Many-body Entanglement: Topological Orders, Tensor Networks and Superconductivity

by

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Submitted to the Department of Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Abstract

In this thesis, we discuss the characterization and application of quantum many-body entanglements. We try to establish a non-local “order parameter” description of different patterns of many-body entanglement, which are also named topological orders. In 2+1D, we show that this could be achieved by calculating the non-Abelian geometric phase (S, T)-matrices from the fixed-point wave functions, obtained in the string-net approach by Levin and Wen and the local unitary transformation approach by Chen, Gu and Wen. In doing so, (S, T)-matrices act as our non-local “order parameter” and give a full characterization of 2+1D exact topological orders (topological orders that have a gappable edge). For a generic non-fixed-point wave function, however, obtaining the (S, T)-matrices is numerically formidable. To go around this problem, we introduce a new tensor-network method that works on any generic wave function, and obtain the “environment matrix” as a less powerful “order parameter” description. The “environment matrix” can characterize topological orders described by any gauge theory as well as 1D symmetry protected topological (SPT) orders. As an application of both the concept of many-body entanglement and the new tensor-network method developed earlier, in the last chapter of the thesis, we propose a non-BCS mechanism for superconductivity, in which the driving force is not traditional pair-attraction, but statistical confusion of charge carrier induced by strong many-body entanglement. This may open new doors for identifying and constructing new superconducting states.

Thesis Supervisor: Xiao-Gang Wen
Title: Professor of Physics
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To my wife, Fan Jiang, for always being there with me, through the best and the worst times.

"It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair."

- Charles Dickens
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Chapter 1

Introduction

1.1 Characterizing Long-range Entanglement

The history of condensed matter physics has been continuously marked by the discovery and understanding of new phases of matter. From the first experimental discovery of superconductors in 1911 [50] to the discovery of superfluidity in liquid Helium in 1937 [1], then to the theoretical understanding of superfluidity by Landau in 1941 [41] and the microscopic understanding of superconductivity through the famous BCS theory in 1957 [3], new phases of matter have inspired generations of physicists to try and unveil the mystery of our mother nature.

In trying to understand different phases of matter, one of the most important questions one may ask is, how should we characterize and classify these different phases of matter? Landau's symmetry-breaking theory [40, 21] provides a very elegant answer to this question. In Landau's theory, different phases are characterized by different symmetries. Thus by classifying all different symmetries in the system, we can get a classification of different phases, and the phase transitions between these phases will be marked by certain symmetry being broken, hence the name "symmetry-breaking theory". The symmetry-breaking theory soon gained its popularity, and it once seemed that all phases of matter and their transitions could be explained by it.

Nature however never stops testing the boundary of our understanding. As is with all great theories, though extremely powerful, Landau's theory soon failed to
explain many new states found in later experiments, including the famous Fractional Quantum Hall (FQH) states. These new states thus bring back once again the old question: how should we classify these new phases of matter?

It was soon realized that the key lies in the concept of entanglement. A closer look at Landau’s symmetry-breaking theory reveals that the theory only describes direct-product states up to some small local perturbations (see [11]). Since these small perturbations can only modify the direct-product states locally, Landau’s theory can only describe states with “short-range entanglement”. Intuitively, these states can only account for a small fraction of all possible quantum many-body states. Thus what seems to be missing in Landau’s theory is a large class of states with “long-range entanglement”. Those states were named topologically ordered states in 1989 even before the concept of entanglements became popular. Thus it seems that different quantum phases could then be characterized by different patterns of long-range-entanglement/topological-order.

But what are these patterns of long-range entanglement? What are topological orders? It is hard to answer these questions since those patterns/orders are new concepts that do not even have a label to characterize them. So to study topological-order/long-range-entanglement, we first need to invent labels or mathematical symbols for them. The first label/symbol invented for 2+1D topological orders is the ground state degeneracy \( D_{\text{deg}}(g) \) on torus (with genus \( g = 1 \)) and other Riemann surfaces. But it was clear from the beginning that this is not a very good label, since the same \( D_{\text{deg}}(g) \) can correspond to many different topological orders.

A much better and very powerful label for topological order was later developed by Wen and collaborators in two equivalent formulations, namely the string-net approach of [46] and the local unitary transformation approach of [11]. They provide a systematic description of all 2+1D exact topological orders (i.e. the topological orders that can have gapped boundary) by systematically labeling the corresponding “long-range entangled” many-body wave functions through a set of fixed-point tensors. Here the term “fixed-point tensors” refers to the set of data characterizing the string-net wave function in [46], or equivalently the fixed-point wave function under
the renormalization scheme in [11]. Thus once we know the fixed-point tenors, we know the fixed-point wave function. These fixed-point tensors form a mathematical structure called unitary fusion category (UFC) theory. Since the many-body wave functions described by the fixed-point tensors are explicitly known, we can even construct exactly soluble Hamiltonians to realize each topological order described by a fixed-point tensor, and both theories reached great success. In this paper, we will refer to both of them as the fixed-point tensor approach.

Although the fixed-point tensor approach (i.e. the string-net or the local-unitary-transformation approach) successfully characterized different patterns of long-range entanglement through the use of fixed-point tensors, it is known that different fixed-point tensors can correspond to the same quantum phase. Mathematically, since different fixed-point tensors describe different UFC’s, it turns out that two different UFC’s will always describe the same $2 + 1D$ topological order if they have the same Drinfeld center.[37, 39] Thus the fixed-point tensor approach does not provide a one-to-one label for different topological orders.

In chapter 2 of this thesis, to obtain a even better label, we follow a very different route from the above by considering the non-Abelian geometric phases. In [71, 33], it was conjectured that the non-Abelian geometric phases[75] of degenerate ground states on Riemann surfaces can completely characterize different topological orders. We try to make this idea fully concrete by introducing the $T$- and $S$- matrices, to be introduced in chapter 2, and propose a new theory for exact topological orders in $2 + 1D$. We believe that the $T$- and $S$- matrices give a one-to-one label for different exact topological orders, thus providing a full characterization of different patterns of long-range entanglement.

To draw a connection between our new theory and the existing fixed-point tensor approach, in chapter 2, we also detail a procedure to calculate the $T$- and $S$- matrices from the fixed-point wave functions (obtained from the fixed-point tensors). This allows us to identify various previously discovered topological orders by giving them the new $T$- and $S$- matrices label, and in doing so obtain many physical properties of them. In particular, we can obtain the number of quasiparticle types, the quasiparticle
statistics and quantum dimensions, etc. directly from the $T$- and $S$- matrices. Thus when given any fixed-point wave function, we can directly tell which topological order it belongs to through the $T$- and $S$- matrices, making our new characterization very useful in identifying different phases of matter.

1.2 Short-range Entanglement and Non-fixed-point Wave Function

In the previous discussion, we mainly focused on the characterization of patterns of long-range entanglement. A natural question to ask is, how about short-ranged entangled states? Are they all described by Landau’s symmetry-breaking theory?

It turns out that even short-ranged entangled states can go beyond Landau. The recently discovered topological insulator is a perfect example of non-trivial short-range entangled state [31, 8, 47, 19, 35, 56]. In this state, the bulk behaves as an insulator but the surface is conducting, and the surface states are protected by certain symmetries. These non-trivial short-range entangled phases are named Symmetry Protected Topological (SPT) Orders [26].

Much progress has been made recently in the study of SPT, and a full classification of SPT orders has been reached through tensor network and group cohomology[10, 32, 73, 26]. In particular, in the tensor network approach [26], different SPT phases are again characterized by fixed-point tensors, much like the fixed-point tensor approach discussed in last section for describing long-range entanglements. In this tensor network approach, the fixed-point tensors used to describe different SPT phases are also obtained through a tensor renormalization scheme.

Now that we can systematically characterize both long-range entangled phases (i.e. topological orders) and non-trivial short-range entangled phases (i.e. SPT orders) for any fixed-point wave function, the next natural question to ask is, what about non-fixed-point wave functions? How do we determine the topological order or SPT order for a generic wave function?
Lots of attempts have been made in trying to answer the above question, many of which apply different tensor-network renormalization schemes in the hope of getting the fixed-point wave function [71, 33, 69, 78, 14, 77, 62, 11, 54, 30, 70, 48, 29]. Though a lot of them reached great success in 1D, a higher-dimension generalization is still a numerically formidable task, with many 2D tensor-network renormalization schemes facing the infamous “corner double-line” problem [24, 26]. When this happens, the tensor-network renormalization quickly breaks down after just a few iterations, making it impossible to obtain the fixed-point wave function. Thus a computationally efficient tensor-network method to implement RG is badly desired.

In chapter 3 of this thesis, we try to tackle this problem by explore a different way of thinking. There, we introduce a very efficient “mean-field” tensor-network approach based on the concept of “environment matrix”. Through studying the symmetry structure of the environment matrix, a large class of topological orders and SPT phases could be obtained from generic wave functions, making it very useful in identifying patterns of both long-range and short-range entanglement. Here, we want to point out that our “mean-field” tensor-network approach is by no means comprehensive: it can only be used to identify topological orders that correspond to gauge theories and SPT in 1D. Thus future work still needs to be done to fully determine the topological order or SPT order for a generic wave function.

1.3 From Entanglement To Superconductivity

Superconductivity, the central topic that almost spanned through the history of condensed matter physics, has no doubt long been a focus of the entire field. At the beginning of this chapter, we have already mentioned the first experimental discovery of superconductors in 1911 [50] and the magnificent microscopic BCS theory in 1957 [3]; later on, the experimental discovery of High-temperature superconductors in 1986 [7] inspired yet another wave of debates on the mechanism of High-Tc, some of them are still on-going even today. In this process, the brilliant BCS theory was so successful that it now becomes the golden standard for all theories describing su-
perconductivity, and the “cooper pair” condensation idea is now deeply rooted into our understanding of superconductivity.

But is pair condensation the only means to superconductivity? In chapter 4 of this thesis, as an application of the entanglement study done in previous chapters, we explore a different possible mechanism for superconductivity where the driving force is not pair condensation, but strong entanglements in the system. The root of this idea goes all the way back to 1987, when Anderson himself suggested the use of doped resonating valence bond (RVB) state to study High-Tc [6]. In this thesis, however, we certainly do not attempt to answer the question of what mechanism is truly behind High-Tc; we merely want to explore the possibility of having superconductivity driven by statistical confusions.

The idea of statistical confusion of charge carrier is intimately related to the notion of spin-charge separation [36]. When we dope a system with strong entanglement, the charge carrier (doped holes) may change its statistics. When the charge carriers behave like bosons, they may directly Bose-Einstein condense, leading to superconductivity without the need of pair attractions and pair condensation. It is precisely this change in statistics that drives the superconducting state, which is directly due to the entanglements in the system. This gives us a new perspective in the understanding of superconducting states, and may open new doors towards identification and construction of new superconducting states.
Chapter 2

A New Characterization of $2 + 1D$ Topological Orders

2.1 Outline of the chapter

Following the discussion in chapter 1, in this chapter, we introduce a new characterization for topological orders in $2 + 1D$ through the non-Abelian geometric phases. The non-Abelian geometric phases contain a universal non-Abelian part\cite{74, 33} and a path dependent Abelian part\cite{74, 69}. The non-Abelian part carries information about the $T$- and $S$-matrices while the Abelian part carries information about the chiral central charge $c$. It is believed that $(S, T, c)$ form a complete and one-to-one description of $2+1D$ topological orders.

Since $(S, T, c)$ completely describe $2+1D$ topological orders, we can view $(S, T, c)$ as some kind of “non-local order parameters”. Therefore, the concept of topological order (at least in $2+1D$) is fully defined through two physical properties: 1) robust ground state degeneracy (called topological degeneracy),\cite{70, 74} 2) robust geometric phases induced by the modular transformation of the degenerate ground states,\cite{74, 33} much like the concept of superfluid order was characterized by physical properties of zero viscosity and quantized vorticity. These will then form a new characterization of the concept of topological order (and long-range entanglements).

This chapter is structured as follows: In section 2.2, we outline a theory for exact
2+1D topological order based on \((S, T, c = 0)\). In section 2.3, we review what the fixed-point tensors are and how they are related to the many-body wave functions. In section 2.4, we explain in details how \(T\)- and \(S\)-matrices could be obtained from the fixed-point tensors, and give a list of many topological orders with their corresponding \(T\)- and \(S\)-matrices labels. Section 2.5 is the summary.

### 2.2 Theory of exact topological orders based on \((S, T)\)

To outline a general theory of exact 2+1D topological order, we plan to address two issues in this section: 1) how to compute \((S, T)\) from many-body ground state wave functions, 2) which \((S, T)\) can be realized by physical many-body systems.

#### 2.2.1 Non-Abelian-geometric-phase description of topological order

The non-Abelian-geometric-phase description of topological order based \(T\)- and \(S\)-matrices was first introduced in [74, 33]. Together, they generate a projective representation of the modular transformation on a torus, thus we also refer to them as the "modular transformations". We will first make some general remarks about the modular transformations.

The non-Abelian geometric phases are obtained from two types of transformations named \(T\)- and \(S\)-transformation, both defined on a torus. The "\(T\)-transformation" can be defined by shear-twisting a ground state around an axis (let us say the \(x\)-axis) by 360°; whereas the "\(S\)-transformation" can be defined by rotating a ground state by 90°. A pictorial representation will come in (2.24) and (2.26) when we go into details in section 2.4. Essentially they are all discrete deformations of the ground states on a torus. As hinted from the one-to-one correspondence between the ground state degeneracy on a torus and the number of different types of quasi-particles, the information of the quasi-particles is always encoded in the geometry of the torus. Here again, through the non-Abelian geometric phases of the ground states, we can
“decode” the information about their quasi-particles. This gives a natural motivation why $T$- and $S$-matrices give a better description of topological orders.

The non-Abelian geometric phases for some canonical choices of path are calculated for 2+1D bosonic Abelian topological order described by $K$-matrix:[69]

\[
\tilde{T}_{\alpha\beta} = e^{i\frac{\pi}{2} (N^T K N - K^T d N)} e^{i\pi (\alpha^T K \alpha - \frac{c}{2})} \delta_{\alpha\beta}
\]

\[
\tilde{S}_{\alpha\beta} = (-)^{(N^T K N - K^T d N)/2} \frac{e^{-i2\pi \beta^T K \alpha}}{\sqrt{|\det(K)|}},
\]

(2.1)

where $c$ is the difference in the numbers of positive and negative eigenvalues of $K$, $K^T_d = (K^{11}, K^{22}, \cdots, K^{\kappa\kappa})$ is two times the spin vector of the Abelian FQH state, and $N^T = (N_1, N_2, \cdots, N_{\kappa})$ are the numbers of the bosonic “electrons” in each layer.[72]

We see that the $U(1)$ factors depend on the number of electrons and are not universal. But we can isolate the universal non-Abelian part of the non-Abelian geometric phases to taking the limit $N \to 0$, and find

\[
T_{\alpha\beta} = e^{-i2\pi \gamma_0} e^{i\pi \alpha^T K \alpha} \delta_{\alpha\beta}
\]

\[
S_{\alpha\beta} = \frac{e^{-i2\pi \beta^T K \alpha}}{\sqrt{|\det(K)|}} = S_{\alpha\beta}.
\]

(2.2)

We see that the universal non-Abelian part of the non-Abelian geometric phases determines $S$ and a combination of $T$ and $c$.

Eqn. (2.2) is a general result for any Abelian 2+1D topological orders. In this chapter, we will discuss in detail how to compute $S$ and $T$ for non-Abelian topological orders described by string-net wave functions.

\section*{2.2.2 A classifying theory of topological orders}

Now let us consider the second question, which $S$ and $T$ matrices describe existing exact topological orders (with $c = 0$). Here we will list the known conditions on $S$ and $T$:

27
1. $S$ is symmetric and unitary, and satisfies the Verlinde Formula:

\[ N_{\alpha\beta}^\gamma = \sum_{\lambda=0}^{N} \frac{S_{\lambda\alpha}S_{\lambda\beta}(S_{\lambda\gamma})^*}{S_{0\lambda}} = \text{non-negative integer}, \quad (2.3) \]

where $\alpha, \beta, \cdots = 0, 1, \cdots, N$. $N_{\alpha\beta}^\gamma$ is called fusion coefficient, which gives the fusion rule for quasi-particles.

2. Let

\[ d_\alpha = \frac{S_{0\alpha}}{S_{00}} \quad (2.4) \]

which is called quantum dimension. Then $d_\alpha \geq 1$ is the largest eigenvalue of the matrix $N_\alpha$, whose elements are $(N_\alpha)_{\beta\gamma} = N_{\alpha\beta}^\gamma$.

3. $T$ is unitary and diagonal:

\[ T_{\alpha\beta} = e^{i\theta_\alpha} e^{-i2\pi \frac{c}{D}} \delta_{\alpha\beta}. \quad (2.5) \]

Here $\theta_\alpha/2\pi$ is called topological spin ($\theta_\alpha$ is also called statistical angle). $c$ is the chiral central charge which is zero for exact 2+1D topological orders.

4. $S$ and $T$ satisfy:

\[ (ST)^3 = S^2 = C, \quad C^2 = 1, \quad C_{\alpha\beta} = N_{\alpha\beta}^0. \quad (2.6) \]

Thus $S$ and $T$ generate a unitary representation of $SL(2, \mathbb{Z})$.

5. $S$ and $T$ also satisfy [see Eq. (223) in [34]]:

\[ S_{\alpha\beta} = \frac{1}{D} \sum_{\gamma} N_{\alpha\beta}^\gamma e^{i(\theta_\alpha + \theta_\beta - \theta_\gamma)} d_{\gamma}, \quad (2.7) \]

where $D = \sqrt{\sum_\alpha d_{\alpha}^2}$. 

28
6. We also have:[55, 68]

\[ e^{i \sum_{\sigma'} M_{\sigma'}} = e^{i \theta_4 \frac{4}{3} \sum_{\sigma'} M_{\sigma'}} \]  

(2.8)

where \( M_{\sigma'} = 2N_{\sigma} - N_{\sigma} + N_{\sigma}^* N_{\sigma}^* . \)

7. Let

\[ \nu_{\sigma} = \frac{1}{D^2} \sum_{\sigma', \sigma''} N_{\sigma, \sigma''}^{\gamma} d_{\sigma'} d_{\sigma''} e^{i 2 (\theta_{\sigma'} - \theta_{\sigma''})} , \]  

(2.9)

then[55, 68] \( \nu_{\sigma} = 0 \) if \( \sigma \neq \sigma \), and \( \nu_{\sigma} = \pm 1 \) if \( \sigma = \sigma \).

The above are the necessary condition in order for \( S \) and \( T \) to describe an existing exact 2+1D topological order. It is not clear if they are sufficient.

From the above conditions, we see that, instead of using \( S \) and \( T \), we can also use \( N_{\alpha \beta}^\gamma \) and \( e^{i \theta_\alpha} \) to describe topological orders, since \( S \) and \( T \) can be expressed in terms of \( N_{\alpha \beta}^\gamma \) and \( e^{i \theta_\alpha} \). So we can develop a theory of topological orders based on \( N_{\alpha \beta}^\gamma \) and \( e^{i \theta_\alpha} \), instead of \( S \) and \( T \).

