkn}^H \) for \( \omega_{mkn} = 0 \).

The results we present in Secs. 3.3 and 3.4 will focus on circular, equatorial orbits, for which \( k = n = 0 \). The zero-frequency modes in this limit have \( m = 0 \), for which \( \gamma = 0 \). The Wronskian simplifies further:

\[
W_{(m=0)} = -\frac{2(2l + 1)!}{l!(l + 2)!} (r_+ - r_-)^{(2-l)}. 
\]

For generic orbit geometries, there will exist cases that have \( \omega_{mkn} = 0 \) with \( m \neq 0 \), akin to the “resonant” orbits studied at length in Refs. [144, 145]. We defer discussion of this possibility to a later analysis which will go beyond circular and equatorial orbits.
3.2.3 The operator $\bar{\delta}\bar{\delta}$

The operator $\bar{\delta}$, when acting on a quantity $\eta$ of spin-weight $s$, takes the following form:

$$\bar{\delta}\eta = [\bar{\delta} - (\alpha - \bar{\beta})]\eta ;$$  \hspace{1cm} (3.60)

$\bar{\delta}\eta$ is then a quantity of spin-weight $s - 1$. The quantities $\alpha$ and $\beta$ are both Newman-Penrose spin coefficients, and $\bar{\delta}$ is a Newman-Penrose derivative operator. These quantities are all related to the tetrad legs $\mathbf{m}, \bar{\mathbf{m}}$:

$$\bar{\delta} = \bar{m}^\mu \partial_\mu ,$$ \hspace{1cm} (3.61)

$$\alpha - \bar{\beta} = \frac{1}{2} \bar{m}^\nu (m^\mu \nabla_\nu \bar{m}_\mu - \bar{m}^\mu \nabla_\nu m_\mu) .$$ \hspace{1cm} (3.62)

We do this calculation using the Hawking-Hartle tetrad; details are given in Appendix 3.A. The result for general black hole spin $a$ is

$$\bar{\delta}\eta = \frac{1}{\sqrt{2}(r_+ - ia \cos \theta)} \left( L_+^a - am\Omega_H \sin \theta - \frac{isa \sin \theta}{r_+ - ia \cos \theta} \right) \eta .$$ \hspace{1cm} (3.63)

The operator$^5$ $L_+^a$ lowers the spin-weight of the spherical harmonics by 1:

$$L_+^a Y_{lm} = (\partial_\theta + s \cot \theta + m \csc \theta) Y_{lm}$$

$$= \sqrt{(l + s)(l - s + 1)} Y_{l-1} .$$ \hspace{1cm} (3.64)

In a few places, we will need to evaluate $L_+^a [\cos \theta \eta]$ and $L_+^a [\sin \theta \eta]$. This requires that we rewrite $\cos \theta$ and $\sin \theta$ in a form that properly indicates their spin weight. We treat $\cos \theta$ as spin-weight zero, writing

$$\cos \theta = \sqrt{\frac{4\pi}{3}} Y_{10} .$$ \hspace{1cm} (3.65)

$^5$This operator is denoted $\tilde{\delta}_0$ in Ref. [27]. We will use the symbol $\bar{\delta}_0$ to instead denote the Schwarzschild limit of $\bar{\delta}$. 
Likewise, we treat $\sin\theta$ as spin-weight $-1$, writing
\[ \sin\theta = -\sqrt{\frac{8\pi}{3}} Y_{-10}. \] (3.66)

This accounts for the fact that $\sin\theta$ always appears in our calculation inside operators that lower spin-weight.

With this, the following identities follow:

\[
L_- \left[ \cos \theta \eta \right] = \sqrt{\frac{4\pi}{3}} L_- \left[ 0Y_{10} \eta \right]
\]
\[= \sqrt{\frac{4\pi}{3}} \left( 0Y_{10} L^s \eta + \eta L^s_0 Y_{10} \right) \]
\[= \sqrt{\frac{4\pi}{3}} \left( 0Y_{10} L^s \eta + \eta \sqrt{2} Y_{10} \right) \]
\[= \cos \theta L^s_\eta - \sin \theta \eta; \] (3.67)

\[
L_- \left[ \sin \theta \eta \right] = -\sqrt{\frac{8\pi}{3}} L_- \left[ -1Y_{10} \eta \right]
\]
\[= -\sqrt{\frac{8\pi}{3}} \left( -1Y_{10} L^s \eta + \eta L^s_{-1} Y_{10} \right) \]
\[= -\sqrt{\frac{8\pi}{3}} Y_{10} L^s \eta \]
\[= \sin \theta L^s_\eta. \] (3.68)

We used the fact that $L^s_\eta$ applied to $-1Y_{10}$ yields zero.

Using these results, it follows that
\[
L^s_\left( 1 - \frac{ia \cos \theta}{r_+} \right)^{-s} \eta = \left( 1 - \frac{ia \cos \theta}{r_+} \right)^{-s} \left( L^s_\left( \frac{ia \sin \theta}{r_+ - ia \cos \theta} \right)^{-s} \right) \eta. \] (3.69)

We can next rewrite Eq. (3.63) as
\[
\delta\eta = \frac{1}{\sqrt{2r_+}} \left( 1 - \frac{ia \cos \theta}{r_+} \right)^{s-1} \left( L^s_\left( a\Omega_H \sin \theta \right) \left( 1 - \frac{ia \cos \theta}{r_+} \right)^{-s} \right) \eta. \] (3.70)
When \( a = 0 \), this reduces to

\[
\delta \tilde{\eta} = \frac{1}{2 \sqrt{2M}} L_- \eta \equiv \delta_0 .
\]  

(3.71)

When \( \eta \) is of spin-weight 2, Eq. (3.70) tells us that

\[
\delta \tilde{\eta} = \frac{1}{2r_+^2} (L_+ - a m \Omega_H \sin \theta)^2 \left( 1 - \frac{ia \cos \theta}{r_+} \right)^{-2} \eta .
\]  

(3.72)

For \( a \ll M \), Eq. (3.72) reduces to

\[
\delta \tilde{\eta} = \frac{1}{8M^2} L_- L_+ \left( 1 + \frac{ia \cos \theta}{M} \right) \eta ,
\]  

(3.73)

which reproduces Eq. (4.19) of Ref. [27].

We will apply \( \delta \tilde{\eta} \) to the spheroidal harmonic \( S_{lm}^+ (\theta) \). Following Ref. [137], we compute this function by expanding it using a basis of spherical harmonics, writing

\[
S_{lm}^+ (\theta) = \sum_{q=q_{\min}}^{\infty} b_q^l (a \omega_{mkn} + 2 Y_{qm}(\theta) ,
\]  

(3.74)

where \( q_{\min} = \min(2, |m|) \). Efficient algorithms exist to compute the expansion coefficients \( b_q^l (a \omega_{mkn}) \) (cf. Appendix A of Ref. [137]). Expanding Eq. (3.72) puts it into a form very useful for our purposes:

\[
\delta \tilde{\eta} = \frac{1}{2(r_+ - ia \cos \theta)^2} \left[ L_- L_+ + A_1 L_+ + A_2 \right] \eta ,
\]  

(3.75)

where

\[
A_1 = -2a \sin \theta \left[ m \Omega_H + \frac{2i}{r_+ - ia \cos \theta} \right] ,
\]  

(3.76)

\[
A_2 = a^2 \sin^2 \theta \left[ m^2 \Omega_H^2 + \frac{4im \Omega_H}{r_+ - ia \cos \theta} - \frac{6}{(r_+ - ia \cos \theta)^2} \right] .
\]  

(3.77)
Combining Eqs. (3.74) and (3.75), and making use of Eq. (3.64), we finally obtain

\[ \delta \delta \mathcal{S}^+_{lm} = \frac{1}{2(r_+ - ia \cos \theta)^2} \sum_{q=q_{\text{min}}}^{\infty} b_q(a \omega_{mkn}) \]
\[ \times \left[ \sqrt{(q + 2)(q + 1)q(q - 1)} \right]_0 Y_{qm} \]
\[ + \mathcal{A}_1 \sqrt{(q + 2)(q - 1)}_1 Y_{qm} + \mathcal{A}_2 Y_{qm} \]  

(3.78)

This equation is simple to evaluate using the techniques presented in Appendix A of Ref. [137].

### 3.2.4 The phase of the tidal bulge

As we will see when we examine the geometry of distorted event horizons in detail in Secs. 3.3 and 3.4, a major effect of tides on a black hole is to cause the horizon to bulge. As has been described in detail in past literature (e.g., [11]), the result is not so different from the response of a fluid body to a tidal driving force, albeit with some counterintuitive aspects thanks to the teleological nature of the event horizon.

In this section, we describe three ways to characterize the tidal bulge of the distorted event horizon. Two of these methods are based on comparing the position at which the horizon is most distorted to the position of the orbit. Because the orbit and the horizon are at different locations, comparing their positions requires us to map from one to the other. The notion of bulge phase that follows then depends on the choice of map we use. As such, any notion of bulge phase built from comparing orbit position to horizon geometry must be somewhat arbitrary, and can only be understood in the context of the mapping that has been used.

We use two maps from orbit to horizon. The first is a “null map.” Following Hartle [27], we connect the orbit to the horizon using an inward-going, zero-angular-momentum null geodesic. This choice is commonly used in the literature, and so is useful for comparing our results with past work. The second is an “instantaneous map.” We compare the horizon geometry to the orbit position on a slice of constant ingoing time coordinate \( v \). This is particularly convenient for showing figures of the
distorted horizon.

The third method of computing bulge phase directly compares the horizon's response to the applied tidal field. Since both quantities are defined on the horizon, no mapping is necessary, and no arbitrary choices are needed. We do not use this notion of bulge phase very much in this analysis, but anticipate using it in future work which will examine more complicated cases than the circular, equatorial orbits that are our focus here.

Relative position of orbit and bulge I: Null map

In his original examination of black hole tidal distortion, Hartle [27] connects the orbit to the horizon with a zero angular momentum ingoing light ray. Choosing our origins appropriately, the orbiting body is at angle

$$\phi_o = \Omega_{\text{orb}} t$$  \hspace{1cm} (3.79)

in Boyer-Lindquist coordinates. We convert to ingoing coordinates using Eq. (3.9):

$$\psi_o = \Omega_{\text{orb}} (v - r^*_o) + \bar{r}_o$$

$$\equiv \Omega_{\text{orb}} v + \Delta\psi(r_o) ,$$  \hspace{1cm} (3.80)

where $\bar{r}_o \equiv \bar{r}(r_o)$ and $r^*_o \equiv r^*(r_o)$ are given by Eqs. (3.11) and (3.10), and where

$$\Delta\psi(r_o) \equiv \bar{r}_o - \Omega_{\text{orb}} r^*_o$$  \hspace{1cm} (3.81)

is, for each orbital radius $r_o$, a fixed angular offset associated with the transformation from Boyer-Lindquist to ingoing coordinates.

The orbit's location mapped onto the horizon is then

$$\psi^\text{NM}_o = \Omega_{\text{orb}} v + \Delta\psi(r_o) + \delta\psi^\text{null} ,$$  \hspace{1cm} (3.82)

where $\delta\psi^\text{null}$ is the axial shift accumulated by the ingoing null ray as it propagates
from the orbit to the horizon. This shift must in general be computed numerically, but to leading order in $a$ (which will be sufficient for our purposes) it is given by

$$\delta \psi^\text{null} = -\frac{a}{2M} + \frac{a}{r_0} = 2M \Omega_H \left( \frac{2M}{r_0} - 1 \right).$$

(3.83)

The second form uses $\Omega_H = a/4M^2$ for small $a$ to rewrite this formula, which will be useful when we compare our results to previous literature for small spin. (One should also correct the ingoing time, $v \rightarrow v + \delta v$, to account for the time it takes for the ingoing null ray to propagate from the orbit to the horizon. However, at leading order $\delta v \propto a^2$, so we can neglect it for the applications we will use in this chapter.)

Let $\psi^\text{bulge}$ be the angle at which $B_H^{(1)}$ is maximized. This value varies from mode to mode, but is easy to read off once $B_H^{(1)}$ is computed. The offset of the orbit and bulge using the null map is then

$$\delta\psi^{\text{OB-NM}} \equiv \psi^\text{bulge} - \psi_o^{\text{NM}}$$

$$= \psi^\text{bulge} - \Omega_{\text{orb}} v - \Delta \psi(r_o) - \delta \psi^\text{null}.$$  (3.84)

A positive value for $\delta\psi^{\text{OB-NM}}$ means that the bulge leads the orbit.

**Relative position of orbit and bulge II: Instantaneous map**

Consider next a mapping that is instantaneous in ingoing time coordinate $v$. This choice is useful for making figures that show both bulge and orbit, since we simply show their locations at a given moment $v$. This mapping neglects the term $\delta \psi^\text{null}$, but is otherwise identical to the null map:

$$\psi_o^{\text{IM}} = \psi_o = \Omega v + \Delta \psi(r_o).$$

(3.85)

The offset of the orbit and bulge in this mapping is

$$\delta\psi^{\text{OB-IM}} \equiv \psi^\text{bulge} - \psi_o^{\text{IM}}$$

$$= \psi^\text{bulge} - \Omega_{\text{orb}} v - \Delta \psi(r_o).$$

(3.86)
Since $\delta\psi_{\text{null}} = 0$ for $a = 0$, the null and instantaneous maps are identical for Schwarzschild black holes.

Before concluding our discussion of the tidal bulge phase, we emphasize again that the phase in both the null map and the instantaneous map follow from arbitrary choices, and must be interpreted in the context of those choices. Other choices could be made. For example, one could make a map that is instantaneous in a different time coordinate, or that is based on a different family of ingoing light rays (e.g., the principle ingoing null congruence, along which $v, \psi, \text{and} \theta$ are constant; such a map would be identical to the instantaneous map). These two maps are good enough for our purposes — the null map allows us to compare with other papers in the literature, and the instantaneous map is excellent for characterizing the plots we will show in Secs. 3.3 and 3.4.

Relative phase of tidal field and response

Our third method of characterizing the tidal bulge is to use the relative phase of the horizon distortion $R^{(1)}_{\text{H},\text{lmkn}}$ and distorting tidal field $\psi_0$. For our frequency-domain study, this phase is best understood on a mode-by-mode basis. Begin by re-examining Eq. (3.16):

$$ R^{(1)}_{\text{H},\text{lmkn}} = -4 \Im \left[ \frac{\delta \delta \psi_{0,\text{lmkn}}^{\text{HH}}}{p_{\text{mkn}}(ip_{\text{mkn}} + 2\epsilon)} \right] $$

$$ \equiv \Im [R^c_{\text{lmkn}}] . \quad (3.87) $$

Let us define the phase $\delta\psi^{\text{TB}}_{\text{lmkn}}$ by

$$ \frac{R^c_{\text{lmkn}}}{\psi^{\text{HH}}_{0,\text{lmkn}}} = \frac{|R^c_{\text{lmkn}}|}{|\psi^{\text{HH}}_{0,\text{lmkn}}|} e^{-i\delta\psi^{\text{TB}}_{\text{lmkn}}} . \quad (3.88) $$

As with $\delta\psi^{\text{OB-NM}}_{\text{lmkn}}$ and $\delta\psi^{\text{OB-IM}}_{\text{lmkn}}$, $\delta\psi^{\text{TB}}_{\text{lmkn}} > 0$ means that the horizon’s response leads the tidal field.
Using Eq. (3.29), we see that

\[ \frac{R_{lmkn}^c}{\psi_{0,lmkn}^{HH}} = -\frac{4}{p_{mkn}(ip_{mkn} + 2\epsilon)} \delta \delta S_{lmkn}^+ . \]  

(3.89)

With a few definitions, this form expedites our identification of \( \delta \psi_{lmkn}^{TB} \). First, note that \( p_{mkn} \) and \( S_{lmkn}^+ \) are both real, so the phase arises solely from the factor \( 1/(ip_{mkn} + 2\epsilon) \) and the operator \( \delta \delta \). The first factor is easily rewritten in a more useful form:

\[ \frac{1}{ip_{mkn} + 2\epsilon} = \frac{e^{-i \arctan(p_{mkn}/2\epsilon)}}{\sqrt{p_{mkn}^2 + 4\epsilon^2}} . \]  

(3.90)

To clean up the phase associated with \( \delta \delta \), we make a definition:

\[ \frac{\delta \delta S_{lmkn}^+}{S_{lmkn}^+} = \Sigma_{lmkn}(\theta) e^{-i S_{lmkn}(\theta)} . \]  

(3.91)

The amplitude ratio \( \Sigma_{lmkn}(\theta) \) and phase \( S_{lmkn}(\theta) \) must in general be determined numerically. We will show expansions for small \( a \) and slow motion in Sec. 3.4. We include \( S_{lmkn}^+ \) in this definition because it may pass through zero at a different angle than \( \delta \delta S_{lmkn}^+ \) passes through zero. This will appear as a change by \( \pi \) radians in the phase \( S_{lmkn} \).

Combining Eqs. (3.88) – (3.91) and using the fact that \( \epsilon = \kappa/2 \) (where \( \kappa \) is the black hole surface gravity), we at last read out

\[ \delta \psi_{lmkn}^{TB} = \arctan(p_{mkn}/\kappa) + S_{lmkn}(\theta) . \]  

(3.92)

Recall that the wavenumber \( p_{mkn} = \omega_{mkn} - m\Omega_H \). In geometrized units, \( \kappa^{-1} \) is a timescale which characterizes how quickly the horizon adjusts to an external disturbance (cf. Sec. VI C 5 of Ref. [11] for discussion). The first term in Eq. (3.92) is thus determined by the wavenumber times this characteristic horizon time. For a circular, equatorial orbit which has \( \Omega_{orb} = \Omega_H \), this term is zero, in accord with the Newtonian intuition that the tide and the response are exactly aligned when the spin and orbit frequencies are identical. This intuition does not quite hold up thanks to
the correcting phase $S_{lmkn}(\theta)$. We will examine the impact of this correction in Sec. 3.4.

The phase $\delta\psi_{lmkn}^{TB}$ is particularly useful for describing the horizon's response to complicated orbits where the relative geometry of the horizon and the orbit is dynamical. For example, Vega, Poisson, and Massey [128] use a measure similar to $\delta\psi_{lmkn}^{TB}$ to describe how a Schwarzschild black hole responds to a body that comes near the horizon on a parabolic encounter, demonstrating that the horizon's response leads the applied tidal field (cf. Sec. 5.2 of Ref. [128]). We will examine $\delta\psi_{lmkn}^{TB}$ briefly for the circular, equatorial orbits we focus on in this chapter, but will use it in greater depth in a follow-up analysis that looks at tides from generic orbits.

When $a = 0$, the operator $\bar{\delta}\bar{\delta}$ is real, and $S_{lmkn}(\theta) = 0$. We have $\rho_{mkn} = \omega_{mkn}$ and $\kappa = 1/4M$ in this limit, so

$$\delta\psi_{lmkn}^{TB} \bigg|_{a=0} \rightarrow \delta\phi_{mkn}^{TB} = \arctan \left(4M\omega_{mkn}\right). \quad (3.93)$$

We will show in Sec. 3.3 that this agrees with the phase shift obtained by Fang and Lovelace [130]. It also agrees with the results of Vega, Poisson, and Massey [128], though in somewhat different language. They work in the time domain, showing that a Schwarzschild black hole's horizon response leads the field by a time interval $\kappa^{-1}_{Schw} = 4M$. For a field that is periodic with frequency $\omega$, this means that we expect the response to lead the field by a phase angle $4M\omega$. This agrees with Eq. (3.93).

3.2.5 The down-horizon flux

Although not needed for this chapter, we now summarize how one computes the down-horizon flux. Our purpose is to show that the coefficients $Z_{lmkn}^H$ which characterize the geometry of the deformed event horizon also characterize the down-horizon gravitational-wave flux, showing that the "deformed horizon" and "down-horizon flux" pictures are just different ways of interpreting how the horizon interacts with the orbit.

Our discussion follows Teukolsky and Press [121], which in turn follows Hawking
and Hartle [135], modifying the presentation slightly to follow our notation. The starting point is to note that a tidal perturbation shears the generators of the event horizon. This shear, \( \sigma \), causes the area of the event horizon to grow:

\[
\frac{d^2 A}{d\Omega dt} = \frac{2Mr_+}{\epsilon} |\sigma|^2.
\]  

(3.94)

We also know the area of a black hole’s event horizon,

\[
A = 8\pi \left( M^2 + \sqrt{M^4 - S^2} \right),
\]  

(3.95)

where \( S = aM \) is the black hole’s spin angular momentum. Using this, we can write the area growth law as

\[
\frac{d^2 A}{d\Omega dt} = \frac{8\pi}{\sqrt{M^4 - S^2}} \left( 2M^2 r_+ \frac{d^2 M}{d\Omega dt} - S \frac{d^2 S}{d\Omega dt} \right).
\]  

(3.96)

Consider now radiation going down the horizon. Radiation carrying energy \( dE^H \) and angular momentum \( dL_z^H \) into the hole changes its mass and spin by

\[
dM = dE^H, \quad dS = dL_z^H.
\]  

(3.97)

Angular momentum and energy carried by the radiation are related according to

\[
dL_z = \frac{m}{\omega_{mkn}} dE.
\]  

(3.98)

Putting all of this together and using Eq. (3.19), we find

\[
\frac{d^2 E^H}{dtd\Omega} = \frac{\omega_{mkn} Mr_+}{2\pi p_{mkn}} |\sigma|^2,
\]  

(3.99)

\[
\frac{d^2 L_z^H}{dtd\Omega} = \frac{mMr_+}{2\pi p_{mkn}} |\sigma|^2.
\]  

(3.100)

So to compute the down-horizon flux, we just need to know the shear \( \sigma \). It is
simply computed from the tidal field $\psi_0^{HH}$. First, expand $\sigma$ as

$$\sigma = \sum_{lmkn} \sigma_{lmkn}^+ G_{lmkn}^+ (\theta) e^{i[mp_0 - \omega_{lmkn} t - mK(a)]}. \quad (3.101)$$

The shear mode amplitudes $\sigma_{lmkn}$ are related to the tidal field mode $\psi_0^{HH}$ by $[121]$:

$$\sigma_{lmkn} = \frac{i \psi_0^{HH,mkn}}{p_{mkn} - 2i\epsilon}. \quad (3.102)$$

Combine Eq. (3.102) with Eqs. (3.27), (3.33), and (3.34). Integrate over solid angle, using the orthogonality of the spheroidal harmonics. Equations (3.99) and (3.100) become

$$\left(\frac{dE}{dt}\right)^H = \sum_{lmkn} \alpha_{lmkn} \left| \frac{Z_{lmkn}^H}{4\pi \omega_{mkn}^2} \right|^2, \quad (3.103)$$

$$\left(\frac{dL_z}{dt}\right)^H = \sum_{lmkn} \alpha_{lmkn} \frac{m \left| Z_{lmkn}^H \right|^2}{4\pi \omega_{mkn}^3}. \quad (3.104)$$

The coefficient

$$\alpha_{lmkn} = \frac{256(2\pi r)^5}{p_{mkn}^3} \left| \frac{c_{lmkn}}{c_{lmkn}} \right|^2 \times \left( p_{mkn}^2 + 4\epsilon^2 \right) \left( p_{mkn}^2 + 16\epsilon^2 \right), \quad (3.105)$$

with $|c_{lmkn}|^2$ given by Eq. (3.35), comes from combining the various prefactors in the relations that lead to Eqs. (3.103) and (3.104). Notice that $\alpha_{lmkn} \propto p_{mkn}$. This means that $\alpha_{lmkn} = 0$ when $\omega_{mkn} = m\Omega_H$. The down-horizon fluxes (3.103) and (3.104) are likewise zero for modes which satisfy this condition.

It is interesting to note that the shear $\sigma_{lmkn}$ and the tidal field $\psi_0^{HH}$ are both proportional to $p_{mkn}$, and hence both vanish when $\omega_{mkn} = m\Omega_H$. The horizon’s Ricci curvature $R_{H,lmkn}^{(1)}$ does not, however, vanish in this limit. Mathematically, this is because $R_{H,lmkn}^{(1)}$ includes a factor of $1/p_{mkn}$ which removes this proportionality [cf. Eq. (3.16)]. Physically, this is telling us that when $\Omega_H = \Omega_{orb}$, the horizon is deformed, but this deformation is static in the horizon’s reference frame. This static deformation
does not shear the generators, and does not carry energy or angular momentum into the hole.

Equations (3.103) and (3.104) illustrate the point of this section: The fluxes of $E$ and $L_z$ into the horizon are determined by the same numbers $Z_{lmkn}^H$ used to compute the horizon’s deformed geometry, Eq. (3.39).

### 3.3 Results I: Schwarzschild

Using the formalism we have assembled, we now examine the tidally deformed geometry of black hole event horizons. In this chapter, we will only study the circular, equatorial limit: The orbiting body is at $r = r_o$, $\theta = \pi/2$, and $\phi = \Omega_{\text{orb}} t$. Harmonics of $\Omega_\theta$ and $\Omega_r$ can play no role in any physics arising from these orbits, so the index set $\{lmkn\}$ reduces to $\{lm\}$, and the mode frequency $\omega_{mkn}$ to $\omega_m$. We will consider general orbits in a later analysis.

Before tackling general black hole spin, it is useful to examine Eq. (3.39) for Schwarzschild black holes. Several simplifications occur when $a = 0$:

- The radius $r_+ = 2M$; the frequency $\Omega_H = 0$, so the wavenumber $p_m = \omega_m$; the factor $\epsilon = 1/8M$; the phase factor $K(a) = 0$ [cf. Eq. (3.13)]; and the ingoing axial coordinate $\psi = \phi$.

- The spin-weighted spheroidal harmonic becomes a spin-weighted spherical harmonic: $+2S_{lm}(\theta) \rightarrow +2Y_{lm}(\theta)$. The eigenvalue of the angular function therefore simplifies, as does the complex number $c_{lm}$: $\mathcal{E} = l(l+1)$, and $c_{lm} = (l+2)(l+1)\sqrt{l(l-1)} + 12iM\omega_m$.