Again not all \( N_{\alpha \beta}^\gamma \) and \( e^{i \theta_\alpha} \) describe existing exact 2+1D topological orders. Here we list the necessary conditions on \( N_{\alpha \beta}^\gamma \) and \( e^{i \theta_\alpha} \):

1. \( N_{\alpha \beta}^\gamma \) are non-negative integers that satisfy

\[ N_{\alpha \beta}^\gamma = N_{\beta \alpha}^\gamma, \quad N_{0 \alpha}^\beta = \delta_{\alpha \beta}, \quad \sum_{\gamma = 0}^{N} N_{\alpha \gamma}^0 N_{\gamma \beta}^0 = \delta_{\alpha \beta}, \]  

(2.10)

\[ \sum_{\mu = 0}^{N} N_{\alpha \beta}^\mu N_{\mu \gamma}^\lambda = \sum_{\nu = 0}^{N} N_{\alpha \nu}^\lambda N_{\beta \gamma}^\nu , \]

where \( \alpha, \beta, \cdots = 0, 1, \cdots, N \). In fact \( N_{\alpha \beta}^0 \) defines a charge conjugation \( \alpha \rightarrow \bar{\alpha} \):

\[ N_{\alpha \beta}^0 = \delta_{\alpha \beta} . \]  

(2.11)

2. \( N_{\alpha \beta}^\gamma \) and \( e^{i \theta_\alpha} \) satisfy:[55, 68]

\[ e^{i \sum_{\sigma'} \theta_{\sigma'} M_{\sigma'}} = e^{i \theta_4 \frac{4}{3} \sum_{\sigma'} M_{\sigma'}} \]  

(2.12)
where $M_{\sigma\sigma'} = 2(N + 1)N_{\sigma\sigma'}N_{\sigma\sigma'} + N_{\sigma\sigma}N_{\sigma\sigma'}$.

3. Let $d_\alpha$ be the largest eigenvalue of the matrix $N_\alpha$, whose elements are $(N_\alpha)_{\beta\gamma} = N_{\alpha\beta}^\gamma$. Let

$$S_{\alpha\beta} = \frac{1}{\sqrt{\sum_\alpha d_\alpha^2}} \sum_\gamma N_{\alpha\beta}^\gamma e^{i(\theta_\alpha + \theta_\beta - \theta_\gamma)} d_\gamma. \quad (2.13)$$

Then, $S$ is unitary, and satisfies [63]

$$N_{\alpha\beta}^\gamma = \sum_\chi S_{\lambda\alpha} S_{\lambda\beta} (S_{\lambda\gamma})^*. \quad (2.14)$$

4. Let

$$T_{\alpha\beta} = e^{i\theta_\alpha} e^{-i2\pi \frac{\nu}{2\lambda}} \delta_{\alpha\beta}. \quad (2.15)$$

Then

$$(ST)^3 = S^2 = C, \quad C^2 = 1. \quad (2.16)$$

In fact $C_{\alpha\beta} = N_{\alpha\beta}^0$.

5. Let

$$\nu_\sigma = \frac{1}{\sum_\alpha d_\alpha^2} \sum_{\sigma'\sigma''} N_{\sigma'\sigma''}e^{i2(\theta_{\sigma'} - \theta_{\sigma''})}. \quad (2.17)$$

Then[55, 68] $\nu_\sigma = 0$ if $\sigma \neq \bar{\sigma}$, and $\nu_\sigma = \pm 1$ if $\sigma = \bar{\sigma}$.

If the above conditions are sufficient for $N_{\alpha\beta}^\gamma$ and $e^{i\theta_\alpha}$ to describe an existing exact 2+1D topological order, then the above will represent a classifying theory of exact 2+1D topological orders.

Form the above discussion, we see that to calculate/determine the topological order in a ground state wave function, is to calculate/determine $(S, T)$ or $(N_{\alpha\beta}^\gamma, e^{i\theta_\alpha})$.

In the following, we show how to determine $(S, T)$ from the ideal string-net wave functions labeled by a set of fixed-point tensors.
Figure 2-1: A quantum state on a graph. There are \( N + 1 \) states on each edge, labeled by \( i = 0, \cdots, N \). There are \( N_{ijk} \) states on each vertex which are labeled by \( \alpha = 1, \cdots, N_{ijk} \), with \( i, j, k \) labeling the three edges of a vertex. Note that the indices \( \alpha = 1, \cdots, N_{ijk} \), with \( i, j, k \) labeling the three edges of a vertex. Note that the indices \( i \rightarrow j \rightarrow k \) runs anti-clockwise.

2.3 Fixed-point tensor labels of the string-net states

In both the string-net [46] and the local-unitary-transformation [11] approach, exact topological orders are labeled by a set of fixed-point tensors. In this section, we would like to briefly review what these fixed-point tensor labels are. The question of how to obtain these fixed-point tensor labels is beyond the scope of this thesis, and readers should refer to [46, 11] for details; for the convenience of the readers, we summarized some of the key derivations in Appendix A.

The fixed-point tensor label contains a set of four tensors, \( (N_{ijk}, P_{klm}^{i}, \lambda, \rho_{k}, \alpha, \beta, A_{i}) \). Together they describe the string-net states [46] or equivalently the fixed-point wave function under local unitary transformation [11]. (Note: in this thesis, these two terms will be used interchangeably.) These tensor labels could be best understood through a set of graphs, so before we go into the details of these tensors, we will first introduce the graphical representation of a quantum state.

Fig. 2-1 shows a quantum state on a graph, with degrees of freedom living on both edges and vertices of the graph. In this chapter, we focus on general trivalence graphs. Each edge has \( N + 1 \) states, labeled by \( i = 0, \cdots, N \), and the edge degrees of freedom have orientations, with \( i^{*} \) representing the inverse of \( i \), and we have trivially that \( (i^{*})^{*} = i \). Each vertex has \( N_{ijk} \) states, labeled by \( \alpha = 1, \cdots, N_{ijk} \). In all examples of this chapter, we have that \( N_{ijk} = 0 \) or 1, thus tensor \( N_{ijk} \) acts like a branching rule, determining whether a particular branching structure \( ijk \) is allowed or not at a vertex, and is often referred to as the“fusion rule”.

A string-net state (or equivalently the fixed-point state in [11]) is obtained by
summing over all possible graph configurations with appropriate amplitudes, and tensors $F_{klm,\gamma\lambda}^{ijm,\alpha\beta}$ and $P_{i}^{kj,\alpha\beta}$ describe precisely the local constraints that these amplitudes must satisfy. These local constraints are expressed through local deformations named “F-move” and “P-move”. “F-move” corresponds to locally deform

\[ i \quad \beta \quad j \]

to

\[ i \quad \beta \quad j \]

(here only relevant part of the graph is drawn), and “P-move” corresponds to locally deform

\[ i \quad \beta \quad j \]

to

\[ i \quad \beta \quad j \]

The two corresponding local constraints on the string-net wave functions (or the fixed-point wave functions) are the following:

\[
\Phi_{\text{fix}} \left( \begin{array}{c} \beta \\ i \\ j \end{array} \right) \sim \sum_{n=0}^{N} \sum_{\chi=1}^{N_{\chi}} \sum_{\delta=1}^{N_{\delta}} F_{klm,\gamma\lambda}^{ijm,\alpha\beta} \Phi_{\text{fix}} \left( \begin{array}{c} i \\ j \\ k \end{array} \right),
\]

\[
\Phi_{\text{fix}} \left( \begin{array}{c} \beta \\ i \\ \alpha \\ j \end{array} \right) \sim P_{i}^{kj,\alpha\beta} \Phi_{\text{fix}} \left( \begin{array}{c} i \\ k \end{array} \right).
\]  

(2.18)

As shown above, the $F_{klm,\gamma\lambda}^{ijm,\alpha\beta}$ and $P_{i}^{kj,\alpha\beta}$ tensors relate the string-net wave function amplitudes on different graph configurations.

The remaining tensor label $A^{i}$ could be easily understood as the amplitude of string-net state (a.k.a. fixed-point state) on a closed loop:

\[
\Phi_{\text{fix}} \left( \begin{array}{c} i \\ \circ \end{array} \right) = A^{i} = A^{i*}
\]

(2.19)

Note that starting from $A^{i}$, we can re-construct the entire string-net wave function (a.k.a. fixed-point wave function) using “F-move” and “P-move” introduced above, and obtain the amplitudes of the string-net state on all different graph configurations; thus the four tensors $(N_{ijk}, F_{klm,\gamma\lambda}^{ijm,\alpha\beta}, P_{i}^{kj,\alpha\beta}, A^{i})$ completely characterize a string-net state (a.k.a. fixed-point state).

With the above tensor label for the manybody wave function, the next natural questions to ask are: Are they in one-to-one correspondence with different topological orders? How can we physically understand these tensors? Are there any physical
quantities that we can at least numerically measure?

In fact, the non-Abelian geometric phases $T$ and $S$ (up to unitary transformations) provide an answer to the above questions. Not only can $T$ and $S$ give a one-to-one label for different topological orders, but they can also provide a link between the string-net states (a.k.a. fixed-point ground states) and their corresponding quasiparticle excitations. We believe that the description given by the $T$- and $S$-matrices is complete, meaning that they can completely characterize different non-chiral topological orders. Further calculations of various topological properties thus can all be obtained from $T$- and $S$-matrices.

2.4 Modular transformations and a list of topological orders

First, let us make a general remark. Mathematically, the relationship between $(S, T)$-matrix labels and fixed-point-tensor labels $(N_{ijk}, F^{ijm,\alpha\beta}_{\kappa\eta}, P^{k\eta,\alpha\beta}_i, A^i)$ can be understood within unitary fusion category theory. On one hand, the $T$- and $S$-matrices describe a set quasiparticle excitations, which may have fractional and/or non-Abelian statistics and are described by a modular tensor category $T$. On the other hand, the fixed-point tensors correspond to the many-body wave functions of exact topological orders (string-net states), which are described by a unitary fusion category $F$. Thus to calculate the $T$- and $S$-matrices from the fixed-point tensors is to calculate the quasiparticle excitations from the many-body wave functions, which corresponds to calculating the modular tensor category $T$ from a unitary fusion category $F$.[39] Mathematically, calculating $T$ from $F$ corresponds to taking the “Drinfeld center” of the unitary fusion category $F$. Our calculation of $T$- and $S$-matrices from the fixed-point tensors corresponds exactly to taking the “Drinfeld center” of the unitary fusion category $F$.

This picture agrees perfectly with our calculations: we perform modular transformations on several fixed-point states (2.4.1, 2.4.2, 2.4.3, 2.4.5, 2.4.6), all having
the structure of unitary tensor category; our resulting $T$- and $S$-matrices for the quasi-particle excitations are all of the structure of modular tensor category. This result further strengthened our belief that tensor category theory is the mathematical structure behind topological order.

In the following subsections, we first follow the method of [11] to obtain a large list of fixed-point tensor labels $(N_{ijk}, F_{kin,\gamma\lambda}, P_{i,\alpha\beta}, A^i)$ corresponding to different string-net states, including many new ones (2.4.3, 2.4.4, 2.4.6, 2.4.7). Note that the list is incomplete for any $N$ (recall that $N$ represents the number of state on each edge), because exhausting all possible fixed-point tensor labels is very tedious. We then apply "modular transformations" to most of these string-net states (2.4.1, 2.4.2, 2.4.3, 2.4.5, 2.4.6) to get their $T$ and $S$-matrices, describing their corresponding quasi-particles.

Note that there's a special state, i.e., the "chiral" case 2.4.8. In that case, the labeling fixed-point tensors form the so-called "multi-fusion category" and its fusion rule breaks the "chiral" symmetry. Although the resulting fixed-point wave function has a trivial topological order without any symmetry, we anticipate that the "chiral" case can be highly non-trivial after introducing certain symmetries.

### 2.4.1 $N = 1$ loop states – the $\mathbb{Z}_2$ states

Let us first consider a system where there are only two states $|0\rangle$ and $|1\rangle$ on each link of the graph ($N = 1$). We choose $i^* = i$ and the simplest fusion rule of a string-net state is the following (recall that tensor $N_{ijk}$ is referred to as the "fusion rule"):

$$
N_{000} = N_{110} = N_{101} = N_{011} = 1,
$$

other $N_{ijk} = 0.$

Since $N_{ijk} \leq 1$, there are no states on the vertices. So the indices $\alpha, \beta, \cdots$ labeling the states on a vertex can be suppressed.

The above fusion rule corresponds to the fusion rule of the $N = 1$ loop states discussed in [46], thus the name $N = 1$ loop states. Note further that the three edge
labels of \( N \) also form a \( \mathbb{Z}_2 \) group: for example, \( N_{110} = 1 \) represents the group action \( 1 \otimes 1 = 0 \) (Here our group action is addition). Thus we call the states obtained in this section the \( \mathbb{Z}_2 \) states.

We now give the \( F_{kn}^{ijm} \) tensor of the \( \mathbb{Z}_2 \) states. There are only four independent non-zero components, denoted as \( f_0,...,f_3 \):

\[
\begin{align*}
F_{000}^{000} &= f_0 \\
F_{111}^{000} &= (F_{100}^{011})^* = (F_{010}^{101})^* = f_1 \\
F_{011}^{011} &= (F_{101}^{101})^* = f_2 \\
F_{110}^{110} &= f_3
\end{align*}
\] (2.21)

We note that \( F_{kn}^{ijm} \) in (2.18) relates wave functions on two graphs. In the above we have drawn the two related graphs right after the \( F \) tensor. The first graph following \( F \) corresponds to the graph on the left-hand side of (2.18) and the second one corresponds to the graph on the right-hand side of (2.18). The dotted line corresponds to the \( |0\)-state on the link and the solid line corresponds to the \( |1\)-state on the link.

There are four non-zero components in \( P_{kj}^{ij} \), which are denoted by \( p_0,...,p_3 \):

\[
\begin{align*}
P_{00}^0 &= p_0, & P_{01}^0 &= p_1, & P_{00}^0 &= p_2, & P_{10}^0 &= p_3.
\end{align*}
\] (2.22)

Without going into details, we give here two isolated fixed-point labels parameterized by \( \eta = \pm 1 \):

\[
\begin{align*}
f_0 &= f_1 = f_2 = 1, & f_3 &= \eta, \\
p_0 &= p_2 = \frac{1}{\sqrt{2}}, & p_1 &= p_3 = \frac{\eta}{\sqrt{2}}, \\
A^0 &= \frac{1}{\sqrt{2}}, & A^1 &= \frac{\eta}{\sqrt{2}}.
\end{align*}
\] (2.23)

The above tensors \( (N_{ijk}, F_{kn,\gamma}^{ijm,\alpha\beta}, P_{ij}^{kl,\gamma\lambda}, A^i) \) with \( \eta = \pm 1 \) both correspond to fixed-
Figure 2-2: (Color online) Two equivalent representations of non-contractable graphs on a torus. On the left, the inner square and the outer square should be identified to form a torus. On the right, the torus is drawn as a square, with opposite edges being identical. The graph on the right could be viewed as a zoom-in version of the graph on the left. We use both representations in this thesis.

Figure 2-3: (Color online) The 4 characteristic non-contractable graphs on a torus. Any other fixed-point graphs can always be reduced to the above four by F-moves and P-moves, but the four types cannot be changed into each other. The dotted line represents the $|0\rangle$-state on the edges, and the solid line represents the $|1\rangle$-state on the edges. The three integers $ijk$ at the lower right corner of each graph are the edge labels following the convention of (2.24).

point states. The $\eta = 1$ state is an equal weight superposition of all the graphic states that satisfy the fusion rule, whereas the $\eta = -1$ state is also a superposition of all fusion-rule-satisfying graphic states, but some with coefficient 1 and some others with coefficient $-1$.

Having obtained the abstract fixed-point tensors, we now want to physically understand the results by introducing the concept of “modular transformations”. Modular transformations are defined on a torus (i.e. a planar graph with periodic boundary conditions in both $x$- and $y$-directions). Notice that after putting the above states onto a torus, there will be four different types of non-contractable graphs (see Fig. 2-2,
2-3). They correspond to four types of fixed-point states that are linearly independent (meaning they cannot be transformed into each other through $F$ or $P$-moves) and form the degenerate ground-state subspace of a local Hamiltonian (See Appendix B for the form of the Hamiltonian). The modular transformations can be defined within this ground-state subspace. We will first introduce the “$T$-transformation”, also known as the method of Dehn twist. As in [25], we define the Dehn twist formally by requiring it to map a fixed-point state $\Phi_{\text{fix}}^{\alpha \beta}(i, j, k, \alpha, \beta) = \Phi_{\text{fix}}^{\alpha \beta}$ on a torus to another fixed-point state defined on a different graph $\tilde{\Phi}_{\text{fix}}^{\alpha \beta}(i, j, k, \alpha, \beta) = \tilde{\Phi}_{\text{fix}}^{\alpha \beta}$.

Then we can use an $F$-move to deform the graph to $\tilde{\Phi}_{\text{fix}}^{\alpha \beta}$, which leads to a unitary transformation $T$ between the four fixed-point states on the torus, called the “$T$-transformation”. The deformation process could be seen more clearly from the following:

\[
\Phi_{\text{fix}}^{\alpha \beta} \begin{array}{c}
    \begin{array}{c}
        \text{Dehn twist} \\
        \text{Twist}
    \end{array}
\end{array} \Phi_{\text{fix}}^{\alpha \beta} = \sum_{l, \lambda \delta} F_{ij, \alpha \beta}^{ijk, \alpha \beta} \Phi_{\text{fix}}^{\lambda \delta}
\]

\[
= \sum_{l, \lambda \delta} F_{ij, \alpha \beta}^{ijk, \alpha \beta} \Phi_{\text{fix}}^{\lambda \delta} \begin{array}{c}
    \begin{array}{c}
        \text{(F - move)} \ \\
        \text{F - move}
    \end{array}
\end{array} = \sum_{l, \lambda \delta} F_{ij, \alpha \beta}^{ijk, \alpha \beta} \Phi_{\text{fix}}^{\lambda \delta}
\]

\[
= \sum_{l, \lambda \delta} F_{ij, \alpha \beta}^{ijk, \alpha \beta} \Phi_{\text{fix}}^{\lambda \delta} \begin{array}{c}
    \begin{array}{c}
        \text{(Deformation)} \\
        \text{Deformation}
    \end{array}
\end{array} = \sum_{l, \lambda \delta} F_{ij, \alpha \beta}^{ijk, \alpha \beta} \Phi_{\text{fix}}^{\lambda \delta}
\]

\[
= \sum_{l} F_{ij, \alpha \beta}^{ijk} \Phi_{\text{fix}}^{\lambda \delta}. \quad (2.24)
\]

In the last step above, we have suppressed the vertex indices $\alpha, \beta, \delta, \chi$ because a maximum of one state is allowed on each vertex.
Writing "T-transformation" in the basis of the four non-contractable fixed point states on the torus, we get a 4 by 4 unitary matrix with elements $F_{i^*j^*}^{ijk}$. This matrix is called as "T-matrix". The basis of the 4 by 4 matrix is defined as follows:

\[
\begin{align*}
\Phi_{\text{fix}} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} &= (1, 0, 0, 0), \\
\Phi_{\text{fix}} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} &= (0, 1, 0, 0), \\
\Phi_{\text{fix}} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} &= (0, 0, 1, 0), \\
\Phi_{\text{fix}} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} &= (0, 0, 0, 1).
\end{align*}
\]

We now apply Dehn twist to all the four non-contractable states (recall Fig. 2-3) to get the $T$-matrix:

\[
\begin{align*}
\Phi_{\text{fix}} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} &= F_{000}^{000} \Phi_{\text{fix}} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \\
\Phi_{\text{fix}} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} &= F_{101}^{101} \Phi_{\text{fix}} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \\
\Phi_{\text{fix}} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} &= F_{011}^{011} \Phi_{\text{fix}} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\end{align*}
\]
We thus obtain the 4 by 4 $T$-matrix for the $\mathbb{Z}_2$ states as follows:

$$T = \begin{pmatrix}
F_{000}^0 & 0 & 0 & 0 \\
0 & F_{101}^1 & 0 & 0 \\
0 & 0 & 0 & F_{110}^{110} \\
0 & 0 & F_{011}^0 & 0 \\
\end{pmatrix}.$$  \hspace{1cm} (2.25)

To complete the story, we will also introduce the “$S$-transformation”. Unlike $T$-transformation which is generated by Dehn twist on the torus, $S$-transformation is generated by $90^\circ$ rotations. We define the $90^\circ$ rotation formally by mapping a fixed-point state $\Phi_{\text{fix}}^{\alpha\beta}(i, j, k, \alpha, \beta) = \Phi_{\text{fix}}^{\alpha\beta}(\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array})$ on a torus to its counter-clockwise $90^\circ$ rotation state on a different graph $\Phi_{\text{fix}}^{\alpha\beta}(i, j, k, \alpha, \beta) = \Phi_{\text{fix}}^{\alpha\beta}(\begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array})$. Then we can again use an $F$-move to deform the graph back to , which leads to a unitary transformation $S$ between the four fixed-point states on torus. This deformation process can be seen more clearly through the following graph:

$$\sum_{\lambda, \chi, \delta} F_{ij, \alpha, \beta}^{\lambda, \chi, \delta} \Phi_{\text{fix}}^{\lambda\chi\delta} = \sum_{\lambda} F_{ij}^{\lambda} \Phi_{\text{fix}}^{\lambda}.$$  \hspace{1cm} (2.26)
Again, we have suppressed the vertex indices $\alpha, \beta, \delta, \chi$ in the last step above.