- The angular operator $\tilde{\delta} \equiv \tilde{\delta}_0 = 1/(2\sqrt{2M})L_z^\ast$. Using Eq. (3.64), we have

$$L_+^\ast L_-^\ast +2Y_{lm}(\theta) = \sqrt{(l+2)(l+1)l(l-1)}_0 Y_{lm}, \quad (3.106)$$
which tells us that
\[
\delta S^+_{lm}(\theta) = \frac{1}{8M^2} \sqrt{(l+2)(l+1)(l-1)} Y_{lm}
\]
(3.107)
for \(a = 0\).

Putting all of this together, for \(a = 0\) we have
\[
P^{(1)}_{H,lm} = \text{Im} \left[ C_{lm} Z_{lm} e^{i\Phi_m} \right] \times \frac{1}{8M^2} \sqrt{(l+2)(l+1)(l-1)} Y_{lm}(\theta),
\]
(3.108)
where
\[
C_{lm} = \frac{1024M^2(i M \omega_m - 1/4)(M \omega_m + i/2)}{(l+2)(l+1)(l-1) + 12i M \omega_m},
\]
(3.109)
\[
\Phi_m = m\phi - \omega_m v.
\]
(3.110)

### 3.3.1 Slow motion: Analytic results

We begin our analysis of the Schwarzschild tidal deformations by expanding all quantities in orbital speed \(u \equiv (M/r_0)^{1/2}\). We take all relevant quantities to \(O(u^5)\) beyond the leading term; this is far enough to see how the curvature behaves for multipole index \(l \leq 4\). These results should be accurate for weak-field orbits, when \(u \ll 1\). In the following subsection, we will compare with numerical results that are good into the strong field.

Begin with \(C_{lm}\). Expanding Eq. (3.109), we find
\[
C_{2m} = -\frac{16i}{3} M^2 \exp \left( -\frac{13}{2} i M u^3 \right),
\]
(3.111)
\[
C_{3m} = -\frac{16i}{15} M^2 \exp \left( -\frac{61}{10} i M u^3 \right),
\]
(3.112)
\[
C_{4m} = -\frac{16i}{45} M^2 \exp \left( -\frac{181}{30} i M u^3 \right).
\]
(3.113)

To perform this expansion, we used the fact that, for \(a = 0\), \(M\Omega_{\text{orb}} = u^3\), so \(M\omega_m = \ldots\)
Next, we construct analytic expansions for the amplitudes $Z_{lm}^H$, following the algorithm described in Sec. 3.2.2. All the results which follow are understood to neglect contributions of $O(u^6)$ and higher.

For $l = 2$, the amplitudes are

\begin{align*}
Z_{20}^H &= \sqrt{\frac{3\pi \mu}{10 r_o^3}} \left( 1 + \frac{7}{2} u^2 + \frac{561}{56} u^4 \right), \quad (3.114) \\
Z_{21}^H &= -3i \sqrt{\frac{\pi \mu}{5 r_o^3}} \left( u + \frac{8}{3} u^3 + \frac{10i}{3} u^4 + \frac{152}{21} u^5 \right) \\
&= -3i \sqrt{\frac{\pi \mu}{5 r_o^3}} \left( u + \frac{8}{3} u^3 + \frac{152}{21} u^5 \right) \times \exp \left( \frac{10}{3} i u^3 \right), \quad (3.115) \\
Z_{22}^H &= -\frac{3}{2} \sqrt{\frac{\pi \mu}{5 r_o^3}} \left( 1 + \frac{3}{2} u^2 + \frac{23i}{3} u^3 + \frac{1403}{168} u^4 + \frac{473i}{30} u^5 \right) \\
&= -\frac{3}{2} \sqrt{\frac{\pi \mu}{5 r_o^3}} \left( 1 + \frac{3}{2} u^2 + \frac{1403}{168} u^4 \right) \times \exp \left[ i \left( \frac{23}{3} u^3 + \frac{64}{15} u^5 \right) \right]. \quad (3.116)
\end{align*}

For $l = 3$,

\begin{align*}
Z_{30}^H &= -i \sqrt{\frac{30\pi \mu}{7 r_o^3}} \left( u^3 + 4u^5 \right), \quad (3.117) \\
Z_{31}^H &= -\frac{3}{2} \sqrt{\frac{5\pi \mu}{14 r_o^3}} \left( u^2 + \frac{13}{3} u^4 + \frac{43i}{30} u^5 \right) \\
&= -\frac{3}{2} \sqrt{\frac{5\pi \mu}{14 r_o^3}} \left( u^2 + \frac{13}{3} u^4 \right) \exp \left( \frac{43}{30} i u^3 \right), \quad (3.118) \\
Z_{32}^H &= 5i \sqrt{\frac{\pi \mu}{7 r_o^3}} \left( u^3 + 4u^5 \right), \quad (3.119) \\
Z_{33}^H &= \frac{5}{2} \sqrt{\frac{3\pi \mu}{14 r_o^3}} \left( u^2 + 3u^4 + \frac{43i}{10} u^5 \right) \\
&= \frac{5}{2} \sqrt{\frac{3\pi \mu}{14 r_o^3}} \left( u^2 + 3u^4 \right) \exp \left( \frac{43}{10} i u^3 \right). \quad (3.120)
\end{align*}
Finally, for \( l = 4 \),

\[
\begin{align*}
Z_{40}^H &= -\frac{9}{14} \sqrt{\frac{5\pi}{2}} \frac{\mu}{r_0^3} u^4, \\
Z_{41}^H &= \frac{45i}{14} \sqrt{\frac{\pi}{2}} \frac{\mu}{r_0^3} u^5, \\
Z_{42}^H &= \frac{15}{14} \sqrt{\frac{\pi}{r_0^3}} u^4, \\
Z_{43}^H &= -\frac{15i}{2} \sqrt{\frac{\pi}{14 r_0^3}} u^5, \\
Z_{44}^H &= -\frac{15}{4} \sqrt{\frac{\pi}{7 r_0^3}} u^4.
\end{align*}
\]

Note that \( Z_{l-m}^H = (-1)^l \bar{Z}_{lm}^H \), where overbar denotes complex conjugation.

It is particularly convenient to combine the modes in pairs, examining \( R_{H,l-m}^{(1)} + R_{H,lm}^{(1)} \). Doing so, we find for \( l = 2 \),

\[
\begin{align*}
R_{H,20}^{(1)} &= -\frac{\mu}{r_0^3} (3 \cos^2 \theta - 1) \left( 1 + \frac{7}{2} u^2 + \frac{561}{56} u^4 \right), \\
R_{H,1-1}^{(1)} + R_{H,11}^{(1)} &= 0, \\
R_{H,2-2}^{(1)} + R_{H,22}^{(1)} &= \frac{3\mu}{r_0^3} \sin^2 \theta \left( 1 + \frac{3}{2} u^2 + \frac{1403}{168} u^4 \right) \\
&\times \cos \left[ 2 \left( \phi - \Omega_{\text{orb}} v - \frac{8}{3} u^3 + \frac{32}{5} u^5 \right) \right].
\end{align*}
\]

For \( l = 3 \), we have

\[
\begin{align*}
R_{H,30}^{(1)} &= 0, \\
R_{H,3-1}^{(1)} + R_{H,31}^{(1)} &= \frac{3\mu}{2 r_0^3} \sin \theta \left( 1 - 5 \cos^2 \theta \right) u^2 \left( 1 + \frac{13}{3} u^2 \right) \\
&\times \cos \left[ \phi - \Omega_{\text{orb}} v - \frac{14}{3} u^3 \right], \\
R_{H,3-2}^{(1)} + R_{H,32}^{(1)} &= 0, \\
R_{H,3-3}^{(1)} + R_{H,33}^{(1)} &= \frac{5\mu}{2 r_0^3} \sin^3 \theta u^2 \left( 1 + 3 u^2 \right) \cos \left[ 3 \left( \phi - \Omega_{\text{orb}} v - \frac{14}{3} u^3 \right) \right].
\end{align*}
\]
And, for \( l = 4 \),

\[
R_{H,40}^{(1)} = \frac{9}{56 r_0^3} \mu \left( 3 - 30 \cos^2 \theta + 35 \cos^4 \theta \right) u^4,
\]

\[
R_{H,4-1}^{(1)} + R_{H,41}^{(1)} = 0,
\]

\[
R_{H,4-2}^{(1)} + R_{H,42}^{(1)} = \frac{15}{14 r_0^3} \mu \sin^2 \theta \left( 1 - 7 \cos^2 \theta \right) u^4 
\times \cos \left[ 2 \left( \phi - \Omega_{\text{orb}} v - \frac{181}{30} u^3 \right) \right],
\]

\[
R_{H,4-3}^{(1)} + R_{H,43}^{(1)} = 0,
\]

\[
R_{H,4-4}^{(1)} + R_{H,44}^{(1)} = \frac{15}{8 r_0^3} \mu \sin^4 \theta u^4 \cos \left[ 4 \left( \phi - \Omega_{\text{orb}} v - \frac{181}{30} u^3 \right) \right].
\]

In the next section, we will compare Eqs. (3.126) – (3.137) with strong-field numerical calculations. Before doing so, we examine some consequences of these results and compare with earlier literature.

**Nearly static limit**

In Ref. [27], Hartle examines the deformation of a black hole due to a nearly static orbiting moon. To reproduce his results, consider the \( u \to 0 \) limit of Eqs. (3.126) – (3.137). Only the \( l = 2, m = 0, m = \pm 2 \) contributions remain when \( u \to 0 \). Adding these contributions, we find

\[
R_{H}^{(1)} = -\frac{\mu}{r_0^3} \left[ 3 \cos^2 \theta - 3 \sin^2 \theta \cos (2\phi') - 1 \right],
\]

where \( \phi' = \phi - \Omega_{\text{orb}} v \) is the azimuthal coordinate of the orbiting moon. Hartle writes\(^6\) his result

\[
R_{\text{Hartle}}^{(1)} = \frac{4\mu}{r_0^3} P_2(\cos \chi) = \frac{2\mu}{r_0^3} \left( 3 \cos^2 \chi - 1 \right),
\]

where \( \chi \) is the angle between the point of interest and the direction to the moon" [Ref. [27], text following Eq. (4.32)]. The angle \( \chi \) can be interpreted as \( \theta \) if we place

---

\(^6\)Note that the result Hartle presents in Ref. [27] contains a sign error. This can be seen by computing the curvature associated with the metric he uses on the horizon [Eqs. (5.10) and (5.13) of Ref. [26]]. The embedding surface Hartle uses, Eq. (4.33) of Ref. [27] [or (5.14) of Ref. [26]] is correct given this metric.
Hartle’s moon at $\theta_{\text{moon}} = 0$. To compare the two solutions, we must rotate. One way to do this rotation is to note that the equatorial plane in our calculation ($\theta = \pi/2$) should vary with $\phi'$ as Hartle’s result varies with $\chi$. Put $\theta = \pi/2$ and $\phi' = \chi$ in Eq. (3.138):

$$R_{H}^{(1)}|_{\theta=\pi/2,\phi'=\chi} = \frac{\mu}{r_{o}^{3}} (3 \cos 2 \chi + 1)$$

$$= \frac{2\mu}{r_{o}^{3}} (3 \cos^{2} \chi - 1). \quad (3.140)$$

Another way to compare is to note that the $\phi' = 0$ circle should vary with angle in a way that duplicates Hartle’s result, modulo a shift in angle, $\theta = \chi + \pi/2$:

$$R_{H}^{(1)}|_{\theta=\chi+\pi/2,\phi'=0} = \frac{-\mu}{r_{o}^{3}} \left[ 3 \cos^{2}(\chi + \pi/2) - 3 \sin^{2}(\chi + \pi/2) + 1 \right]$$

$$= \frac{-\mu}{r_{o}^{3}} (3 \sin^{2} \chi - 3 \cos^{2} \chi + 1)$$

$$= \frac{2\mu}{r_{o}^{3}} (3 \cos^{2} \chi - 1). \quad (3.141)$$

Both forms reproduce Hartle’s static limit.

Embedding the quadrupolar distortion

At various places in this chapter, we will examine the geometry of a distorted horizon by embedding it in a 3-dimensional space. The details of this calculation are given in Appendix 3.B; equivalent discussion for Schwarzschild, where the results are particularly clean, is also given in Ref. [128]. Briefly, a Schwarzschild horizon that has been distorted by a tidal field has the scalar curvature of a spheroid of radius

$$r_{E} = 2M \left[ 1 + \sum_{lm} \varepsilon_{lm}(\theta, \phi) \right], \quad (3.142)$$
where, as shown in Appendix 3.B.1 and Ref. [128],

\[ \varepsilon_{lm} = \frac{2M^2}{(l + 2)(l - 1)} R_{H,lm}^{(1)} . \]  

(3.143)

By considering a Schwarzschild black hole embedded in a universe endowed with post-Newtonian tidal fields, Taylor and Poisson [125] compute \( \varepsilon_{lm} \) in a post-Newtonian framework. Specializing to the tides appropriate to a binary system, they find

\[ \sum_m \varepsilon_{2m}(\theta, \phi) = \frac{\mu M^2}{b^3} \left( 1 + \frac{1}{2} u^2 \right) (1 - \cos^2 \theta) + \frac{3\mu M^2}{b^3} \left( 1 - \frac{3}{2} u^2 \right) \sin^2 \theta \cos [2(\phi - \Omega_{\text{orb}} v)] . \]  

(3.144)

This is Eq. (8.8) of Ref. [125], with \( M_2 \to \mu, M_1 \to M, v_{\text{rel}}/c \to u \), and expanded to leading order in \( \mu \). Their parameter \( b \) is the separation of the binary in harmonic coordinates. Using the fact that \( r_H = r_S - M \) (with "S" and "H" subscripts denoting harmonic and Schwarzschild, respectively), it is easy to convert to \( r_o \), our separation in Schwarzschild coordinates:

\[ \frac{1}{b^3} = \frac{1}{r_o^3 (1 - M/r_o)^3} \simeq \frac{1}{r_o^3} \left( 1 + 3u^2 \right) . \]  

(3.145)

Replacing \( b \) for \( r_o \) and truncating at \( O(u^2) \), Eq. (3.144) becomes

\[ \sum_m \varepsilon_{2m}(\theta, \phi) = \frac{\mu M^2}{r_o^3} \left( 1 + \frac{7}{2} u^2 \right) (1 - \cos^2 \theta) + \frac{3\mu M^2}{r_o^3} \left( 1 - \frac{3}{2} u^2 \right) \sin^2 \theta \cos [2(\phi - \Omega_{\text{orb}} v)] . \]  

(3.146)

Comparing with Eqs. (3.126) and (3.128) and correcting for the factor \( M^2/2 \) which converts curvature \( R_{H,2m}^{(1)} \) to \( \varepsilon_{2m} \), we see agreement to \( O(u^2) \).

**Phase of the tidal bulge**

Using these analytic results, let us examine the notions of bulge phase introduced in Sec. 3.2.4. First consider the position of the bulge versus the position of the orbit
according to the null and instantaneous maps (which are identical for Schwarzschild), Eq. (3.84). The various modes which determine the shape of the horizon all peak at angle \( \phi = \Omega_{\text{orb}}v + \delta \phi(u) \), where \( \delta \phi(u) \) can be read out of Eqs. (3.126)–(3.137). For Schwarzschild \( \bar{r} = 0 \), and the ingoing angle \( \psi = \phi \). The orbit’s position mapped onto the horizon is \( \phi_o^{NM} = \phi_o = \Omega_{\text{orb}}v + \Delta \phi(r_o) \), where

\[
\Delta \phi(r_o) = -\Omega_{\text{orb}}r^*_o
\]

is Eq. (3.81) for \( a = 0 \). The result for the bulge’s offset from the orbit is

\[
\begin{align*}
\delta \phi_{22}^{OB} &= \frac{8}{3} u^3 - \frac{32}{5} u^5 - \Delta \phi(r_o), \\
\delta \phi_{31}^{OB} &= \delta \phi_{33}^{OB} = \frac{14}{3} u^3 - \Delta \phi(r_o), \\
\delta \phi_{42}^{OB} &= \delta \phi_{44}^{OB} = \frac{181}{30} u^3 - \Delta \phi(r_o).
\end{align*}
\]

For the multipoles which we do not include here, no useful notion of bulge position exists: for \( m = 0 \) the bulge is axisymmetric, and for the others, the bulge’s amplitude is zero to this order. Our results for \( l = |m| = 2 \) agree with Fang and Lovelace; cf. Eq. (4) of Ref. [130].

Consider next the relative phase of the tidal bulge and the perturbing field, Eq. (3.93). For small \( u \), we have

\[
\delta \phi_m^{TD} = 4mu^3.
\]

This again agrees with Fang and Lovelace — compare Eq. (6) of Ref. [130], bearing in mind that \( m \) is built into their definition of the offset angle [their Eq. (50)], and that they fix \( m = 2 \).

In both cases, note that the bulge’s offset is a positive phase. This indicates that the bulge leads both the orbiting body’s instantaneous position, as well as the tidal field that sources the tidal deformation. As discussed in the Introduction, this is consistent with past work, and is a consequence of the horizon’s teleological nature.
Figure 3-1: Convergence of contributions to the horizon’s tidal distortion. We show $R_{H,lm}^{(l)}$ summed over $m$ for a given $l$. Red curve is for $l = 2$, green is $l = 3$, blue is $l = 4$, magenta is $l = 5$, and cyan is $l = 6$. (Higher order contributions are omitted since their variations cannot be seen on the scale of this plot.) These curves are for a circular orbit at $r_o = 6M$, which has $u = 0.41$, the largest value for the Schwarzschild cases we consider. As such, this case has the slowest convergence among Schwarzschild orbits. The falloff with $l$ is more rapid for all other cases.

3.3.2 Fast motion: Numerical results

Our numerical results for Schwarzschild black holes are summarized by Figs. 3-1, 3-2, and 3-3. We compute $R_{H}^{(l)}$ by solving for $Z_{lm}^{H}$ numerically as described in Sec. 3.2.2, and then applying Eq. (3.108). We include all contributions up to $l = 15$ in the sum. Figure 3-1 shows that contributions to the horizon’s scalar curvature converge quite rapidly. The contributions from $l = 15$ are about $10^{-9}$ of the total for the most extreme case we consider here, $r_o = 6M$.

Figure 3-2 compares the analytic predictions for $R^{H}$ [Eqs. (3.126)–(3.137)] with numerical results for $l = 2$, $l = 3$, and $l = 4$, and for two different orbital radii
(\(r_o = 50M\) and \(6M\)). The agreement is outstanding for the large radius orbit. Our numerical and analytic predictions can barely be distinguished at \(l = 2\) and \(l = 3\), and differ by about 10% at maximum for \(l = 4\) (where our analytic formula includes only the leading contribution to the curvature). The agreement is much poorer at small radius. At \(r_o = 6M\), disagreement is several tens of percent for \(l = 2\), rising to a factor \(\sim 5\) for \(l = 4\). For both the large and small radius cases we show, the sum over modes is dominated by the contribution from \(l = 2\). The phase agreement between analytic and numerical formulas is quite good all the way into the strong field, even when the amplitudes differ significantly.

**Figure 3-2**: Comparison of numerically computed scalar curvature perturbation \(R^{(1)}_H\) for Schwarzschild with the analytic expansion given in Eqs. (3.126)–(3.137). The four panels on the left compare numerical (red curves) and analytic (green) results for an orbit at \(r_o = 50M\). Panels on the right are for \(r_o = 6M\). In both cases, we plot \(\left(r^3_o/\mu\right)R^{(1)}_H\), scaling out the leading dependence on orbital radius and the orbiting body’s mass. We show contributions for \(l = 2\), \(l = 3\), and \(l = 4\), plus the sum of these modes. For \(r_o = 50M\), we have \(u = 0.14\), and we see very good agreement between the numerical and analytic formulas. In several cases, the numerical data lie on top of the analytic curves. For \(r_o = 6M\), \(u = 0.41\), and the agreement is not as good. Although the amplitudes disagree in the strong field (especially for large \(l\), the two computations maintain good phase agreement well into the strong field.

Figure 3-3 shows distorted black holes by embedding the horizon in a 3-dimensional Euclidean space, as discussed in Sec. 3.3.1. Now, we do not truncate at \(l = 2\), but
include all moments that we calculate. We show the equatorial slices of our embeddings for several different circular orbits \((r_o = 50M, 20M, 10M, \text{ and } 6M)\). In all of our plots, we scale the horizon distortion \(\varepsilon_{\ell m}\) by a factor proportional to \(r_o^3/\mu\) so that the tide’s impact is of roughly the same magnitude for all orbital separations.

The embeddings are shown in a frame that corotates with the orbit at an instant \(v = \text{constant}\). The \(x\)-axis is at \(\phi = 0\), so the orbiting body sits at \(\phi = \Delta\phi(r_o) = -\Omega_{\text{orb}} r_o^*\). In each panel, we have indicated where the radius of the embedding is largest (green dashed line, showing the angle of greatest tidal distortion) and the angular position of the orbiting body (black dotted line). In all cases, the bulge leads the orbiting body’s position, just as predicted in Sec. 3.3.1. The numerical value of the bulge’s position relative to the orbit, \(\delta\phi^{\text{num}}\), agrees quite well with \(\delta\phi_{22}^{\text{OB}}\), Eq. (3.148) From Fig. 3-3, we have

\[
\delta\phi^{\text{num}} = \begin{align*}
9.56^\circ & \quad r_o = 50M , \\
17.3^\circ & \quad r_o = 20M , \\
27.8^\circ & \quad r_o = 10M , \\
37.6^\circ & \quad r_o = 6M .
\end{align*}
\]

Equation (3.148) tells us

\[
\delta\phi_{22}^{\text{OB}} = \begin{align*}
9.54^\circ & \quad r_o = 50M , \\
17.1^\circ & \quad r_o = 20M , \\
26.8^\circ & \quad r_o = 10M , \\
35.0^\circ & \quad r_o = 6M .
\end{align*}
\]

In all cases, the true position of the bulge is slightly larger than \(\delta\phi_{22}^{\text{OB}}\). This appears to be due in large part to the contribution of modes other than \(l = |m| = 2\); the agreement improves if we calculate \(\delta\phi^{\text{num}}\) using only the \(l = 2\) contribution to the embedding.
3.4 Results II: Kerr

Now consider non-zero black hole spin. We begin with slow motion and small black hole spin, expanding Eq. (3.39) using \( u \equiv (M/r_0)^{1/2} \ll 1 \) and \( q \equiv a/M \ll 1 \), and derive analytic results which are useful points of comparison to the general case. We then show numerical results which illustrate tidal deformations for strong-field orbits.

### 3.4.1 Slow motion: Analytic results

Here we present analytic results, expanding in powers of \( u = (M/r_0)^{1/2} \) and \( q = a/M \). We take all relevant quantities to order \( u^5 \) and \( q \) beyond the leading term; this is far enough to see how quantities behave for \( l \leq 4 \). We compare with strong-field numerical results in the following subsection.

Begin again with \( C_{lm} \). Neglect the \( k \) and \( n \) indices which are irrelevant for circular, equatorial orbits, and expand \( \lambda = \lambda_0 + (a \omega_m) \lambda_1 \), with \( \lambda_0 \) and \( \lambda_1 \) given by Eqs. (3.316) and (3.318) for \( s = -2 \) [recall that \( \lambda \) comes from the harmonic \( S_{lm}^{-}(\theta) \)]. Finally, expand to \( O(u^5) \) and \( O(q) \). Doing so, Eq. (3.40) yields

\[
C_{2m} = -\frac{16i}{3} M^2 \left( 1 - \frac{13}{3} q m^2 u^3 \right) \exp \left[ -i m \left( \frac{13}{2} u^3 - \frac{3}{2} q \right) \right], \quad (3.154)
\]

\[
C_{3m} = -\frac{16i}{15} M^2 \left( 1 - \frac{14}{3} q m^2 u^3 \right) \exp \left[ -i m \left( \frac{61}{10} u^3 - \frac{3}{2} q \right) \right], \quad (3.155)
\]

\[
C_{4m} = -\frac{16i}{45} M^2 \left( 1 - \frac{24}{5} q m^2 u^3 \right) \exp \left[ -i m \left( \frac{181}{30} u^3 - \frac{3}{2} q \right) \right]. \quad (3.156)
\]

These reduce to the Schwarzschild results, Eqs. (3.111) – (3.113), when \( q \to 0 \).

Next, the amplitudes \( Z_{lm}^H \), again following the algorithm described in Sec. 3.2.2. These results should be understood to neglect contributions of \( O(u^5) \), \( O(q^2) \) and
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higher. For \( l = 2 \), we have

\[
Z_{20}^H = \sqrt{\frac{3\pi \mu}{10 r_0^3}} \left( 1 + \frac{7}{2} u^2 - 4qu^3 + \frac{561}{56} u^4 - 18qu^5 \right), \tag{3.157}
\]

\[
Z_{21}^H = -3i \sqrt{\frac{\pi \mu}{5 r_0^3}} \left\{ \left( 1 - \frac{i}{2} q \right) u - \frac{2}{3} qu^2 + \left( \frac{8}{3} - \frac{4i}{3} q \right) u^3 + \left[ \frac{10i}{3} + \left( \frac{1}{6} - \frac{\pi^2}{3} \right) q \right] u^4 + \left[ \frac{152}{21} - \frac{368}{63} q \right] u^5 \right\} \exp \left[ i \left( \frac{10}{3} u^3 - \frac{q}{2} \right) \right], \tag{3.158}
\]

\[
Z_{22}^H = -3 \sqrt{\frac{\pi \mu}{2 r_0^3}} \left\{ \left( 1 - \frac{i}{2} q \right) u - \frac{2}{3} qu^2 + \frac{8}{3} u^3 - \left( \frac{3}{2} + \frac{\pi^2}{3} \right) qu^4 + \frac{152}{21} u^5 \right\} \times \exp \left[ i \left( \frac{10}{3} u^3 - \frac{q}{2} \right) \right], \tag{3.159}
\]

For \( l = 3 \),

\[
Z_{30}^H = -i \sqrt{\frac{30\pi \mu}{7 r_0^3}} \left( u^3 - \frac{3}{4} qu^4 + 4u^5 \right), \tag{3.160}
\]

\[
Z_{31}^H = -\frac{3}{2} \sqrt{\frac{5\pi \mu}{14 r_0^3}} \left\{ \left( 1 - \frac{i}{6} q \right) u^2 + \left[ \frac{13}{3} - \frac{13i}{18} q \right] u^4 + \left[ \frac{43i}{30} - \left( \frac{247}{180} + \frac{\pi^2}{3} \right) q \right] u^5 \right\} \exp \left[ i \left( \frac{43}{30} u^3 - \frac{q}{6} \right) \right], \tag{3.161}
\]

\[
Z_{32}^H = 5i \sqrt{\frac{\pi \mu}{7 r_0^3}} \left( u^3 - \frac{3}{4} qu^4 + 4 \left( 1 - \frac{i}{3} q \right) u^5 \right) \exp \left[ -iq/3 \right], \tag{3.162}
\]

\[
Z_{33}^H = \frac{5}{2} \sqrt{\frac{3\pi \mu}{14 r_0^3}} \left\{ \left( 1 - \frac{i}{2} q \right) u^2 + \left( \frac{3}{2} - \frac{3i}{2} q \right) u^4 + \left[ \frac{43i}{10} + \left( \frac{393}{20} - 3\pi^2 \right) q \right] u^5 \right\} \exp \left[ i \left( \frac{43}{10} u^3 - \frac{q}{2} \right) \right]. \tag{3.163}
\]
And for $l = 4$, 

$$Z_{40}^H = -\frac{9}{14} \sqrt{\frac{5\pi}{2}} \frac{\mu}{r_0^3} u^4, \quad (3.164)$$

$$Z_{41}^H = \frac{45i}{14} \sqrt{\frac{\pi}{2}} \frac{\mu}{r_0^3} \left[ \left( 1 + \frac{i}{12} q \right) u^5 \right] = \frac{45i}{14} \sqrt{\frac{\pi}{2}} \frac{\mu}{r_0^3} u^5 \exp \left( i q / 12 \right), \quad (3.165)$$

$$Z_{42}^H = \frac{15}{14} \sqrt{\frac{\pi}{2}} \frac{\mu}{r_0^3} \left[ \left( 1 + \frac{i}{6} q \right) u^4 \right] = \frac{15}{14} \sqrt{\frac{\pi}{2}} \frac{\mu}{r_0^3} u^4 \exp \left( i q / 6 \right), \quad (3.166)$$

$$Z_{43}^H = -\frac{15i}{2} \sqrt{\frac{\pi}{2}} \frac{\mu}{r_0^3} \left[ \left( 1 + \frac{i}{4} q \right) u^3 \right] = -\frac{15i}{2} \sqrt{\frac{\pi}{2}} \frac{\mu}{r_0^3} u^3 \exp \left( i q / 4 \right), \quad (3.167)$$

$$Z_{44}^H = -\frac{15}{4} \sqrt{\frac{\pi}{2}} \frac{\mu}{r_0^3} \left[ \left( 1 + \frac{i}{3} q \right) u^2 \right] = -\frac{15}{4} \sqrt{\frac{\pi}{2}} \frac{\mu}{r_0^3} u^2 \exp \left( i q / 3 \right). \quad (3.168)$$

Equations (3.157) – (3.168) reduce to Eqs. (3.114) – (3.125) when $q \to 0$. Modes for $m < 0$ can be obtained using the rule $Z_{i-m}^H = (-1)^l \overline{Z_{im}^H}$, with overbar denoting complex conjugate.

Lastly, we need the angular function $\bar{\bar{\delta}} \delta S_{lm}^+$ to leading order in $q$. Using Eqs. (3.67), (3.68), (3.73), and the condition $q \ll 1$, we have

$$\bar{\bar{\delta}} \delta S_{lm}^+ = \frac{1}{8M^2} L_+^s L_-^s (1 + iq \cos \theta) S_{lm}^+ = \frac{1}{8M^2} \left[ (1 + iq \cos \theta) L_+^s L_-^s S_{lm}^+ - 2iq \sin \theta L_-^s S_{lm}^+ \right]. \quad (3.169)$$

Following the analysis in Appendix 3.C, the spheroidal harmonic to this order is

$$S_{lm}^+ = 2Y_{lm} + qM \omega_m \left[ c_{lm}^{l+1} Y_{(l+1)m} + c_{lm}^{l-1} Y_{(l-1)m} \right], \quad (3.170)$$

where

$$c_{lm}^{l+1} = \frac{2}{(l + 1)^2} \sqrt{\frac{(l + 3)(l - 1)(l + m + 1)(l - m + 1)}{(2l + 3)(2l + 1)}}, \quad (3.171)$$

$$c_{lm}^{l-1} = \frac{2}{l^2} \sqrt{\frac{(l + 2)(l - 2)(l + m)(l - m)}{(2l + 1)(2l - 1)}}. \quad (3.172)$$

Using Eq. (3.64) with Eqs. (3.169) and (3.170) and expanding to leading order in $q$, ...
we find

\[ \tilde{\alpha} \tilde{\alpha} S^+_{lm} = \frac{1}{8M^2} \left[ (1 + \imath q \cos \theta) \sqrt{(l+2)(l+1)l(l-1)} Y_{lm} - 2 \imath q \sin \theta \sqrt{(l+2)(l-1)} Y_{lm} \right. \\
\left. + q M \omega_m \left( c_{lm}^{i+1} \sqrt{(l+3)(l+2)(l+1)} Y_{l+1,m} \right. \\
\left. + c_{lm}^{i-1} \sqrt{(l+1)(l-1)(l-2)} Y_{l-1,m} \right) \right]. \]  

(3.173)

As in Sec. 3.3.1, it is convenient to combine modes in pairs. For \( l = 2 \), we find

\[ R_{H,20}^{(1)} = -\frac{\mu}{r_0^3} (3 \cos^2 \theta - 1) \]
\[ \times \left( 1 + \frac{7}{2} u^2 - 4qu^3 + \frac{561}{56} u^4 - 18qu^5 \right), \]  

(3.174)

\[ R_{H,21}^{(1)} + R_{H,21}^{(1)} = \frac{4\mu}{r_0^3} (5 \cos^2 \theta - 1) \sin \theta q \left( u + \frac{8}{3} u^3 + \frac{152}{21} u^5 \right) \]
\[ \times \cos \left( \psi - \Omega_{\text{orb}} v - \frac{3}{2} u^3 + 6M\Omega_H \right), \]  

(3.175)

\[ R_{H,22}^{(1)} + R_{H,22}^{(1)} = \frac{3\mu}{r_0^3} \sin^2 \theta \left[ 1 + \frac{3}{2} u^2 - \left( 10 + \frac{4\pi^2}{3} \right) q u^3 + \frac{1403}{168} u^4 \right. \\
\left. - \left( \frac{131}{9} + 2\pi^2 \right) q u^5 \right] \]
\[ \times \cos \left[ 2 \left( \psi - \Omega_{\text{orb}} v - \frac{8}{3} u^3 + \frac{32}{5} u^5 + \frac{14}{3} M\Omega_H \right) \right]. \]  

(3.176)
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For $l = 3$,

\[
R_{H,30}^{(1)} = -\frac{\mu}{r_o^3} (1 - 12 \cos^2 \theta + 15 \cos^4 \theta) qu^3(1 + 4u^2), \tag{3.177}
\]

\[
R_{H,31}^{(1)} + R_{H,31}^{(1)} = \frac{3 \mu}{2r_o^3} \sin \theta (1 - 5 \cos^2 \theta) u^2 \left[ 1 + \frac{13}{3} u^2 - \left( \frac{113}{12} + \frac{\pi^2}{2} \right) qu^3 \right]
\times \cos \left[ \psi - \Omega \omega v - \frac{14}{3} u^3 + \frac{20}{3} M \Omega_H \right], \tag{3.178}
\]

\[
R_{H,32}^{(1)} + R_{H,32}^{(1)} = \frac{5 \mu}{3r_o^3} q(u^3 + 4u^5) \left[ 9 \cos^2 \theta \cos \left( 2 \left( \psi - \Omega \omega v - \frac{56}{10} u^3 + \frac{22}{3} M \Omega_H \right) \right) \right.
- \cos \left[ 2 \left( \psi - \Omega \omega v - \frac{158}{30} u^3 + \frac{22}{3} M \Omega_H \right) \right] \right], \tag{3.179}
\]

\[
R_{H,33}^{(1)} + R_{H,33}^{(1)} = \frac{5 \mu}{2r_o^3} \sin^3 \theta u^2 \left[ 1 + 3u^2 - \left( \frac{49}{2} + 3\pi^2 \right) qu^3 \right]
\times \cos \left[ 3 \left( \psi - \Omega \omega v - \frac{14}{3} u^3 + \frac{20}{3} M \Omega_H \right) \right]. \tag{3.180}
\]

And for $l = 4$,

\[
R_{H,40}^{(1)} = \frac{9 \mu}{56 r_o^3} (3 - 30 \cos^2 \theta + 35 \cos^4 \theta) u^4, \tag{3.181}
\]

\[
R_{H,41}^{(1)} + R_{H,41}^{(1)} = -\frac{9 \mu}{28 r_o^3} \sin \theta qu^5 \left [ 98 \cos^4 \theta \cos \left( \psi - \Omega \omega v - \frac{169}{30} u^3 + \frac{25}{3} M \Omega_H \right) \right. 
- 57 \cos^2 \theta \cos \left( \psi - \Omega \omega v - \frac{1568}{285} u^3 + \frac{25}{3} M \Omega_H \right) 
+ 3 \cos \left( \psi - \Omega \omega v - \frac{77}{15} u^3 + \frac{25}{3} M \Omega_H \right) \right], \tag{3.182}
\]

\[
R_{H,42}^{(1)} + R_{H,42}^{(1)} = \frac{15 \mu}{14 r_o^3} \sin^2 \theta (1 - 7 \cos^2 \theta) u^4
\times \cos \left[ 2 \left( \psi - \Omega \omega v - \frac{181}{30} u^3 + \frac{119}{15} M \Omega_H \right) \right], \tag{3.183}
\]

\[
R_{H,43}^{(1)} + R_{H,43}^{(1)} = \frac{3 \mu}{4r_o^3} \sin^3 \theta qu^5 \left [ 14 \cos^2 \theta \cos \left( 3 \left( \psi - \Omega \omega v - \frac{169}{30} u^3 + \frac{25}{3} M \Omega_H \right) \right) \right. 
- \cos \left[ 3 \left( \psi - \Omega \omega v - \frac{77}{15} u^3 + \frac{25}{3} M \Omega_H \right) \right] \right], \tag{3.184}
\]

\[
R_{H,44}^{(1)} + R_{H,44}^{(1)} = \frac{15 \mu}{8 r_o^3} \sin^4 \theta u^4 \cos \left[ 4 \left( \psi - \Omega \omega v - \frac{181}{30} u^3 + \frac{119}{15} M \Omega_H \right) \right]. \tag{3.185}
\]

In writing these formulas, we have used the fact that $\Omega_H = q/4M$ in the $q \ll 1$ limit to rewrite certain terms in the phases using $\Omega_H$ rather than $q$. For example, in
Eq. (3.176) our calculation yields a term $7q/6$ in the argument of the cosine, which we rewrite $14M\Omega_H/3$. We have found that this improves the match of Eqs. (3.174) – (3.185) with the numerical results we discuss in Sec. 3.4.2.

Phase of the tidal bulge: Null map

We begin by examining the bulge-orbit offset using the null map, Eq. (3.84). The horizon’s geometry is dominated by contributions for which $l+m$ is even; modes with $l+m$ odd are suppressed by $qu$ relative to these dominant modes (thus vanishing in the Schwarzschild limit). The dominant modes peak at $\psi_{lm}^{\text{bulge}} = \Omega_{\text{orb}}u + \delta\psi_{lm}(u) + \delta\psi_{lm}(q)$, where $\delta\psi_{lm}(u)$ and $\delta\psi_{lm}(q)$ can be read out of Eqs. (3.174) – (3.185). The orbit mapped onto the horizon in the null map is given by Eq. (3.82). Following discussion in Sec. 3.2.4, the offset phases in the null map for the dominant modes, to $O(u^5)$ and $O(q)$, are

\[
\begin{align*}
\delta\psi_{22}\text{--NM} &= \frac{8}{3} \left( u^3 - M\Omega_H \right) - \frac{32}{5} u^5 - \frac{4M^2\Omega_H}{r_o} - \Delta\psi(r_o), \\
\delta\psi_{31}\text{--NM} &= \frac{14}{3} \left( u^3 - M\Omega_H \right) - \frac{4M^2\Omega_H}{r_o} - \Delta\psi(r_o), \\
\delta\psi_{42}\text{--NM} &= \frac{181}{30} u^3 - \frac{89}{15} M\Omega_H - \frac{4M^2\Omega_H}{r_o} - \Delta\psi(r_o).
\end{align*}
\]

We again see agreement with Fang and Lovelace for $l = m = 2$, who correct a sign error in Hartle’s [27] treatment of the bulge phase; compare Eq. (61) and footnote 6 of Ref. [130] and associated discussion. In contrast to the Schwarzschild case, the Kerr offset phases can be positive or negative, depending on the values of $r_o$ and $q$. To highlight this further, let us examine Eq. (3.186) for very large $r_o$: we drop the term in $u^5$, and expand $\Delta\psi(r_o)$. The result is

\[
\delta\psi_{22}\text{--NM} \sim \frac{8}{3} \left( u^3 - M\Omega_H \right) + \sqrt{\frac{M}{r_o}}.
\]
As \( r_o \to \infty \), we see that this bulge lags the orbit by \( \delta_{22}^{\text{OB-NM}} = -8 \Omega_H / 3 \), which reproduces Hartle’s finding for a stationary moon orbiting a slowly rotating Kerr black hole [Eq. (4.34) of Ref. [27], correcting the sign error discussed in footnote 6 of Ref. [130]]. We discuss this point further in Sec. 3.5.

**Phase of the tidal bulge: Instantaneous map**

Consider next the instantaneous-in-v map discussed in Sec. 3.2.4. The position of the orbit on the horizon in this mapping is given by Eq. (3.86). To \( O(u^5) \) and \( O(q) \), the offset phase for the dominant modes in this map is

\[
\begin{align*}
\delta \psi_{22}^{\text{OB-IM}} &= \frac{8}{3} u^3 - \frac{14}{3} \Omega_H - \frac{32}{5} u^5 - \Delta \psi(r_o), \\
\delta \psi_{31}^{\text{OB-IM}} &= \delta \psi_{33}^{\text{OB-IM}} \\
&= \frac{14}{3} u^3 - \frac{20}{3} \Omega_H - \Delta \psi(r_o), \\
\delta \psi_{42}^{\text{OB-IM}} &= \delta \psi_{44}^{\text{OB-IM}} \\
&= \frac{181}{30} u^3 - \frac{119}{15} \Omega_H - \Delta \psi(r_o).
\end{align*}
\]

As in the null map, these phases can be positive or negative, depending on the values of \( r_o \) and \( q \). As we’ll see when we examine numerical results for the horizon geometry, Eq. (3.190) does a good job describing the angle of the peak horizon bulge for small values of \( q \).

**Phase of the tidal bulge: Tidal field versus tidal response**

Finally, let us examine the relative phase of tidal field modes \( \psi_{l,m}^{\text{HH}} \) and the horizon’s response \( R_{H,l,m}^{(1)} \). For \( q \ll 1 \), we have \( \kappa^{-1} = 4M + O(q^2) \). Expanding in the weak-field limit, Eq. (3.92) becomes

\[
\delta \psi_{lm}^{\text{TB}} = 4m(u^3 - \Omega_H) + S_{lm}(\pi/2).
\]

For the modes with \( l + m \) even which dominate the horizon’s response, it is not difficult to compute \( S_{lm}(\pi/2) \) to leading order in \( q \). Equation (3.173) and the definition (3.91)
yield
\[ S_{lm}(\pi/2) = \frac{2q}{\sqrt{l(l+1)}} y_{lm}(\pi/2) + O(q^2). \] (3.194)

We also know [cf. Eq. (A8) of Ref. [137]] that
\[ y_{lm}(\theta) = -\frac{1}{\sqrt{l(l+1)}} (\partial_\theta - m \csc \theta)y_{lm}(\theta). \] (3.195)

For \( l+m \) even, \( \partial_\theta y_{lm} = 0 \) at \( \theta = \pi/2 \). Plugging the resulting expression for \( y_{lm}(\pi/2) \) into Eq. (3.194), we find
\[ S_{lm}(\pi/2) = \frac{2mq}{l(l+1)} = \frac{8mM\Omega_H}{l(l+1)}, \] (3.196)

where in the last step we again used \( q = 4M^2\Omega_H \), accurate for \( q \ll 1 \). With this, Eq. (3.193) becomes
\[ \delta\psi_{lm}^{TB} = 4m \left[ u^3 - M\Omega_H \left( 1 - \frac{2}{l(l+1)} \right) \right] 
= 4m \left[ u^3 - M\Omega_H \frac{(l+2)(l-1)}{l(l+1)} \right]. \] (3.197)

Just as with the offset phases of the bulge and the orbit for Kerr, this tidal bulge phase can be either positive or negative depending on \( r_o \) and \( q \), and so the horizon’s response can lead or lag the applied tidal field.

### 3.4.2 Fast motion: Numerical results

Figures 3-4, 3-5, and 3-6 present summary data for our numerical calculations of tidally distorted Kerr black holes. Just as in Sec. 3.3.2, we compute \( R_H^{(1)} \) by solving for \( Z_{lm}^H \) as described in Sec. 3.2.2, and then apply Eq. (3.39). As in the Schwarzschild case, we find rapid convergence with mode index \( l \). All the data we show are for the equatorial plane, \( \theta = \pi/2 \). We typically include all modes up to \( l = 15 \) (increasing this to 20 and 25 in a few very strong field cases). Contributions beyond this are typically at the level of \( 10^9 \) or smaller, which is accurate enough for this exploratory
Figure 3-4 is the Kerr analog of Fig. 3-2, comparing numerical results for $R_{\ell m}^H$ with analytic predictions for selected black hole spins, mode numbers, and orbital radii. For all modes we show here, we see outstanding agreement in both phase and amplitude for $q = 0.1$ and $r_o = 50M$; in some cases, the numerical data lies almost directly on top of the analytic prediction. The amplitude agreement is not quite as good as we increase the spin to $q = 0.2$ and move to smaller radius ($r_o = 10M$), though the phase agreement remains quite good for all modes.

Figures 3-5 and 3-6 show equatorial slices of the embedding of distorted Kerr black holes for a range of orbits and black hole spins. These embeddings are similar to those we used for distorted Schwarzschild black holes (as described in Sec. 3.3.1), with a few important adjustments. The embedding surface we use has the form

$$r_E = r_E^0(\theta) + r_+ \sum_{\ell m} \varepsilon_{\ell m}(\theta, \psi).$$

Both the undistorted radius $r_E^0(\theta)$ and the tidal distortion $\varepsilon_{\ell m}(\theta, \psi)$ are described in Appendix 3.B; see also Ref. [131]. The background embedding reduces to a sphere of radius $2M$ when $a = 0$, but is more complicated in general. The embedding’s tidal distortion is linearly related to the curvature $R_{H,\ell m}^{(1)}$, but in a way that is more complicated than the Schwarzschild relation (3.143). In particular, mode mixing becomes important: Different angular basis functions are needed to describe the curvature $R_{H,\ell m}^{(1)}$ and the embedding distortion $\varepsilon_{\ell m}$ when $a \neq 0$. Hence, the $\ell = 2$ contribution to the horizon’s shape has contributions from all $l$ curvature modes, not just $l = 2$. See Appendix 3.B for detailed discussion.

In this chapter, we only generate embeddings for $a/M \leq \sqrt{3}/2$. For spins greater than this, the horizon cannot be embedded in a globally Euclidean space. Although a “belt” from $\pi - \theta_L \leq \theta \leq \theta_L$ can be embedded in 3-dimensional Euclidean space, one must embed the “polar cones” $0 \leq \theta < \theta_L$ and $\pi - \theta_L < \theta \leq \pi$ in a Lorentzian geometry (where $\theta_L$ is related to the root of a function used in the embedding; see Appendix 3.B for details). This is closely related to the change in sign of $R_H^{(0)}$ near the poles
for $a/M > \sqrt{3}/2$ discussed near Eq. (3.15). For $a/M > \sqrt{3}/2$, one can embed the tidally distorted horizon in a piecewise fashion, separately embedding the equatorial belt and the two polar cones in appropriate spaces. Although it is straightforward to do this exercise, its technical details are quite lengthy, and do not add much insight beyond what the embeddings for $a/M \leq \sqrt{3}/2$ already teach us. In Appendix 3.B, we briefly describe what must be done to embed the distorted horizon for $a/M > \sqrt{3}/2$, but defer a detailed presentation to future work.

As with the Schwarzschild embeddings shown in Fig. 3-3, the Kerr embeddings we show are all plotted in a frame that corotates with the orbit at a moment $v = \text{constant}$. The $x$-axis is at $\psi = 0$, and the orbiting body sits at $\psi = \Delta\psi(r_o) = \bar{r}_o - \Omega_{\text{orb}}r_o^*$. As in Fig. 3-3, the green dashed line labels the horizon’s peak bulge, and the black dotted line shows the position of the orbiting body.

For small $q$, we find that the numerically computed bulge offset agrees quite well with the $l = 2$ analytic expansion in the instantaneous map, Eq. (3.190). For $q = 0.1$, our numerical results are

$$
\begin{align*}
\delta\psi^{\text{num}} &= 3.01^\circ \quad r_o = 50M , \\
&= 10.8^\circ \quad r_o = 20M , \\
&= 21.6^\circ \quad r_o = 10M , \\
&= 33.7^\circ \quad r_o = 5.669M .
\end{align*}
$$

These are within a few percent of predictions based on the weak-field, slow spin expansion:

$$
\begin{align*}
\delta\psi_{22}^{\text{OB-IM}} &= 2.95^\circ \quad r_o = 50M , \\
&= 10.7^\circ \quad r_o = 20M , \\
&= 20.6^\circ \quad r_o = 10M , \\
&= 30.0^\circ \quad r_o = 5.669M .
\end{align*}
$$

As we move to larger spin, the agreement rapidly becomes worse. Terms which we
neglect in our expansion become important, and the mode mixing described above becomes very important. For $q = 0.4$, the agreement degrades to a few tens of percent in most cases:

$$\delta \psi^{\text{num}} = -13.5^\circ \quad r_o = 50M ,$$
$$= -6.17^\circ \quad r_o = 20M ,$$
$$= 3.55^\circ \quad r_o = 10M ,$$
$$= 21.4^\circ \quad r_o = 4.614M ; \quad (3.201)$$

and

$$\delta \psi^{\text{OB-IM}} = -17.9^\circ \quad r_o = 50M ,$$
$$= -9.76^\circ \quad r_o = 20M ,$$
$$= 0.82^\circ \quad r_o = 10M ,$$
$$= 13.6^\circ \quad r_o = 4.614M . \quad (3.202)$$

The agreement gets significantly worse as $q$ is increased further. Presumably, $q \sim 0.3$ is about as far as the leading order expansion in $q$ can reasonably be taken.

To conclude this section, we show two examples of embeddings for the entire horizon surface, rather than just the equatorial slice. The left-hand panel of Fig. 3-7 is an example of a relatively mild tidal distortion. The black hole has spin $a = 0.3 M$, and the orbiting body is at $r = 20M$. The distortion is strongly dominated by the $\ell = 2$ contribution, and we see a fairly simple prolate ellipsoid whose bulge lags the orbit. The right-hand panel shows a much more extreme example. The black hole here has $a = 0.866 M$, and the orbiting body is at $r = 1.8 M$. The horizon's shape has strong contributions from many multipoles; and so is bent in a rather more complicated way than in the mild case. The connection between the orbit and the horizon geometry is quite unusual here. Note that this extreme case corresponds to an unstable circular orbit, and so one might question whether this figure is physically relevant. We include it because we expect similar horizon distortions for very strong
field orbits of black holes with $a/M > \sqrt{3}/2$, and that such a horizon geometry will be produced transiently from the closest approach of eccentric orbits around black holes with $a/M \lesssim \sqrt{3}/2$. Both of these cases will be investigated more thoroughly in later works.

### 3.5 Lead or lag?

We showed in Sec. 3.2.5 that the orbital energy evolves due to horizon coupling according to $dE^H/dt \propto (\Omega_{\text{orb}} - \Omega_H)$. As discussed in the Introduction, it is simple to build an intuitive picture of this physics for Newtonian tides acting on a fluid body. In this limit, when $\Omega_{\text{orb}} > \Omega_H$, tides raise a bulge on the body which leads the orbit’s position. This bulge exerts a torque which transfers energy from the body’s spin to the orbit. When $\Omega_H < \Omega_{\text{orb}}$, the bulge lags the orbit, and the torque transfers energy from the orbit to the body’s spin. When $\Omega_H = \Omega_{\text{orb}}$, $dE^H/dt = 0$, and the Newtonian fluid expectation is that there should be no offset between the bulge and the orbit. The tidal bulge should point directly at the orbiting body, locking the body’s tide to the orbit.

When $\Omega_{\text{orb}} \gg \Omega_H$ (e.g., the Schwarzschild limit) and $\Omega_H \gg \Omega_{\text{orb}}$ (large radius orbits of Kerr black holes), this Newtonian fluid intuition is consistent with our results (modulo the switch of “lead” and “lag” thanks to the teleological nature of the event horizon). However, it is not so clear that this intuition holds up when $\Omega_{\text{orb}}$ and $\Omega_H$ are comparable in magnitude.

Let us investigate this systematically. Begin with the weak-field $l = m = 2$ offset angles in the null and instantaneous maps, Eqs. (3.186) and (3.190). Dropping terms of $O(u^8)$ and noting that $u^\phi = M\Omega_{\text{orb}} + O(qu^6)$, we solve for the conditions under which $\delta \psi_{22}^{\text{OB-NM}}$ and $\delta \psi_{22}^{\text{OB-IM}}$ are zero. In the null map, we find

$$\Omega_{\text{orb}} = \Omega_H + \frac{3M\Omega_H}{2r_o} + \frac{3\Delta \psi_o}{8M} \quad (3.203)$$

The bulge leads the orbit when the equals in the above equation is replaced by greater
3.5. LEAD OR LAG?

than, and lags when replaced by less than. In the instantaneous map,

\[ \Omega_{\text{orb}} = \frac{7}{4} \Omega_H + \frac{3\Delta \psi_o}{8M}, \]  \hspace{1cm} (3.204)

with the same replacements indicating lead or lag.