Writing “$S$-transformation” in the same basis as “$T$-transformation”, we get a 4 by 4 unitary matrix with elements $F_{ij}^{i,j}$. This matrix is called as the “$S$-matrix”. We now apply 90° rotations to all the four non-contractable states (recall Fig. 2-3) to get the $S$-matrix:

\[
\Phi_{\text{fix}} \begin{pmatrix}
0
0
0
0
\end{pmatrix} = F_{000}^{000} \Phi_{\text{fix}} \begin{pmatrix}
0
0
0
0
\end{pmatrix}
\]

\[
\Phi_{\text{fix}} \begin{pmatrix}
1
0
0
1
\end{pmatrix} = F_{101}^{101} \Phi_{\text{fix}} \begin{pmatrix}
1
0
0
1
\end{pmatrix}
\]

\[
\Phi_{\text{fix}} \begin{pmatrix}
0
1
1
0
\end{pmatrix} = F_{011}^{011} \Phi_{\text{fix}} \begin{pmatrix}
0
1
1
0
\end{pmatrix}
\]

\[
\Phi_{\text{fix}} \begin{pmatrix}
1
1
0
1
\end{pmatrix} = F_{110}^{110} \Phi_{\text{fix}} \begin{pmatrix}
1
1
0
1
\end{pmatrix}
\]

We can thus obtain the 4 by 4 $S$-matrix as follows:

\[
S = \begin{pmatrix}
F_{000}^{000} & 0 & 0 & 0 \\
0 & 0 & F_{011}^{011} & 0 \\
0 & F_{101}^{101} & 0 & 0 \\
0 & 0 & 0 & F_{110}^{110}
\end{pmatrix}.
\tag{2.27}
\]

The $T$ and $S$-matrix obtained above together give us the modular transformations.
When $\eta = 1$, we have:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (2.28)$$

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.29)$$

We can choose a new basis to make the $T$ and $S$-matrices be in a more standard form. In this new basis, we require that $(S, T)$ to satisfy the conditions on $S$ and $T$. It is conjectured that the $T$ and $S$-matrices satisfying those requirements are unique (up to permutations of the basis), and thus the basis of $T$ and $S$-matrices are completely fixed through those requirements (up to permutations.) Numerical simulations have been done in several cases in the later sections of the chapter, all agreeing with the conjecture.

Now, diagonalize the $T$-matrix and make the $S$-matrix satisfy the conditions on $S$ and $T$, we have:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (2.30)$$
Numerical simulations show that the above $T$ and $S$-matrices are unique (up to permutation of basis.) The above $T$ and $S$-matrices match exactly with the corresponding matrices in the “Toric Code modular tensor category” in [55]. From the dimension of the $T$ and $S$-matrices, we can see that there are four different types of quasi-particles. From the conditions on $S$ and $T$, we can get the fusion rules between the quasi-particle excitations. The quasi-particle statistical angles are given by the eigenvalues of $T$-matrix.\cite{74,18} The quantum dimensions of these four quasi-particles are given by the first row (column) of the $S$-matrix.

Note that the $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ $U(1) \times U(1)$ Chern-Simons (CS) theory also has 4 types of quasiparticles with statistical angles

$$(e^{i\theta}) = (1, 1, 1, -1).$$

Therefore, the $\eta = 1$ $\mathbb{Z}_2$ state can be described by the above CS theory.

When $\eta = -1$, we have:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$ (2.32)
\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\] (2.33)

Again, we can diagonalize the \( T \)-matrix and make the \( S \)-matrix satisfy the conditions on \( S \) and \( T \) under a proper basis; numerical simulations show that such a basis is unique (up to permutations). In such a basis, we have:

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (2.34)

\[
S = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\] (2.35)

In this particular case, the above \( T \) and \( S \) matrices can be further reduced to the following forms:

\[
T = \begin{pmatrix}
1 & 0 \\
0 & i
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & -i
\end{pmatrix}
\] (2.36)

\[
S = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\] (2.37)

which shows the “doubled” structure of the quasi-particles. The above \( T \) and \( S \)-matrices match exactly with the “doubled Semion” modular tensor category in [55].
Again, we can see that there are four different types of quasi-particles from the dimension of the $T$ and $S$-matrices. From calculating the conditions on $S$ and $T$, we can get the fusion rules between the quasi-particle excitations. The quasi-particle statistics are given by the eigenvalues of $T$-matrix, whereas the quantum dimensions of different quasi-particles are given by the elements on the first row of $S$-matrix.

Notice that the eigenvalues of $T$-matrix again match with the $K = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$ CS theory (which is equivalent to the $K = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ CS theory), which has 4 types of quasiparticles with statistical angles

$$(e^{i\theta_i}) = (1, 1, i, -i).$$

Also, the “doubled” structure hints that the effective theory can be “decoupled”, which is exactly what we get here: $K = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ can indeed be decoupled. Therefore, the $\eta = -1 \mathbb{Z}_2$ state can be described by the $K = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$ CS theory.

### 2.4.2 $N = 1$ string-net state – the “Fibonacci” state

To obtain another class of simple solutions, we modify the fusion rule to

$$N_{000} = N_{110} = N_{101} = N_{011} = N_{111} = 1,$$

other $N_{ijk} = 0.$ \hspace{1cm} (2.38)

while keeping everything the same.

The new fusion rule corresponds to the fusion rule for the $N = 1$ string-net state discussed in [46], thus the name $N = 1$ string-net state. Notice the additional fusion rule can be written as $1 \otimes 1 = 0 \oplus 1$ which looks like Fibonacci’s golden rule, we also refer to the state as the “Fibonacci State”.

Now comes the $F_{kin}^{ijm}$ tensor. Now there are seven independent non-zero compo-
nents which are denoted as $f_0,\ldots,f_6$:

\[
F_{000}^{000} \cdot (\ldots) = f_0
\]
\[
F_{111}^{000} \cdot (\ldots) = (F_{100}^{101} \cdot (\ldots))^* = (F_{010}^{101} \cdot (\ldots))^*
\]
\[
= F_{001}^{110} \cdot (\ldots) = f_1
\]
\[
F_{011}^{011} \cdot (\ldots) = (F_{101}^{101} \cdot (\ldots))^* = f_2
\]
\[
F_{111}^{011} \cdot (\ldots) = (F_{111}^{101} \cdot (\ldots))^* = F_{011}^{111} \cdot (\ldots)
\]
\[
= (F_{101}^{111} \cdot (\ldots))^* = f_3
\]
\[
F_{110}^{110} \cdot (\ldots) = f_4
\]
\[
F_{111}^{111} \cdot (\ldots) = (F_{110}^{111} \cdot (\ldots))^* = f_5
\]
\[
F_{111}^{111} \cdot (\ldots) = f_6
\]

(2.39)

There are five potentially non-zero components in $P^{k\ell}_{ij}$, which are denoted by $p_0,\ldots,p_4$:

\[
P_0^{000} = p_0, \quad P_0^{011} = p_1, \quad P_1^{000} = p_2, \quad P_1^{011} = p_3, \quad P_1^{111} = p_4.
\]

(2.40)

Again we give here the fixed-point tensors without going into details. Let $\gamma$ be the positive solution of $\gamma^2 + \gamma = 1$: $\gamma = \frac{\sqrt{5} - 1}{2}$. The above can be written as

\[
f_0 = f_1 = f_2 = f_3 = 1, \quad f_4 = -f_6 = \gamma, \quad f_5 = \sqrt{\gamma},
\]
\[
p_0 = p_2 = \frac{\gamma}{\sqrt{\gamma^2 + 1}}, \quad p_1 = p_3 = \frac{1}{\sqrt{\gamma^2 + 1}}, \quad p_4 = 0,
\]
\[
A^0 = \frac{\gamma}{\sqrt{\gamma^2 + 1}}, \quad A^1 = \frac{1}{\sqrt{\gamma^2 + 1}}.
\]

(2.41)

The fixed-point tensors above $(N_{ijk}, F^{ijm,\alpha\beta}_{\kappa\ell\gamma\lambda}, P^{k\ell,\alpha\beta}_{ij}, A^i)$ corresponds to the $N = 1$ string-net condensed state[46].

Following the logic from last section, we will now apply the modular transforma-
tions. Let us first consider Dehn twist on a torus. Note that we now formally have 5 possible non-contractable graphs on a torus rather than 4, due to the additional fusion rule. (Recall Fig. 2-3; the additional graph would be a graph with all solid lines.) Doing the Dehn twist for all 5 graphic states using (2.24), we have:

\[
\phi_{\text{fix}} \left( \begin{array}{c}
\text{state 1} \\
\text{state 2}
\end{array} \right) = F_{000} \phi_{\text{fix}} \left( \begin{array}{c}
\text{state 3} \\
\text{state 4}
\end{array} \right)
\]

\[
\phi_{\text{fix}} \left( \begin{array}{c}
\text{state 1} \\
\text{state 2}
\end{array} \right) = F_{110} \phi_{\text{fix}} \left( \begin{array}{c}
\text{state 5} \\
\text{state 6}
\end{array} \right) + F_{111} \phi_{\text{fix}} \left( \begin{array}{c}
\text{state 7} \\
\text{state 8}
\end{array} \right)
\]

\[
\phi_{\text{fix}} \left( \begin{array}{c}
\text{state 1} \\
\text{state 2}
\end{array} \right) = F_{101} \phi_{\text{fix}} \left( \begin{array}{c}
\text{state 9} \\
\text{state 10}
\end{array} \right)
\]

\[
\phi_{\text{fix}} \left( \begin{array}{c}
\text{state 1} \\
\text{state 2}
\end{array} \right) = F_{011} \phi_{\text{fix}} \left( \begin{array}{c}
\text{state 11} \\
\text{state 12}
\end{array} \right)
\]

\[
\phi_{\text{fix}} \left( \begin{array}{c}
\text{state 1} \\
\text{state 2}
\end{array} \right) = F_{110} \phi_{\text{fix}} \left( \begin{array}{c}
\text{state 13} \\
\text{state 14}
\end{array} \right) + F_{111} \phi_{\text{fix}} \left( \begin{array}{c}
\text{state 15} \\
\text{state 16}
\end{array} \right)
\]
We thus obtain the 5 by 5 $T$-matrix for the Fibonacci state as follows:

$$T = \begin{pmatrix}
F_{000} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F_{011} & 0 \\
0 & 0 & F_{101} & 0 & 0 \\
0 & F_{110} & 0 & 0 & F_{111} \\
0 & F_{111} & 0 & 0 & F_{111}
\end{pmatrix} \quad (2.42)$$

Similarly, we can also get the $S$-matrix by applying $90^\circ$ rotations to all five non-contractable states on a torus:

$$\Phi_{\text{fix}} \begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
1
\end{pmatrix} = F_{000}^{000} \Phi_{\text{fix}} \begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
1
\end{pmatrix}$$

$$\Phi_{\text{fix}} \begin{pmatrix}
0 \\
0 \\
1 \\
1 \\
1
\end{pmatrix} = F_{110}^{110} \Phi_{\text{fix}} \begin{pmatrix}
0 \\
0 \\
1 \\
1 \\
1
\end{pmatrix} + F_{111}^{110} \Phi_{\text{fix}} \begin{pmatrix}
0 \\
0 \\
1 \\
1 \\
1
\end{pmatrix}$$

$$\Phi_{\text{fix}} \begin{pmatrix}
1 \\
0 \\
1 \\
1 \\
0
\end{pmatrix} = F_{101}^{101} \Phi_{\text{fix}} \begin{pmatrix}
1 \\
0 \\
1 \\
1 \\
0
\end{pmatrix}$$

$$\Phi_{\text{fix}} \begin{pmatrix}
1 \\
0 \\
0 \\
1 \\
1
\end{pmatrix} = F_{011}^{011} \Phi_{\text{fix}} \begin{pmatrix}
1 \\
0 \\
0 \\
1 \\
1
\end{pmatrix}$$
From which we can get the 5 by 5 $S$-matrix for the Fibonacci state:

$$
S = \begin{pmatrix}
F^{000}_{000} & 0 & 0 & 0 & 0 \\
0 & F^{110}_{110} & 0 & 0 & F^{111}_{111} \\
0 & 0 & 0 & F^{011}_{011} & 0 \\
0 & 0 & F^{101}_{101} & 0 & 0 \\
0 & F^{110}_{110} & 0 & 0 & F^{111}_{111}
\end{pmatrix}
$$

Now we can substitute in the values of the $F$ tensors, and thus obtain:

$$
T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & \gamma & 0 & 0 & \sqrt{\gamma} \\
0 & \sqrt{\gamma} & 0 & 0 & -\gamma
\end{pmatrix}
$$

$$
S = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \gamma & 0 & 0 & \sqrt{\gamma} \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & \sqrt{\gamma} & 0 & 0 & -\gamma
\end{pmatrix}
$$

where $\gamma = \frac{\sqrt{5} - 1}{2}$.

There is, however, a small complication here compared to the previous $\mathbb{Z}_2$ case. Although there are now five possible non-contractable graphs on a torus, they are no
longer linearly independent (meaning they can be transformed into each other through $F$ or $P$-moves) and are not all fixed-point graphs. Thus they don’t all correspond to ground states of a local Hamiltonian. As can be checked using the Hamiltonian construction in Appendix B, the ground state subspace in this case is only 4-fold degenerate. We thus need to restrict our modular transformations to be within this ground state subspace. In order to get the $T$ and $S$-matrix within the ground states, we need to project the above obtained matrices onto its $4 \times 4$ ground-state subspace. Such a process can be easily done by diagonalizing the Hamiltonian and projecting the $T$ and $S$-matrices only to the ground-state subspace; see Appendix B for details.

After the projection, we can again diagonalize the $T$-matrix and make the $S$-matrix satisfy the conditions on $S$ and $T$ under a proper basis; numerical simulations again show that such a basis is unique (up to permutations). We thus have the final form of $T$ and $S$:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i \frac{2\pi}{5}} & 0 & 0 \\ 0 & 0 & e^{-i \frac{4\pi}{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (2.46)

$$S = \frac{5 - \sqrt{5}}{10} \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & -1 & \frac{3+\sqrt{5}}{2} & -\frac{1-\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & -1 & -\frac{1-\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} & -\frac{1-\sqrt{5}}{2} & -\frac{1-\sqrt{5}}{2} & 1 \end{pmatrix}$$  \hspace{1cm} (2.47)

As can be easily seen, the above $S$-matrix is real and symmetric. Also in this particular case, the above $T$ and $S$ matrices can be further reduced to the following forms:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i \frac{4\pi}{5}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i \frac{4\pi}{5}} \end{pmatrix} \hspace{1cm} (2.48)$$
\[
S = \frac{5 - \sqrt{5}}{10} \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & -1 \end{pmatrix}
\]  
(2.49)

which shows the "doubled" structure of the quasi-particles. The above \(T\) and \(S\)-matrices match exactly with the doubled "Fibonacci" modular tensor category in [55]. As before, we can tell there are four different types of quasi-particles from the dimension of \(T\) and \(S\); we can also get the fusion rules between quasi-particle excitations from calculating the conditions on \(S\) and \(T\). The quasi-particle statistical angles are given by the eigenvalues of \(T\), whereas the quantum dimensions of different quasi-particles are given by the first row (column) of the \(S\)-matrix.

Following the last section, here we want to also comment on the eigenvalues of \(T\)-matrix. These eigenvalues correspond exactly to the quasiparticle statistical angles of the doubled \(SO(3)_3\) Chern-Simons gauge theory:

\[
(e^{i\theta_i}) = (1, 1, e^{i\frac{4\pi}{7}}, e^{-i\frac{4\pi}{7}}).
\]

Thus the "Fibonacci" state can be described by the doubled \(SO(3)_3\) CS theory.

### 2.4.3 An \(N = 2\) string-net state - the "Pfaffian" state

Here we will give a more complicated example of a non-orientable string-net state. We choose \(N = 2, 0^* = 0, 1^* = 1, 2^* = 2\), and

\[
N_{000} = N_{011} = N_{110} = N_{101} = N_{022} = N_{202} = N_{220} = N_{112} = N_{121} = N_{211} = 1,
\]

other \(N_{ijk} = 0\).  
(2.50)

Similar to the \(Z_2\) case, the three edge states of \(N\) form a structure called a “tensor category” with the following fusion rules between its three elements: \(1 \otimes \sigma = \sigma,\ 1 \otimes \psi = \psi,\ \psi \otimes \psi = 1,\ \sigma \otimes \sigma = 1 \oplus \psi\). In the above fusion rules, “1” represents state 0 on the edge, “\(\sigma\)” represents state 1 and “\(\psi\)” represents state 2. (For example,
Now comes the $F_{ijm}^{klm}$-tensor. There are fourteen independent potentially non-zero components which are denoted as $f_0,...,f_{13}$:

$$F_{000}^{000} = f_0$$
$$F_{111}^{000} = (F_{101}^{011})^* = (F_{010}^{101})^* = F_{010}^{101} = f_1$$
$$F_{222}^{000} = (F_{200}^{022})^* = (F_{020}^{202})^* = F_{020}^{202} = f_2$$
$$F_{011}^{111} = (F_{101}^{101})^* = f_3$$
$$F_{122}^{111} = (F_{201}^{211})^* = (F_{012}^{121})^* = F_{012}^{121} = f_4$$

$$F_{211}^{011} = (F_{121}^{101})^* = (F_{101}^{211})^* = F_{101}^{211} = f_5$$
$$F_{022}^{022} = (F_{202}^{202})^* = f_6$$
$$F_{111}^{022} = (F_{212}^{101})^* = (F_{021}^{112})^* = F_{021}^{112} = f_7$$
$$F_{110}^{110} = f_8$$
$$F_{112}^{110} = (F_{110}^{112})^* = f_9$$
\[ F^{112}_{112} = f_{10} \]
\[ F^{110}_{221} = (F^{121}_{210} F^{111}_{120})^* = (F^{211}_{120} F^{111}_{120})^* = F^{220}_{111} = f_{11} \]
\[ F^{121}_{121} = (F^{211}_{211} F^{111}_{121})^* = f_{12} \]
\[ F^{220}_{220} = f_{13} \]  

(2.51)

There are ten potentially non-zero components in \( P^{ij} \), which are denoted by \( p_0, ..., p_9 \):

\[ P^{00}_0 = p_0, \quad P^{01}_0 = p_1, \quad P^{02}_0 = p_2, \quad P^{00}_1 = p_3, \quad P^{01}_1 = p_4, \]
\[ P^{21}_1 = p_5, \quad P^{02}_1 = p_6, \quad P^{00}_2 = p_7, \quad P^{01}_2 = p_8, \quad P^{02}_2 = p_9. \]  

(2.52)

Without going into details, we present the fixed-point tensors as follows:

\[ f_0 = f_1 = ... = f_7 = f_{11} = f_{13} = 1, \]
\[ f_8 = f_9 = -f_{10} = \frac{1}{\sqrt{2}}, \quad f_{12} = -1, \]
\[ p_1 = p_4 = p_8 = \frac{1}{\sqrt{2}}, \quad p_5 = 0, \]
\[ p_0 = p_2 = p_3 = p_6 = p_7 = p_9 = \frac{1}{2}, \]
\[ A^0 = A^2 = \frac{1}{2}, \quad A^1 = \frac{1}{\sqrt{2}}. \]  

(2.53)

The above gives the full details of the fixed-point tensors \( (N_{ijk}, F^{ijm, \alpha \beta}_{kln, \gamma \lambda}, P^{ij}_{k \lambda}, A^{i}) \).