Neither of these conditions are consistent with \( \Omega_{\text{orb}} = \Omega_H \) indicating zero bulge-orbit offset. In both the null and instantaneous maps, we find \( \Omega_{\text{orb}} \ll \Omega_H \) when the bulge angle is zero. For example, for \( a = 0.3 \) (roughly the largest \( a \) for which the small spin expansion is trustworthy), Eq. (3.203) has a root at \( r_o = 35.9M \), for which \( M\Omega_{\text{orb}} = 0.00464, M\Omega_H = 0.0768 \). (A second root exists at \( r_o = 2.15M \), but this is inside the photon orbit.) Using the instantaneous map changes the numbers, but not the punchline: for \( a = 0.3M \), the root moves to \( r_o = 16.7M \), with \( M\Omega_{\text{orb}} = 0.0146 \). Changing the spin changes the numbers, but leaves the message the same: zero offset in these maps does not correspond to \( \Omega_{\text{orb}} = \Omega_H \).

Equations (3.203) and (3.204) were derived using a small spin expansion. Before drawing too firm a conclusion from this, let us examine the situation using numerical data good for large spin. In Fig. 3-8, we examine a sequence of "corotating" orbits — orbits for which \( \Omega_H = \Omega_{\text{orb}} \), so that \( dE^H/dt = 0 \). For very small spins, the orbit leads the bulge. As the black hole's spin increases, the lead becomes a lag. This lead gets smaller as the spin gets larger. Since the lag becomes a lead as the spin is changed from \( a = 0.1M \) to \( a = 0.2M \), there must be a spin value between \( a = 0.1M \) and \( a = 0.2M \) for which the lead angle is zero for the corotating orbit. Our data also suggest that the lead angle may approach zero as the spin gets very large. But this suggests that the horizon locks to the orbit for at most only two spin values, in this map — a set of measure zero. We do not find any systematic connection between the geometry and the horizon for these orbits.

Before concluding, let us examine the relative phase of the tidal field and the horizon's curvature, Eq. (3.197). Setting \( \delta \psi_{Tm}^{TB} = 0 \) yields

\[ \Omega_{\text{orb}} = \Omega_H \frac{(l + 2)(l - 1)}{l(l + 1)}. \]  \hspace{1cm} (3.205)
We again see $\Omega_{\text{orb}} \neq \Omega_{\text{H}}$ when the field and the response are aligned (although $\Omega_{\text{orb}} \rightarrow \Omega_{\text{H}}$ as $l$ gets very large).

The analytical expansions and numerical data indicate that the Newtonian fluid intuition for the geometry of tidal coupling simply does not work well for strong-field black hole binaries, even accounting for the teleological swap of "lag" and "lead." Only in the extremes can we make statements with confidence: when $\Omega_{\text{H}} \gg \Omega_{\text{orb}}$, the tidal bulge will lag the orbit; when $\Omega_{\text{orb}} \gg \Omega_{\text{H}}$, the bulge will lead the orbit. But when $\Omega_{\text{orb}}$ and $\Omega_{\text{H}}$ are of similar magnitude, we cannot make a clean prediction.

The tidal bulge is not locked to the orbit when $dE^H/dt = 0$, at least using any scheme to define the lead/lag angle that we have examined.

### 3.6 Conclusions

In this chapter, we have presented a formalism for computing tidal distortions of Kerr black holes. Using black hole perturbation theory, our approach is good for fast motion, strong field orbits, and can be applied to a black hole of any spin parameter.

We have also developed tools for visualizing the distorted horizon by embedding its 2-dimensional surface in a 3-dimensional space. For now, our embeddings are only good for Kerr spin parameter $a/M \leq \sqrt{3}/2$, the highest value for which the entire horizon can be embedded in a globally Euclidean space. Higher spins require a piecewise embedding of an equatorial "belt" in a Euclidean space, and a region near the "poles" in a Lorentzian space. We will present tools for embedding spins $\sqrt{3}/2 < a/M \leq 1$ in later works.

Although our formalism is good for arbitrary bound orbits, we have focused on circular and equatorial orbits for this first analysis. This allowed us to validate this formalism against existing results in the literature, and to explore whether there is a simple connection between the tidal coupling of the hole to the orbit, and the relative geometry of the orbit and the horizon's tidal bulge. We find that there is no such simple connection in general. Perhaps not surprisingly, strong-field black hole systems are more complicated than Newtonian fluid bodies.
We plan two followup analyses to extend the work we have done here. First, as mentioned above, we will extend the work on embedding horizons to $a/M > \sqrt{3}/2$, the domain for which we cannot use a globally Euclidean embedding. Though straightforward, this calculation is rather lengthy and not especially enlightening. As such, we left it out of this initial presentation of this formalism. Second, we examine tidal distortions from generic — inclined and eccentric — Kerr orbits in chapter 4. The circular equatorial orbits we have studied in this chapter are stationary, as are the tidal fields and tidal responses that arise from them. If one examines the system and the horizon’s response in a frame that corotates with the orbit, the tide and the horizon will appear static. This will not be the case for generic orbits. Even when viewed in a frame that rotates at the orbit’s mean $\phi$ frequency, the orbit will be dynamical, and so the horizon’s response will likewise be dynamical. Similar analyses for Schwarzschild have already been presented by Vega, Poisson, and Massey [128]; it will be interesting to compare with the more complicated and less symmetric Kerr case.

An extension of our analysis may be useful for improving initial data for numerical relativity simulations of merging binary black holes. One source of error in such simulations is that the black holes typically have the wrong initial geometry — unless the binary is extremely widely separated, we expect each hole to be distorted by their companion’s tides. Accounting for this in the initial data requires matching the near-horizon geometry to the binary’s spacetime metric. Such matchings have been done for the case of inspiraling Schwarzschild black holes [146, 147]. With some effort (in order to get the geometry in a region near the horizon, not just on the horizon), we believe it should be possible to use this work as a foundation for extending the matching procedure to the important case of inspiraling Kerr black holes.

Appendix 3.A  Details of computing $\bar{\delta}$

In this appendix, we present details regarding the operator $\bar{\delta}$ in the form that we need it for our analysis.
3.A.1 The Newman-Penrose tetrad legs

A useful starting point is to write out the Newman-Penrose tetrad legs $l$, $n$, and $m$. In much of the literature on black hole perturbation theory, we use the Kinnersley form of these tetrad legs in Boyer-Lindquist coordinates:

\begin{align}
(l^\mu)^{_{\text{BL}}} & \doteq \frac{1}{\Delta} \left[(r^2 + a^2), \Delta, 0, a\right], \\
(n^\mu)^{_{\text{BL}}} & \doteq \frac{1}{2\Sigma} \left[(r^2 + a^2), -\Delta, 0, a\right], \\
(m^\mu)^{_{\text{BL}}} & \doteq \frac{1}{\sqrt{2}(r + ia\cos \theta)} \left[ia \sin \theta, 0, 1, i\csc \theta\right];
\end{align}

\begin{align}
(l^\mu)^{_{\text{BL}}} & \doteq [-1, \Sigma/\Delta, 0, a \sin^2 \theta], \\
(n^\mu)^{_{\text{BL}}} & \doteq \frac{1}{2\Sigma} \left[-\Delta, -\Sigma, 0, a\Delta \sin^2 \theta\right], \\
(m^\mu)^{_{\text{BL}}} & \doteq \frac{1}{\sqrt{2}(r + ia\cos \theta)} \left[-ia \sin \theta, 0, \Sigma, i(r^2 + a^2) \sin \theta\right].
\end{align}

The components of the fourth leg, $\mathbf{m}$, are related to the components of $\mathbf{m}$ by complex conjugation. The notation $\left[b^\mu\right]^{_{\text{BL}}} \doteq (b^t, b^r, b^\theta, b^\phi)$ means “the components of the 4-vector $\mathbf{b}$ in Boyer-Lindquist coordinates are represented by the array on the right-hand side,” and similarly for the 1-form components $\left(b^\mu\right)^{_{\text{BL}}}$.

Because our analysis focuses on the Kerr black hole event horizon, we will find it useful to transform to Kerr ingoing coordinates $(v, r', \theta, \psi)$. Using Eqs. (3.5) – (3.6), we transform tetrad components between the two coordinate systems with the matrix elements

\begin{align}
\frac{\partial v}{\partial t} = 1, & \quad \frac{\partial v}{\partial r} = \frac{r^2 + a^2}{\Delta}, & \quad \frac{\partial \psi}{\partial r} = \frac{a}{\Delta}, & \quad \frac{\partial \psi}{\partial \phi} = 1, & \quad \frac{\partial r'}{\partial r} = 1.
\end{align}

All elements which could connect $(t, r, \phi)$ and $(v, r', \psi)$ which are not explicitly listed here are zero. (The angle $\theta$ is the same in the two coordinate systems.) The matrix
elements for the inverse transformation are

\[
\frac{\partial t}{\partial v} = 1, \quad \frac{\partial t}{\partial r'} = -\frac{(r^2 + a^2)}{\Delta}, \quad \frac{\partial \Psi}{\partial r'} = -\frac{a}{\Delta}, \quad \frac{\partial \phi}{\partial \psi} = 1, \quad \frac{\partial r}{\partial r'} = 1.
\]

(3.213)

As noted in the Introduction, \(r\) and \(r'\) are identical; we just maintain a notational distinction for clarity while transforming between these two different coordinate systems.

With these, it is a simple matter to transform the tetrad components to their form in Kerr ingoing coordinates:

\[
(l^\mu)_{\text{IN}} \doteq \frac{1}{\Delta} \left[ 2(r^2 + a^2), \Delta, 0, 2a \right],
\]

(3.214)

\[
(n^\mu)_{\text{IN}} \doteq \frac{1}{2\Sigma} \left[ 0, -\Delta, 0, 0 \right],
\]

(3.215)

\[
(m^\mu)_{\text{IN}} \doteq \frac{1}{\sqrt{2(r + ia\cos \theta)}} \left[ ia \sin \theta, 0, 1, i \csc \theta \right],
\]

(3.216)

\[
(l_\mu)_{\text{IN}} \doteq \left[ -1, 2\Sigma/\Delta, 0, a \sin^2 \theta \right],
\]

(3.217)

\[
(n_\mu)_{\text{IN}} \doteq \frac{1}{2\Sigma} \left[ -\Delta, 0, 0, a\Delta \sin^2 \theta \right],
\]

(3.218)

\[
(m_\mu)_{\text{IN}} \doteq \frac{1}{\sqrt{2(r + ia\cos \theta)}} \left[ -ia \sin \theta, 0, \Sigma, i(r^2 + a^2) \sin \theta \right].
\]

(3.219)

The notation \((b^\mu)_{\text{IN}} \doteq (b^v, b^{r'}, b^\theta, b^{\phi'})\) means “the components of the 4-vector \(\vec{b}\) in Kerr ingoing coordinates are represented by the array on the right-hand side,” and similarly for the 1-form components \((b_\mu)_{\text{IN}}\).

Changing coordinates is not enough to fix various pathologies associated with the behavior of quantities on the event horizon. To ensure that quantities we examine are well behaved there, we next change to the Hawking-Hartle tetrad, which is related to
Kinnersley's tetrad as follows:

\[
\begin{align*}
1_{HH} &= \frac{\Delta}{2(r^2 + a^2)} \mathbf{l}, \\
n_{HH} &= \frac{2(r^2 + a^2)}{\Delta} \mathbf{n}, \\
m_{HH} &= \mathbf{m} - \frac{ia \sin \theta}{\sqrt{2(r + ia \cos \theta)}} \mathbf{l}.
\end{align*}
\]  

(3.220)  
(3.221)  
(3.222)

With this, we finally obtain the tetrad elements that we need for this analysis:

\[
\begin{align*}
(l^\mu)_{HH, IN} &= \frac{1}{(r^2 + a^2)} \left[ r^2 + a^2, \frac{\Delta}{2}, 0, a \right], \\
n(\nu)_{HH, IN} &= \left[ 0, -(r^2 + a^2)/\Sigma, 0, 0 \right], \\
m(\nu)_{HH, IN} &= \frac{1}{\sqrt{2(r + ia \cos \theta)}} \left[ 0, -ia \Delta \sin \theta, 1, i \csc \theta - \frac{ia^2 \sin \theta}{r^2 + a^2} \right];
\end{align*}
\]  

(3.223)  
(3.224)  
(3.225)

\[
\begin{align*}
(l_\mu)_{HH, IN} &= \frac{1}{2(r^2 + a^2)} \left[ -\Delta, 2\Sigma, 0, a \Delta \sin^2 \theta \right], \\
n(\nu)_H H, IN &= \frac{(r^2 + a^2)}{\Sigma} \left[ -1, 0, 0, a \sin^2 \theta \right], \\
m(\nu)_{HH, IN} &= \frac{1}{\sqrt{2(r + ia \cos \theta)}} \times \frac{1}{2(r^2 + a^2)} \times \\
&\left[ -ia(2r^2 + 2a^2 - \Delta) \sin \theta, -2ia \Sigma \sin \theta, 2(r^2 + a^2) \Sigma, \\
&\ i \left( 2(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \right) \sin \theta \right].
\end{align*}
\]  

(3.226)  
(3.227)  
(3.228)

In the remainder of this appendix, we will use the Hawking-Hartle components in ingoing coordinates, and will drop the "HH, IN" subscript.

### 3.A.2 Constructing $\bar{\delta}$

Here we derive the form of the operator $\bar{\delta}$, acting at the radius of the Kerr event horizon, $r = r_+$. Following Hartle [27], $\bar{\delta}$ acting upon a quantity $\eta$ of spin-weight $s$ is given by

\[
\bar{\delta} \eta = \left[ \delta - s(\alpha - \bar{\beta}) \right] \eta.
\]  

(3.229)
The operator $\bar{\delta} = \bar{m}^\mu \partial_\mu$. Evaluating this at $r = r_+$ [using the fact that $\Delta = 0$ there, and that $a/(r_+^2 + a^2) = a/(2Mr_+) = \Omega_H$] we find

$$\bar{\delta} = \frac{1}{\sqrt{2(r_+ - ia \cos \theta)}} \left[ \partial_\theta - i(a \Omega_H \sin \theta) \partial_\psi \right]. \quad (3.230)$$

Next consider the Newman-Penrose spin coefficients $\alpha$ and $\beta$. They are given by

$$\alpha = \frac{1}{2} \bar{m}^\nu (n^\mu \nabla_\nu l_\mu - \bar{m}^\mu \nabla_\nu m_\mu), \quad (3.231)$$
$$\beta = \frac{1}{2} \bar{m}^\nu (n^\mu \nabla_\nu l_\mu - \bar{m}^\mu \nabla_\nu m_\mu). \quad (3.232)$$

This means that

$$\alpha - \bar{\beta} = \frac{1}{2} \bar{m}^\nu (m^\mu \nabla_\nu l_\mu - \bar{m}^\mu \nabla_\nu m_\mu). \quad (3.233)$$

Using ingoing coordinates, we find

$$\left. (\alpha - \bar{\beta}) \right|_{r=r_+} = \frac{(a^2 - 2Mr_+) \cot \theta + iar_+ \csc \theta}{\sqrt{2}r_+(r_+ - ia \cos \theta)^2}$$
$$= \frac{1}{\sqrt{2}(r_+ - ia \cos \theta)^2} \left[ \frac{(a^2 - 2Mr_+)}{r_+} \cot \theta + ia \csc \theta \right]. \quad (3.234)$$
Finally, we combine Eqs. (3.230) and (3.234) to build $\tilde{\theta}$. Assume that $\eta$ is a function of spin-weight $s$ with an axial dependence $e^{im\psi}$:

$$\tilde{\theta}\eta = \frac{1}{\sqrt{2}(r_+ - ia\cos\theta)} \left[ \partial_{\theta} - i(csc\theta - a\Omega_H \sin\theta)\partial_{\psi} \right]$$

$$- \frac{s}{(r_+ - ia\cos\theta)} \left[ \frac{(a^2 - 2Mr_+)}{r_+} \cot\theta + ia\csc\theta \right] \eta$$

$$= \frac{1}{\sqrt{2}(r_+ - ia\cos\theta)} \left[ \partial_{\theta} + s\cot\theta + m\csc\theta - a\Omega_H \sin\theta \right]$$

$$- s\cot\theta - \frac{s}{(r_+ - ia\cos\theta)} \left[ \frac{(a^2 - 2Mr_+)}{r_+} \cot\theta + ia\csc\theta \right] \eta$$

$$= \frac{1}{\sqrt{2}(r_+ - ia\cos\theta)} \left[ L^s_{-} - am\Omega_H \sin\theta \right]$$

$$- \frac{s}{(r_+ - ia\cos\theta)} \left[ \frac{(a^2 + r_+^2 - 2Mr_+ - iar_+\cos\theta)}{r_+} \cot\theta + ia\csc\theta \right] \eta$$

$$= \frac{1}{\sqrt{2}(r_+ - ia\cos\theta)} \left[ L^s_{-} - am\Omega_H \sin\theta - \frac{s}{(r_+ - ia\cos\theta)} (ia\csc\theta - ia\cos\theta \cot\theta) \right] \eta$$

$$= \frac{1}{\sqrt{2}(r_+ - ia\cos\theta)} \left[ L^s_{-} - am\Omega_H \sin\theta - \frac{i\alpha \sin\theta}{(r_+ - ia\cos\theta)} \right] \eta$$

$$= \frac{1}{\sqrt{2}r_+} \left( 1 - \frac{ia\cos\theta}{r_+} \right)^{s-1} \left[ L^s_{-} - am\Omega_H \sin\theta \right] \left( 1 - \frac{ia\cos\theta}{r_+} \right)^{-s} \eta$$

(3.235)

In going from line 1 to line 2 of Eq. (3.235), we used the fact that $\eta \propto e^{im\psi}$; we also added and subtracted $s\cot\theta$ inside the square brackets. In going from line 2 to line 3, we recognized that the first three terms inside the brackets are just the operator $L^s_{-}$; cf. Eq. (3.64). We also moved the negative $s\cot\theta$ term inside the second set of square brackets. In going from line 3 to line 4, we used the fact that $r_+^2 + a^2 = 2Mr_+$. We then used $\csc\theta - \cot\theta \cos\theta = \sin\theta$ to go from line 4 to line 5, and finally Eq. (3.69) to obtain our final form for this operator. This last line is identical to Eq. (3.70).

**Appendix 3.B  Visualizing a distorted horizon**

Following Hartle [26, 27], we visualize distorted horizons by embedding the two-surface of the horizon on a constant time surface in a flat three-dimensional space. The embedding is a surface $r_E(\theta, \psi)$ that has the same Ricci scalar curvature as
the distorted horizon. For unperturbed Schwarzschild black holes, \( r_E = 2M \); for an unperturbed Kerr hole, \( r_E \) is a more complicated function that varies with \( \theta \). In the general case, we write

\[
 r_E(\theta, \psi) = r_E^{(0)}(\theta) + r_E^{(1)}(\theta, \psi) .
\]  

(3.236)

In this chapter, we focus on cases where the entire horizon can be embedded in a Euclidean space, which means that we require \( a/M \leq \sqrt{3}/2 \). (We briefly discuss considerations for \( a/M > \sqrt{3}/2 \) at the end of this appendix.) To generate the embedding, we define Cartesian coordinates on the horizon as usual:

\[
 X(\theta, \psi) = r_E(\theta, \psi) \sin \theta \cos \psi ,
\]  

(3.237)

\[
 Y(\theta, \psi) = r_E(\theta, \psi) \sin \theta \sin \psi ,
\]  

(3.238)

\[
 Z(\theta, \psi) = r_E(\theta, \psi) \cos \theta .
\]  

(3.239)

We compute the line element

\[
 ds^2 = dX^2 + dY^2 + dZ^2
\]

\[
 \equiv g^E_{\theta \theta} d\theta^2 + 2g^E_{\theta \psi} d\theta d\psi + g^E_{\psi \psi} d\psi^2 ,
\]  

(3.240)

and then the Ricci scalar corresponding to the embedding metric \( g_{\alpha \beta}^E \) to linear order in \( r_E^{(1)} \). We require this to equal the scalar curvature computed using Eq. (3.39), and then read off the distortion \( r_E^{(1)}(\theta, \psi) \).

### 3.B.1 Schwarzschild

Thanks to the spherical symmetry of the undistorted Schwarzschild black hole, results for this limit are quite simple. The metric on an embedded surface of radius

\[
 r_E = 2M + r_E^{(1)}(\theta, \phi)
\]  

(3.241)
is given by

\[ ds^2 = (2M)^2 \left[ 1 + \frac{r_E^{(1)}(\theta, \phi)}{M} \right] (d\theta^2 + \sin^2 \theta \, d\phi^2) \]  
(3.242)

(Recall that \( \psi = \phi \) for \( a = 0 \).) It is a straightforward exercise to compute the scalar curvature associated with the metric (3.242); we find

\[ R_E = \frac{1}{2M^2} - \left[ 2 + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{m^2}{\sin^2 \theta} \right] \frac{r_E^{(1)}}{4M^3} \]  
(3.243)

Let us expand \( r_E^{(1)} \) in spherical harmonics:

\[ r_E^{(1)}(\theta, \phi) = 2M \sum_{lm} \varepsilon_{lm} \rho Y_{lm}(\theta)e^{im\phi} \]  
(3.244)

Using this, Eq. (3.243) simplifies further:

\[ R_E = \frac{1}{2M^2} \left[ 1 + \sum_{lm} \varepsilon_{lm} (l + 2)(l - 1) \rho Y_{lm}(\theta)e^{im\phi} \right] \]  
(3.245)

The scalar curvature we compute using black hole perturbation theory takes the form

\[ R_H = R_H^{(0)} + \sum_{lmkn} R_H^{(1),lmkn} \]  
(3.246)

where \( R_H^{(0)} = 1/2M^2 \). Equating this to \( R_E \), we find

\[ \varepsilon_{lm} \rho Y_{lm}(\theta)e^{im\phi} = \sum_{kn} \frac{2M^2 R_H^{(1),lmkn}}{(l + 2)(l - 1)} \]  
(3.247)

or

\[ r_E^{(1)}(\theta, \phi) = \sum_{lmkn} \frac{4M^3 R_H^{(1),lmkn}}{(l + 2)(l - 1)} \]  
(3.248)

Equation (3.248) is identical (modulo a slight change in notation) to the embedding found in Ref. [128]; compare their Eqs. (4.33) and (4.34). We use \( r_E^{(1)}(\theta, \phi) \) to visualize distorted Schwarzschild black holes in Sec. 3.3.2 (dropping the indices \( k \) and \( n \) since we only present results for circular, equatorial orbits in this chapter).
3.B.2 Kerr

Embedding a distorted Kerr black hole is rather more complicated. Indeed, embedding an undistorted Kerr black hole is not trivial: as discussed in Sec. 3.2.1, the scalar curvature $R_h$ of an undistorted Kerr black hole changes sign near the poles for spins $a/M > \sqrt{3}/2$. A hole with this spin cannot be embedded in a global Euclidean space, and one must instead use a Lorentzian embedding near the poles \[131\]. We briefly describe how to embed a tidally distorted black hole with $a/M > \sqrt{3}/2$ at the end of this appendix, but defer all details to future work. For now, we focus on the comparatively simple case $a/M \leq \sqrt{3}/2$.

Undistorted Kerr

We begin by reviewing embeddings of the undistorted case. Working in ingoing coordinates, the metric on the horizon is given by

$$ds^2 = g_{xx} \, dx^2 + g_{\psi\psi} \, d\psi^2,$$

with

$$g_{xx} = \frac{r_+^2 + a^2 x^2}{1 - x^2}, \quad g_{\psi\psi} = \frac{4M^2 r_+^2 (1 - x^2)}{r_+^2 + a^2 x}.$$

Equation (3.249) is the metric on a spheroid of radius

$$r^{(0)}(x) = \sqrt{r_\perp(x)^2 + Z(x)^2},$$

where

$$r_\perp(x) = \eta \sqrt{f(x)},$$

$$Z(x) = \eta \int_0^x \sqrt{\frac{4 - (df/dx)^2}{4f(x')}} \, dx'.$$
with

\[ f(x) = \frac{1 - x^2}{1 - \beta^2(1 - x^2)} , \tag{3.253} \]
\[ \eta = \sqrt{r_+^2 + a^2} , \tag{3.254} \]
\[ \beta = a/\eta . \tag{3.255} \]

Using Eqs. (3.253) – (3.255), we can rewrite \( Z(x) \) as

\[ Z(x) = \int_0^x \frac{H(x')}{[r_+^2 + a^2(x')^2]^{3/2}} dx' , \tag{3.256} \]

where

\[ H(x) = \left[ r_+^8 - 6a^4 r_+^4 x^2 - 4a^6 r_+^2 x^2 (1 + x^2) \right. \]
\[ \left. - a^8 x^4 (1 + x^2 + x^4) \right]^{1/2} . \tag{3.257} \]

For \( a/M > \sqrt{3}/2 \), \( H(x) = 0 \) at some value \( |x| = x_L \). This means that \( H(x) \) is imaginary for \( |x| > x_L \) for this spin; \( Z \) is imaginary over this range as well. This defines the region over which the horizon must be embedded in a Lorentzian space rather than a Euclidean one. For all \( a \), the equator \( (x = 0) \) is a circle of radius \( 2M \).

The scalar curvature associated with this metric is

\[ R_E^{(0)} = R_H^{(0)} = \frac{2}{r_+^2} \frac{(1 + a^2/r_+^2)(1 - 3a^2x^2/r_+^2)}{(1 + a^2x^2/r_+^2)^3} . \tag{3.258} \]

**Distorted Kerr:** \( a/M \leq \sqrt{3}/2 \)

For this calculation, it will be convenient to use Dirac notation to describe the dependence on \( x \). We write the spin-weighted spherical harmonics as a ket,

\[ sY_{lm}(x) \rightarrow |slm \rangle , \tag{3.259} \]
and define the inner product

\[ \langle skm|f(x)|slm \rangle = 2\pi \int_{-1}^{1} Y_{km}(x)f(x)Y_{lm}(x)dx. \] (3.260)

These harmonics are normalized so that

\[ \delta_{kl}\delta_{nm} = \int_{0}^{2\pi} d\psi \int_{-1}^{1} dx \ Y_{kn}(x)Y_{lm}(x)e^{i(m-n)\psi} \]
\[ = \delta_{nm}\langle skn|slm \rangle. \] (3.261)

The $2\pi$ prefactor in Eq. (3.260) means that

\[ \langle skm|slm \rangle = \delta_{kl}. \] (3.262)

Using this notation, let us now consider the curvature of a tidally distorted Kerr black hole. Begin with the curvature from black hole perturbation theory, Eq. (3.39). Translating into Dirac notation, we have

\[ |R_{H}^{(1)}| = \text{Im} \sum_{lmkn} [C_{lmkn}^{H} e_{lmkn}^{\Phi_{mkn}} |\bar{\otimes}S_{lmkn}^{+} \rangle], \] (3.263)

where $|\bar{\otimes}S_{lmkn}^{+} \rangle$ is given by Eq. (3.78).

We now must assume a functional form for the embedding surface. A key issue is what basis functions we should use to describe the angular dependence of this surface. The basis functions for the angular sector, $\bar{\otimes}S_{lmkn}^{+}$, depend on mode frequency, and so are not useful for describing the embedding surface. Since spherical harmonics are complete functions on the sphere, we use

\[ r_{E}(x, \psi) = r_{E}^{0}(x) + r_{E}^{(1)}(x, \psi), \] (3.264)

where

\[ r_{E}^{(1)}(x, \psi) = r_{+} \sum_{\ell m} \varepsilon_{\ell m} Y_{\ell m}(x)e^{i\psi}, \] (3.265)

and where $r_{E}^{0}(\theta)$ is given by Eq. (3.250). Note that the index $\ell$ used in Eq. (3.265)
is not the same as the index $l$ used in Eq. (3.263). It is important to maintain a distinction between the indices that are used on the spheroidal and the spherical harmonics.

Using Eqs. (3.264) and (3.265), we find that the embedding surface yields a metric on the horizon given by

$$ds^2 = (g_{xx} + h_{xx})dx^2 + 2h_{x\psi} dx d\psi + (g_{\psi\psi} + h_{\psi\psi})d\psi^2,$$  

(3.266)

with $g_{xx}$ and $g_{\psi\psi}$ given by Eq. (3.249), and

$$h_{xx} = \frac{2}{(r_+^2 + a^2 x^2)^{3/2}} \left[ \left( H + \frac{4M^2 r_+^2 x^2}{1 - x^2} \right) r^{(1)} + \left( H - 4M^2 r_+^2 \right) \frac{\partial r^{(1)}}{\partial x} \right],$$

(3.267)

$$h_{x\psi} = \frac{(H - 4M^2 r_+^2) x \partial r^{(1)}}{(r_+^2 + a^2 x^2)^{3/2} \partial \psi},$$

(3.268)

$$h_{\psi\psi} = \frac{4M r_+ (1 - x^2)}{(r_+^2 + a^2 x^2)^{3/2}} r^{(1)}.$$  

(3.269)

The function $H = H(x)$ was introduced in the embedding of the undistorted Kerr hole, Eq. (3.257). By restricting ourselves to $a/M \leq \sqrt{3}/2$, we are guaranteed that $H$ is real for this analysis.

Computing the embedding curvature from this metric, we find

$$R_E = R_E^{(0)} + R_E^{(1)},$$  

(3.270)

where $R_E^{(0)}$ is given by Eq. (3.258), and

$$R_E^{(1)} \longrightarrow |R_E^{(1)}| = \sum_{\ell m} \varepsilon_{\ell m} \left[ C(x) |0\ell m\rangle + D(x) \frac{d}{dx} |0\ell m\rangle \right] e^{i\ell \psi}.$$  

(3.271)
The functions $C(x)$ and $D(x)$ are given by

$$C(x) = \frac{1}{2HM^2 r_+(r_+^2 + a^2x^2)^{11/2}} \left[ \sum_{j=0}^{8} c_{0,j} a^{2j} + \sum_{j=0}^{5} c_{1,j} a^{2j} \right] ,$$  

$$D(x) = \frac{1}{HM(r_+^2 + a^2x^2)^{11/2}} \sum_{j=0}^{7} d_j a^{2j} .$$

The coefficients introduced in Eqs. (3.272) and (3.273) are

$$c_{0,0} = 2r_+^5 \left\{ \ell(\ell + 1)HM \left[ r_+^6 + 4M^2(H - 4M^2 r_+^2)x^2 \right] - r_+^{11} \right. 
+ 2r_+^5 (r_+^6 - 4HM^2)x^2 \} ,$$  

$$c_{0,1} = 8\ell(\ell + 1)H^2 M^3 r_+^3 x^4 - r_+^{14} (6 - 23x^2 + 9x^4) 
- 2HM r_+^5 \left\{ 16\ell(\ell + 1)M^4 x^4 + 6Mr_+^3 x^2 (4 - 3x^2) 
+ r_+^4 [1 - (4 + 5\ell(\ell + 1))x^2] \right\} ,$$  

$$c_{0,2} = r_+^6 \left\{ 4 [6 + 5\ell(\ell + 1)]HM r_+ x^4 - 12HM^2 x^2 (4 - 9x^2) 
- r_+^6 (6 - 63x^2 + 57x^4) \right\} ,$$  

$$c_{0,3} = r_+^4 \left\{ 4HM r_+ x^4 \left[ 3 + [6 + 5\ell(\ell + 1)]x^2 \right] - 4HM^2 x^2 (4 - 27x^2) 
- r_+^6 (2 - 103x^2 + 181x^4 - 24x^6) \right\} ,$$  

$$c_{0,4} = r_+^2 x^2 \left\{ 36HM^2 x^2 + 2HM r_+ x^4 \left[ 8 + [4 + 5\ell(\ell + 1)]x^2 \right] 
+ r_+^6 (104 - 332x^2 + 67x^4 + 21x^6) \right\} ,$$  

$$c_{0,5} = r_+ x^2 \left\{ 2HM x^6 \left[ 3 + \ell(\ell + 1)x^2 \right] 
+ r_+^3 (63 - 355x^2 + 56x^4 + 62x^6 + 6x^8) \right\} ,$$  

$$c_{0,6} = r_+^4 x^2 (21 - 217x^2 + 6x^4 + 60x^6 + 18x^8) ,$$  

$$c_{0,7} = r_+^2 x^2 (3 - 71x^2 - 8x^4 + 18x^6 + 18x^8) ,$$  

$$c_{0,8} = -x^4 (10 + x^2 + x^4 - 6x^6) ;$$
CHAPTER 3. STRONG-FIELD TIDAL DISTORTIONS OF ROTATING BLACK HOLES:
FORMALISM AND RESULTS FOR CIRCULAR, EQUATORIAL ORBITS

\[ c_{1,0} = 2m^2Mr_+^3(2HMr_+^7 - Hr_+^8 - 4H^2M^2r_+^2x^2 + 16HM^4r_+^4x^2 \]
\[ -2HMr_+^7x^2 + 8M^3r_+^9x^2) , \] (3.283)

\[ c_{1,1} = -2Hm^2Mr_+^3r_+^2x^2 [5r_+^6 + 4M^2(H - 4M^2r_+^2)x^2] , \] (3.284)

\[ c_{1,2} = -4m^2Mr_+^4x^4 [H(6M + 5r_+) - 6M(H - 4M^2r_+^2)x^2] , \] (3.285)

\[ c_{1,3} = -4m^2Mr_+^4x^6 [H(8M + 5r_+) - 8M(H - 4M^2r_+^2)x^2] , \] (3.286)

\[ c_{1,4} = -2m^2Mr_+^2x^8 [H(6M + 5r_+) - 6M(H - 4M^2r_+^2)x^2] , \] (3.287)

\[ c_{1,5} = -2Hm^2Mr_+x^{10} ; \] (3.288)

\[ d_0 = 2r_+^8x \left\{ -2r_+^6(1 - x^2) + HM \left[ r_+ + M(6 - 8x^2) \right] \right\} , \] (3.289)

\[ d_1 = r_+^6x \left\{ -r_+^6(8 - 17x^2 + 9x^4) \right. \]
\[ +4HM \left[ 6M - 2(9M - r_+^2)x^2 + 9Mx^4 \right] \} , \] (3.290)

\[ d_2 = -4r_+^4x \left\{ r_+^6(1 - 13x^2 + 12x^4) \right. \]
\[ -3HM \left[ M - 8Mx^2 + (6M + r_+^2)x^4 \right] \} , \] (3.291)

\[ d_3 = r_+^2x^3 \left\{ r_+^6(89 - 113x^2 + 24x^4) + 4HM \left[ 2r_+^4x^4 - M(10 - 9x^2) \right] \right\} \} (3.292)

\[ d_4 = 2HMr_+x^9 + r_+^6x^3(75 - 149x^2 + 53x^4 + 21x^6) , \] (3.293)

\[ d_5 = r_+^4x^3(30 - 114x^2 + 35x^4 + 43x^6 + 6x^8) , \] (3.294)

\[ d_6 = r_+^2x^3(5 - 47x^2 + 7x^4 + 23x^6 + 12x^8) , \] (3.295)

\[ d_7 = -x^5(8 - x^2 - x^4 - 6x^6) . \] (3.296)

The term in \( C(x) \) that is proportional to \( 1/(1 - x^2) \) is written so that \( C(x) \) is well
behaved in the limit \( x \to \pm 1 \):

\[ \lim_{x \to \pm 1} \sum_{j=0}^{5} \frac{c_{1,j}a^{2j}}{1 - x^2} = 128a^2m^2M^6 \left[ 64M^7r_+ - 16a^2M^5(3r_+ + 2M) - 8a^4M^3(r_+ - 2M) \right. \]
\[ +a^6M(5r_+ + 6M) - a^8 \] . (3.297)

This ensures that this function is well-behaved in all of our numerical applications.
For small spin, the functions $C$ and $D$ become

\[
C(x) = \frac{(\ell + 2)(\ell - 1) + \ell(\ell + 1)(5x^2 - 2) + 2(18x^2 + m^2 - 7)}{2M^2} \left( \frac{a}{M} \right)^2 - \frac{498x^4 - 764x^2 + 168 - 4m^2(5x^2 + 6) - \ell(\ell + 1)(51x^4 - 100x^2 + 16)}{256M^2}
\times \left( \frac{a}{M} \right)^4 + \ldots \tag{3.298}
\]

\[
D(x) = \frac{5x(1 - x^2)}{8M^2} \left( \frac{a}{M} \right)^2 + \frac{x(1 - x^2)(58 - 67x^2)}{64M^2} \left( \frac{a}{M} \right)^4 + \ldots. \tag{3.299}
\]

The two expressions for the deformed hole's curvature, Eqs. (3.263) and (3.271), must equal one another. These expressions both vary with $\psi$ as $e^{im\psi}$, so we can examine them for each value of $m$. To facilitate this comparison, break up the phase function $\Phi_{mkn}(v, \psi)$ [Eq. (3.30)] as

\[
\Phi_{mkn}(v, \psi) = \Phi_m(v, \psi) - (k\Omega_\theta + n\Omega_r)v. \tag{3.300}
\]

We have defined

\[
\Phi_m(v, \psi) = m[\psi - \Omega_\phi v - K(a)]. \tag{3.301}
\]

Recall $K(a)$ is defined in Eq. (4.27).

With all of this in hand, we can now compare our two expressions for the curvature for each $m$:

\[
|R_{H,m}^{(1)}| = \text{Im} \left[ e^{i\Phi_m} \sum_{lkn} C_{lmkn} \bar{Z}_{lmkn}^H e^{-i(k\Omega_\theta + n\Omega_r)v} \right], \tag{3.302}
\]

\[
|R_{E,m}^{(1)}| = \sum_{\ell} \varepsilon_{\ell m} \left[ C(x)|0\ell m\rangle + D(x) \frac{d}{dx}|0\ell m\rangle \right] e^{im\psi}. \tag{3.303}
\]

Left multiply both of these expressions by $\langle 0qm |$. Define the vector $\tilde{mR}$ as the object whose components are

\[
_{mR_q} = \sum_{lkn} C_{lmkn} \bar{Z}_{lmkn}^H e^{-i(k\Omega_\theta + n\Omega_r)v} \langle 0qm | \bar{\delta} \delta S_{lmkn}^+. \tag{3.304}
\]
Likewise define the matrix $mD$ as the object whose components are

$$
m_{Dq\ell} = \langle 0qm|C(x)|0\ell m \rangle + \langle 0qm|D(x)\frac{d}{dx}|0\ell m \rangle. \tag{3.305}
$$

Notice that $C(x)$ is an even function of $x$, and $D(x)$ is odd. Because of this, the only non-zero elements of $mD$ are those for which $q$ and $\ell$ are either both even or both odd.

Finally, define $m\vec{e}$ as the vector whose components are $\varepsilon_{\ell m}$. Requiring that

$$
\langle 0qm|R_{H,m}^{(1)} \rangle = \langle 0qm|R_{E,m}^{(1)} \rangle
$$

yields the matrix equation

$$
\text{Im} \left[ e^{i\Phi_m} m\vec{R} \right] = e^{im\psi} mD \cdot m\vec{e}. \tag{3.306}
$$

To solve this, first consider the similar equation

$$
m\vec{R} = mD \cdot m\vec{e}_c. \tag{3.307}
$$

The formal solution of this is

$$
m\vec{e}_c = mD^{-1} \cdot m\vec{R}. \tag{3.308}
$$

Comparing Eqs. (3.306) and (3.307), we see that

$$
m\vec{e} = \text{Im} \left[ e^{i\Phi_m} m\vec{e}_c \right] e^{-im\psi}. \tag{3.309}
$$

With the components $\varepsilon_{\ell m}$ in hand, we assemble $r_{E}^{(1)}(x, \psi)$ using Eq. (3.265).

At least for the circular, equatorial orbits we have studied so far, we find that both the vector $mR_q$ and the matrix $mD_{q\ell}$ converge quickly. Consider first convergence of the terms which contribute to $mR_q$. Strictly speaking, the sum over $\ell$ in Eq. (3.304) goes to infinity. We find that this sum is dominated by the term with $q = \ell$; other terms are reduced from this peak term by a factor $\sim \varepsilon^{q-\ell}$, with $\varepsilon$ ranging from 0 for Schwarzschild (only terms with $q = \ell$ are non-zero in that case), to about 0.1 – 0.2 for orbits near the ISCO for spin $a/M = \sqrt{3}/2$. We have found that taking the sum to $l_{\text{max}} = 15$ is sufficient to ensure fractional accuracy of about $10^{-9}$ or better in the
components $mR_q$ for small spins ($a \lesssim 0.4M$) for all the orbits we have considered; we take the sums to $l_{\text{max}} = 20$ or $l_{\text{max}} = 25$ to achieve this accuracy for small radius orbits at spins $a/M = 0.7$ and $\sqrt{3}/2$.

Next consider the components of $m\tilde{R}$ and $mD$ themselves. Formally, we should treat both $m\tilde{R}$ and $mD$ as infinite dimensional objects. However, their contributions to the tidal distortion falls off quite rapidly as $q$ and $\ell$ become large. We find that $mR_q$ is dominated by the $q \equiv q_{\text{peak}} = \max(2, |m|)$ component. Components beyond this peak fall off as $\epsilon^{|q-q_{\text{peak}}|}$, with $\epsilon \sim 0.1$ across a wide range of spins. The matrix components $mD_{q\ell}$ are dominated by those with $q = \ell$, but fall off with a similar power law form as we move away from the diagonal. We have found empirically that our results are accurate to about $10^{-9}$ including terms out to $q = \ell = 15$ for small spin, but need to go as high as $q = \ell = 25$ for large spin, strong-field orbits.

**Distorted Kerr: Considerations for $a/M > \sqrt{3}/2$**

The techniques described above do not work when $a/M > \sqrt{3}/2$. For these spins, $H(x) = 0$ at $|x| = x_L$, and is imaginary for $|x| > x_L$. To handle this spin range, we must introduce separate embeddings to cover the domains $|x| \leq x_L$ and $|x| > x_L$. Special care must be taken at the boundaries $|x| = x_L$, since factors of $1/H$ in the embedding curvature $R^{(1)}_E$ introduce singularities there. The basis functions used to expand the embedding function $r^{(1)}_E(\theta, \psi)$ must be chosen so that these singularities are canceled out, leaving the embedding curvature smooth and well behaved.

Although straightforward to do in principle, getting the details of this analysis correct is quite involved, and does not add substantially to the core physics we wish to present (although it would add substantially to the already rather large number of long equations in this chapter). We defer a detailed analysis of horizon embeddings for $a/M > \sqrt{3}/2$ in a later paper.
Appendix 3.C  Spin-weighted spheroidal harmonics to linear order in $a/M$

In Sec. 3.4.1, we derive analytic results for the tidal distortion to leading order in $q \equiv a/M$, and to order $u^5$ (where $u \equiv \sqrt{M/r}$). As part of that analysis, we need analytic expressions for the spin-weighted spheroidal harmonics $+2S_{lm}$ to leading order in $q$. We also need the eigenvalue $\lambda$ for $s = -2$ to the same order. Here we derive the relevant results for arbitrary spin-weight $s$. Similar results for $s = -2$ can be found in Refs. [148, 149]; much of this approach is laid out (and intermediate steps provided) in Ref. [122].

The equation governing the spin-weighted spheroidal harmonics for spin-weight $s$ and black hole spin $a$ is

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS}{d\theta} \right) + \left[ \lambda - a^2 \omega^2 \sin^2 \theta \right.
+ 2a\omega (m - s \cos \theta) - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} + s \Big] S_{lm}(\theta) = 0. \tag{3.310}
\]

The parameter $\lambda$ appearing here is one form of the eigenvalue for this equation; another common form is $\mathcal{E} = \lambda + 2am\omega + a^2\omega^2 + s(s + 1)$. We write both $\lambda$ and the harmonic as expansions in $a\omega$:

\[
\lambda = \lambda_0 + (a\omega)\lambda_1, \tag{3.311}
\]
\[
_s S_{lm}(\theta) = _s Y_{lm}(\theta) + (a\omega)_s S_{lm}^1(\theta). \tag{3.312}
\]

This could be taken to higher order (for example, Ref. [148] does so to $O(a^2\omega^2)$ for $s = -2$), but linear order is enough for our purposes.

Begin by defining the operator

\[
\mathcal{L}_0 \equiv \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) + \left[ s - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} \right]. \tag{3.313}
\]
Equation (3.310) can then be decomposed order by order, becoming

\[(L_0 + \lambda_0) \, sY_{lm} = 0, \quad (3.314)\]

\[(L_0 + \lambda_0) \, sS_{lm}^1 = (2s \cos \theta - 2m - \lambda_1) \, sY_{lm}. \quad (3.315)\]

Equation (3.314) tells us that

\[\lambda_0 = (l - s)(l + s + 1). \quad (3.316)\]

Multiply Eq. (3.315) by \(2\pi_s Y_{lm} \sin \theta\) and integrate both sides with respect to \(\theta\) from 0 to \(\pi\). Integrating by parts, using Eqs. (3.262) and (3.314) and the fact that

\[2\pi \int_0^\pi \, sY_{lm}(\theta) \cos \theta \, sY_{lm}(\theta) \sin \theta \, d\theta = -\frac{sm}{l(l + 1)}, \quad (3.317)\]

we find

\[\lambda_1 = -2m \left[1 + \frac{s^2}{l(l + 1)}\right]. \quad (3.318)\]

To compute \(sS_{lm}^1\), put

\[sS_{lm}^1 = \sum_{l' = \min(|s|, |m|)}^{\infty} c_{lm}' \, sY_{l'm}. \quad (3.319)\]

Inserting this into Eq. (3.315), multiplying by \(2\pi_s Y_{l'm} \sin \theta\) and integrating, we find

\[c_{lm}' = \frac{4\pi s}{\lambda_0(l) - \lambda_0(l')} \int_0^\pi \, sY_{l'm}(\theta) \cos \theta \, sY_{lm}(\theta) \sin \theta \, d\theta \quad \text{for} \quad (l' \neq l), \quad (3.320)\]

\[= 0 \quad \text{for} \quad (l' = l). \quad (3.321)\]

Using the fact that this integral can be expressed using Clebsch-Gordan coefficients,
we see that $c'_{lm}$ is non-zero only for $l' = l \pm 1$. We find

$$c'_{lm} = -\frac{s}{(l+1)^2} \times$$

$$\sqrt{\frac{(l+s+1)(l-s+1)(l+m+1)(l-m+1)}{(2l+3)(2l+1)}}$$

(3.322)

$$c_{lm}^{l-1} = \frac{s}{l^2} \sqrt{\frac{(l+s)(l-s)(l+m)(l-m)}{(2l+1)(2l-1)}}$$

(3.323)

For $s = -2$, these reproduce the values given in Appendix A of Ref. [148].

**Appendix 3.D  Glossary of notation changes**

Previous work by one of the present authors and various collaborators (e.g., Ref. [138]) has used notation for various quantities related to the Teukolsky equation and its solutions which differs from that used by Fujita and Tagoshi and their collaborators [142, 132, 100]. We have recently switched our core numerical engine to one that is based on the Fujita-Tagoshi method, and as such have found it to be much more convenient to follow their conventions in our work.

Begin by examining how Eqs. (3.50)–(3.53) appear in the previous notation:

$$R^H_{lm\omega}(r \rightarrow r_+) = B^\text{hole}_{lm\omega} \Delta^2 e^{-i\omega r^*},$$

(3.324)

$$R^H_{lm\omega}(r \rightarrow \infty) = B^\text{out}_{lm\omega} r^3 e^{i\omega r^*} + \frac{B^\text{in}_{lm\omega}}{r} e^{-i\omega r^*},$$

(3.325)

$$R^\infty_{lm\omega}(r \rightarrow r_+) = D^\text{out}_{lm\omega} r^3 e^{i\omega r^*} + D^\text{in}_{lm\omega} \Delta^2 e^{-i\omega r^*},$$

(3.326)

$$R^\infty_{lm\omega}(r \rightarrow \infty) = D^\infty_{lm\omega} r^3 e^{i\omega r^*}.$$  

(3.327)

[These are Eqs. (3.15a–d) in Ref. [138].] As discussed in Sec. 3.2.2, we use these homogeneous solutions to assemble a Green's function, and then define a general solution

$$R_{lm\omega}(r) = Z^H_{lm\omega}(r) R^\infty_{lm\omega}(r) + Z^\infty_{lm\omega}(r) R^H_{lm\omega}(r),$$

(3.328)
where

\[ Z^\text{H}_{\text{lmw}}(r) = \frac{1}{\mathcal{W}} \int_{r_+}^{r} \frac{R^\text{H}_{\text{lmw}}(r') T_{\text{lmw}}(r')}{\Delta(r')^2} \, dr', \quad (3.329) \]

\[ Z^\infty_{\text{lmw}}(r) = \frac{1}{\mathcal{W}} \int_{r}^{\infty} \frac{R^\infty_{\text{lmw}}(r') T_{\text{lmw}}(r')}{\Delta(r')^2} \, dr', \quad (3.330) \]

where \( \mathcal{W} \) is the Wronskian associated with \( R^\text{H}_{\text{lmw}}, R^\infty_{\text{lmw}} \). We then define

\[ Z^\text{H}_{\text{lmw}} \equiv Z^\text{H}_{\text{lmw}}(r \to \infty), \quad (3.331) \]

\[ Z^\infty_{\text{lmw}} \equiv Z^\infty_{\text{lmw}}(r \to r_+). \quad (3.332) \]

These amplitudes define the fluxes of energy and angular momentum into the black hole's event horizon and carried to infinity. Unfortunately, they have the rather annoying property that their connection to these fluxes is "backwards": \( Z^\text{H}_{\text{lmw}} \) encodes information about the fluxes at infinity, and \( Z^\infty_{\text{lmw}} \) encodes fluxes on the horizon. Although the labels defined by Eqs. (3.331) and (3.332) follow logically from their connection to the homogeneous solutions \( R^\text{H}_{\text{lmw}} \) and \( R^\infty_{\text{lmw}} \), they connect rather illogically to the fluxes that they ultimately encode.

To switch to the notation that is used in Refs. [142, 132, 100], we rename various functions and coefficients. For the fields that are regular on the horizon, we put

\[ R^\text{H}_{\text{lmw}} \to R^\text{in}_{\text{lmw}}, \quad (3.333) \]

\[ B^\text{hole}_{\text{lmw}} \to B^\text{trans}_{\text{lmw}}, \quad (3.334) \]

\[ B^\text{out}_{\text{lmw}} \to B^\text{ref}_{\text{lmw}}, \quad (3.335) \]

\[ B^\text{in}_{\text{lmw}} \to B^\text{inc}_{\text{lmw}}. \quad (3.336) \]
and for fields that are regular at infinity,

\begin{align}
R_{\text{in}}^\infty &\rightarrow R_{\text{up}}^\infty, \quad (3.337) \\
D_{\text{in}}^\infty &\rightarrow C_{\text{in}}^\text{trans}, \quad (3.338) \\
D_{\text{out}}^\infty &\rightarrow C_{\text{up}}^\infty, \quad (3.339) \\
D_{\text{in}}^\infty &\rightarrow C_{\text{ref}}^\infty. \quad (3.340)
\end{align}

The general solution which follows from this is our Eq. (3.44), with functions \(Z_{\text{in}}^\infty(r)\) and \(Z_{\text{up}}^\infty(r)\) defined exactly as \(Z_{\text{in}}^\text{H}(r)\) and \(Z_{\text{up}}^\infty(r)\) are defined in Eqs. (3.45) and (3.46).

As described in Sec. 3.2.2, we then define

\begin{align}
Z_{\text{in}}^\text{H} &\equiv Z_{\text{in}}^\text{up}(r \rightarrow r_+), \quad (3.341) \\
Z_{\text{in}}^\infty &\equiv Z_{\text{in}}^\text{in}(r \rightarrow \infty). \quad (3.342)
\end{align}

This definition reverses the labels that were introduced in Eqs. (3.331) and (3.332), so that fluxes on the horizon are encoded by \(Z_{\text{in}}^\text{H}\), and those to infinity by \(Z_{\text{in}}^\infty\). It is also in accord with the notation used in Refs. [142, 132, 100].