Now we are ready to apply the Modular transformations. First we try to get the \( T \)-matrix by applying the Dehn twists. In the "Pfaffian" case we have altogether 10 non-contractable graphs on a torus as can be seen in below. Applying Dehn twists to
all of them using (2.24), we have:

\[ \Phi_{\text{fix}} ^{\text{000}} = F_{\text{000}} ^{\text{000}} \Phi_{\text{fix}} ^{\text{000}} \]

\[ \Phi_{\text{fix}} ^{\text{110}} = F_{\text{110}} ^{\text{110}} \Phi_{\text{fix}} ^{\text{110}} + F_{\text{112}} ^{\text{112}} \Phi_{\text{fix}} ^{\text{112}} \]

\[ \Phi_{\text{fix}} ^{\text{220}} = F_{\text{220}} ^{\text{220}} \Phi_{\text{fix}} ^{\text{220}} \]

\[ \Phi_{\text{fix}} ^{\text{011}} = F_{\text{011}} ^{\text{011}} \Phi_{\text{fix}} ^{\text{011}} \]

\[ \Phi_{\text{fix}} ^{\text{211}} = F_{\text{211}} ^{\text{211}} \Phi_{\text{fix}} ^{\text{211}} \]

\[ \Phi_{\text{fix}} ^{\text{121}} = F_{\text{121}} ^{\text{121}} \Phi_{\text{fix}} ^{\text{121}} \]
We thus obtain the 10 by 10 $T$-matrix for the Pfaffian state:

$$T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \quad (2.54)$$

Similarly, we can apply the 90° rotations and obtain the $S$-matrix. Without going
into too much details, we give here the final result (under the same basis):

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\] (2.55)

As in the "Fibonacci" case, the above 10 non-contractable graphs are not all fixed-point graphs and do not all correspond to ground states of a local Hamiltonian. Using the Hamiltonian construction in Appendix B, it's easy to check that the true ground state subspace is actually 9-fold; we can thus project the above matrices onto its ground state subspace. By carefully choosing the basis, we can again make the resulting \(9 \times 9\) \(T\)-matrix diagonalized and the \(9 \times 9\) \(S\)-matrix satisfy the conditions on \(S\) and \(T\) at the same time, as shown below. Numerical simulations again show that such a basis is unique (up to permutations.)

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-i\frac{\pi}{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{i\frac{\pi}{8}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\frac{\pi}{8}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\] (2.56)
\[ S = \frac{1}{4} \begin{pmatrix} 1 & \sqrt{2} & 1 & \sqrt{2} & 2 & \sqrt{2} & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} & 2 & 0 & -2 & \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 & \sqrt{2} & -2 & \sqrt{2} & 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & -2 & -\sqrt{2} \\ 2 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 2 \\ \sqrt{2} & -2 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 & -\sqrt{2} & -2 & -\sqrt{2} & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} & -2 & 0 & 2 & \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 & -\sqrt{2} & 2 & -\sqrt{2} & 1 & -\sqrt{2} & 1 \end{pmatrix} \]  

The above S-matrix is again real and symmetric. Also in this particular case, the above T and S matrices can be further reduced to the following forms:

\[ T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\pi}{8}} & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{8}} & 0 \\ 0 & 0 & -1 \end{pmatrix} \]  

\[ S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \otimes \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \]  

which shows the "doubled" structure of the quasi-particles. The above T and S matrices match exactly with the doubled "Ising" modular tensor category in [55]. As before, we can tell that there are 9 types of different quasi-particles from the dimension of T and S and through calculating the Verlinde Formula in conditions on S and T, we can get the fusion rule between them. The quasi-particle statistics can be obtained from the eigenvalues of T-matrix and the quantum dimensions of these quasiparticles are given by the elements on the first row of S-matrix.
2.4.4 Another $N = 2$ string-net state – the $S_3$ state

By adding an additional fusion rule on top of the previous result, we can get yet another example of a non-orientable string-net state. Here we still choose $N = 2$, $0^* = 0$, $1^* = 1$, $2^* = 2$, and

\[
N_{000} = N_{011} = N_{110} = N_{101} = N_{022} = N_{202} = N_{220} = N_{111} = N_{112} = N_{121} = N_{211} = 1,
\]

other $N_{ijk} = 0$. \hspace{1cm} (2.60)

These fusion rules correspond to the $N = 2$ string-net model with non-orientable strings in [46]. Following that paper, we will also call this state the $S_3$ state. As can be seen in the following, after gauge fixing, the fixed-point solutions completely agree with the “local rules” obtained in [46], thus we re-obtain another string-net state from local unitary transformation point of view. (Note that edge labels 1 and 2 are reversed in [46] compared to this thesis.)

Now comes the $F_{kln}^{ijm}$ tensor. There are nineteen independent potentially non-zero
components which are denoted as $f_0, ..., f_{18}$:

\[
\begin{align*}
F_{000}^{000} &= f_0 \\
F_{111}^{000} &= (F_{100}^{011})^* = (F_{010}^{101})^* = F_{110}^{110} = f_1 \\
F_{222}^{000} &= (F_{200}^{022})^* = (F_{020}^{202})^* = F_{220}^{220} = f_2 \\
F_{011}^{011} &= (F_{101}^{101})^* = f_3 \\
F_{111}^{011} &= (F_{111}^{111})^* = F_{011}^{111} = f_4 \\
F_{111}^{111} &= (F_{101}^{112})^* = F_{012}^{112} = f_5 \\
F_{211}^{011} &= (F_{121}^{121})^* = (F_{101}^{121})^* = F_{211}^{211} = f_6 \\
F_{022}^{022} &= (F_{202}^{202})^* = f_7 \\
F_{111}^{022} &= (F_{212}^{112})^* = F_{021}^{112} = f_8 \\
F_{110}^{110} &= f_9 \\
F_{111}^{110} &= (F_{110}^{111})^* = f_{10} \\
F_{112}^{110} &= (F_{110}^{112})^* = f_{11}
\end{align*}
\]
\[ F^{111}_{111} \langle \chi \rangle = f_{12} \]
\[ F^{111}_{112} \langle \chi \rangle = (F^{112}_{111} \langle \chi \rangle)^* = f_{13} \]
\[ F^{112}_{112} \langle \chi \rangle = f_{14} \]
\[ F^{111}_{121} \langle \chi \rangle = (F^{111}_{211} \langle \chi \rangle)^* = F^{121}_{111} \langle \chi \rangle = (F^{211}_{111} \langle \chi \rangle)^* = f_{15} \]
\[ F^{110}_{221} \langle \chi \rangle = (F^{121}_{210} \langle \chi \rangle)^* = (F^{211}_{120} \langle \chi \rangle)^* = F^{220}_{111} \langle \chi \rangle = f_{16} \]
\[ F^{121}_{121} \langle \chi \rangle = (F^{211}_{211} \langle \chi \rangle)^* = f_{17} \]
\[ F^{220}_{220} \langle \chi \rangle = f_{18} \]  

(2.61)

There are ten potentially non-zero components in \( P^k_i \), which are denoted by \( p_0, \ldots, p_9 \):

\[ P^{00}_0 = p_0, \quad P^{01}_0 = p_1, \quad P^{02}_0 = p_2, \quad P^{00}_1 = p_3, \quad P^{01}_1 = p_4, \]
\[ P^{11}_1 = p_5, \quad P^{21}_1 = p_6, \quad P^{02}_1 = p_7, \quad P^{00}_2 = p_8, \quad P^{01}_2 = p_9, \]
\[ P^{02}_2 = p_{10}. \]  

(2.62)
Without going into details, we present the fixed-point tensors as follows:

\[ f_0 = f_1 = \ldots = f_8 = f_{16} = f_{17} = f_{18} = 1, \]
\[ f_9 = f_{11} = f_{14} = \frac{1}{2}, \quad f_{12} = 0, \]
\[ f_{10} = -f_{13} = \frac{1}{\sqrt{2}}, \quad f_{15} = -1, \]
\[ p_1 = p_4 = p_9 = \frac{2}{\sqrt{6}}, \quad p_5 = p_6 = 0, \]
\[ p_0 = p_2 = p_3 = p_7 = p_8 = p_{10} = \frac{1}{\sqrt{6}}, \]
\[ A^0 = A^2 = \frac{1}{\sqrt{6}}, \quad A^1 = \frac{2}{\sqrt{6}}. \quad (2.63) \]

The above gives the full details of the fixed-point tensors \((N_{ijk}, F^{ijm,\alpha\beta}_k, P^{ij,\alpha\beta}_k, A_i)\). A careful comparison with [46] shows that the F-tensors obtained above perfectly match the \(F\) local rules for the \(N = 2\) non-orientable string-net state in [46] (after switching the edge labels 1 and 2). Thus the above state corresponds to the standard lattice gauge theory with gauge group \(S_3\) – the permutation group of 3 objects.

### 2.4.5 An \(N = 2\) string-net state – the \(Z_3\) state

The above four examples all correspond to non-orientable string-net states. Here we will give an example of an orientable string-net state. We choose \(N = 2\), \(0^* = 0\), \(1^* = 2\), \(2^* = 1\), and

\[ N_{000} = N_{012} = N_{120} = N_{201} = N_{021} = N_{102} = N_{210} = N_{111} = N_{222} = 1, \]
\[ \text{other } N_{ijk} = 0. \quad (2.64) \]

This state also appeared in [46] as the other example of \(N = 2\) string-net state. As in the \(Z_2\) case, the three edge labels of \(N\) form a \(Z_3\) group after we switch the definition of positive direction in the third label of \(N\). For example, \(N_{012}\) becomes \(N_{011}\) after switching, which will then correspond to group action \(0 \otimes 1 = 1\). We will
call such a state the $\mathbb{Z}_3$ state.

Now comes the $F_{km}^{ij}$ tensor. There are eight independent potentially non-zero components which are denoted as $f_0,\ldots, f_7$:

$$
F_{000}^{000} \equiv f_0
$$

$$
F_{111}^{000} = (F_{000}^{011})^\ast = F_{000}^{120} = f_1
$$

$$
F_{222}^{000} = (F_{000}^{022})^\ast = (F_{010}^{101})^\ast = f_2
$$

$$
F_{011}^{001} = (F_{001}^{022})^\ast = f_3
$$

$$
F_{122}^{011} = (F_{121}^{101})^\ast = f_4
$$

$$
F_{211}^{022} = (F_{212}^{202})^\ast = f_5
$$

$$
F_{210}^{112} = (F_{221}^{210})^\ast = f_6
$$

$$
F_{110}^{120} = (F_{220}^{210})^\ast = f_7
$$

There are nine potentially non-zero components in $P_i^{kj}$, which are denoted by $p_0,\ldots, p_8$:

$$
P_0^{00} = p_0, \quad P_0^{01} = p_1, \quad P_0^{02} = p_2, \quad P_0^{10} = p_3, \quad P_0^{11} = p_4,
$$

$$
P_1^{02} = p_5, \quad P_1^{00} = p_6, \quad P_1^{01} = p_7, \quad P_1^{10} = p_8.
$$
Without going into details, we present here the fixed-point tensors:

\[ f_i = 1, \quad i = 0, 1, ..., 7, \]
\[ p_i = \frac{1}{\sqrt{3}}, \quad i = 0, 1, ..., 8, \]
\[ A^0 = A^1 = A^2 = \frac{1}{\sqrt{3}}. \]  

(2.67)

The above fixed-point tensors \((N_{ijk}, F_{\alpha\beta}^{ijm}, P_{\alpha\beta}^{kij}, A^i)\) correspond to the \(Z_3\) string-net condensed state[46].

The process of applying the Modular transformations are exactly the same as in 2.4.1. By applying Dehn twists to all the 9 non-contractable fixed-point states on a torus, we can get the \(T\)-matrix. Note that as in 2.4.1, here all the 9 non-contractable graphs correspond to fixed-point states and are ground states of a certain local Hamiltonian. Without going into too much details, we present here the resulting 9 by 9 \(T\)-matrix:

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]  

(2.68)

Similarly, \(S\)-matrix can be obtained by applying the 90° rotations to all the 9 fixed-
point states. The resulting 9 by 9 $S$-matrix is then:

$$S = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}.$$ 

(2.69)

As before, we can choose a particular basis in which $S$ and $T$-matrices satisfy conditions on $S$ and $T$. Without going into too much details, we present here the resulting $T$ and $S$-matrix under a change of basis:

$$T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \xi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \xi^* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi^* & 0 \\
\end{pmatrix}. 
$$ 

(2.70)
and

\[
S = \frac{1}{3} \left( \begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \xi^* & \xi^* & \xi^* & \xi & \xi \\
1 & 1 & \xi & \xi & \xi & \xi^* & \xi & \xi^* \\
1 & \xi^* & \xi & \xi & \xi & \xi^* & \xi & \xi^* \\
1 & \xi^* & \xi^* & \xi & 1 & \xi & (\xi)^2 & 1 \\
1 & \xi^* & \xi & \xi & 1 & (\xi)^2 & \xi^* & 1 \\
1 & \xi & \xi^* & 1 & \xi & \xi^* & 1 & \xi^* & \xi \\
1 & \xi & \xi & (\xi)^2 & 1 & \xi^* & \xi & 1 \\
1 & \xi & \xi^* & \xi^* & 1 & \xi & \xi & 1 & (\xi)^2 \\
\end{array} \right)
\]

(2.71)

where \( \xi = e^{i \frac{2\pi}{3}} \). As before, all information of the quasi-particles can be obtained from the above \( T \) and \( S \) matrices.

Note that the corresponding eigenvalues of \( T \) are:

\[
(1, 1, 1, 1, 1, e^{i \frac{2\pi}{3}}, e^{i \frac{2\pi}{3}}, e^{-i \frac{2\pi}{3}}, e^{-i \frac{2\pi}{3}}).
\]

which exactly correspond to the statistical angles of the \( U(1) \times U(1) \) Chern-Simons theory\[46, 38] \( \]with \( K = K_{ij} l_{ij} a_{ij} e^{\mu \lambda} \).

(2.72)

with \( K = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \). This is also equivalent to \( Z_3 \) gauge theory.

2.4.6 \( N = 3 \) string-net states – the \( Z_4 \) states

Now we will allow four states on each edge of the graph, namely \( |0, 1, 2, 3 \). We will first give an example of orientable string-net state. We choose \( N = 3, 0^* = 0, \)
\[ 1^* = 3, \quad 2^* = 2, \quad 3^* = 1, \text{ and} \]

\[ N_{000} = N_{013} = N_{130} = N_{301} = N_{022} = N_{202} = N_{220} \]
\[ = N_{031} = N_{103} = N_{310} = N_{112} = N_{121} = N_{211} \]
\[ = N_{233} = N_{323} = N_{332} = 1, \]
\[ \text{other } N_{ijk} = 0. \tag{2.73} \]

We will see that one of the solutions to be obtained corresponds to the \( \mathbb{Z}_4 \) gauge theory. Again, as in the \( \mathbb{Z}_3 \) case, the three edge labels of \( N \) form a \( \mathbb{Z}_4 \) group after switching the positive direction of the third label of \( N \): for example, \( N_{121} \) becomes \( N_{123} \) which corresponds to group action \( 1 \otimes 2 = 3 \). Thus, we will call states obtained from the above fusion rules the \( \mathbb{Z}_4 \) states.

Again comes the \( F_{klm}^{ij} \) tensor. There are now twenty independent potentially non-zero components which are denoted as \( f_0, \ldots, f_{19} \):

\[
F_{000}^{000} = f_0
\]
\[
F_{111}^{000} = (F_{000}^{011})^* = F_{003}^{130} = f_1
\]
\[
F_{222}^{000} = (F_{000}^{022})^* = (F_{020}^{202})^* = f_2
\]
\[
F_{333}^{000} = (F_{000}^{033})^* = (F_{010}^{101})^* = f_3
\]
\[
F_{011}^{011} = (F_{011}^{033})^* = (F_{030}^{303})^* = f_4
\]
\begin{align*}
F_{122}^{011} & = \left( F_{103}^{112} \right)^{\ast} = F_{032}^{123} \\
& = \left( F_{131}^{202} \right)^{\ast} = f_5 \\
F_{233}^{011} & = \left( F_{121}^{101} \right)^{\ast} = \left( F_{103}^{123} \right)^{\ast} \\
& = F_{031}^{213} = f_6 \\
F_{022}^{022} & = \left( F_{202}^{102} \right)^{\ast} = f_7 \\
F_{133}^{022} & = \left( F_{232}^{101} \right)^{\ast} = F_{021}^{112} \\
& = \left( F_{102}^{213} \right)^{\ast} = f_8 \\
F_{311}^{022} & = \left( F_{302}^{231} \right)^{\ast} = \left( F_{212}^{303} \right)^{\ast} \\
& = F_{023}^{332} = f_9 \\
\end{align*}

\begin{align*}
F_{211}^{033} & = \left( F_{013}^{231} \right)^{\ast} = F_{323}^{303} \\
& = \left( F_{301}^{321} \right)^{\ast} = f_{10} \\
F_{322}^{033} & = \left( F_{313}^{202} \right)^{\ast} = F_{012}^{321} \\
& = \left( F_{201}^{332} \right)^{\ast} = f_{11} \\
F_{132}^{112} & = f_{12} \\
F_{310}^{112} & = \left( F_{332}^{130} \right)^{\ast} = \left( F_{112}^{310} \right)^{\ast} \\
& = F_{332}^{130} = f_{13} \\
F_{210}^{123} & = \left( F_{331}^{220} \right)^{\ast} = F_{120}^{231} \\
& = \left( F_{223}^{310} \right)^{\ast} = f_{14}
\end{align*}
\[ F_{321} \cdot \gamma = (F_{213} \cdot \gamma) = (F_{231} \cdot \gamma) \]
\[ = F_{123} \cdot \gamma = f_{15} \]
\[ F_{130} \cdot \gamma = (F_{310} \cdot \gamma) = f_{16} \]
\[ F_{221} \cdot \gamma = (F_{213} \cdot \gamma) = F_{220} \cdot \gamma \]
\[ = (F_{230} \cdot \gamma) = f_{17} \]
\[ F_{220} \cdot \gamma = f_{18} \]
\[ F_{312} \cdot \gamma = f_{19} \]

There are now sixteen potentially non-zero components in \( P_i^{kj} \), which are denoted by \( p_0, \ldots, p_{15} \):

\[
\begin{align*}
P_0^{00} &= p_0, & P_0^{01} &= p_1, & P_0^{02} &= p_2, & P_0^{03} &= p_3, & P_0^{00} &= p_4; \\
P_1^{01} &= p_5, & P_1^{02} &= p_6, & P_1^{03} &= p_7, & P_1^{00} &= p_8, & P_1^{01} &= p_9; \\
P_2^{02} &= p_{10}, & P_2^{03} &= p_{11}, & P_2^{00} &= p_{12}, & P_2^{01} &= p_{13}; \\
P_3^{02} &= p_{14}, & P_3^{03} &= p_{15}.
\end{align*}
\] (2.75)

Without going into details, we present here two fixed-point tensor solutions:

\[ f_{12} = f_{15} = f_{19} = \eta, \]
\[ f_i = 1, \quad i = \text{otherwise}, \]
\[ p_i = \frac{1}{2}, \quad i = 0, 1, \ldots, 15, \]
\[ A^0 = A^1 = A^2 = A^3 = \frac{1}{2} \] (2.76)

where \( \eta = \pm 1 \). The above gives the full details of the fixed-point tensors \( (N_{ijk}, F_{kln,\gamma\lambda}^{ijm,\alpha\beta}, P_i^{kj,\alpha\beta}, A^i) \).

Similar to the \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) cases, we can obtain the \( T \) and \( S \)-matrices of the two states above using modular transformations. By relating the eigenvalues of the \( T- \)
matrices to quasi-particle statistics, we can also find the corresponding effective theory for these two states. First, by applying Dehn twists to all the 16 fixed-point states on a torus, we obtain the following $T$-matrix:

$$T = \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}$$

(2.77)

Similarly, we can apply the 90° rotations and obtain the $16 \times 16 S$-matrix on the same basis:
As before, we can choose a basis so that $S$ and $T$-matrices satisfy conditions on
\( S \) and \( T \). For \( \eta = 1 \), we have the resulting \( T \) and \( S \)-matrices:

\[
T = \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
-1 \\
-1 \\
1 \\
1 \\
-i \\
i \\
-i \\
i \\
-1 \\
1 \\
-i \\
i \\
-1
\end{pmatrix}
\]  

(2.79)
All information of the quasi-particles can be obtained from the above $T$ and $S$ matrices.

Note that the eigenvalues of $T$-matrix are:

$$(1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, i, i, -i, -i)$$

which correspond exactly to the statistical angles of the $\mathbb{Z}_4$ gauge theory, or equivalently the $U(1) \times U(1)$ Chern-Simons gauge theory:

$$\mathcal{L} = \frac{1}{4\pi} K^I J a_{I\mu} \partial_{\nu} a_{J\lambda} \epsilon^{\mu\nu\lambda}, \quad (2.81)$$

with $K = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$.

On the other hand, when $\eta = -1$, by properly choosing the basis, we have the
following $T$ and $S$-matrices:

$$T = \begin{pmatrix}
1 \\
-1 \\
e^{-i\frac{3\pi}{4}} \\
e^{-i\frac{3\pi}{4}} \\
-1 \\
1 \\
e^{i\frac{\pi}{4}} \\
e^{i\frac{\pi}{4}} \\
e^{i\frac{3\pi}{4}} \\
e^{-i\frac{\pi}{4}} \\
1 \\
1 \\
e^{i\frac{3\pi}{4}} \\
e^{-i\frac{\pi}{4}} \\
1 \\
1
\end{pmatrix} \quad (2.82)$$
In this particular case, again, the above $T$ and $S$ matrices can be further reduced to the following forms:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & e^{i\frac{3\pi}{4}} & 0 \\ 0 & 0 & e^{i\frac{3\pi}{4}} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & e^{-i\frac{3\pi}{4}} & 0 \\ 0 & 0 & e^{-i\frac{3\pi}{4}} & 0 \end{pmatrix}$$

(2.84)

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \\ 1 & -1 & -i & i \end{pmatrix} \otimes \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \\ 1 & -1 & i & -i \end{pmatrix}$$

(2.85)

which shows the “doubled” structure of the quasi-particles. As before, all information of the quasi-particles can be obtained from the above $T$ and $S$ matrices.
Note that the eigenvalues of $T$ now become:

$\left(1, 1, 1, 1, 1, -1, -1, e^{\pm i\frac{\pi}{4}}, e^{\pm i\frac{3\pi}{4}}\right)$

which correspond exactly to the statistical angles of the $U(1) \times U(1)$ Chern-Simons gauge theory

$$\mathcal{L} = \frac{1}{4\pi} \left(4a_{1\mu} \partial_\nu a_{2\lambda} \epsilon^{\mu\nu\lambda} - 4a_{2\mu} \partial_\nu a_{1\lambda} \epsilon^{\mu\nu\lambda}\right),$$

(2.86)
or equivalently the $U(1) \times U(1)$ Chern-Simons gauge theory:

$$\mathcal{L} = \frac{1}{4\pi} K^{ij} a_{i\mu} \partial_\nu a_{j\lambda} \epsilon^{\mu\nu\lambda},$$

(2.87)

with $K = \begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix}$.

### 2.4.7 More $N = 3$ string-net states – the $\mathbb{Z}_2 \times \mathbb{Z}_2$ states

Now we will give an example of a non-orientable string-net state with four edge states. That is to set $N = 3$, $0^* = 0$, $1^* = 1$, $2^* = 2$, $3^* = 3$. We then have the following fusion rules for $N$:

$$N_{000} = N_{011} = N_{101} = N_{110} = N_{022} = N_{202} = N_{220} = N_{033} = N_{303} = N_{330} = N_{123} = N_{231} = N_{312} = N_{132} = N_{321} = N_{213} = 1,$$

other $N_{ijk} = 0$.

(2.88)

As before, we can recognize from the above that the three edge labels of $N$ form a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group if we flip the positive direction of the third label. Since the state is non-orientable, switching the direction has no real effects. Note that any element square will give unity, which is precisely the property for group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Now comes the $F_{klm}^{ij}$ tensor. There are now twenty-two independent potentially
non-zero components which are denoted as $f_0, \ldots, f_{21}$:

\[ F_{000}^{000} = f_0 \]
\[ F_{111}^{000} = (F_{011}^{011})^* = (F_{101}^{101})^* \]
\[ = F_{001}^{110} = f_1 \]
\[ F_{222}^{000} = (F_{022}^{022})^* = (F_{202}^{202})^* \]
\[ = F_{002}^{220} = f_2 \]
\[ F_{333}^{000} = (F_{033}^{033})^* = (F_{303}^{303})^* \]
\[ = F_{003}^{330} = f_3 \]
\[ F_{011}^{011} = (F_{101}^{101})^* = f_4 \]

\[ F_{233}^{011} = (F_{301}^{123})^* = F_{013}^{231} \]
\[ = (F_{303}^{121})^* = f_5 \]
\[ F_{322}^{011} = (F_{201}^{132})^* = (F_{202}^{202})^* \]
\[ = F_{012}^{321} = f_6 \]
\[ F_{022}^{022} = (F_{202}^{202})^* = f_7 \]
\[ F_{133}^{022} = F_{023}^{132} = (F_{302}^{213})^* \]
\[ = (F_{212}^{303})^* = f_8 \]
\[ F_{311}^{022} = (F_{232}^{101})^* = (F_{102}^{231})^* \]
\[ = F_{021}^{312} = f_9 \]

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There are now sixteen potentially non-zero components in $P^k_i$, which are denoted
by $p_0, ..., p_{15}$:

\[
\begin{align*}
    P^{00}_0 &= p_0, & P^{01}_0 &= p_1, & P^{02}_0 &= p_2, & P^{03}_0 &= p_3, & P^{00}_1 &= p_4, \\
    P^{01}_1 &= p_5, & P^{02}_1 &= p_6, & P^{03}_1 &= p_7, & P^{00}_2 &= p_8, & P^{01}_2 &= p_9, \\
    P^{02}_2 &= p_{10}, & P^{03}_2 &= p_{11}, & P^{00}_3 &= p_{12}, & P^{01}_3 &= p_{13}, \\
    P^{02}_3 &= p_{14}, & P^{03}_3 &= p_{15}.
\end{align*}
\]  

(2.90)

Again, without going into details, we present here four sets of resulting fixed-point tensors:

**I.** $f_i = 1$, $i = 0, 1, ..., 21$,
\[
    p_i = \frac{1}{2}, \quad i = 0, 1, ..., 15,
\]
\[
    A^0 = A^1 = A^2 = A^3 = \frac{1}{2}.
\]

**II.** $f_{13} = f_{15} = f_{17} = f_{18} = f_{19} = f_{20} = -1$,
\[
    f_i = 1, \quad i = \text{otherwise},
\]
\[
    p_0 = p_3 = p_4 = p_7 = p_8 = p_{11} = p_{12} = p_{15} = \frac{1}{2},
\]
\[
    p_1 = p_2 = p_5 = p_6 = p_9 = p_{10} = p_{13} = p_{14} = \frac{1}{2},
\]
\[
    A^0 = -A^1 = -A^2 = A^3 = \frac{1}{2}.
\]
III. \( f_{13} = f_{15} = f_{17} = f_{21} = -1, \)
\[ f_i = 1, \ i = otherwise, \]
\[ p_0 = p_2 = p_4 = \ldots = p_{14} = \frac{1}{2}, \]
\[ p_1 = p_3 = p_5 = \ldots = p_{15} = -\frac{1}{2}, \]
\[ A^0 = -A^1 = A^2 = -A^3 = \frac{1}{2}. \]

IV. \( f_{18} = f_{19} = f_{20} = f_{21} = -1, \)
\[ f_i = 1, \ i = otherwise, \]
\[ p_0 = p_1 = p_4 = p_5 = p_8 = p_9 = p_{12} = p_{13} = \frac{1}{2}, \]
\[ p_2 = p_3 = p_6 = p_7 = p_{10} = p_{11} = p_{14} = p_{15} = -\frac{1}{2}, \]
\[ A^0 = A^1 = -A^2 = -A^3 = \frac{1}{2}. \]  

(2.91)

We can further check if the results obtained above are consistent with the ones in the \( \mathbb{Z}_2 \) case (c.f. 2.4.1). A quick check can be done as follows: In 2.4.1, we have
\[ f'_0 = 1 \text{ and } f'_3 = \eta. \]

(We use \( f' \) to differentiate it from the \( f \) obtained in this section.) Since \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) can be seen as two copies of \( \mathbb{Z}_2 \), we expect
\[ f_0 = f'_0 \times f'_0 = 1, \ f_{13} = f'_3 \times f'_0 = \eta^a, \]
\[ f_{18} = f'_0 \times f'_3 = \eta^b, \ f_{21} = f'_3 \times f'_3 = \eta^a \times \eta^b. \]

Thus \( f_{13}, f_{18}, f_{21} \) would either all be one, or two out of them would be \(-1\). This precisely corresponds to the four solutions obtained above. A more detailed check also shows consistency between the two results.
2.4.8 An $N = 3$ string-net state – the “Chiral” state

Finally, we present here a new state which looks like a non-string-net state, but is actually a string-net state. In this state, interestingly enough, the chiral symmetry of the fusion rule is spontaneously broken, as can be seen in the following. We choose $N = 3$, $0^* = 0$, $1^* = 3$, $2^* = 2$, $3^* = 1$, and

$$
N_{000} = N_{013} = N_{130} = N_{301} = N_{123} = N_{231} = N_{312} = N_{222} = 1, \\
\text{other } N_{ijk} = 0.
$$

(2.92)

Note that e.g. $N_{310} = 0$, thus breaking the chiral symmetry.

This state has several interesting features. Besides its chiral symmetry being spontaneously broken, the three edge labels of $N$ form a structure called a “multi-fusion category”. This mathematical structure has the following fusion rule between its four elements:

$$
M_{ij} \times M_{kl} = \delta_{jk} M_{il}, \quad \text{where } i, j, k, l = 1 \text{ or } 2.
$$

After making the following mapping, $M_{11} = state \ 0$, $M_{12} = state \ 1$, $M_{22} = state \ 2$, $M_{21} = state \ 3$, we can see that the fusion rule above is exactly equivalent to (2.92) (again, we need to flip the positive direction of the third label of $N$). Although the fusion rule is fairly non-trivial, we will see that at least in the no-symmetry case which we are studying in this thesis, the corresponding fixed-point state is trivial.

Now comes tensor $F^{ijm}_{kln}$. There are now six independent potentially non-zero
components which are denoted as \( f_0, \ldots, f_5 \):

\[
\begin{align*}
F_{000}^{000} &= f_0 \\
F_{333}^{000} &= (F_{100}^{033})^* = (F_{010}^{101})^* = f_1 \\
F_{223}^{033} &= (F_{323}^{132})^* = f_2 \\
F_{110}^{132} &= (F_{332}^{310})^* = f_3 \\
F_{222}^{132} &= (F_{322}^{211})^* = (F_{232}^{222})^* = f_4 \\
F_{222}^{222} &= f_5
\end{align*}
\]  

(2.93)

There are eight potentially non-zero components in \( P_i^{kj} \), which are denoted by
\( p_0, \ldots, p_7 \):

\[
\begin{align*}
P_0^{00} &= p_0, & P_0^{03} &= p_1, & P_1^{00} &= p_2, & P_1^{03} &= p_3, & P_2^{21} &= p_4, \\
P_2^{22} &= p_5, & P_3^{21} &= p_6, & P_3^{22} &= p_7.
\end{align*}
\]  

(2.94)

Here, without going into too much details, we give the resulting fixed-point tensors.

Note that the solution contains an undetermined parameter \( \eta \in [0, 1] \):

\[
\begin{align*}
f_i &= 1, & i &= 0, 1, \ldots, 5, \\
p_0 &= p_2 = p_4 = p_6 = \eta, \\
p_1 &= p_3 = p_5 = p_7 = \sqrt{1 - \eta^2}, \\
A^0 &= \eta^2, & A^2 &= 1 - \eta^2, \\
A^1 &= A^3 = \eta \sqrt{1 - \eta^2}.
\end{align*}
\]  

(2.95)

The above gives the full details of the fixed-point tensors \( (N_{ijk}, F_{km, \gamma\lambda}^{ijm, \alpha\beta}, P_i^{kj, \alpha\beta}, A^i) \).
It can be shown that the above state corresponds to a trivial loop state with adjustable loop weights. By introducing the “double-line rule” and associating different weights to the “dashed-line” loops & “solid-line” loops, the state reduced to a state with independent loops. This can be seen very clearly in Fig. 2-4.

2.5 Summary

In this chapter, we computed the $T$ and $S$-matrices that was introduced to define topological orders in 2D.[74, 33] We argued that this new label can be used to fully describe topological orders. We show explicitly how to obtain these labels from their fixed-point states by applying the “modular transformations”. The resulting $T$ and $S$-matrices are uniquely determined after a proper choice of basis and can thus be used as labels for topological orders.

The “modular transformations” can be defined through two transformations, i.e. the “Dehn twist” and “90° rotation”. After applying both transformations in the ground-state subspace, we can get two matrices, $T$ and $S$, describing “modular transformations”. We can then change the basis to get the proper form of these two matrices.
Applying the two transformations in the ground-state subspace can be achieved by projection. We first define $T$ and $S$ for all non-contractable graphs on a torus, then apply unitary transformation to diagonalize the Hamiltonian. We then project $T$ and $S$ only onto the subspace with the smallest eigenvalue of the Hamiltonian, thus obtaining $T$ and $S$ for the ground states.

To get the “proper form” of the two labeling matrices, we require them to satisfy the conditions on $S$ and $T$. After this unitary transformation, the form of $T$ and $S$ will be uniquely determined only up to permutations of basis. We note that Y. Zhang et al. [78] proposed a different way of getting the “proper form” of the modular matrices by using the “minimum entropy states”. We expect the two methods to agree, which is indeed the case for the $\mathbb{Z}_2$ Toric Code states calculated in both papers (see 2.4.1).

The resulting $T$ and $S$-matrices are unique labels for different topological orders. We believe they contain full information of the phase and can fully characterize different topological orders. Future work can be done to further generalize this method to obtain $T$ and $S$ matrices for general tensor-product states or other many-body states, so as to determine the topological order of an arbitrary many-body state. Even further, using this $T$ and $S$-matrices description, we may eventually be able to describe phase transitions between different topological orders.
Chapter 3

A Tensor Network Method for Topological Orders

3.1 Outline of the chapter

In the previous chapter, we introduced a new order-parameter-like label for different topological orders (namely the $T$- and $S$- matrices) and introduced in details how to obtain these new labels from an ideal fixed-point wave function. In this chapter, we present a tensor-network method to identify different topological orders directly from a generic non-fixed-point wave function. The calculation of $T$- and $S$- matrices directly from a generic wave function is of course a numerically formidable task; so here we present a less ambitious approach and characterize only topological orders that correspond to gauge theories.

This chapter is structured as follows: In section 3.2, we first introduce the concept of "environment matrix" and outline how this method is applied in 1$D$, with the example of the transverse Ising model. In section 3.3, we make some detailed emphasis on the symmetry structure of the environment matrix, which leads to a characterization of different phases. In section 3.4 and 3.5, we detail how to detect different phases without knowing the symmetry structure, which makes our method immediately applicable to existing 1$D$ numerical methods in identifying different SPT phases. Finally in section 3.6 and 3.7, we generalize this method to 2$D$, and apply
Figure 3-1: A matrix-product state. All the physical sites are represented by dots, and physical/internal degrees of freedom by vertical/horizontal lines.

it to both the transverse Ising model and the Toric-Code model with a B-field on a honeycomb lattice.

3.2 1D environment tensor method

The environment tensor method has been widely applied in 1D systems through the study of Matrix-Product States (MPS). Consider the 1D transverse Ising model on an infinite lattice:

$$H = -J \sum_{<i,j>} \sigma_i^x \sigma_j^x - h \sum_i \sigma_i^z$$  \hspace{1cm} (3.1)

where $\sigma$'s are the regular Pauli matrices. Recall that the wave function of a 1D system could always be written into a matrix-product form; in particular, if the system is translationally symmetric, then we have (in Figure 3-1):

$$|\Psi\rangle = \sum_{\{m_i\}} \sum_{\{\alpha_i\}} \text{Tr}[\prod_i M_{\alpha_i,\alpha_{i+1}}^{m_i}] |\{m_i\}\rangle$$  \hspace{1cm} (3.2)

where matrices $M$'s are independent of the site labels $i$ and are labeled by the physical degrees of freedoms $m_i$.

In 1D, the environment tensor method is essentially a variational calculation based on the above matrix-product state, with matrices $M$'s as variational parameters. We use the matrices $M$'s to obtain the average energy (see the top of Fig. 3-2). We minimize the the average energy to obtain $M$'s.

The calculation of average energy is actually a finite calculation. The key is to use "environment matrix" to capture the contributions from far-away sites. This
Figure 3-2: Approximate the energy through the environment matrix, $E$. Note that because the system is translation invariant, we only need to calculate $H$ for two neighboring sites. In 1D, by assuming $M$ to be left-right symmetric, the environment matrices on the left and right will be the same.

Figure 3-3: A self-consistent condition that environment matrix $E$ must satisfy, where $\lambda$ is a scaling factor. The part enclosed by the shaded area is the so-called double-tensor $T_{b\beta,aa}$.

is graphically shown in the bottom of Fig. 3-2. As can be seen, there are two environment matrices, one on each side of the 1D chain.

So in the actual environment tensor method, we use the matrices $M$’s to obtain the environment matrices $E$, and then use the matrices $M$’s and environment matrices $E$’s to obtain the average energy. We then minimize the average energy to obtain $M$’s (and $E$’s).

For fixed matrices $M$’s, the environment matrix could be obtained through iterations. As shown in Figure 3-3, starting from some random initial values $E_0$ that satisfies $\text{Tr} E_0^\dagger E_0 = 1$, we can update the environment matrix using a “double-tensor”, which is formed by two $M$ matrices with physical indices contracted. After applying the “double-tensor”, $E_0$ is changed to $\lambda_1 E_1$ where $E_1$ satisfies $\text{Tr} E_1^\dagger E_1 = 1$ and $\lambda_1$ is a scaling factor. After iterating enough number of times, a final stable “environment
matrix" $E_\infty = E$ and a final stable scaling factor $\lambda_\infty = \lambda$ would be reached. Note that this process, after viewing the environment as a vector and the double-tensor as an operator, is essentially equivalent to picking out the eigenvector with the largest absolute value of the eigenvalues of the double tensor. In this way, for each $M$, we can obtain the corresponding environment matrix $E$ through iterations, and by applying $E$ both on the left and on the right (see Figure 3-2), we can get the total energy. The variational calculation could then be carried out for different values of $h/J$, and a phase diagram could then be obtained.

More specifically, we require our matrix-product state to have a $Z_2$ symmetry that corresponds to spin up-down flipping, for both the symmetry-breaking and the symmetric phases. So even in symmetry breaking phase, we choose the ground state to be, say, $(|\uparrow\uparrow\ldots\rangle + |\downarrow\downarrow\ldots\rangle)$ when $h/J \to 0$. Note that this is different from traditional symmetry breaking description, where the ground state spontaneously picks one of the ferromagnetic states.