References


REFERENCES


Figure 3-3: Equatorial section of the embedding of a distorted Schwarzschild horizon. Each panel shows the distortion for a different orbital radius, varying from $r_o = 50M$ to $r_o = 6M$. The black circles are the undistorted black hole, and the red curves are the distorted horizons, embedded with Eq. (3.143). These plots are in a frame that corotates with the orbit, and are for a slice of constant ingoing time $v$. The green dashed line in each panel shows the angle at which the tidal distortion is largest; the black dotted line shows the orbit’s position. Notice that the bulge leads the orbit in all cases, with the lead angle growing as the orbit moves to smaller orbital radius. We have rescaled the horizon’s tidal distortion by a factor $\propto r_o^3/\mu$ so that, at leading order, the magnitude of the distortion is the same in all plots.
Figure 3-4: Comparison of selected modes for the numerically computed scalar curvature perturbation $R^{(1)}_{H,l,m}$ with the analytic expansion given in Eqs. (3.174) – (3.185). The four panels on the left are for orbits of a black hole with $a = 0.1M$ at $r_o = 50M$; those on the right are for orbits of a black hole with $a = 0.2M$ at $r_o = 10M$. The mode shown is indicated by $(l, m)$ in the upper right corner of each panel [we actually show the contributions from $(l, m)$ and $(l, -m)$]. In all cases, we plot $(r_o^3/\mu)R^{(1)}_{H,l,m}$, scaling out the leading dependence on orbital radius and the orbiting body’s mass. Curves in green are the analytic results, those in red are our numerical data. Agreement for the large radius, low spin cases is extremely good, especially for small $l$ where the numerical data lies practically on top of the analytic predictions. As we increase $q$ and decrease $r_o$, the amplitude agreement becomes less good, though the analytic formulas still are within several to several tens of percent of the numerical data. The phase agreement is outstanding in all of these cases.
Figure 3-5: Equatorial section of the embedding of distorted Kerr black hole event horizons, $a = 0.1M$ and $a = 0.4M$. Each panel represents the distortion for a different radius of the orbiting body, varying from $r_o = 50M$ to the ISCO ($r_o = 5.669M$ for $a = 0.1M$, $r_o = 4.614M$ for $a = 0.4M$). As in Fig. 3-3, the green dashed line shows the angle at which the tidal distortion is largest, and the black dotted line shows the position of the orbit. In contrast to the Schwarzschild results, the bulge does not lead the orbit in all cases here. The amount by which the bulge leads the orbit grows as the orbit moves to small orbital radius (in some cases, changing from a lag to a lead as part of this trend).
Figure 3-6: Equatorial section of the embedding of distorted Kerr black hole event horizons, $a = 0.7M$ and $a = 0.866M$. Each panel represents the distortion for a different radius of the orbiting body, varying from $r_o = 50M$ to the ISCO ($r_o = 3.393M$ for $a = 0.7M$, $r_o = 2.537M$ for $a = 0.866M$), with the green dashed and black dotted lines labeling the locations of maximal distortion and position of the orbit, respectively. The bulge lags the orbit in most cases we show here, with the lag angle getting smaller and converting to a small lead as the orbit moves to smaller and smaller orbital radius.
Figure 3-7: Two example embeddings of the tidally distorted horizon’s surface. Both panels show the 3-dimensional embedding surface, $r_{E}(\theta, \psi)$; the color scale indicates the horizon’s distortion relative to an isolated Kerr black hole. Red indicates stretching of the horizon (i.e., $r_{E}$ increased by the tides relative to an isolated hole); blue indicates squeezing ($r_{E}$ decreased by tides). On the left, we show a relatively gentle deformation around a moderately spinning black hole: $a = 0.3M$, $r_{o} = 20M$. The distortion here is dominated by a quadrupolar deformation of the horizon (lagging the orbiting body, whose angular position is indicated by the small blue ball). On the right, we show a rather extreme case: $a = 0.866M$, $r_{o} = 1.75M$. The deformation here is much more complicated, as many multipoles beyond $l = 2$ contribute to the shape of the horizon.
Figure 3-8: Embedding of distorted Kerr black hole event horizons for a corotating orbit — i.e., an orbit for which $\Omega_{\text{orb}} = \Omega_H$. As in Figs. 3-3, 3-5, and 3-6, the green dashed line points along the direction of greatest horizon distortion, and the black dotted line points to the orbiting body. At very small spins (for which the corotating orbital radius is very large), the bulge lags the orbit slightly, but the bulge leads for all other spins.
Chapter 4

Strong-field tidal distortions of rotating black holes:
II. Horizon dynamics from eccentric and inclined orbits

Abstract

In chapter 3, we developed tools for studying the horizon geometry of a tidally distorted Kerr black hole, using techniques that require large mass ratios, but can be applied to any bound orbit and allow for arbitrary black hole spin. We now apply these techniques to generic Kerr black hole orbits. This allows us to investigate horizon dynamics: The tidal field perturbing the horizon’s geometry varies over a generic orbit, with significant variations for eccentric orbits. In this chapter, we apply these techniques to study dynamical horizon deformations arising from eccentric and inclined black hole orbits. We find that many of the features of the horizon’s behavior found in our previous analysis carry over to the dynamical case in a natural way. In particular, we find significant offsets between the applied tide and the horizon’s response, which leads to “bulging” in the horizon’s geometry which can lag or lead the orbit, depending upon the hole’s rotation and the orbit’s geometry. One interesting new feature we find are small amplitude coherent wiggles in the horizon’s shear response to the applied tide, which we explain using a teleological Green’s function relating the shear to the tide. These wiggles are clearest when the black hole’s spin is very large ($a \geq 0.99M$). Future work which extends our abilities to embed the horizon to large spins will look for this signature in the distorted horizon’s geometry.
4.1 Introduction

The study of relativistic tidal deformations and their impact on the dynamics of compact binaries has received a great deal of attention in recent years. Much of this recent activity was kicked off by studies of tides in systems containing neutron stars [112, 113, 114, 115, 116, 117, 118, 119]. Older work had already demonstrated that tidal coupling was quite important in systems containing black holes, although using language that clouded the role of tides, using instead a dual description of tidal coupling as “radiation down the event horizon” [120, 121, 137, 99]. Recent papers focusing on black holes in binaries have examined in detail how to quantify how tides distort black holes and their near-hole geometry, focusing largely on non-rotating [116, 126, 125, 127, 128] and slowly rotating [130, 150] black holes.

Our contribution to this body of work has been to develop numerical tools, good for strong-field orbits and arbitrary black hole spins, for characterizing tidally distorted black holes in binaries. These tools are based on black hole perturbation theory, and so assume an extreme mass ratio: The mass $\mu$ of the small body which is the source of the tide is much less than the mass $M$ of the black hole that is tidally distorted. In chapter [ch:circular], we developed tools for characterizing the tidal field that acts on a Kerr black hole. The tools are designed in order to adapt pre-existing codes which have been used to study gravitational wave emission from extreme mass ratio binaries (e.g., [138]). We also developed tools to visualize a tidally distorted black hole by embedding the two-dimensional horizon at each moment in some time slicing in a flat three-dimensional Euclidean space. These embeddings are only good for Kerr spin parameter $a/M \leq \sqrt{3}/2$; for higher spins, the horizon cannot be globally embedded in a Euclidean space even in the absence of a distorting tide [131].

Chapter 3 presented results for tidal distortions arising from circular and equatorial orbits. By focusing on this relatively simple case, we were able to examine some of the key aspects of event horizon physics in a particularly clean limit. For example, chapter 3 examined in some detail the phase offset between the angle at which the horizon is maximally distorted (the location of its “tidal bulge”) and the position of
the orbit. As has been amply discussed in past literature [135, 26, 27, 11], the event horizon acts in many ways like the surface of a gravitating fluid body. The horizon is deformed by tidal stresses, tending to bulge toward the “moon” which is the source of the tide. The bulging response is, however, not synchronous with the applied tide. For a fluid body, viscosity causes the fluid’s response to lag the applied tidal force. As a consequence, if the moon’s orbit is faster than the body’s spin, the bulge lags the orbit’s position. Conversely, if the orbit is slower than the spin, then the bulge leads the orbit’s position.

At least for very slowly varying tidal fields, this picture describes the geometry of the black hole’s tidal bulge with respect to the orbit — provided we swap “lead” and “lag.” Tides from a moon which orbits faster than the hole’s spin raise a bulge which leads the orbit’s position; tides from a moon which orbits more slowly lag the orbit’s position. The swap of “lead” and “lag” as compared to the fluid star is due to the teleological nature of the event horizon: How the horizon depends at some moment in a given time slicing depends upon the stresses that it will feel in the future. Though this counterintuitive behavior would seem to violate causality, it is a simple consequence of how the horizon is defined: Whether an event is inside or outside the horizon depends on that event’s future. See chapter 3 and references therein (as well as the references cited above) for much more detailed discussion of the horizon’s teleological nature, and its consequences.

Although circular and equatorial orbits were useful for testing our tidal distortion toolkit, this limit does not show the full range of horizon dynamics that can be expected from tidal interactions. Indeed, the horizon’s distortion is stationary in this case, showing no variation at all in a frame that co-rotates with the orbit. The purpose of this chapter is to go beyond this limit and to explore how the horizon responds to generic — inclined and eccentric — orbits. Generic orbits and the tides they produce are dynamical even when examined in a frame that corotates at the orbit’s axial frequency. Eccentricity is particularly important: At leading order, the tidal field varies as $1/r^3$, so as the orbit’s radius varies from $r_{\text{max}}$ to $r_{\text{min}}$, the tidal field varies by a factor $r_{\text{max}}^3/r_{\text{min}}^3$. Even circular but inclined orbits show some horizon
dynamics, at least for spinning black hole orbits, since the tidal field varies even for constant radius orbits in the non-spherical spacetime of a rotating black hole.

The remainder of this chapter is organized as follows. We begin in Sec. 4.2 by presenting a summary of the formalism that we developed in chapter 3. Section 4.2.1 introduces the notation and conventions that we use, and carefully defines several quantities that are critical to our analysis, such as the Newman-Penrose basis legs, the tidal field $\psi_0$, and the horizon's shear $\sigma$. This section also briefly describes the techniques we use to compute these quantities; further details are given in chapter 3. Appendix 4.A supplements this material, demonstrating that the complex fields we use for various quantities needed to describe the horizon's distortion are equivalent to certain 2nd-rank tensors defined on the horizon which other authors have used (notably Ref. [128], hereafter VPM11). Section 4.2.2 summarizes how these quantities are used to understand the distorted horizon's geometry, providing a brief synopsis of material that is discussed at much greater length in chapter 3.

We show our results in Secs. 4.3, 4.4, and 4.5. We first examine with some care the Schwarzschild limit, $a = 0$. This limit is spherically symmetric, so the horizon distortions must exhibit certain symmetries as an orbit is inclined from equatorial to some arbitrary inclination $\theta_{\text{inc}}$. We demonstrate that this is the case. This is not a surprise, since the black hole perturbation theory code on which our analysis is based has previously been shown to handle this limit correctly [137, 138], though it is reassuring to see that the modifications we made to analyze distorted horizons have not broken this behavior.

In Sec. 4.4, we next compare certain important aspects of the applied tidal field $\psi_0$ to the horizon shear $\sigma$ that arises from this field. We first (Sec. 4.4.1) look at the relative phase of the tide and the shear, an analysis quite similar to one that we undertook in chapter 3, focusing for simplicity on equatorial orbits. In the Schwarzschild limit, the tide and the shear are very similar. Much of the difference between the two quantities is due to a simple temporal offset of $\kappa^{-1} = 4M$ (where $\kappa$ is the event horizon's surface gravity). This offset can be understood by examining the equation relating $\psi_0$ to $\sigma$ in the frequency domain. The difference becomes much less simple
as the black hole's spin is increased.

It's worth noting that some of the physics associated with the offset between the orbiting body and the horizon's distortion that we discussed above is reproduced in the tide-shear analysis. In particular, we find that the shear response leads the applied tidal field for \( a = 0 \), but lags it for large black hole spin — just as the horizon bulge always leads the orbit in Schwarzschild, but lags the orbit for rapidly spinning Kerr. Because the tide and the shear are evaluated at the same coordinate radius, many of the ambiguities associated with comparing the position of the horizon's bulge with the position of the orbit disappear. This helps to put notions of which quantities "lead" and "lag" on a very firm footing.

We then examine in some detail some interesting behavior which arises from the teleological relation between the tide and the shear. In Sec. 4.2.2, we show that one can relate the tide and the shear to one another with a simple Green's function. Although we do all of our calculations in the frequency domain, this function is written in a time-like domain. Let \( \lambda \) denote affine parameter along a generator of the event horizon. Then, the Green's function that we introduce relates the shear along a generator at parameter \( \lambda \) to the tide that this generator will experience for \( \lambda' \geq \lambda \). The Green's function shows that the tide's future influence is important over an interval \( \Delta \lambda \simeq a \times \kappa^{-1} \), where \( \kappa \) is the horizon's surface gravity. We thus expect the future influence of the tide to be particularly important when \( \kappa^{-1} \) is very large, which happens as \( a \to M \). In Sec. 4.4.2 we identify a very interesting small amplitude, periodic structure in the perturbed horizon's shear which is a very clear signature of this behavior.

Finally, we examine the dynamics of horizon embeddings in Sec. 4.5. In all the cases that we examine, we show a sequence of still images taken from an animation that combines the behavior of the small body's orbit with the dynamics of the horizon embedding. Those animations can be found at http://gmunu.mit.edu/viz/embed_viz/embed_viz.html. Although we have endeavored to describe the dynamics as clearly as possible using the stills we show in Sec. 4.5, some of these results are particularly clear when examined with the animations. We first consider orbits that
are circular but inclined in Sec. 4.5.1, examining in detail orbits of a Schwarzschild black hole and of a Kerr hole with \( a = 0.6M \). The Schwarzschild results confirm our expectations from Sec. 4.3 about how the horizon should behave in this spherically symmetric example; the non-spherical Kerr results show more interesting shape dynamics. We then consider eccentric orbits in Sec. 4.5.2. As expected, the horizon’s distortion varies considerably as an orbit moves from \( r_{\text{max}} \) to \( r_{\text{min}} \) and back. We examine in some detail two highly eccentric \( (e > 0.7) \) orbits: one that is equatorial, and one inclined at \( \theta_{\text{inc}} = 30^\circ \). The generic case combines features that we see from the inclined circular and the eccentric equatorial limits.

4.2 Summary of formalism

4.2.1 Tools, notation, and conventions

All of our calculations are performed in the spacetime of a Kerr black hole with mass \( M \) and spin angular momentum \( J \). Throughout this analysis, we work in ingoing coordinates \((v, r, \theta, \psi)\) which are well behaved on the black hole’s event horizon. In these coordinates, the spacetime’s line element is given by

\[
\begin{align*}
    ds^2 &= - \left( 1 - \frac{2Mr}{\Sigma} \right) dv^2 + 2dv dr - 2a \sin^2 \theta dr d\psi \\
    &\quad - \frac{4Mar \sin^2 \theta}{\Sigma} dv d\psi + \Sigma d\theta^2 \\
    &\quad + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\Sigma} d\psi^2 , \quad (4.1)
\end{align*}
\]

with \( a = J/M \). (Here and throughout the chapter, we use units in which \( G = 1 = c \).) Equation (4.1) introduces the functions \( \Delta = r^2 - 2Mr + a^2 \) and \( \Sigma = r^2 + a^2 \cos^2 \theta \). The event horizon is at coordinate radius \( r = r_+ = M + \sqrt{M^2 - a^2} \), the large root of \( \Delta \). Although not needed here, for completeness we note that ingoing coordinates are simply related to the more commonly used Boyer-Lindquist coordinates \((t, r, \theta, \phi)\): The coordinates \( r \) and \( \theta \) are identical, and the ingoing time \( v \) and angle \( \psi \) are related
to Boyer-Lindquist time $t$ and angle $\phi$ via

$$
\frac{dv}{dt} = \frac{r^2 + a^2}{\Delta} dr , \tag{4.2}
$$

$$
\frac{d\psi}{d\phi} = \frac{a}{\Delta} dr . \tag{4.3}
$$

The tidal field which distorts the black hole's horizon arises from a small body on a bound Kerr geodesic; detailed discussion of these orbits, with an emphasis on the properties relevant to this analysis, is given in Refs. [151, 152]. Such geodesics are parameterized by three conserved integrals: The orbital energy $E$, related to the spacetime's timelike Killing vector; the axial angular momentum $L_z$, related to the spacetime's axial Killing vector; and the Carter constant $Q$, related to the Kerr spacetime's Killing tensor. Once $E$, $L_z$, and $Q$ have been selected, the orbit's motion is determined up to initial conditions. A particularly important feature of bound Kerr orbits is that they are triperiodic [151]. Each orbit has a frequency $\Omega_r$ which describes radial oscillations, a frequency $\Omega_\theta$ which describes polar oscillations, and a frequency $\Omega_\phi$ which describes rotations about the black hole's spin axis. Once $E$, $L_z$, and $Q$ are known, it is not too difficult to compute $\Omega_r$, $\Omega_\theta$, and $\Omega_\phi$ [151, 152].

We remap the motion in $r$ and $\theta$ to the parameters $p$ (semi-latus rectum), $e$ (eccentricity), and $\theta_m$, defined by

$$
r = \frac{p}{1 + e \cos \psi} , \tag{4.4}
$$

$$
\cos \theta = \cos \theta_m \cos(\chi + \chi_0) . \tag{4.5}
$$

With this parameterization, the geodesic equations for the coordinates $r$ and $\theta$ become equations for the angles $\psi$ and $\chi$. Note that we could include an offset phase $\psi_0$ in Eq. (4.4). We have set $\psi_0 = 0$, which is equivalent to choosing the origin of our time coordinate to the moment that the orbit passes through periapsis. References [151, 152] give easy-to-use expressions relating the $(E, L_z, Q)$ and $(p, e, \theta_m)$ parameterizations. For much of our analysis, we use the angle $\theta_{\text{inc}}$ introduced in Ref. [138]
in place of $\theta_m$:

$$\theta_{\text{inc}} = \pi/2 - \sgn(L_z)\theta_m.$$  

(4.6)

This angle varies smoothly from 0 to $\pi$ as the orbit varies from prograde equatorial ($\theta_m = \pi/2$, $L_z > 0$) to retrograde equatorial ($\theta_m = \pi/2$, $L_z < 0$).

The tidal field is quantified by the scalar field $\psi_0$, a complex scalar built from the Weyl curvature tensor:

$$\psi_0 = -C_{\mu\alpha\nu\beta}l^\mu m^\alpha l^\nu m^\beta.$$  

(4.7)

The vectors used here are the Newman-Penrose null legs in the Hawking-Hartle representation [135]:

$$l^\mu \doteq \left[1, \frac{\Delta}{2\varpi^2}, 0, \frac{a}{\varpi^2}\right],$$  

(4.8)

$$n^\mu \doteq \frac{1}{\Sigma} \left[-a^2 \sin^2 \theta/2, -\varpi^2 + \frac{a^2 \Delta \sin^2 \theta}{4\varpi^2}, 0, -a + \frac{a^3 \sin^2 \theta}{2\varpi^2}\right],$$  

(4.9)

$$m^\mu \doteq \frac{1}{\sqrt{2}(r + ia \cos \theta)} \left[0, -\frac{ia\Delta \sin \theta}{\varpi^2}, 1, i \csc \theta - \frac{ia^2 \sin \theta}{\varpi^2}\right].$$  

(4.10)

The notation $\doteq$ means “the components of the quantity on the left-hand side are represented by the array on the right-hand side in ingoing Kerr coordinates.” For brevity, we have introduced $\varpi^2 = r^2 + a^2$. These legs satisfy

$$l^\mu n_\nu = -1, \quad m^\mu \bar{m}_\mu = 1,$$  

(4.11)

with overbar denoting complex conjugate; all other inner products between legs vanish.

Following VPM11, the Weyl curvature on the horizon is completely described by a two-dimensional trace-free symmetric tensor $C_{AB}$, where capital Roman indices denote components associated with coordinates on the horizon. Such a tensor has only two independent components, which we can describe as “tidal polarizations,” and
denote \( C_+ \) and \( C_x \). These polarizations are simply related to the curvature scalar \( \psi_0 \):

\[
C_+ + iC_x = -\psi_0(r_+) .
\]

(4.12)

See App. 4.A for further details and a proof of Eq. (4.12). We use the polarizations \( C_{+,x} \) in much of our presentation of results, especially in Sec. 4.4.

The tidal field can be decomposed into harmonics of the three fundamental Kerr frequencies, allowing us to write its value at \( r = r_+ \) as

\[
\psi_0(v, \theta, \psi) = \frac{1}{16M^2r_+^2} \sum_{lmkn} W_{lmkn}^H S_{lmkn}^+(\theta) e^{i\Phi_{mkn}(v, \psi)} .
\]

(4.13)

The function

\[
S_{lmkn}^+(\theta) = +2S_{lm}(\theta; a\omega_{mkn})
\]

(4.14)
is a spheroidal harmonic of spin-weight +2; detailed discussion of this function and how it is computed can be found in Ref. [137]. The frequency \( \omega_{mkn} \) is a harmonic of the orbital frequencies,

\[
\omega_{mkn} = m\Omega_\phi + k\Omega_\theta + n\Omega_r .
\]

(4.15)
The product \( a\omega_{mkn} \) sets the "oblateness" associated with \( S_{lmkn}^+(\theta) \). We describe the phase \( \Phi_{mkn}(v, \psi) \) in more detail below.

The amplitude \( W_{lmkn}^H \) can be found by solving the Teukolsky equation [120]. In practice, we compute the field \( \psi_4 \), a different projection of the Weyl curvature. In the limits \( r \to r_+ \) and \( r \to \infty \), the fields \( \psi_4 \) and \( \psi_0 \) can be related to one another without too much trouble [121]. As \( r \to r_+ \), \( \psi_4 \) takes the form

\[
\psi_4 = \frac{\Delta^2}{(r - ia\cos\theta)^4} \sum_{lmkn} Z_{lmkn}^H S_{lmkn}^-(\theta) e^{i\Phi_{mkn}(v, \psi)} .
\]

(4.16)

Detailed discussion of how to compute the amplitude \( Z_{lmkn}^H \) using the Teukolsky equation is given in Ref. [138]. The function

\[
S_{lmkn}^-(\theta) = -2S_{lm}(\theta; a\omega_{mkn})
\]

(4.17)
is a spheroidal harmonic of spin-weight $-2$; see [137] for detailed discussion.

The Starobinsky-Churilov identities [140] connect the amplitudes of these two curvature scalars:

$$ W_{lmkn}^H = \beta_{lmkn} Z_{lmkn}^H , \quad (4.18) $$

where

$$ \beta_{lmkn} = \frac{64(2Mr_+)^4 p_{mkn}(p_{mkn}^2 + \kappa^2)(p_{mkn} + 2i\kappa)}{c_{lmkn}} , \quad (4.19) $$

$$ |c_{lmkn}|^2 = \{[(\lambda + 2)^2 + 4ma\omega_{mkn} - 4a^2\omega_{mkn}^2]\} $$
$$ \times (\lambda^2 + 36ma\omega_{mkn} - 36a^2\omega_{mkn}^2) $$
$$ + (2\lambda + 3)(96a^2\omega_{mkn}^2 - 48ma\omega_{mkn}) \} $$
$$ + 144\omega_{mkn}^2(M^2 - a^2) , \quad (4.20) $$

$$ \text{Im } c_{lmkn} = 12M\omega_{mkn} , \quad (4.21) $$

$$ \text{Re } c_{lmkn} = +\sqrt{|c_{lmkn}|^2 - 144M^2\omega_{mkn}^2} . \quad (4.22) $$

In these equations, the quantity

$$ p_{mkn} = \omega_{mkn} - m\Omega_H , \quad (4.23) $$

with $\Omega_H = a/2Mr_+$ the angular frequency associated with the Kerr event horizon.

The real number $\lambda$ is related to the eigenvalue of the spheroidal harmonic:

$$ \lambda = \mathcal{E} - 2am\omega_{mkn} + a^2\omega_{mkn}^2 - s(s + 1) , \quad (4.24) $$

where $s = -2$, and $\mathcal{E}$ is the eigenvalue$^1$ associated with the $s = -2$ spheroidal harmonic. In the limit $a = 0$, $\mathcal{E} = l(l + 1)$.

Other important quantities appearing in these equations are the black hole's sur-

---

$^1$Multiple conventions for this eigenvalue can be found in the literature. Another common one puts $\lambda = A - 2am\omega_{mkn} + a^2\omega_{mkn}^2$; they are related by $A = \mathcal{E} - s(s + 1)$. 
face gravity,

\[ \kappa = \frac{\sqrt{M^2 - a^2}}{2Mr_+} , \quad (4.25) \]

and the phase

\[ \Phi_{mkn}(v, \psi) = m[\psi - K(a)] - (m\Omega_\phi + k\Omega_\theta + n\Omega_r)v , \quad (4.26) \]

where

\[
K(a) = \frac{a}{2M(Mr_+ - a_2)} \left\{ a^2 - Mr_+ + 2Mr_+ \arctanh \left( \sqrt{1 - a^2/M^2} \right) + M\sqrt{M^2 - a^2} \ln \left[ \frac{a^2}{4(M^2 - a^2)} \right] \right\} . \quad (4.27)
\]

In Eqs. (4.13) and (4.16), the sum over \( l \) goes from 2 to \( \infty \); the sum over \( m \) from \(-l \) to \( l \); and the sums over \( k \) and \( n \) from \(-\infty \) to \( \infty \). We abbreviate this set of indices using \( \Lambda = \{l, m, k, n\} \). With this, Eq. (4.13) becomes

\[
\psi_0(v, \theta, \psi) = \frac{1}{16M^2r_+^2} \sum_{\Lambda} W^H_{\Lambda} S^+_{\Lambda}(\theta)e^{i\Phi_{\Lambda}(v, \psi)}
\]

\[
\equiv \sum_{\Lambda} \psi_{0,\Lambda} S^+_{\Lambda}(\theta)e^{i\Phi_{\Lambda}(v, \psi)} , \quad (4.28)
\]

where we've introduced

\[
\psi_{0,\Lambda} = \frac{W^H_{\Lambda}}{16M^2r_+^2} = \frac{64M^2r_+^2p_{\Lambda}(p_{\Lambda}^2 + \kappa^2)(p_{\Lambda} + 2i\kappa)Z^H_{\Lambda}}{c_{\Lambda}} . \quad (4.29)
\]

Note that the phase \( \Phi_{\Lambda} \equiv \Phi_{mkn} \) and wavenumber \( p_{\Lambda} \equiv p_{mkn} \) don't actually depend on the index \( l \). Using \( \Lambda \) as a label for these quantities is thus somewhat redundant, though this redundancy is harmless.
4.2.2 The geometry of a distorted event horizon

The shear to the horizon’s generators

The first tool we need to understand how the tidal field affects the horizon’s geometry is the shear $\sigma$ of the horizon’s generators. It is given by

$$\sigma = m^\mu m^\nu \nabla_\mu l_\nu ,$$  \hspace{1cm} (4.30)

evaluated at $r = r_+$. (Note that, for an unperturbed black hole, $l^\mu$ is tangent to the generators at $r = r_+$. ) As written, $\sigma$ is a complex function, defined on the hole’s event horizon. Just as the complex Weyl scalar $\psi_0$ can be written using polarizations $C_{+,x}$ of the on-horizon Weyl tensor, the shear can be written in terms of polarizations $\sigma_{+,x}$ of an on-horizon shear tensor:

$$\sigma = \sigma_+ + i\sigma_x .$$  \hspace{1cm} (4.31)

See App. 4.A for further details and a proof of Eq. (4.31). We will use $\sigma_{+,x}$ in much of our discussion of results, especially in Sec. 4.4.