Recall that on-site symmetry of the ground state requires matrices $M$'s to transform in a special way w.r.t. symmetry:[51]

$$
\sum_{m'} g_{mm'} M^{m'} = e^{i\theta_g} U_g \, M^m U_g 
$$

Here, $m$ is the spin index, matrix $g_{mm'}$ represents the on-site symmetry and acts on the spin basis, $\theta$ is a phase factor (set to 0 in this thesis), $U_g$ is a unitary matrix acting on internal degrees of freedom, and forms a (projective) representation of the symmetry group $G$.[53, 12, 57]

For internal dimension $D$ (dimension of $M$) being 2, we can choose $U_g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $g$ is the non-trivial element of $Z_2$. (Note that the key here is to choose $U_g$ to be a reducible representation of $Z_2$.) Then equation (3.3) reduces to

$$
M^\dagger = U_g^\dagger M^+ U_g, 
$$

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Figure 3-4: Energy as a function of $h/J$. The $h$- and $J$- terms are also individually plotted in the graph, so the phase transition could be easily spotted at $h/J = 0.83$. This result is obtained when internal dimension $D=2$, and we’ve chosen the grid so that sample points are denser close to the transition point.

and we thus have: $M^\uparrow = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $M^\downarrow = \begin{pmatrix} a & -b \\ -b & c \end{pmatrix}$. Here $M$’s are symmetric because of left-right symmetry, and $a, b, c$ are free variational parameters.

With the above symmetry analysis in mind, numerical simulation could be run on our 1D Ising model. Following the previous discussion, for each $h/J \in [0, \infty]$ in equation (3.1), we minimize the energy by varying $M$’s satisfying equation (3.4). By plotting the two energy terms, a phase diagram is obtained (see Fig. 3-4). For internal dimension $D = 2$, the phase transition occurred at $h/J = 0.83$, with an error of 17%.

We could easily improve the result by increasing the internal dimension. For internal dimension $D = 4$, we can choose $U_q$ in (3.4) to be $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, where $I$ is the $2 \times 2$ Identity matrix (again we need a reducible representation of $\mathbb{Z}_2$). The most general symmetric $M$ satisfying (3.4) has 10 variational parameters. Following the same variational procedure, we can get the energy plot shown in Fig. 3-5. The phase transition occurred at $h/J = 0.97$, with a mere 3% error.
Figure 3-5: Energy plot when internal dimension increased to D=4. We can see the phase transition point occurred around $h/J \approx 0.97$, a big improvement from Fig. 3-4.

Note that in both calculations, we used symmetric matrices $M$'s with all real parameters. The typical runtime on a laptop was just a few seconds in both cases.

### 3.3 Symmetry structure of the environment matrix

From the above plot, we see that there is a phase transition at $h/J \approx 0.83$, at least when we restrict to the manifold of MPS's with $M$ of the symmetric form specified before. To understand the phases on the two sides of the transition, let us choose another basis

$$
\tilde{M}^\dagger = WM^\dagger W^\dagger = \frac{1}{2} \begin{pmatrix}
    a + c + 2b & a - c \\
    a - c & a + c - 2b
\end{pmatrix}
$$

$$
\tilde{M} = WM^\dagger W^\dagger = \frac{1}{2} \begin{pmatrix}
    a + c - 2b & a - c \\
    a - c & a + c + 2b
\end{pmatrix}
$$

where $W = 2^{-1/2} \begin{pmatrix}
    1 & 1 \\
    1 & -1
\end{pmatrix}$. In the new basis the meaning of the $\tilde{M}$'s is more clear.
When $b = 0$, $\tilde{M}^\dagger = \tilde{M}^\dagger$, and the MPS is a pure product state $(|\uparrow\rangle + |\downarrow\rangle)^{\otimes N}$ that does not break the $\mathbb{Z}_2$ symmetry. When $b \neq 0$ and $a = c$, the MPS is a symmetry breaking state of the form $|\Psi\rangle + U|\Psi\rangle$ where $U$ is $\mathbb{Z}_2$ symmetry transformation. But when $b \neq 0$ and $a \neq c$, what is the nature of the MPS?

To answer such a question, we would like to study the symmetry structure of the environment matrix $E$. We find that, depends on which phase we are in, the symmetry structure of $E$ will be very different.

As we mentioned before, the environment matrix $E$ and the associated scaling factor $\lambda$ is calculated via the iteration (or the self consistent condition) in Fig. 3-3. In general, there can be many environment matrices $E$ that satisfy the self consistent condition. Here we choose those with largest absolute value of the scaling factor $\lambda$. If there are many environment matrices with the degenerate largest absolute value of the scaling factor, we then choose the environment matrices with minimal “entropy”

$$S = \sum_i -s_i \ln s_i,$$

(3.6)

where $s_i$ is the singular values of $E$. This will fix the basis and give us a set of environment matrices $\{E\}$ that have very nice properties to be shown later. We want to emphasize that this special need to fix the basis only appears in 1D, and is in general uneccesary in higher dimension.

Next, we want to point out that environment matrix has only internal indices, so for $E$, the symmetry transformation (3.3) translates to:

$$E \rightarrow U_g^\dagger \cdot E \cdot U_g.$$  

(3.7)

If the $M$'s are invariant under the symmetry transformation (3.3), then the set of environment matrices $\{E\}$ will be invariant under the above transformation (3.7).

If the action of transformation (3.7) is trivial on the set of environment matrices $\{E\}$, then the MPS does not have symmetry breaking. If the action is non-trivial (i.e. generate a permutation of the set $\{E\}$), then the MPS, in general, has a symmetry
The entanglement density matrix can be calculated from the environment matrix. (Note that technically, the above may not be the full entanglement density matrix, which should really be calculated from the total the environment tensor $E^0 \otimes E^0 + E^1 \otimes E^1$.)

breaking; but this is not guaranteed.

The reason for the complication is that there are zero-measure possibilities that some internal bond degrees of freedom (called “phantom degrees of freedom”) can completely decouple from the physical degrees of freedom. When this happens, the transformation (3.7) could act non-trivially within those phantom degrees of freedom, which does not correspond to symmetry breaking. To fix this problem, we may consider the entanglement density matrix $\rho_{m_1m_2\ldots,m'_1m'_2\ldots}$ defined in Fig. 3-6. We say that two environment matrices are equivalent if they generate the same $\rho_{m_1m_2\ldots,m'_1m'_2\ldots}$. Let us use $\{E\}/\sim$ to denote the equivalent class of the environment matrices. Then if the action (3.7) is non-trivial on $\{E\}/\sim$, the MPS has a true symmetry breaking.

With the above general discussion, we now go to our numerical results for internal dimension $D = 2$. For the symmetric phase, using the iteration method in Figure 3-3 and after energy minimization, we obtain a final $E = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$, which gives an invariant $E$ under eqn (3.7). As a result, the total environment tensor has a pure tensor product form

$$E^{tot} = E \otimes E.$$  \hspace{0.5cm} (3.8)

For the symmetry-breaking phase, after minimizing entropy according to eqn. (3.6), depending on the initial values of $E$, the iteration method would give either
Figure 3-7: The off-diagonal term in environment matrix is plotted here as a function of $a - c$ and $b$. We can see the delta-function-like behavior when $a = c$. Note the dip at $a = c$ and $b = 0$: this point corresponds to the four-fold degeneracy in the double-tensor, which is in the symmetric phase.
\[ E^{g_1} = \begin{pmatrix} p & p \\ p & p \end{pmatrix} \] or \[ E^{g_2} = \begin{pmatrix} p & -p \\ -p & p \end{pmatrix} \], which transforms into each other under eqn (3.7). Both of these correspond to environment matrix of the fixed-point wavefunction, to be explained later in this section. If we construct the total environment tensor

\[ E^{tot} = E^{g_1} \otimes E^{g_1} + E^{g_2} \otimes E^{g_2}, \] (3.9)

then the \( \mathbb{Z}_2 \) symmetry is restored, but now the the total environment tensor does not have a pure tensor product form.

Note that the concept of total environment tensor is particularly helpful in higher dimensions, when the contribution from surrounding states are inter-connected. We will see later that in higher dimension, the total environment tensor still have the simple tensor-product decomposition form in the symmetric phase, whereas it could not be written in a pure tensor product form for the symmetry-breaking phase.

We are now in the position to answer the question raised at the beginning of this section. We now know that symmetry-breaking phase is signatured by a non-zero off-diagonal term in the environment matrix. As shown in Fig. 3-7, if we plot this off-diagonal term as a function of \( a - c \) and \( b \) in \( M \) (see eqn (3.5)), then we see that the system is only in symmetry breaking state when \( a = c \) and \( b \neq 0 \). So when \( b \neq 0 \) and \( a \neq c \), the state is in the symmetric phase.

Here we've also plotted the magnetization as a function of \( h/J \) in Fig. 3-8. Note that in the graph, we get a first-order phase transition for both \( D = 2 \) and \( D = 4 \). This is because we required our matrices \( M \)'s in the MPS to have the \( \mathbb{Z}_2 \) symmetry (recall eqn. (3.5)). This symmetry requirement favors the symmetric phase, because symmetry-breaking phase requires \( a = c \), so \( M \) is block-diagonalized, reducing its effective internal dimension. Thus the phase transition point is shifted leftwards, leading to a first-order transition. As we increase the internal dimension, we expect the phase transition point to approach \( h/J = 1 \) from the left.

In Fig. 3-8, we've also plotted the order parameter as a function of \( h/J \). Note that these "order parameters" do not vary as we change \( h/J \). This is because the
Figure 3-8: A plot of magnetization and the “fixed-point” order parameter as a function of $h/J$. When we increase the internal dimension, we get a more accurate phase transition point. In both cases, we get a first-order phase transition. When internal dimension $D = 2$, order parameter is just the off-diagonal term of the environment matrix (represented by blue dots). When $D = 4$, order parameter is the norm of the $2 \times 2$ off-diagonal block in the environment matrix (represented by red circles). The black line curve is $\langle \sigma_i^z \rangle = \pm (2 - 2h)^{1/8}$. 
environment matrix is obtained from enough iterations that it really corresponds to the fully renormalized wavefunction. The order parameter obtained from the environment matrix then corresponds actually to the order parameter at the fixed point, thus is always the same until hitting the phase transition point.

3.4 Detecting phases of MPS without knowing the transformation property of the matrices

In the above, we have assumed that the matrices in the MPS has the symmetry and studied how to use the symmetry breaking of the environment matrix to detect the spontaneous symmetry breaking in MPS. However, in many calculations, such as the density-matrix-renormalization-group (DMRG) calculation, the resulting matrices in the MPS do not have the symmetry in the symmetry breaking phase, and in general we do not even know how the matrices transform under the symmetry transformation (since we do not know how the internal indices should transform under the symmetry). In this section, we will discuss how to detect the spontaneous symmetry breaking in MPS, without knowing how the matrices in the MPS transforms under the symmetry transformation.

Assume we have already obtained the matrices in the MPS. We first calculate the environment matrix $E$ and the scaling factor $\lambda$ using Fig. 3-3. In general, the environment matrix $E$ is unique even in the symmetry breaking state, since the matrices in the MPS obtained from DMRG in general already break the symmetry.
Next, we insert the symmetry transformation $g$ (see eqn (3.3)) in the double-tensor to obtain a twisted double-tensor. The corresponding twisted environment matrix is denoted as $E^g$ and the twisted scaling factor as $\lambda^g$ (see Fig. 3-9) [54].

If $|\lambda^g| < |\lambda|$, then the wave function overlap between MPS and the twisted MPS is smaller than 1, and the corresponding MPS have a spontaneous symmetry breaking. In fact, there is a more direct way to detect symmetry breaking. Let $P$ and $P^g$ be the matrices defined via the $n$th power of double-tensor (where $T$ and $T^g$ are viewed as matrices)

$$P_{ab,\alpha\beta} = T^n_{\beta\beta,aa}, \quad P^g_{ab,\alpha\beta} = (T^g)^n_{\beta\beta,aa}.$$ \hspace{1cm} (3.10)

If $P$ and $P^g$ have different singular values in large $n$ limit, then the corresponding MPS break the symmetry explicitly.

If $|\lambda^g| = |\lambda|$, then the wave function overlap between MPS and the twisted MPS = 1, and the MPS is invariant under symmetry transformation eqn (3.3). In this case, the two environment matrices $E^g$ and $E$ are related by the symmetry transformation (see eqn (3.3)) [54]

$$E = E^g U_g, \quad \text{or} \quad E^{-1} E^g = U_g^\dagger.$$ \hspace{1cm} (3.11)

In fact, we have

$$E = U_g^\dagger E U_g.$$ \hspace{1cm} (3.12)

If $U_g$ forms a projective representation of the symmetry group $G$, then the corresponding MPS does not break the symmetry and has a non-trivial SPT order protected by the on-site symmetry. If $U_g$ forms a 1D representation of the symmetry group $G$, then the corresponding MPS does not break the symmetry and has a non-trivial SPT order protected by translation symmetry (and the on-site symmetry).

Let us apply the above approach to a MPS state of spin-1 chain, where the matrix $M^l, l = x, y, z$ are given by the Pauli matrices: $M^l = \sigma^l$. The double-tensor is given
by (see Fig. 3-3)

\[ T_{b\beta,aa} = \sigma_{ba}^{x} \sigma_{b\alpha}^{x} - \sigma_{ba}^{y} \sigma_{b\alpha}^{y} + \sigma_{ba}^{z} \sigma_{b\alpha}^{z} \]  

(3.13)

The action of the double-tensor \( T_{b\beta,aa} \) on the environment matrix \( E_{b\beta} \rightarrow T_{b\beta,aa}E_{aa} \) can be written in a matrix form

\[ E \rightarrow \sum_{l} \sigma^{l} E \sigma^{l}. \]  

(3.14)

We see that \( E = 2^{-1/2} \sigma^{0} \) (the 2-by-2 identity matrix) is the non-degenerate environment matrix with \( \lambda = 3 \).

Now, let us show that the MPS has a \( \mathbb{Z}_{2}^{x} \times \mathbb{Z}_{2}^{z} \) symmetry where \( \mathbb{Z}_{2}^{x} \) is generated by \( R_{x} \) – the 180° spin rotation in \( S^{x} \)-direction and \( \mathbb{Z}_{2}^{z} \) is generated by \( R_{z} \) – the 180° spin rotation in \( S^{z} \)-direction. Under the symmetry twists \( R_{x} \) and \( R_{z} \), the corresponding double-tensors are

\[ T_{b\beta,aa}^{R_{x}} = -\sigma_{ba}^{x} \sigma_{b\alpha}^{x} - \sigma_{ba}^{y} \sigma_{b\alpha}^{y} + \sigma_{ba}^{z} \sigma_{b\alpha}^{z}, \]

\[ T_{b\beta,aa}^{R_{z}} = +\sigma_{ba}^{x} \sigma_{b\alpha}^{x} - \sigma_{ba}^{y} \sigma_{b\alpha}^{y} - \sigma_{ba}^{z} \sigma_{b\alpha}^{z}. \]  

(3.15)

The corresponding twisted environment matrices are given by

\[ E^{R_{x}} = 2^{-1/2} \sigma^{x}, \quad E^{R_{z}} = 2^{-1/2} \sigma^{z}. \]  

(3.16)

with \( \lambda^{R_{x}} = \lambda^{R_{z}} = -3 \). We see that

\[ U_{R_{x}} = \sigma^{x}, \quad U_{R_{z}} = \sigma^{z}. \]  

(3.17)

Since \( |\lambda^{R_{x}}| = |\lambda^{R_{z}}| = |\lambda| \) and

\[ E = U_{R_{x}}^{\dagger} E U_{R_{x}}, \quad E = U_{R_{z}}^{\dagger} E U_{R_{z}}, \]  

(3.18)

we found that the \( \mathbb{Z}_{2}^{x} \times \mathbb{Z}_{2}^{z} \) symmetry is not broken. We also see that \( U_{R_{x}}, U_{R_{z}} \)
generate a projective representation of $Z^x_2 \times Z^z_2$. So the MPS is a SPT state protected by $Z^x_2 \times Z^z_2$.

### 3.5 A tensor network approach for 1D model

In this section, we are going to use an infinite time-evolving block decimation (iTEBD) approach\[64\] to study 1D models, such as the transverse Ising model (3.1). We are going study symmetry breaking by testing if $|\lambda^g| = |\lambda|$ or $|\lambda^g| < |\lambda|$.

#### 3.5.1 The iTEBD method

The iTEBD method is a tensor network version of the DMRG approach. The fundamental idea behind the iTEBD method is to use imaginary time evolution to get the ground state of a two-body Hamiltonian, and to use Singular Value Decomposition (SVD) to control the internal dimensions.

Consider any 1D Hamiltonian with only nearest-neigbour interactions, we can always separate it into two parts, labeled by $H_A$ and $H_B$:

$$H = \sum_i H_{i,i+1} = \sum_{i \text{ odd}} H_{i,i+1} + \sum_{i \text{ even}} H_{i,i+1}$$

$$= H_A + H_B. \quad (3.19)$$

This way, either $H_A$ or $H_B$ would have no overlapping terms within itself. When the time step $\delta t$ is very tiny, we have:

$$e^{-\delta t H} \approx e^{-\delta t H_A} e^{-\delta t H_B} \equiv W. \quad (3.20)$$
We could then apply imaginary-time evolution layer by layer, as shown in Fig. 3-10. Now within each layer, time-evolution only operates on non-overlapping neighboring sites. Thus the entire problem reduces to a two-site problem.

The two-site time-evolution is done through Singular Value Decomposition, see Fig. 3-11. We first apply the time-evolution operator (labeled by \( W_{i,i+1} \)) on two sites, resulting in a rank-4 tensor, \( T_{m'\alpha,k'\gamma} \):

\[
T_{m'\alpha,k'\gamma} = W_{m'k',mk} A_{m}^{\alpha\beta} B_{\beta\gamma}^{k}.
\]  

(3.21)

Then we do SVD to split the rank-4 tensor:

\[
T_{m'\alpha,k'\gamma} = U_{m'\alpha,\beta} S'_{\beta\lambda} V_{k'\lambda,\gamma}.
\]  

(3.22)

Note that after applying SVD, the internal dimension has grown on the inner link. We could get back our original internal dimension by keeping only the \( D \) largest singular values of \( S' \). We'll call the truncated matrix \( S' \). Finally, we absorb diagonal matrix \( S \) into the two on-site matrices, thus completing one step of evolution:

\[
\tilde{A}_{m'}^{\alpha\beta} = U_{m'\alpha,\beta} \sqrt{S'_{\beta}}, \quad \tilde{B}_{\beta\gamma}^{k'} = V_{k'\beta,\gamma} \sqrt{S_{\beta}}.
\]  

(3.23)

This completes one step in time evolution.

One improvement can be made on the above time-evolution step. Note that in the truncation process above, we implicitly assumed that all bond indices are equally

---

Figure 3-11: Time-evolution on two sites. Step 1: apply time-evolution operator. Step 2: apply SVD and truncate the singular matrix to only contain \( D \) largest singular values (\( \tilde{S} \) denotes the singular matrix after truncation). Step 3: separate the singular values into the two sites.
important; however, we know that’s not the case. The “environment indices” $\alpha, \gamma$ do not contribute equally, and their weights could be naturally included by using the singular values $S_\alpha, S_\gamma$ from the previous time-evolution step. (This is because in the previous step, $\alpha$ and $\gamma$ were inner link indices, and each index naturally carries a weight according to the previous step of SVD.)

Thus the improved time-evolution step works as follows: 1. First, we scale the “environment indices” using singular values $S_\alpha$ obtained from last step:

$$A_{\alpha\beta}^m \rightarrow \sqrt{S_\alpha} A_{\alpha\beta}^m, \quad B_{\beta\gamma}^m \rightarrow \sqrt{S_\gamma} A_{\beta\gamma}^m.$$  

(3.24)

2. Then apply the time-evolution step described before. 3. Lastly, we scale the “environment indices” back, by doing the following:

$$\tilde{A}_{\alpha\beta}^m \rightarrow \sqrt{S_\alpha^{-1}} \tilde{A}_{\alpha\beta}^m, \quad \tilde{B}_{\beta\gamma}^m \rightarrow \sqrt{S_\gamma^{-1}} \tilde{B}_{\beta\gamma}^m.$$  

(3.25)
Figure 3-13: The difference of the scaling factors $|\lambda_1 - \lambda_2|$ of transverse Ising model as a function of $h$ with $J = 1$. The ‘+’ points are for $D = 2$, ‘x’ for $D = 4$, ‘•’ for $D = 8$, and ‘□’ for $D = 16$.

Figure 3-14: The magnetization of transverse Ising model $\langle \sigma_i^z \rangle$ as a function of $h$ with $J = 1$. The ‘+’ points are for $D = 2$, ‘x’ for $D = 4$, ‘•’ for $D = 8$, ‘□’ for $D = 16$, and ‘■’ for $D = 32$. The line curve is $\langle \sigma_i^z \rangle = \pm (2 - 2h)^{1/8}$. 
3.5.2 The iTEBD results for transverse Ising model

After applying the imaginary-time evolution steps described above, we will eventually obtain the tensor that describes the ground state wave function very well. The next issue is to identify the symmetry breaking order and/or SPT order in the ground state, using the method discussed before.

For the transverse Ising model, Fig. 3-12 describes the calculated $|\lambda| - |\lambda_0|$ and the magnetization $\langle \sigma_i^z \rangle$, using the iTEBD approach with various $D$. Fig. 3-13 and Fig. 3-14 are the results near the transition point. The transition point is found to be $h_c \approx 1.08$ for $D = 2$, $h_c \approx 1.0188$ for $D = 4$, $h_c \approx 1.0101$ for $D = 8$, $h_c \approx 1.0047$ for $D = 16$, and $h_c \approx 1.0015$ for $D = 32$. The exact transition point is at $h_c = 1$. We see that “order parameter” $|\lambda| - |\lambda_0|$ works very well, in identify symmetry breaking transitions.

3.5.3 The iTEBD calculation of 1D model with symmetry-breaking and/or SPT orders

In this section, we are going to use the iTEBD approach to study spin-1 model with $\mathbb{Z}_2^x \times \mathbb{Z}_2^z$ symmetry:

$$H = \sum_i [J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z + J_{zz} (S_i^z)^2]$$  \hspace{1cm} (3.26)
Figure 3-16: The phase diagram for the spin-1 model in \( J_x - J_{zz} \) plane, with \( J_y = J_z = 1 \), calculated by the iTEBD method with \( D = 8 \). The points mark which \( \frac{|\Lambda| - |\Lambda_0|}{|\Lambda|} = 0 \). The “+” points are for symmetry twist \( g = R_x \), “x” for \( g = R_y \), and “□” for \( g = R_z \). The green shaded area and the white area at the top have the full \( R_x, R_y, R_z \) symmetry. The gold shaded area has only the \( R_y \) symmetry, and the blue shaded area has only the \( R_x \) symmetry.

The \( \mathbb{Z}_2^x \) is generated by 180° spin-rotation \( R_x \) around the \( S_x \) axis. The \( \mathbb{Z}_2^z \) is generated by 180° spin-rotation \( R_z \) around the \( S_z \) axis. The \( R_x R_z = R_y \) is the 180° spin-rotation around the \( S_y \) axis.

We choose \( D = 8 \) and calculated the tensor \( M_{\alpha\beta}^m \) for the ground state. To determine if \( M_{\alpha\beta}^m \) describes a symmetry breaking state or not, it is not correct to directly test if \( M_{\alpha\beta}^m \) has the symmetry or not. This is because even when \( M_{\alpha\beta}^m \) is not invariant under any symmetry transformation of the form eqn (3.3), \( M_{\alpha\beta}^m \) can still describe a symmetric state.

So to determine symmetry of the ground state, we instead calculated the quantity \( \frac{|\Lambda| - |\Lambda_0|}{|\Lambda|} \) for symmetry twists \( g = R_x, R_y, R_z \) (see Fig. 3-15). We determined the phase diagram by examine where and which \( \frac{|\Lambda| - |\Lambda_0|}{|\Lambda|} \) vanishes. The gold shaded area in Fig. 3-16 only has the \( R_x \) symmetry, since only \( \frac{|\Lambda| - |\Lambda_0|}{|\Lambda|} = 0 \). The blue shaded area in Fig. 3-16 only has the \( R_y \) symmetry since only \( \frac{|\Lambda| - |\Lambda_0|}{|\Lambda|} = 0 \). The green shaded area and the white area have \( \frac{|\Lambda| - |\Lambda_0|}{|\Lambda|} = 0 \) for \( g = R_x, R_y, R_z \) and have the full \( R_x, R_y, R_z \) symmetry. In fact the green area is a phase with a non-trivial SPT order (the Haldane
3.6 Application to 2D model with symmetry breaking transition

Now we want to generalize the above simple picture to 2D. Consider the transverse Ising model on an infinite honeycomb lattice, with spins living on vertices. The Hamiltonian remains the same as equation (3.1). Since the ground state of a gapped system in 2D could be faithfully described by a tensor-network state, [76, 45, 65, 59, 52] for a translation invariant system, we have (see Figure 3-17):

$$|\Psi\rangle = \sum_{\{m_i\}} \sum_{\{a,\beta,\gamma\}} t\text{Tr}[\otimes_i M_{M_i}^{m_i}] |\{m_i\}\rangle$$

(3.27)

where tensors $M$’s are again labeled by the physical degrees of freedoms $m_i$, and $t\text{Tr}$ (tensor trace) contracts over all internal degrees of freedom on connected links labeled by $\alpha, \beta$ and $\gamma$. Again, we want to do variational calculations with a simple picture involving the total environment tensor $E^{tot}$, which now consists of four environment matrices, see Figure 3-18.

The key question now is how do we obtain a good environment matrix, as we did in 1D (recall Figure 3-3)? Here we introduce a simple yet powerful iteration process: assume we have a three-fold rotational symmetry for tensor $M$, then the iteration needs two input matrices, and gives out only one output, see Figure 3-
As before, after enough numbers of iterations, we would reach a final stable “environment matrix”.

It might be surprising, at first sight, why such a naive iteration process would give a reliable environment matrix. The key however is to realize that this iteration actually gives an environment matrix for the infinite Bethe lattice, which is a very good first approximation for our honeycomb lattice (see Figure 3-20). As shown in the graph, the iteration process is actually equivalent to a self-consistent update for a large cluster of lattice points, and thus its legitimacy.

Just like in the 1D case, here we would also like to require our tensor-product state to have a $\mathbb{Z}_2$ symmetry corresponding to the spin up-down symmetry. Similar to the matrix-product state, on-site symmetry of the ground state also requires tensor $M$ in the tensor-network state to transform in a special way:

$$
\sum_{m'} g_{mm'} M_{\alpha,\beta,\gamma}^{m'} = e^{i\theta_g} \sum_{\alpha',\beta',\gamma'} M_{\alpha',\beta',\gamma'}^m U^\alpha_{g} U^\beta_{g'} U^\gamma_{g'}.
$$

This is just a tensor generalization of condition eqn (3.3). As before, $m$ is the physical spin label, $g_{mm'}$ represents the on-site symmetry and acts in the spin space, $U_g$ forms a projective representation of the symmetry group $g$, and is a unitary matrix acting
Figure 3-20: The iteration process on Bethe lattice. Note that this is a top-down view, and we have only shown one layer of tensor-network state. Starting from an environment surrounding $C$ tensor, we could iterate to get environment for $B$, and then to $A$. The circled region is the region of interest.

on internal degrees of freedom labeled by $\alpha, \beta$ and $\gamma$.

For internal dimension of $D = 2$, we can choose $U_g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The most general symmetric tensor $M^{m_i}_{\alpha,\beta,\gamma}$ satisfying eqn. (3.28) has 4 variational parameters and looks like the following:

$$M^{\uparrow,\alpha=1}_{\beta,\gamma} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad M^{\uparrow,\alpha=2}_{\beta,\gamma} = \begin{pmatrix} b & c \\ c & d \end{pmatrix},$$

$$M^{\downarrow,\alpha=1}_{\beta,\gamma} = \begin{pmatrix} a & -b \\ -b & c \end{pmatrix} \quad M^{\downarrow,\alpha=2}_{\beta,\gamma} = \begin{pmatrix} -b & c \\ c & -d \end{pmatrix}. \quad (3.29)$$

Here we again assume $M$’s to be symmetric, because of rotational symmetry.

With the above $M$ tensors, numerical simulation could again be run on the 2D Ising model. Following what we did in 1D, we vary $h/J$ in equation (3.1) and minimize the energy for each value of $h/J$. By plotting the two energy terms, a phase diagram could also be obtained (see Fig. 3-21). For internal dimension $D = 2$, the phase transition point occured at $h/J = 2.09$, which was within 2% error from Quantum Monte Carlo prediction of $h/J = 2.13$.\[9\] The typical runtime on a laptop was just a few seconds. If we increase the internal dimension to $D = 4$ and use a completely
Figure 3-21: Energy as a function of $h/J$ for the 2D transverse Ising model. As in the 1D case, h- and J- terms are individually plotted in the graph as well. We can see from the graph that our simulation shows a weak first-order phase transition, with phase transition point at $h/J = 2.09$.

Following what we did in 1D, here we would also like to comment on the symmetry structure of the environment matrix $E$. Recall that $E$ is obtained through iterations (or self consistent condition) in Fig. 3-19, which picks out the $E$ with the largest absolute value of scaling factor $\lambda$. One important difference/simplification in 2D is that unlike in 1D, in general, we do not have any degeneracies for $E$ through the iteration equation (Fig. 3-19), since the equation is non-linear. Thus in general $E$ obtained is unique, and we do not need eqn. (3.6) to fix the basis.

With the above discussion, we can go into the symmetry structure for our environment matrix $E$ (See Fig. 3-22). For internal dimension being 2, using the iteration method mentioned above and after energy minimization, we have in the symmetric phase $E = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$, which gives an invariant $E$ under eqn. (3.7). As a result, the
Figure 3-22: Magnetization and order parameter as a function of $h/J$ for the $2D$ transverse Ising model. When $D = 2$, the order parameter plotted (represented by red crosses) is $p/(s + t)$, which goes to zero in the symmetric phase.

The total environment tensor is just the tensor product of them:

$$E^{\text{tot}} = E \otimes E \otimes E \otimes E.$$  

(3.30)

As for the symmetry breaking phase, depending on the initial values of $E$, we get either $E^{g_1} = \begin{pmatrix} \tilde{s} & p \\ p & \tilde{t} \end{pmatrix}$ or $E^{g_2} = \begin{pmatrix} \tilde{s} & -p \\ -p & \tilde{t} \end{pmatrix}$, which transforms into each other under eqn. (3.7). If we construct the total environment tensor

$$E^{\text{tot}} = \sum_g E^g \otimes E^g \otimes E^g \otimes E^g,$$  

(3.31)

then the $\mathbb{Z}_2$ symmetry is restored, but now again the total environment tensor does not have a pure tensor product form, as was the case in $1D$. 

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3.7 Application to 2D model with topological order

We now move on to the non-trivial example of Toric-Code model in a B-field, with spins living on links of an infinite 2D honeycomb lattice. The Hamiltonian is as follows:

\[ H = -A \sum_v \prod_{i \in v} \sigma_i^z - B \sum_p \prod_{j \in p} \sigma_j^x - h \sum_k \sigma_k^z \]  

(3.32)

where \( \sigma \)'s are the usual Pauli matrices. We will first consider the phase diagram by fixing \( A \to \infty \) and varying \( h/B \) between \([0, \infty]\). When \( h/B = 0 \), we have the original Toric-Code model, whose ground state is an equal-weight superposition of all closed loops of down-spins (in the background of up-spins). When \( h/B \to \infty \), we have the spin-polarized state where all spins are pointing up.

Later in this section, we will also consider the case when \( h/B \to -\infty \) so the ground state is the fully packed loop state, which is an equal weight superposition of all loop configurations that are fully packed (every vertex has a loop passing through). The question of whether the fully packed loop state has topological order or not will then be explored.

As in the previous example, we now try to use a tensor-network state to represent the ground state of the above Hamiltonian. Here since all spins live on links of the lattice, we will need two tensors \( T \) and \( M \) to represent our variational ground state.
Figure 3-24: Variational energy for Toric-Code model in a B-field.

(see Figure 3-23):

$$|\Psi\rangle = \sum_{\{m_l\}} \sum_{\{\alpha,\beta,\gamma,\delta,\eta\}} t\text{Tr}[\otimes_v T_{\alpha,\beta,\gamma} \otimes_l M^{m_l}_{\delta,\eta}] |\{m_l\}\rangle$$  \hspace{1cm} (3.33)

where $v$ labels different vertices, $l$ labels different links, $\alpha, \beta, \gamma, \delta, \eta$ label internal degrees of freedom, $m_l$ label physical degrees of freedom of link $l$, and $t\text{Tr}$ contracts over all connected internal indices. Note that due to the $B$ term in the Hamiltonian 3.32, we will need to include an entire plaquette in our variational calculation, as shown in Figure 3-24.

Now we start by introducing our tensor ansatz in the simple case of internal dimension 2. In order to enforce the condition that $A \rightarrow \infty$ and the rotational symmetry of the system, we need the following tensors $T$ and $M$:

$$T_{\alpha,\beta,\gamma} = \begin{cases} 1, & \text{if } \alpha = \beta = \gamma = 0; \\ x, & \text{else if } \alpha + \beta + \gamma = 0 \pmod{2}; \\ 0, & \text{otherwise}; \end{cases}$$

$$M_{\delta,\eta}^\uparrow = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{\delta,\eta}^\downarrow = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \hspace{1cm} (3.34)$$

where spin-up and spin-down’s are labeled by arrows. Note that when $x = 1$, it represents the regular Toric-Code ground state[23], whereas when $x = 0$, it represents the all-spins-up state.
Figure 3-25: Energy as a function of $h/B$ for the 2D Toric Code model in a magnetic field. Here we only plotted the region when $h/B > 0$. We can see from the graph that our simulation shows a weak first-order phase transition, with phase transition point at $h/J = 0.3$.

Before going into our variational calculations, we first note that our model in equation (3.32) could be mapped into a transverse Ising model by introducing a new plaquette spin operator $\mu_p$, where spins live on the plaquettes and $p$ is the plaquette label.[60] By doing the following mapping: $\prod_{j \in p} \sigma_j^x \rightarrow \mu_p^x$, $\sigma_i^z \rightarrow \mu_p^z \mu_{p'}^z$, and consider only the $A \rightarrow \infty$ sector, our Hamiltonian reduces to:

$$H = -B \sum_p \mu_p^x - h \sum_{\langle p, p' \rangle} \mu_p^z \mu_{p'}^z, \quad (3.35)$$

which is the familiar transverse Ising model. Note that this Ising model is now on a 2D triangular lattice.

With the above tensor network ansatz, we could run our variational scheme on the Toric-Code model. Just like in the Ising model cases, we vary $h/B$ in eqn (3.32) (recall that we hold $A \rightarrow \infty$) and minimize the energy for each value of $h/B$. The environment tensor was calculated in the same way as before (See Fig. 3-19). The only difference here is that since the Hamiltonian (3.32) have a six-body interaction...
Figure 3-26: Internal $\mathbb{Z}_2$ symmetry of the tensors $T$ and $M$ for our Toric-Code model. Here $T$ has a $\sigma_z \otimes \sigma_z \otimes \sigma_z$ symmetry, and $M$ has a $\sigma_z \otimes \sigma_z$ symmetry, both of which square to identity. Note that this symmetry transformation could independently act on top and bottom layers of the tensor-network, thus the environment matrix transforms under a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group.

As a result, we have to include more sites into our mean-field calculation (See Figure 3-24). By plotting the two energy terms as a function of $h/B$, we get a phase diagram, which is plotted in Fig. 3-25. For internal dimension $D = 2$, we got a phase transition point at $B/h = 3.33$, with an error of 30% to the Quantum Monte Carlo result of $B/h = 4.768$.[9] This is not surprising as we only have one variational parameter. With internal dimension of 3 and only two variational parameters, our result quickly improved to a phase transition point at $B/h = 4.407$, with an error of less than 8% to the Quantum Monte Carlo result. Note that the Quantum Monte Carlo value was obtained on the mapped equivalent model (see equation (3.35)) on a 2D triangular lattice.

Now in order to understand the above result better and to further explore the case when $h/B < 0$, we need to understand the symmetry structure of both our tensor-product state and the environment tensor obtained. Note here that although the ground state doesn’t have a physical $\mathbb{Z}_2$ symmetry, the tensor ansatz $T$ and $M$ (3.34) still need to have an internal $\mathbb{Z}_2$ symmetry, namely the “necessary symmetry condition”[13] (See Figure 3-26):

$$T_{\alpha,\beta,\gamma} = \sum_{\alpha',\beta',\gamma'} T^{\alpha,\beta',\gamma'} \sigma_z^{\alpha' \alpha} \sigma_z^{\beta' \beta} \sigma_z^{\gamma' \gamma}$$

$$M^{m}_{h,n} = \sum_{\delta',\eta'} M^{\delta',\eta'} \sigma_z^{\delta' \delta} \sigma_z^{\eta' \eta}.$$  (3.36)

Here, the internal symmetry is represented by $\sigma_z \otimes \sigma_z \otimes \sigma_z$ for tensor $T$, and $\sigma_z \otimes \sigma_z$ for tensor $M$.
for tensor $M$, where $\sigma_z$ is the Pauli matrix. Since both symmetry actions square to identity, we refer to the above internal symmetry as a $\mathbb{Z}_2$ symmetry.

The physical reason for tensor ansatz to have the above “necessary symmetry condition” is that we want to make sure local variations of the tensors correspond to local perturbations of the Hamiltonian. Tensors that violate the above condition correspond to non-local perturbation in their Hamiltonian and thus cannot be used to describe physical phase transitions [13].

It’s easy to check that tensors in equation (3.34) have the above symmetry. We would then like to ask, with $T$ and $M$ tensors satisfying eqn (3.36), what is the symmetry structure of the environment matrix? Note that unlike the Ising model, here the internal symmetry of the two layers of our tensor-network can act independently, as shown in Figure 3-26. Thus the environment matrix no longer transforms under eqn (3.7), but transforms under a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group:

$$E \rightarrow U_g^\dagger \cdot E, \text{ or } E \cdot U_g, \text{ or } U_g^\dagger \cdot E \cdot U_g.$$  \hspace{1cm} (3.37)

As in the Ising model, we expect that in different phases, the environment matrices $E$’s are either invariant under the above transformation, or undergoes a permutation.

Our numerical result indeed shows the above feature. When internal dimension $D = 2$, we use the tensor ansatz in eqn (3.34) and iteration process (see Fig. 3-19) to get the environment matrix $E$. In the confined phase (including spin-polarized state), we obtain $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which is invariant under eqn. (3.37). As a result, the total environment tensor is just the direct product of them:

$$E^{tot} = E \otimes E \otimes E \otimes E \otimes E \otimes E.$$  

In the deconfined phase (including string-net state), however, we have either $E^{g_1} = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$ or $E^{g_2} = \begin{pmatrix} s & 0 \\ 0 & -t \end{pmatrix}$, which transforms into each other under eqn. (3.37). We
could again construct a total environment tensor

\[ E^{\text{tot}} = \sum_{g} E^{g} \otimes E^{g} \otimes E^{g} \otimes E^{g} \otimes E^{g} \]

that respects the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry, but it does not have a pure tensor product form, as was the case for Ising model.

In doing the above, we have really constructed a numerical way to detect topological orders. In the particular case above, \( \mathbb{Z}_2 \) topological order is signatured by a “symmetry breaking” in the environment matrix, which breaks the original \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry of \( E \) (see eqn (3.37)) down to \( \mathbb{Z}_2 \) (see eqn (3.7)).

With this realization, a natural question to ask is: if we now consider negative magnetic field with \( h/B \to -\infty \), will the fully packed loop state has \( \mathbb{Z}_2 \) topological order? To answer this question, let us first write down the ground state wave function of the fully packed loop state in tensor form:

\[
T_{\alpha,\beta,\gamma} = \begin{cases} 
0, & \text{if } \alpha = \beta = \gamma = 0; \\
1, & \text{else if } \alpha + \beta + \gamma = 0 \text{ (mod 2)}; \\
0, & \text{otherwise}; 
\end{cases}
\]

\[
M^{1}_{\delta,\eta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M^{2}_{\delta,\eta} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]  

(3.38)

Note the difference between this and eqn (3.34): here we require loops to cover each vertex, so \( T_{0,0,0} = 0 \).