With the tetrad and gauge that we use, the perturbed shear is governed by the equation [27]

$$(D - \kappa)\sigma = \psi_0 ,$$  \hspace{1cm} (4.32)

where the derivative operator $D \equiv l^\mu \partial_\mu$. Let us expand $\sigma$ as we expanded $\psi_0$:

$$\sigma(v, \theta, \psi) = \sum_\Lambda \sigma_\Lambda S_\Lambda^+(\theta)e^{i\Phi_\Lambda(v, \psi)} .$$  \hspace{1cm} (4.33)

Using the fact that $D \to \partial_v + \Omega_H \partial_\psi$ on the horizon, we find that Eq. (4.32) is satisfied if

$$\sigma_\Lambda = 64M^2r_+^2c_\Lambda^{-1}p_\Lambda(p_\Lambda + i\kappa)(ip_\Lambda - 2\kappa)Z_\Lambda^H .$$

As was extensively discussed in chapter 3, there is a phase offset between the shear
and the applied tidal field. The phase offset for each mode is simple to calculate:

\[
\frac{\sigma_{\Lambda}}{\psi_{0,\Lambda}} = \frac{i}{p_{\Lambda} - i\kappa} = \exp\left[-i \arctan\left(\frac{p_{\Lambda}}{\kappa}\right)\right] \frac{1}{\sqrt{p_{\Lambda}^2 + \kappa^2}}.
\]  

In other words, for each mode \( \Lambda \), the shear leads the tide by an angle given by the mode’s wavenumber \( p_{\Lambda} \) times the inverse surface gravity \( \kappa^{-1} \). For circular and equatorial orbits, \( p_{\Lambda} \rightarrow m(\Omega_{\phi} - \Omega_{H}) \), so each mode experiences the same phase shift, modulo \( m \). For these orbits, we find a simple, constant offset between the tidal field and the resulting shear. More complicated behavior results for the generic orbits we study in this chapter, since many modes, each with different phase shifts, contribute to \( \psi_{0} \) and \( \sigma \).

Although we do all of our calculations in this chapter in the frequency domain, it is also useful to examine Eq. (4.32) in a “time-like” domain. As mentioned above, the Newman-Penrose leg \( \ell^{\mu} \) is tangent to the unperturbed horizon generators at \( r = r_{+} \). We may therefore write \( D \equiv \partial/\partial\lambda \) on the horizon, where \( \lambda \) is affine parameter along the generator. In this representation, \( \lambda \) is effectively a time measure, albeit a somewhat unusual time, measured by a clock that ticks at a uniform rate as it follows a specific horizon generator.

With this in mind, following Ref. [11] Sec. VI C 6, let us find the Green’s function \( G(\lambda, \lambda') \) for Eq. (4.32):

\[
(D - \kappa)G(\lambda, \lambda') = \delta(\lambda - \lambda').
\]

This equation has the solution

\[
G(\lambda, \lambda') = -e^{\kappa(\lambda - \lambda')} \Theta(\lambda' - \lambda),
\]

where the step function

\[
\Theta(x) = \begin{cases} 
1 & x > 0 \\
0 & x < 0 
\end{cases}
\]

(4.37)
CHAPTER 4. STRONG-FIELD TIDAL DISTORTIONS OF ROTATING BLACK HOLES: II. HORIZON DYNAMICS FROM ECCENTRIC AND INCLINED ORBITS

The solution for the shear along the generator is then

$$\sigma(\lambda) = -\int_{\lambda}^{\infty} e^{\kappa(\lambda' - \lambda')} \psi_0(\lambda') d\lambda', \quad (4.38)$$

or, using Eqs. (4.12) and (4.31),

$$\sigma_{+,\times}(\lambda) = \int_{\lambda}^{\infty} e^{\kappa(\lambda' - \lambda')} C_{+,\times}(\lambda') d\lambda'. \quad (4.39)$$

Notice that the behavior at $\lambda$ depends on the tides to the future of $\lambda$ — a manifestation of the horizon’s teleological nature. What we see is that the shear at $\lambda$ on a particular generator depends on the tide integrated over an interval from $\lambda$ to roughly $\lambda + (a \text{ few}) \times \kappa^{-1}$. This has interesting consequences for the distorted horizon’s dynamics which we will examine in Sec. 4.4.

**The curvature of the distorted horizon**

The tidal field $\psi_0$ on the horizon also tells us the scalar Ricci curvature of the black hole, $R_H$. This is discussed in great detail in chapter 3. Briefly, the scalar curvature of the hole’s event horizon is given by

$$R_H = R_H^{(0)} + R_H^{(1)}, \quad (4.40)$$

where

$$R_H^{(0)} = \frac{2}{r_+^2} \frac{(1 + a^2/r_+^2)(1 - 3a^2 \cos^2 \theta/r_+^2)}{(1 + a^2 \cos^2 \theta/r_+^2)^3} \quad (4.41)$$

describes an undistorted Kerr black hole, and

$$R_H^{(1)} = -4\text{Im} \sum_{\Lambda} \frac{\bar{\Delta} \bar{\Delta} \psi_{0,\Lambda}}{p_{\Lambda}(i p_{\Lambda} + \kappa)} \quad (4.42)$$

is the perturbation to $R_H$ arising from the tidal field $\psi_0$. The operator $\bar{\Delta}$ lowers the spin weight of the angular basis functions. As discussed in Sec. IIC of chapter 3, it is quite simple to evaluate $\bar{\Delta} \bar{\Delta} \psi_{0,\Lambda}$ with the spectral expansion for the spin-weighted spheroidal harmonics that we use. See chapter 3 for detailed discussion.
4.3. HORIZON DYNAMICS I: CONSISTENCY TEST FOR THE SCHWARZSCHILD LIMIT

To visualize the curvature of a distorted horizon, we embed the horizon in a global Euclidean 3-space. This means finding the function

$$r_E(\theta, \psi) = r_E^0(\theta) + r_E^1(\theta, \psi)$$

(4.43)

that has the same Ricci scalar curvature as the distorted horizon. This works well for spins $a/M < \sqrt{3}/2$; for higher spins, global Euclidean embeddings do not exist even for the undistorted event horizon [131]. As such, we confine our embedding visualizations in this chapter to the range $0 \leq a/M < \sqrt{3}/2$. Work in progress indicates that an elegant way to lift this restriction will be to embed the horizon’s distorted geometry in the globally hyperbolic space $H^3$ [153].

Confining our discussion to Euclidean 3-space, a simple analytic solution exists for the undistorted hole’s embedding radius $r_E^0(\theta)$ [131]. To find the perturbation $r_E^1(\theta, \psi)$, we expand in spherical harmonics, writing

$$r_E^1(\theta, \psi) = r_+ \sum_{\ell m} \varepsilon_{\ell m} Y_{\ell m}(\theta, \psi) .$$

(4.44)

Given this functional form, it is a straightforward (although rather lengthy) exercise to construct the scalar curvature associated with $r_E^1$; details are given in Appendix B of chapter 3. This embedding curvature $R_E^1$ depends on the coefficients $\varepsilon_{\ell m}$. By enforcing $R_E^1 = R_H^1$ (i.e., requiring that the embedding curvature equals the curvature due to the tidal perturbation), we read off the embedding coefficients $\varepsilon_{\ell m}$. Full details of the algorithm for doing this are given in Appendix B of chapter 3.

4.3 Horizon dynamics I: Consistency test for the Schwarzschild limit

Schwarzschild black holes, $a = 0$, are spherically symmetric. In this limit, there is no physical distinction between an equatorial orbit ($\theta_{inc} = 0^\circ$) and an orbit of arbitrary inclination. Our representation of these orbits will certainly be different, but this is
simply due to the coordinate orientation we have chosen. (By contrast, when \( a \neq 0 \),
the black hole's spin axis picks out a preferred spatial direction.) We thus expect
that many properties related to black hole perturbations should become invariant
with respect to orbit inclination for \( a = 0 \), or else vary in a simple way.

This limiting behavior has been discussed in past work, in particular describing
how the amplitude of gravitational waves and the energy that they carry varies as the
orbit's inclination varies. As one example, consider the energy carried by gravitational
waves. The total energy carried by a given \( l \)-mode must be constant as a function of
orbital inclination:

\[
\left( \frac{dE}{dt} \right)_l = \sum_{mkn} \left( \frac{dE}{dt} \right)_l^{mkn} = \text{constant with } \theta_{\text{inc}} \quad .
\]

The sum in Eq. (4.45) is taken over \( m \) from \(-l\) to \( l \), and over \( n \) from \(-\infty\) to \( \infty \). The
sum over \( k \) in principle runs from \(-\infty\) to \( \infty \), though many modes do not actually
contribute, as we discuss momentarily.

Although the summed flux \((dE/dt)_l\) does not vary with \( \theta_{\text{inc}} \), the distribution of
gravitational-wave power among the harmonic indices varies with inclination quite
a bit. In the Schwarzschild limit \( \Omega_\theta = \Omega_\phi \). Consider two orbits which are identical
except for inclination. One is at \( \theta_{\text{inc}} = 0^\circ \), the other at \( \theta_{\text{inc}} \neq 0 \). Power in an axial
\( m \)-mode at \( \theta_{\text{inc}} = 0^\circ \) becomes distributed among polar \( k \)-modes and axial modes with
\( m' = (m - k) \) when \( \theta_{\text{inc}} \neq 0 \). The way in which the power is so distributed is easily
deduced from the rotation properties of spherical harmonics:

\[
\frac{(dE/dt)_l^{(m-k)kn}(\theta_{\text{inc}})}{(dE/dt)_l^{m0n}(\theta_{\text{inc}} = 0^\circ)} = |D_{(m-k)m}(\theta_{\text{inc}})|^2 \quad .
\]

Here, \( D_{(m-k)m} \) is a Wigner function, which relates the spherical harmonic \( Y_{l m} \) at \( \theta \)
to the harmonic \( Y_{l(m-k)} \) at \( \theta - \theta_{\text{inc}} \). (Note this relation implies that there is no power in
any mode with \(|m + k| \geq l \).) Further discussion of this relation is given in Refs. [137]
(with a few minor errors) and [138] (which corrects those errors).
What applies to the gravitational wave flux likewise applies to all the quantities which describe tidal distortions of a Schwarzschild black hole's event horizon. We find that, in all cases we have checked, quantities transform under rotation exactly as they should. This is not terribly surprising, since this property of our code has been checked very carefully in previous analyses. It is reassuring, however, that the modifications we have made to compute the horizon's tidal distortion have not broken this behavior.

Figure 4-1 shows one example of a test for the rotational consistency of Schwarzschild horizon distortions. Consider two orbits around a Schwarzschild black hole, both with $p/M = 10$ and $e = 0.5$. One orbit is equatorial with respect to the coordinates we
impose, the other is highly inclined ($\theta_{\text{inc}} = 80^\circ$) in these coordinates. Due to spherical symmetry, the horizon distortion for the equatorial case should be identical to the horizon distortion in the inclined case, correcting for the tilt of $\theta_{\text{inc}}$.

In this figure, we show the perturbation to the radius of the horizon's embedding surface, $r_E^{(1)}$, for these two cases. The solid (blue) line shows the distortion for the inclined case as measured at $\psi = 0, \theta = 10^\circ$ (i.e., rotated $\theta_{\text{inc}}$ from the equator). The dots (red) show the distortion at $\psi = 0$ on the equator for the equatorial orbit. Although the calculations were done using very different orbits, and very different modes enter the expansion, the horizon distortions $r_E^{(1)}$ that we find are essentially identical, only differing due to accumulated round-off error at a level $\lesssim \epsilon$, where $\epsilon$ is a parameter controlling the accuracy of numerical integrals ($\epsilon \approx 10^{-12}$ was used for this plot). If both curves had been plotted as solid lines, they would have been indistinguishible here. This is a typical example of how our code handles this consistency test.

4.4 Horizon dynamics II: Applied tidal field and resulting shear

We now examine horizon dynamics for generic orbits of Kerr black holes. We start our study by examining how the horizon's shear responds to a dynamical driving tide. In chapter 3, the driving tide was stationary, and the difference between the tide and the response amounted to a simple phase shift. For generic orbits, the difference is not so simple. The time-domain response has some rather interesting features, with a structure that reflects the teleological Green's function, Eq. (4.39).

4.4.1 Relative phase of the tide and shear

We will focus our analysis on the horizon's response to an equatorial, eccentric orbit. The applied tidal field in this case varies from quite strong near periapsis [$r = r_{\text{min}} = p/(1 + e)$] to weak near apoapsis [$r = r_{\text{max}} = p/(1 - e)$], giving us a chance to study
4.4. HORIZON DYNAMICS II: APPLIED TIDAL FIELD AND RESULTING SHEAR

Figure 4-2: Applied tidal field versus shear response for an eccentric equatorial orbit of a non-spinning black hole. Data are for an orbit with $p = 8M$, $e = 0.5$, $\theta_{\text{inc}} = 0^\circ$, $a = 0$. We show the on-horizon Weyl curvature polarization $C_+$ (dashed curve), as well as the resulting shear polarization $\sigma_+$ (solid curve). Both fields are plotted at $\theta = \pi/2$, $\psi = 0$. The top panel shows $C_+$ and $\sigma_+$ as functions of ingoing time $v$. Notice that $\sigma_+$ appears to lead $C_+$ by an almost constant time interval. The bottom panel shows the same data, but with $\sigma_+$ shifted by $\Delta v = \kappa^{-1} = 4M$. The shear response lines up perfectly with the driving tide in this panel, showing that the shear $\sigma_+$ leads the tide by $\kappa^{-1}$ in the Schwarzschild limit.

Figures 4-2, 4-3, and 4-4 compare the tidal field and shear in five different situations. In all cases, the orbit has $p = 8M$, $e = 0.5$, $\theta_{\text{inc}} = 0^\circ$, but the black hole spin varies over $a/M \in [0, 0.3, 0.6, 0.9, 0.9999]$. We compare one polarization of the on-horizon Weyl tensor, $C_+$ (dashed curves), to the corresponding polarization of the horizon’s shear, $\sigma_+$. Since the orbits are all equatorial, we examine these quantities in the holes’ equatorial planes: All data are shown at the point $\theta = \pi/2$, $\psi = 0$.

Begin with Fig. 4-2, which shows $C_+$ versus $\sigma_+$ for orbits of a Schwarzschild black
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Figure 4-3: Applied tidal field versus shear response for eccentric equatorial orbits of spinning black holes. Data in both panels are for an orbit with $p = 8M$, $e = 0.5$, $\theta_{\text{inc}} = 0^\circ$; top is for a hole with spin $a = 0.3M$, bottom for $a = 0.6M$. As in Fig. 4-2, we show the on-horizon Weyl curvature polarization $C_+$ (dashed curve) and the shear polarization $\sigma_+$ that results (solid curve). All data are plotted at field point $\theta = \pi/2$, $\psi = 0$, and are functions of ingoing time $v$.

hole. The top panel of this figure shows that the horizon’s response leads the driving tide by what is apparently a constant offset. To understand this, consider again Eq. (4.34):

$$\frac{\sigma_\Lambda}{\psi_{0,\Lambda}} = \frac{\exp[-i\arctan(p_\Lambda/\kappa)]}{\sqrt{p_\Lambda^2 + \kappa^2}}.$$  

(4.47)

In the Schwarzschild limit, $p_\Lambda = \omega_\Lambda$, and $\kappa^{-1} = 4M$. Each mode $\sigma_\Lambda$ of the shear response leads the driving tide $\psi_0$ by $4M\omega_\Lambda$ radians. This is equivalent to $\sigma$ leading $\psi_0$ in time by $4M$. We check this in the bottom panel of Fig. 4-2: this plot is identical to the top panel of Fig. 4-2, but we have shifted $\sigma_+$ by $4M$. Notice that the tide and the shear are almost precisely aligned in this panel, confirming that the responses here differ primarily by a temporal offset of $\kappa^{-1} = 4M$. 
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As the black hole’s spin increases, the shift between the applied tide and the shear response becomes more complicated: the timescale $\tau^{-1}$ becomes larger as $a \to M$, and the wavenumber $p_{\Lambda} = \omega_{\Lambda} - m\Omega_{H}$ which enters the mode ratio (4.34) differs significantly from the frequency $\omega_{\Lambda}$. We can see the impact of this change in Fig. 4-3. In the top panel, we examine the $C_{\pm}$ and $\sigma_{\pm}$ for the same orbit used in Fig. 4-2 ($p = 8M$, $e = 0.5$, $\theta_{\text{inc}} = 0^\circ$), but now about a black hole with spin $a = 0.3M$. In this case, the tide and the shear are nearly coincident as a function of ingoing time $v$. In the bottom panel, we plot $C_{\pm}$ and $\sigma_{\pm}$ for this orbit about a black hole with spin $a = 0.6M$. The tide now leads the shear, and the shapes are not congruent. Empirically, we find that if we shift $\sigma_{\pm}$ by $\delta v \simeq 3.8M$ we can make the largest peaks line up. Other features, however, do not line up so well; the differing behaviors of $C_{\pm}$ and $\sigma_{\pm}$ cannot be ascribed to a simple time shift.

The trend seen in Fig. 4-3 continues in Fig. 4-4, which shows $C_{\pm}$ and $\sigma_{\pm}$ for the same orbit about black holes with $a = 0.9M$ (top) and $a = 0.9999M$ (bottom). We again see that the tidal field $C_{\pm}$ leads the shear response $\sigma_{\pm}$. We can match the largest peaks by shifting $\sigma_{\pm}$ by $\delta v \simeq 8M$ in the case $a = 0.9M$, and by $\delta v \simeq 9M$ in the case $a = 0.9999M$. However, none of the other features align when we do this, indicating that the shift at these large spins cannot be described as a simple shift in time.

One interesting feature that comes across as we review Figs. 4-2 – 4-4 is the transition from shear leading the tide at $a = 0$ to shear lagging the tide for $a > 0.3M$. This transition is reminiscent of the behavior of the tidal bulge that was seen in chapter 3. There, we found for circular equatorial orbits that the tidal bulge leads the applied tide at small spin, and lags the applied tide at large spin. At least in the small $a$ and large $a$ limits, this could be understood in the circular equatorial case as reflecting the relative angular frequencies of the orbit and the black hole. Qualitatively similar behavior clearly shows up for these dynamical situations, although quantifying it is not so straightforward since these orbits have a more complicated time-frequency structure.
Figure 4-4: Applied tidal field versus shear response for eccentric equatorial orbits of spinning black holes. Data in both panels are for an orbit with $p = 8M$, $e = 0.5$, $\theta_{\text{inc}} = 0^\circ$; top is for a hole with spin $a = 0.9M$, bottom for $a = 0.9999M$. As in Figs. 4-2 and 4-3, we show the on-horizon Weyl curvature polarization $C_+$ (dashed curve) and the shear polarization $\sigma_+$ that results (solid curve). All data are plotted at field point $\theta = \pi/2$, $\psi = 0$, and are functions of ingoing time $v$. 
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4.4.1 Detailed study of the horizon's shear response $\sigma_+$ (solid curves) given a driving tidal field $C_+$ (dashed curves). In both panels, we consider equatorial ($\theta_{inc} = 0^\circ$) orbits with eccentricity $e = 0.7$ about a black hole with spin $a = 0.9999M$. In the left-hand panel, the orbit has semi-latus rectum $p = 10M$; in the right, $p = 3M$. For the case $p = 10M$, no particularly noteworthy feature is evident. This behavior is qualitatively similar to the shear response that we see across a wide range of orbits: $\sigma$ has a shape similar to $C$, leading or lagging depending on the black hole's spin. However, for high spins and strong-field orbits, new behavior emerges: Low-amplitude, high-frequency wiggles can be seen in the shear between the high-amplitudes “bursts” corresponding to the orbit’s periastron passage. These wiggles have a period that corresponds to the hole’s horizon frequency $\Omega_H = a/2Mr_+$, and arise from the shear’s teleological response to the driving tide.

4.4.2 Imprint of the teleological Green’s function

As mentioned in Sec. 4.2.2, the relation between the shear response $\sigma$ and the applied tidal field $\psi_0$ can be understood as due to a teleological Green’s function:

$$\sigma_{+\times}(\lambda) = \int_{\lambda}^{\infty} e^{\kappa(\lambda-\lambda')} C_{+\times}(\lambda')d\lambda'. \quad (4.48)$$

Here, $\lambda$ labels an affine parameter along a generator in the event horizon. The teleological nature of the horizon is expressed by the fact that the shear on a particular generator at $\lambda$ depends on the tide integrated along that generator over a future interval, from $\lambda$ to (a few) $\times \kappa^{-1}$.

To see how the nature of the teleological Green’s function manifests itself, let us
Figure 4-6: Zoom onto the low-amplitude, high-frequency wiggles. In all cases, we consider an equatorial orbit ($\theta_{inc} = 0^\circ$) with $p = 3M$, $e = 0.7$, varying black hole spin $a$ as indicated in each panel. We show only $\sigma_+$, zooming in on a portion of the response in which these wiggles can be seen. Notice that the wiggles become more pronounced and longer lasting as $a \rightarrow M$. This can be understood as a consequence of the teleological Green's function which relates $\sigma$ to the tidal field $\psi_0$: For small $a$, the exponential decay of the Green's function suppresses the impact of the tidal field. As $a \rightarrow M$, the decay scale becomes quite long, and the tide's impact becomes much longer lasting.

examine a pair of orbits about a black hole with $a = 0.9999M$, for which $\kappa^{-1} = 143.4M$. Let us begin by examining a relatively large radius orbit: $p = 10M$, $e = 0.7$, $\theta_{inc} = 0^\circ$. Such an orbit has a radial period [151, 152] $T_r = 626.6M$. This can be seen in the dashed curve in the left-hand panel of Fig. 4-5, which shows $C_+$ for this orbit. The solid curve shows the corresponding $\sigma$. Notice that there is a roughly one-to-one correspondence between the features of $C_+$ and those of $\sigma_+$, consistent with $\sigma_+$ arising from $C_+$ integrated over an interval of a few hundred $M$.

An interesting new feature emerges as we move to strong-field orbits. The right-
4.4. HORIZON DYNAMICS II: APPLIED TIDAL FIELD AND RESULTING SHEAR

hand panel of Fig. 4-5 shows $C_+$ (dashed curve) and $\sigma_+$ (solid curve) for an orbit with $p = 3M$, $e = 0.7$, $\theta_{\text{inc}} = 0^\circ$, for which $T_r = 157.4M$. Notice the high-frequency, low amplitude wiggles visible in $\sigma_+$ near $v \sim 20M$ and $v \sim 200M$. To understand this feature, consider the integrand in Eq. (4.48). We expect that, along any given horizon generator, the value of $\sigma$ will be set by the tidal field integrated over an interval of a few hundred $M$, with an influence that decays over an interval $\kappa^{-1} = 143.4M$. For such a strong field orbit, the tidal field will be significant even when the smaller body is at apoapsis. As we integrate along the generator, the imprint of this tide will oscillate on a timescale $\sim T_H = 2\pi/\Omega_H$ as the generator wraps around the black hole and feels the full angular variation of the tidal field.

To test this explanation, let us see what happens when the orbit is kept approximately constant, but we modify the Green's function. In Fig. 4-6, we show $\sigma_+$ for the orbit $p = 3M$, $e = 0.7$, $\theta_{\text{inc}} = 0^\circ$ about 4 different black holes: $a = 0.96M$ (for which $\kappa^{-1} = 9.1M$), $a = 0.97M$ ($\kappa^{-1} = 10.2M$), $a = 0.98M$ ($\kappa^{-1} = 12.1M$), and $a = 0.99M$ ($\kappa^{-1} = 16.2M$). Notice that the decay time $\kappa^{-1}$ varies by a factor of nearly two over this spin range; by contrast, the timescale $T_H$ varies by only about 15%, from $T_H = 16.8M$ at $a = 0.96M$ to $T_H = 14.5M$ at $a = 0.99M$.

The behavior we see in Fig. 4-6 demonstrates that the exponential growth time has a large influence on whether these wiggles can be seen in $\sigma_+$: When $\kappa^{-1}$ is smallest, the wiggles are barely apparent, and vice versa. This is consistent with the fact that the Green’s function quickly decays away when $\kappa^{-1}$ is small, but remains large for a large interval when $\kappa^{-1}$ is large. In particular, wiggles can barely be discerned in the $a = 0.96M$ panel. They become gradually more apparent as the spin is increased, and are quite prominent when $a = 0.99M$. Increasing the spin further just makes the wiggles stand out more clearly, as we see in the right-hand panel of Fig. 4-5. This aspect of the Green’s function is only important for strong-field orbits and for rapid black hole spin, but has a very clear and interesting impact in that case.
4.5 Horizon dynamics III: Horizon embeddings

In this section, we examine dynamical horizon embeddings for several representative orbits\(^2\). Our goal will be to show how the horizon behaves as a function of the orbit’s behavior, so we will show a sequence of figures that show both the tidally distorted horizon and the smaller member of the binary. As discussed at length in chapter 3, there is substantial ambiguity in such a plot, associated with the fact that the horizon and the orbit are at different positions in a curved spacetime. Comparing the horizon and the orbit requires that we carefully define exactly what is shown. Following the choices that we made in chapter 3, our plots are all shown on a slice of constant ingoing time \(v\); this is equivalent to what we called the “instantaneous map” in chapter 3.

In this chapter, we embed the horizon in a Euclidean three-dimensional space, as we did in chapter 3. As discussed in Sec. 4.1, this means we are confined to spin parameter \(a/M \leq \sqrt{3}/2\); for faster spins, even an undistorted horizon cannot be embedded in this geometry. This unfortunately means that we cannot look for signatures related to the small amplitude wiggles that we discussed in Sec. 4.4.2 — they are only strong and clear for nearly maximal Kerr black holes. As mentioned previously, work in progress indicates that embedding the horizon in the globally hyperbolic space \(H^3\), following [153], is a very elegant way to get around this restriction. As this work develops, looking for this signature of the teleological Green’s function will be a priority.

In chapter 3, the embedded horizon plots that we showed were all scaled by a factor \(r_{orb}^3\), compensating for the scaling of tidal quantities with orbital separation. This made it easier to compare tidal distortions at different separations. In this chapter, since we examine eccentric orbits, all of our embeddings are scaled by \(r_{\text{min}}^3 = p^3/(1 + e)^3\). This compensates for the scaling of tidal quantities at periapse, and insures that the maximum distortion we show is roughly the same in all the cases

\(^2\)In the final version of this manuscript (which will be submitted to Physical Review D), we will broaden this analysis to examine the curvature of the horizon \(R_{H}^{(1)}\) and other measures that illustrate the behavior of a dynamically perturbed horizon.
we present. The cases we examine in detail below are associated with Figs. 4-7 -
4-10. These figures are each a series of snapshots taken from animations showing the
combined orbital and embedded horizon dynamics. These animations are available
at http://gmunu.mit.edu/viz/embed_viz/embed_viz.html; the reader may find it
useful to examine these visualizations in concert with the text presented below.

4.5.1 Embeddings from inclined circular orbits

We begin with an especially simple case: an inclined, circular orbit of a Schwarzschild
black hole. Figure 4-7 shows an embedding of the distorted horizon for the case of
a circular orbit with radius $r = 6M$ inclined at $\theta_{\text{inc}} = 60^\circ$. We show 12 frames
illustrating the horizon embedding and particle motion for this orbit; the frames are
evenly spaced over nearly one orbital period ($T_{\text{orb}} = 92.3M$ for a circular orbit at
$r = 6M$ for Schwarzschild). Axes indicate the location of the equatorial plane; they
are static in this sequence, since the horizon of a Schwarzschild black hole is static.
As noted in the preamble to this section, we have scaled the horizon’s distortion by a
factor of $p^3/(1 + e)^3$; this is of course just the orbital radius cubed for a circular orbit.
As such, the distortion here (and in the figures which follow) is somewhat exaggerated
in magnitude.

As should be expected following Sec. 4.3, the results we see in Fig. 4-7 are con-
sistent with the fact that the physics of an inclined orbit is identical to that of an
equatorial orbit in the $a = 0$ limit. In particular, the embedded horizon is identical
to that shown in the $r_{\text{orb}} = 6M$ panel chapter 3’s Fig. 3-3, but with the distortion
centered on a plane that is inclined at $\theta_{\text{inc}} = 60^\circ$ to our chosen equator. The offset
between the orbit and the horizon’s bulge is constant over the orbit (this can be
seen particularly clearly in an animation of the horizon and orbit dynamics), with
the bulge leading the orbiting body’s position by some fixed angle. As we have pre-
viously discussed, this can be understood as due to the spherical symmetry of the
Schwarzschild spacetime — the magnitude of the tidal field does not vary over the
course of the orbit. In chapter 3, the lead angle was purely axial (i.e., purely in the
direction of $\psi$); here the lead is a mixture of the axial and polar angles $\psi$ and $\theta$. As
Figure 4-7: Snapshots of an animation depicting an embedding of the distorted horizon for a circular inclined orbit of a Schwarzschild black hole \((a = 0)\). The orbit is at radius \(r = 6M\), inclined at \(\theta_{inc} = 60^\circ\) to our chosen equatorial plane. The axes shown here indicate the hole’s equatorial plane; we have placed the camera slightly above this plane in order to illustrate the hole’s bulge geometry. The orbiting body is indicated by the small “moon” (dark blue in color plot) located near one of the horizon bulges. The small body’s orbit begins on the side of the black hole near the camera, descends down through the equatorial plane (crossing just after \(v = 7.8M\)), sweeps behind the far side of the hole (moving from right to left as plotted), then comes up through the equatorial plane (crossing just after \(v = 54.6M\)) to pass in front of the side near the camera again. Notice that the horizon’s distortions move in lockstep with the orbiting body, with the horizon’s bulge always leading the orbiting body’s position by a small, constant angle. The animation from which these stills are taken is available at http://gmunu.mit.edu/viz/embed_viz/embed_viz.html.
is discussed extensively in chapter 3 (and briefly in Sec. 4.1), horizon bulge leading the orbit is exactly what we expect for circular orbits in the Schwarzschild limit.

We next consider an inclined, circular orbit of a Kerr black hole. Circular Kerr orbits are defined as those for which the Boyer-Lindquist coordinate radius \( r \) is constant over an orbit. Although they are therefore closely tied to a particular coordinate system, they nonetheless are a well-defined and well-studied subset of Kerr orbits. In addition, it has been shown that the eccentricity \( e \) of Kerr orbits [defined in Eq. (4.4)] decreases over all but the most strong-field orbits due to gravitational-wave driven backreaction [154, 139], and that orbits with \( e = 0 \) remain at \( e = 0 \) [155, 156, 157]. As such, we expect that gravitational-wave emission will drive large mass-ratio binaries toward the constant Boyer-Lindquist radius circular limit.

Figure 4-8 is much like Fig. 4-7, but for an orbit of a black hole with spin parameter \( a = 0.6M \). The orbit again has constant radius \( r = 6M \), and is inclined at \( \theta_{\text{inc}} = 60^\circ \). We again show 12 frames illustrating the horizon and particle motion for this orbit, with frames evenly spaced over nearly one orbital period\(^5\). In this sequence, the axes (which indicate the equatorial plane) are tied to the horizon’s spin, which completes a full rotation in a period \( T_H = 2\pi/\Omega_H = 37.7M \).

Some new horizon dynamics begin to appear in Fig. 4-8. Over the course of a single orbit, the tidal field arising from the small body is not of constant magnitude since the spacetime is no longer spherically symmetric. As a consequence the shape of the embedded horizon varies over the course of the orbit. There is also interesting new behavior associated with the bulge-orbit offset. As discussed at length in chapter 3 (and briefly in Sec. 4.1), for circular, equatorial orbits of rapidly spinning black holes, the horizon bulge tends to lag the position of the orbiting body on a constant \( v \) timeslice. Let us call this “Kerr-like” bulge-orbit behavior, and let us call the opposite behavior (bulge leading the behavior of the orbit on a constant \( v \) timeslice)

\(^5\)It’s worth noting that “orbital period” is somewhat ambiguous in this case: the period to complete a single polar oscillation is \( T_\phi = 98.7M \), and the period to complete a rotation of \( 2\pi \) radians in the axial direction is \( T_\phi = 91.5M \). In this case at least, these two periods differ only by \( \sim 10\% \), so our statement that we show nearly one period is accurate no matter which notion of period we use. Notice that \( T_\phi < T_\phi \); this holds for any inclined Kerr orbit with \( L_\phi > 0 \), and reflects frame dragging by the black hole’s rotation. The orbit moves through more than \( 2\pi \) radians of axial phase over a single polar oscillation.
Figure 4-8: Identical to Fig. 4-7, except that the black hole shown here is spinning with Kerr parameter $a = 0.6M$. As in Fig. 4-7, the axes indicate the hole’s equatorial plane. In this plot, these axes rotate with the event horizon at frequency $\Omega_H = a/2Mr_+ = 1/6M$ (corresponding to a rotation period $T_H = 2\pi/\Omega_H = 37.7M$). In this sequence, the orbiting body (small sphere, dark blue in color) begins near the face close to the camera on the lower right-hand side of the black hole. It then sweeps up, crossing the equator soon after $v = 16.13M$, passes behind the black hole, and then descends downward again, crossing the equator soon after $v = 64.5M$. The horizon’s distortions in this case do not move in lockstep with the orbiting body. Instead, the horizon exhibits mild shape variations. This is because the hole is not spherically symmetric, and so the tidal field acting on the horizon varies slightly over the orbit. Notice that the horizon’s bulge leads the angular position of the orbit in the polar ($\theta$) direction, but lags its position in the axial ($\psi$) direction. The polar behavior is much like what we see for Schwarzschild or very slow rotation; the axial behavior is about the same as the behavior we saw for equatorial circular orbits in chapter 3. The animation from which these stills are taken is available at http://gmunu.mit.edu/viz/embed_viz/embed_viz.html.
"Schwarzschild-like." What we see in Fig. 4-8 is that the bulge behaves in a Kerr-like manner in the $\psi$-direction, but behaves in a Schwarzschild-like manner in the $\theta$-direction.

Let us examine this interesting behavior more carefully. The "camera" which defines the viewpoint used for the sequence of stills in Fig. 4-8 is located slightly above the equatorial plane. With this in mind, consider first the frames at $v = 16.13M$ and $v = 64.50M$. The orbiting body sweeps from below to the equatorial plane on the side of the black hole near the viewer, crossing the plane slightly after $v = 16.13M$. It then sweeps up and behind the hole as seen by the viewer before changing direction and descending, crossing the equatorial plane slightly after $v = 64.50M$. Notice that the bulge in the horizon crosses the equatorial plane slightly before the orbiting body: The frames at $v = 16.13M$ and $v = 64.50M$ show quite clearly that the bulge leads the orbit's position — the bulge leads the orbit's $\theta$ position in a Schwarzschild-like manner. By contrast, many of these frames show that the bulge lags the orbit's $\psi$ position in a Kerr-like manner — the frames at $v = 32.25M$, $40.32M$, $80.63M$, and $88.69M$ make this behavior quite clear.

We have found that this bulge-orbit behavior (Schwarzschild-like with respect to the $\theta$ direction, Kerr-like with respect to the $\psi$ direction) is quite generic: It is quite apparent in all the circular, inclined cases we have examined, and appears in inclined eccentric examples as well. This behavior clearly arises from the fact that the black hole's spin picks out the $\psi$ direction as special. The hole's rotation plus the horizon's teleological nature mixes time and axial angle: a tide that would produce a bulge on a Schwarzschild black hole at $(\theta_{\text{max}}, \psi_{\text{max}})$ will produce a bulge on a Kerr black hole at roughly $(\theta_{\text{max}}, \psi_{\text{max}} - \delta\psi)$, where $\delta\psi$ is (at leading order) proportional to the black hole's spin parameter $a$.

### 4.5.2 Horizon embeddings from eccentric orbits

We conclude this chapter by examining horizon embeddings for highly eccentric black hole orbits. The key point to bear in mind here is that, at leading order, the tidal field varies with orbital separation as $1/r^3$. As such, the tidal field from an orbit
with eccentricity $e$ varies by a factor $(1 + e)^3/(1 - e)^3$ over the course of an orbit. This factor grows very quickly with $e$. The two cases we will examine in detail have $e = 0.7$, for which the tide varies by a factor of about 180. This means that the hole can be essentially unaffected by its companion for much of the orbit, but be highly distorted as the smaller body passes through periapsis.

Figure 4-9 shows this behavior quite clearly; see http://gmunu.mit.edu/viz/embed_viz/embedviz.html for the animation from which these stills were taken. The large black hole used here has spin $a = 0.85M$, nearly the largest value for which a globally Euclidean embedding exists. The orbit is equatorial ($\theta_{inc} = 0^\circ$), quite strong field ($p = 4M$), and highly eccentric ($e = 0.7$). We only show a portion of a full radial cycle, from $r \approx r_{\text{max}}/2$ to $r_{\text{min}} = p/(1 + e)$ back to $r \approx r_{\text{max}}/2$. As in Fig. 4-8, the axes indicating the equatorial plane rotate with the horizon. For $a = 0.85M$, the period of this rotation is $T_H = 22.6M$. We sample our animation every $5.6M$. By coincidence, this is nearly $T_H/4$, so the axes are sampled in a nearly stroboscopic fashion, and appear to be stationary.

The embedded horizon of an undistorted $a = 0.85M$ black hole is an oblate ellipsoid that is nearly flat at the poles. This geometry can be seen in the first and last few frames shown in Fig. 4-9 — the tidal field is so weak in these frames\footnote{Indeed, this is why we show only a fraction of an orbit here. A full radial cycle of this orbit takes $T_r = 229.8M$, with the tide having a large impact only for $r \approx r_{\text{min}}$. The hole is practically undistorted for the majority of the time spent in orbit.} (for which $r \approx r_{\text{max}}/2$) that the horizon is not noticeably distorted by the companion. The distortion becomes quite strong as the orbital approaches periapsis: We see the horizon beginning to change shape at $v = 11.03M$, and is highly distorted over the range $22.06M \leq v \leq 33.10M$. At its peak, the horizon’s distortion is similar to the most distorted horizon embedding shown in chapter 3, the right-hand panel of that chapter’s Fig. 3-7. Notice that the bulge’s position in $\psi$ lags the orbit in all cases. This is quite clear in the $v = 27.58M$ panel. Since the orbit is equatorial, we do not have any lag or lead associated with the angle $\theta$.

Figure 4-10 shows the embedding for a horizon distorted by tides from a generic orbit. We again consider spin $a = 0.85M$, and a very strong-field ($p = 4M$), highly
Figure 4-9: Snapshots of an animation depicting an embedding of the distorted horizon for an equatorial eccentric orbit of a rapidly spinning Kerr black hole ($a = 0.85M$). The spin is nearly the largest value for which a Euclidean embedding of the undistorted horizon exists; the undistorted embedding geometry at this spin is of an axially symmetric oblate spheroid that is nearly flat at its poles. The orbit has semi-latus rectum $p = 4M$ and eccentricity $e = 0.7$, so its orbital radius varies from $r_{\text{max}} = 13.33M$ to $r_{\text{min}} = 2.34M$. As in Fig. 4-8, the axes indicating the equatorial plane rotate at the horizon frequency $\Omega_{H} = a/2Mr_{+}$, corresponding to period $T_{H} = 22.6M$. The axes complete a quarter turn every $5.6M$; this is very close to the cadence with which we sample this animation, so the axes are nearly stationary in this sequence. The orbiting body is shown moving from roughly $r_{\text{max}}/2$ to $r_{\text{min}}$, and back out to roughly $r_{\text{max}}/2$. For the first and last few stills shown here, the embedded horizon is nearly identical to that of an undistorted Kerr black hole. The embedded horizon by contrast is highly distorted in stills corresponding to $r \approx r_{\text{min}}$ ($16.55M \leq v \leq 44.13M$). This reflects the fact that the tidal field varies at leading order as $1/r^3$, which changes by a factor $(1 + e)^3/(1 - e)^3$ over an eccentric orbit. The tidal field thus varies by a factor $\sim 180$ for this orbit; even over this limited segment (for which the orbit only goes out to about $r_{\text{max}}/2$), the tidal field varies by $\sim 180/8 \approx 22$. The horizon bulge lags the particle’s position at all times, consistent with the behavior seen and discussed in chapter 3 for rapidly rotating Kerr black holes. The animation from which these stills are taken is available at http://gmunu.mit.edu/viz/embed_viz/embed_viz.html.
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eccentric \((e = 0.7)\) orbit, but we now take the orbit to be inclined at \(\theta_{\text{inc}} = 30^\circ\). The set of frames we show again corresponds to motion from roughly \(r_{\text{max}}/2\) to \(r_{\text{min}}\) and back to nearly \(r_{\text{max}}/2\). We have moved the "camera" in this sequence to a point slightly above the equatorial plane in order to more clearly see the orbit’s polar motion, and the distortions associated with motion above and below this plane.

The embedding dynamics shown in Fig. 4-10 combines the features found for inclined but circular orbits with those found for eccentric but equatorial orbits. In particular, notice that the embedded horizon geometry is practically undistorted in the first frame, as well as the last two or so frames. This again reflects the large range of the tidal field that acts on the horizon for eccentric orbits; when \(r \gtrsim r_{\text{max}}/2\), the horizon’s distortions are so mild that they cannot be seen with these graphics. A full radial cycle of this orbit takes \(T_r = 255.1M\), so the horizon is practically undistorted for a large fraction of this orbit. As the orbit oscillates above and below the equatorial plane, the horizon’s bulge likewise oscillates above and below the plane. As in Fig. 4-8, we see that the bulge lags the orbit’s \(\psi\) position, but leads its \(\theta\) position. This is the Kerr-like behavior in the axial direction that we see when the hole is rapidly rotating, but the Schwarzschild-like behavior that is seen in the polar direction. Having already examined the equatorial and the circular limits in great detail, it is safe to say that there are no surprises to be found in Fig. 4-10: the interesting behaviors seen in the previously considered cases combine in the generic case in a very logical way.

4.6 Conclusions

In this chapter, we have taken the tools that we introduced in chapter 3 for studying event horizons that are distorted by a strong-field (but small mass ratio) binary companion, and have applied them to eccentric and inclined binaries. For such orbits, the on-horizon tidal field varies significantly over the course of an orbit, leading to dynamical event horizon behavior. We have studied these horizon dynamics with multiple measures, examining the phase offset between the applied tide and the resulting shear to the horizons, as well as examining embeddings of the distorted horizons in a
Figure 4-10: Snapshots of an animation depicting an embedding of the distorted horizon for a generic orbit of a rapidly spinning Kerr black hole ($a = 0.85M$). The system is nearly identical to that used in Fig. 4-9, but we have inclined the orbit to $\theta_{\text{inc}} = 30^\circ$. The orbiting body is again shown moving from roughly $r_{\text{max}}/2$ to $r_{\text{min}}$, and back out to roughly $r_{\text{max}}/2$. The horizon’s dynamics here shares features with both the equatorial case depicted in Fig. 4-9 and the inclined cases in Figs. 4-7 and 4-8. In particular, the horizon varies from nearly undistorted when $r \approx r_{\text{max}}/2$ (roughly first and last stills in this sequence) to highly distorted when $r \approx r_{\min}$ (stills from $28.52M \leq v \leq 57.15M$), in a manner qualitatively similar to the eccentric equatorial case. However, the horizon bulge flexes above and below the plane as the orbital motion oscillates in the polar direction, very much like the circular inclined cases. Careful study of the relative position of the bulge and the orbiting body shows that the bulge lags the body with respect to the axial angle $\psi$, but leads the body with respect to the polar angle $\theta$ — exactly the same offset behavior that was seen in the circular, inclined Kerr case (Fig. 4-8). The animation from which these stills are taken is available at http://gmunu.mit.edu/viz/embed_viz/embed_viz.html.
globally Euclidean 3-space.

Many of the results we have found follow in a fairly natural and logical way from results that were shown in chapter 3. In particular, we find that tidal bulges tend to lead the position of the orbiting body for very slow black hole spin, but lag the orbit for fast black hole spin. This is exactly the teleological tidal behavior that was seen with the simpler orbits we examined in chapter 3. We find an interesting variant of this behavior in the present analysis by looking at orbits that are inclined with respect to the hole’s equatorial plane: The bulge tends to lead the orbit in the $\theta$ direction (“Schwarzschild-like” behavior), but lags the orbit for rapid spin in the $\psi$ direction (“Kerr-like” behavior). The fact that the bulge exhibits different behavior with respect to the two angles is not surprising, since the hole’s spin is along $\psi$.

One interesting and new behavior we have seen here are the low-amplitude wiggles associated with the teleological Green’s function for very rapidly rotating black holes. These wiggles only appear as the spin approaches the maximal Kerr value, values for which the inverse surface gravity $\kappa^{-1}$ (which sets the timescale over which the Green’s function is important) becomes very long. So far, we have only seen this behavior in our comparison of the on-horizon tides to the horizon generators’ shear response. The spins at which these wiggles appear are so large that these black holes cannot be embedded in a globally Euclidean 3-space. We plan to look for this very interesting signature in future work that will generalize our embedding algorithm to a different global space which can accommodate very rapid black hole spins.

Appendix 4.A Newman-Penrose fields and on-horizon tensors

In this chapter, we work with quantities that are based on Newman-Penrose fields such as the complex curvature scalar $\psi_0$. Other works, notably VPM11, use tensors which live in the manifold defined by the black hole’s event horizon. There is a simple one-to-one correspondence between these two representations for the quantities which
are important for our analysis. We develop this correspondence in this appendix.

We begin by defining some notation and background. As elsewhere in this paper, we use ingoing Kerr coordinates \((v, r, \theta, \psi)\) here. Components of tensors in the 2-dimensional manifold of the black hole's event horizon are labeled with upper-case Latin indices; these components range over the set \((0, 4)\). (As elsewhere, Greek indices denote tensors in 4-dimensional spacetime.) Define the projection tensor \(P^A_\alpha\), whose components in ingoing Kerr coordinates are given by the matrix

\[
P^A_\alpha = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};
\]

the components of the inverse tensor \(P^\beta_B\) are the transpose of this. When this operates on tensors at \(r = r_+\), it projects quantities onto a slice of constant \(v\) on the horizon. At a given moment on the horizon, the spacetime's line element (4.1) becomes

\[
ds^2 = g_{AB}dx^A dx^B = \Sigma_+ d\theta^2 + \frac{4M^2r_+^2 \sin^2 \theta}{\Sigma_+} d\psi^2,
\]

where \(\Sigma_+ = r_+^2 + a^2 \cos^2 \theta\). Finally, we will need the Newman-Penrose null legs in the Hawking-Hartle representation \([113, 112]\); these are given in Eqs. (4.8)–(4.10). For \(r \to r_+\),

\[
l^\mu \to [1, 0, 0, a\Omega_H],
\]

\[
m^\mu \to \frac{1}{\sqrt{2(r_+ + ia \cos \theta)}} [0, 0, 1, i (\csc \theta - a\Omega_H \sin \theta)].
\]

(We will not need \(n^\mu\).) At \(r = r_+\), \(l^\mu\) is tangent to the null generators of an unperturbed Kerr hole's horizon. Let us manipulate \(m^\mu(r_+)\): we write

\[
m^A(r_+) \equiv m^\mu(r_+) P^A_\mu = \frac{1}{\sqrt{2}} (\alpha^A + i\beta^A),
\]
where
\begin{align}
\alpha^A &= \frac{1}{\Sigma_+} [r_+, a \cos \theta (\csc \theta - a \Omega_H \sin \theta)] , \\
\beta^A &= \frac{1}{\Sigma_+} [-a \cos \theta, r_+ (\csc \theta - a \Omega_H \sin \theta)] .
\end{align}
(4.54)  (4.55)

Notice that $g_{AB} \alpha^A \alpha^B = g_{AB} \beta^A \beta^B = 1$, $g_{AB} \alpha^A \beta^B = 0$.

The intrinsic geometry of the horizon is governed by the Weyl curvature. In our analysis, we use the Newman-Penrose scalar $\psi_0$, which is given by
\[ \psi_0 = -C_{\mu\nu\rho\sigma} l^\mu m^\nu p^\rho q^\sigma . \] (4.56)

Our focus is on this quantity on the horizon. Let us define
\[ C_{AB} \equiv (C_{\mu\nu\rho\sigma} l^\mu p^\rho q^\sigma)_{r_+} , \] (4.57)
where the subscripted $r_+$ means that all the quantities in parentheses are to be evaluated at $r = r_+$. This definition is identical to that in VPM11 [see text following their Eq. (2.30)]. Using this, on the horizon we have
\begin{align}
\psi_0 &= -C_{AB} m^A m^B \\
&= -\frac{1}{2} C_{AB} (\alpha^A \alpha^B - \beta^A \beta^B + i \alpha^A \beta^B + i \beta^A \alpha^B) \\
&= -C_{AB} (e^A_{+} + ie_{x}^{AB}) . \] (4.58)
\end{align}

On the second line, we used Eq. (4.53); on the third, we introduced the polarization tensors
\begin{align}
e^A_{+} &= \frac{1}{2} (\alpha^A \alpha^B - \beta^A \beta^B) , \\
e_{x}^{AB} &= \frac{1}{2} (\alpha^A \beta^B + \beta^A \alpha^B) .
\end{align} (4.59) (4.60)
We further simplify Eq. (4.58) by defining the Weyl polarization components:

\[ C_+ \equiv C_{AB} e_{+}^{AB} , \]  
\[ C_\times \equiv C_{AB} e_{\times}^{AB} , \]  

yielding

\[ \psi_0 = -C_+ - i C_\times . \]  

In other words, the on-horizon Weyl polarizations are simply the real and imaginary parts of \( \psi_0 \) on the horizon, modulo an overall sign.

Lowering indices on the polarization tensors,

\[ e_{+}^{AB} = g_{AC} g_{BD} e_{+}^{CD} , \]  
\[ e_{\times}^{AB} = g_{AC} g_{BD} e_{\times}^{CD} , \]  

allows us to construct the on-horizon Weyl tensor from the polarization components:

\[ C_{AB} = C_+ e_{+}^{AB} + C_\times e_{\times}^{AB} . \]  

Another important Newman-Penrose quantity which we can analyze in this manner is the spin coefficient

\[ \sigma = m^\mu m^\nu \nabla_\mu l_\nu . \]  

At \( r = r_+ \), this describes the shear of the horizon’s generators. Because \( m^\nu = m^r = 0 \) at \( r = r_+ \), we have

\[ \sigma(r_+) = (m^A m^B \nabla_A l_B)_{r_+} . \]  

Define

\[ \sigma_{AB} = \frac{1}{2} (\nabla_A l_B + \nabla_B l_A)_{r_+} . \]  

Note that \( \sigma_{AB} \) is trace free since \( \nabla_A l^A = 0 \) on the horizon. This definition of the shear tensor for the horizon’s null generators is therefore equivalent to that used in VPM11 [compare their Eqs. (2.11) and (2.15)]. Using Eqs. (4.53), (4.64), and (4.65),
we find

\[ \sigma(r_+) = \sigma_{AB} (e_+^{AB} + ie_x^{AB}) = \sigma_+ + i\sigma_x. \] (4.70)

The shear polarizations written here were introduced by VPM11; they are defined in a manner analogous to \( C_+ \) and \( C_x \), and are just the real and the imaginary parts of the Newman-Penrose quantity \( \sigma \).

References


REFERENCES