Now to see whether this state has \( \mathbb{Z}_2 \) topological order or not, all we need to do is to calculate its environment matrix through iteration (see Fig. 3-19). Depending on the initial condition, we obtain either \( E^{g_1} = \begin{pmatrix} 0.38 & 0 \\ 0 & 0.62 \end{pmatrix} \) or \( E^{g_2} = \begin{pmatrix} 0.38 & 0 \\ 0 & -0.62 \end{pmatrix} \), which again transforms into each other under eqn (3.37). This means that we are still in the deconfined phase, and packed loop state has \( \mathbb{Z}_2 \) topological order.

One may worry that the simple test above would fail to differentiate the “string
Figure 3-27: The above graph shows a “string crystal” state, where blue lines represent spin-downs forming vertical strings, and red lines represent spin-ups forming the background.

crystal” state (see Fig. 3-27) where string configuration is stationary, from the fully packed loop state where the string configurations are fluctuating. This worry turns out to be unnecessary through careful study below.

Consider a “string crystal” state, with vertical strings formed by down-spins (shown in Fig. 3-27). We would like to study the symmetry structure of the environment matrix for this state. Note that unlike the previous tensor-network ansatz in eqn (3.34) and (3.38), here the tensors no longer have three-fold rotational symmetry.

Our method could be easily generalized to non-rotationally symmetric tensors by introducing a three-step iteration process, shown in Fig. 3-28. (Recall that this is different from the symmetric iteration process in Fig. 3-19.) In this three-step iteration process, we introduce three different environment matrices, which then iterate in a cyclic fashion. Now, any physical quantities could again be calculated by sandwiching the operator in between two layers of tensor network states, surrounded by three different types of environment matrices shown in Fig. 3-29, and all of our previous analysis follows. (Again compare this with Fig. 3-18, where there was only one type of environment matrix.)

With the above three-step iteration process, the environment matrix of our “string
Figure 3-28: The self-consistent iteration process for environment matrix when the tensors do not have rotational symmetry. We introduce three different environment matrices, $E^A$, $E^B$ and $E^C$. One cycle of iteration consists of three steps: 1. Input $E^A$ and $E^B$ to update $E^C$; 2. Input $E^B$ and $E^C$ to update $E^A$; 3. Input $E^C$ and $E^A$ to update $E^B$. After iterating for enough number of steps, all three self-consistent equation will be simultaneously satisfied.

Figure 3-29: Expectation value of an operator with non-rotational-symmetric tensor networks. Note here that there are three different types of environment matrices.
crystal" state could then be obtained as follows:

\[ E^A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E^B = E^C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]  

(3.39)

Note that the above \( E^B \) and \( E^C \) do not really break the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry shown in eqn (3.37). This is because the iteration process for \( E^B \) and \( E^C \) are both linear, so an overall minus sign does not affect the iteration result. Thus \( E^B \) and \( E^C \) could only be determined up to a sign, which is a fictitious gauge degree of freedom and have no physical meaning. Thus \( E^B \rightarrow -E^B \) does not correspond to breaking the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry, and “string crystal” state indeed does not possess \( \mathbb{Z}_2 \) topological order.

3.8 Summary

In this chapter, we proposed a new signature for phase transitions between tensor-network states using the environment matrices. Different phases are distinctively labeled by different symmetry structures in the environment matrices. Thus through carefully studying different symmetry structures of the environment matrix, we could identify different phases and obtain the detailed phase boundaries for both symmetry-breaking transitions and topological phase transitions. This greatly helps us in identifying topological orders or SPT orders from a generic tensor-network state.

The environment matrix is obtained through a very simple iteration process using the tensor-network state, in both 1D and 2D. This iteration process provides a self-consistent environment matrix that summarizes the contributions from far away sites, and is like a “mean-field” theory for tensor-networks. In the same line of thinking, the environment matrix serves like an “order parameter”. What’s special about this “mean-field” theory is that it’s suitable for studying long-range entangled states, and is thus suitable for tackling topological phase transitions.

In 1D, we demonstrated that this new signature could be easily combined with existing numerical methods like DMRG or iTEBD to identify SPT phases. We first obtain the ground state in a matrix-product form by applying these 1D numerical
methods. Then we calculate the environment matrix, either through direct iteration process or through a twisted iteration process (where the symmetry transformation $g$ is sandwiched in between the double tensor in the iteration process). By simply comparing the scaling factors in the two iteration process, we could identify which SPT phase we are in, thus providing an easy way to identify SPT orders directly from a matrix-product state.

In 2D, the iteration process gives a very efficient way of calculating variational energies, which in turn leads to a simple numerical methods in obtaining gound state wave function by minimizing the energy. If we require the ground state tensors to have the proper on-site symmetry, iteration process could give us environment matrices that have drastically different symmetry structures, labeling different (topological) phases. Note that the on-site symmetry doesn't have to be a physical symmetry—internal gauge symmetry is also valid.

Here we want to comment on a different but related approach in [58, 27]. There they introduced the concept of “transfer operator” (which is just our “double tensor” looped around the cylinder with internal indices contracted), through which they could also obtain certain signatures for topological orders. Our approach introduces a way to directly tackle the “double tensor”, thus giving rise to a much simpler and more direct signature for topological orders. In addition, by introducing the concept of environment matrix, we also establish a numerically efficient way of calculating variational energy.

This numerical method is very general and could be easily applied to many interesting systems in higher dimension including 3D systems. This will open new doors in numerical study of higher dimensional systems.
Chapter 4

A Non-BCS Mechanism for Superconductivity

4.1 Outline of the chapter

In the previous two chapters, we emphasized several theoretical aspects of many-body entanglement, and demonstrated how to theoretically characterize different patterns of many-body entanglement, both for ideal fixed-point wave functions and generic non-fixed-point wave functions. Here in this chapter, we change our focus to some practical aspects of many-body entanglement, and show how the concept of entanglement could lead to a non-BCS mechanism for superconductivity.

This chapter is structured as follows: In section 4.2, we study possible stable/unstable spin-liquid states, which would work as a parent state for superconductivity upon doping. In section 4.3, we present a simple picture with single hole/double hole in the RVB background, and try to understand the hole statistics by hopping holes around each other. In section 4.4, we try to make the argument more quantitative by adopting a slave-particle calculation with bosonic holes, and show that a superconducting ground state is indeed possible.
4.2 $J_1 - J_2$ model on Honeycomb Lattice

In this section, we study the $J_1 - J_2$ model on the honeycomb lattice, using the tensor network method introduced in chapter 3. We would like to point out that the goal of this section is not to look for actual stable spin liquid states, but to find potential spin-liquid states as competing orders, which could support superconductivity when doping. With this understanding, we now write down the Hamiltonian for spin-$\frac{1}{2}$ systems:

$$
H = J_1 \sum_{<i,j>} S_i \cdot S_j + J_2 \sum_{\ll i,j \gg} S_i \cdot S_j,
$$

(4.1)

where $<i,j>$ goes through all the nearest neighboring pairs, and $\ll i,j \gg$ goes through all the second-nearest neighboring pairs.

It is well known that when $J_2$ is small, the ground state of the above Hamiltonian has anti-ferromagnetic order [17, 49, 20, 79]. The hope then is that as we increase $J_2$, the second-neighbor interaction would frustrate the anti-ferromagnetic order, and eventually favor a resonating valence bond (RVB) order, which is highly entangled [15, 22].

Now before we go into the detailed phase diagram, let us first introduce a tensor-network representation of the nearest neighbor (NN) RVB state. Here for readability reason, we will present the tensor network state using the projected entangled pair state (PEPS) formalism.

In PEPS, the NN RVB state is constructed as follows [67]: On each link, internal bond dimension is $D = 3$, and the internal degrees of freedom form a block diagonal virtual spin singlet state:

$$
|S\rangle = |01\rangle - |10\rangle + |22\rangle,
$$

(4.2)

where 0 and 1 form a virtual singlet, and 2 is just a place-holder. On each physical site, a projector projects one of the internal degrees of freedom into physical space.
while keeping the other two at 2:

$$\mathcal{P} = \sum_{i \neq j \neq k} (\langle \uparrow |0|_i + \langle \downarrow |1|_i \rangle \otimes \langle 22|_{jk},$$  \hfill (4.3)

where $i, j, k$ are internal indices. Now RVB state could be obtained by combining the two:

$$|\Psi\rangle_{RVB} = \prod_v \mathcal{P} \prod_b |S\rangle, \hfill (4.4)$$

where $v$ runs through all physical sites and $b$ runs through all bonds. Note that the above state ensures that each vertex forms one and only one singlet pair with one of its neighboring site.

Now we can introduce our variational ansatz for studying the $J_1 - J_2$ model, where we replace the projector before by

$$\mathcal{P}_A = \sum_{i \neq j \neq k} (\langle \uparrow |0|_i + \alpha \langle \downarrow |1|_i \rangle \otimes \langle 22|_{jk}$$
$$\mathcal{P}_B = \sum_{i \neq j \neq k} (\alpha |\uparrow \rangle \langle 0|_i + |\downarrow \rangle \langle 1|_i \rangle \otimes \langle 22|_{jk}. \hfill (4.5)$$

With the above projector, our ansatz will again be represented by

$$|\Psi\rangle_{ansatz} = \prod_v \mathcal{P}' \prod_b |S\rangle, \hfill (4.6)$$

where $\mathcal{P}' = \mathcal{P}_A$ or $\mathcal{P}_B$ depending on which sub-lattice we are on. Note from the above that when $\alpha = 0$, we only have spin-ups on $A$ sub-lattice and spin-downs on $B$ sub-lattice, corresponding to the anti-ferromagnetic order. On the other hand when $\alpha = 1$, we get back the NN RVB state as shown before. Thus the above ansatz gives a "linear superposition" of both the anti-ferromagnetic state and the NN RVB state.

Using the ansatz above, we can now calculate the variational energy of eqn (4.1) by using the tensor network method introduced in chapter 3. For each value of $J_2 \in [0, \infty]$, we minimize the energy by varying $\alpha$ in eqn (4.5). By plotting the two
energy terms, a phase diagram could be obtained (see Fig. 4-1). We see that when $J_2 = 0.5$, the system experience a phase transition from anti-ferromagnetic phase to NN RVB phase, agreeing with our earlier prediction.

Here we want to make a few important comments. First, although we only have 1 variational parameter, the minimal energy per site we obtained at $J_2 = 0$ was $E/J_1 = -2.13$, a mere 1% error with quantum Monte Carlo calculation of $-2.15$ [15]. (Note here we scaled our spin to $S = 1$ for simplicity of calculation.) This shows that our variational ansatz indeed captures the important physics very well. Second, even deep inside the anti-ferromagnetic phase, our variational parameter $\alpha \approx 0.8$ (See Fig. 4-2), indicating strong entanglement in the system even in the ordered phase. Third, let us emphasize again that our goal here is not to search for a stable spin liquid, but merely to find a competing state that is highly entangled. Thus here we avoid the discussion of stability of $U(1)$ spin liquid entirely, and merely present our NN RVB state as a reasonable competing order.
4.3 Single/Double doped hole in RVB

With NN RVB state being a very promising parent state for superconductivity, we now try to study how holes behave when the system is doped with holes (positive charge carrier). In this section, as a first approach to understanding the statistics of holes, we consider the single/double hole picture, in which we only have one/two doped hole(s), and we assume the spinon background to be unchanged. Our main goal is to examine if there is any possibility for holes to have statistical confusions, leaving room for the existence of bosonic holes.

Consider a $t_2 - J_1 - J_2$ model, where holes only hop between second-nearest-neighbor sites:

$$H = J_1 \sum_{<i,j>} S_i \cdot S_j + J_2 \sum_{<<i,j>>} S_i \cdot S_j - t_2 \sum_{<<i,j>>} (c_{i,\sigma}^\dagger c_{j,\sigma} + h.c.).$$  \hspace{1cm} (4.7)

Here $\sigma$ is the spin index which is contracted, and $h.c.$ represents Hermitian conjugate terms. Note that in the Hamiltonian above, we only consider second neighbor hopping.
Figure 4-3: The first hopping path. Particles $A$ and $B$ start from positions 1 and 2. In the graph above, hole $A$ hops first from position 1 to 3, then from 3 to 4, after which hole $B$ hops from position 2 to 3. Here, original RVB bonds are marked in black, whereas newly created RVB bonds are marked by red ellipses.

where holes hop between sites within the same sub-lattice.

To study the statistics of holes at short distances, we introduce two hopping paths shown in Fig. 4-3 and Fig. 4-4. The hopping operators involved in both paths are the same, but the order of applications are different. Through these different hopping paths, we obtain two final configurations in which the final positions of the two holes are exchanged; thus by calculating the sign of the overlap of the two final configurations, we could get the statistics of the holes. As can be seen in Fig. 4-3 and Fig. 4-4, the overlap reduces to the overlap of just four sites and gives $-1/2$, which seems to suggest that the holes behave like fermions.

Luckily, the above statistical analysis is not the only possibility. Note that although in the last section we chose the RVB state to contain only nearest neighbor singlet pairs, that was for the convenience of our tensor network calculation; the true ground state would always contain singlet pairs between further neighbors due to $J_2$ term in eqn (4.7). This provides a second “channel” when comparing the two final configurations in Fig. 4-3 and Fig. 4-4: if the two configurations are connected through $-J_2 S_i \cdot S_j$, then both of them would have equal weights in the ground state super-
Figure 4-4: The second hopping path. Particles $A$ and $B$ also start from positions 1 and 2. In this path above, hole $B$ hops first from position 2 to 3, then from 3 to 4, after which hole $A$ hops from position 1 to 3. Note that the end position of $A$ and $B$ are exchanged compared to the previous hopping path.

position, which implies that holes behave like bosons. In fact, in the large $N$ limit of $SO(N)$ spins, different RVB configurations would be approximately orthogonal, thus killing the previous direct-overlap channel; in this limit, holes will be purely bosonic. Thus when $J_2$ is large enough and/or in the $SO(N)$ limit, the hole statistics could be bosonic, giving hope for potential superconductivity through direct condensation of holes.

Before we move on to the next section, here we would also like to address the problem of flux experienced by holes when hopping around vertices or plaquettes. The two processes are shown in Fig. 4-5. As can be seen in the graph, both when a hole hops around a vertex or a plaquette, the two configurations before and after hopping are only different on four sites and can be connected to each other through $-J_2 S_i \cdot S_j$. Following the same argument from last paragraph, when $J_2$ is large enough and/or in the $SO(N)$ limit, the two configurations will have the same sign in the ground state superposition, meaning that the holes will not encounter any fluxes with a positive hopping amplitude $t_2$ in eqn (4.7).
Figure 4-5: Single hole hopping around a plaquette. As shown in the graph, the initial and final position for the hole are the same, but the RVB configurations will be modified.

Figure 4-6: Single hole hopping around a vertex. As shown in the graph, the initial and final position for the hole are the same, but the RVB configurations will be modified, similar to the plaquette case.
4.4 A Slave-Boson Approach

The previous single/double hole picture gives a very simple and intuitive understanding on how doped holes could be bosonic. Careful readers may however be unsatisfied with the simple assumptions and the lack of rigor. Therefore here in this section, we present a careful calculation using the slave-boson method [4, 16, 44] to make the argument more quantitative.

In the slave-boson approach, after excluding double occupancy, we write the electron operator as:

\[ c_{i\sigma}^{\dagger} = f_{i\sigma}^{\dagger} b_i, \]  

with the constraint:

\[ f_{i\uparrow}^{\dagger} f_{i\uparrow} + f_{i\downarrow}^{\dagger} f_{i\downarrow} + b_i^\dagger b_i = 1. \]  

Here \( f_{i\sigma} \) are fermion operators and carry the spin index, whereas \( b_i \) are boson operators. Note that the above equations are not operator identities in general, but only holds in the subspace with no double occupancy. We also write spin operators \( S_i \) as:

\[ S_i = \frac{1}{2} f_{i\sigma}^{\dagger} \sigma_{\alpha\beta} f_{i\beta}. \]  

With the above transformations, we can rewrite our \( t_2 - J_1 - J_2 \) model in eqn (4.7) as:

\[ H = -\sum_{i,j} \frac{J_{ij}}{2} (f_{i\alpha}^{\dagger} f_{j\alpha} f_{j\beta}^{\dagger} f_{i\beta}) - t_2 \sum_{<i,j>} (b_i b_j f_{i\sigma}^{\dagger} f_{j\sigma} + h.c.) \]  

where \( J_1 \) and \( J_2 \) terms are combined into \( J_{ij} \).

The above Hamiltonian is obviously impossible to solve. To make real progress in
understanding the physical picture, we apply mean-field approximation as follows:

\[
H = \tilde{J} \sum_{<i,j>} \left[ (-\chi_{ij}^* f^\dagger_{i\sigma} f_{j\sigma} + \Delta_{ij} f^\dagger_{i\alpha} f^\dagger_{j\beta} \epsilon_{\alpha\beta} + h.c.) 
- |\chi_{ij}|^2 - |\Delta_{ij}|^2 \right] - t_2 \sum_{<i,j>} (\chi_{ij} b_i^\dagger b_j^\dagger + h.c.) 
+ \sum_i a_0(i) (f^\dagger_{i\sigma} f_{i\sigma} + b_i^\dagger b_i - 1),
\]

(4.12)

where

\[
\chi_{ij} = \sum_{\sigma} \langle f_{i\sigma}^\dagger f_{j\sigma} \rangle \\
\Delta_{ij} = \sum_{\alpha,\beta} \langle f_{i\alpha}^\dagger f_{j\beta}^\dagger \epsilon_{\alpha\beta} \rangle
\]

(4.13)

are to be solved self-consistently. Here we assume that repeated indices are summed over. Note that \(\tilde{J}\) also needs to be determined self-consistently. Here \(\chi_{ij}\) represents fermion hopping and \(\Delta_{ij}\) represents fermion pairing, \(\epsilon_{\alpha\beta}\) is the antisymmetric Levi-Civita symbol. In the last term, \(a_0(i)\) is a Lagrangian multiplier and is determined by requiring the mean-field ground state to satisfy

\[
\langle f^\dagger_{i\sigma} f_{i\sigma} + b_i^\dagger b_i \rangle = 1,
\]

(4.14)

which is the mean-field version of the constraint equation (4.9).

Instead of solving the self-consistency equation (4.13) directly, an equivalent way of obtaining the self-consistent mean-field solution is the variational approach[]. In this approach, we view the ground state of the mean-field Hamiltonian as a trial wave function, and view \(\chi_{ij}\) and/or \(\Delta_{ij}\) as variational parameters. By minimizing the energy of the actual Hamiltonian (4.11), we can also obtain the self-consistent mean-field solution.

With the above variational approach in mind, we now explore three trial ground states. The first one will be the so-called “pairing state”, which is the ground state of
the following mean-field Hamiltonian:

\[ H_{\text{mean}} = J \sum_{<i,j>} [\Delta f_{i\alpha}^\dagger f_{j\beta}^\dagger \epsilon_{\alpha\beta} + h.c.] + a_0 \sum_i f_{i\sigma}^\dagger f_{i\sigma}, \tag{4.15} \]

and is our mean-field state for superconductivity. The above Hamiltonian has a particularly simple form after applying a particle-hole transformation on B-sub-lattice:

\[
\begin{bmatrix}
  f_A^\uparrow \\
  f_A^\downarrow 
\end{bmatrix} \rightarrow \begin{bmatrix}
  \psi_A^\uparrow \\
  \psi_A^\downarrow 
\end{bmatrix}, \quad \begin{bmatrix}
  f_B^\uparrow \\
  f_B^\downarrow 
\end{bmatrix} \rightarrow \begin{bmatrix}
  -\psi_B^\downarrow \\
  \psi_B^\uparrow 
\end{bmatrix}. \tag{4.16}
\]

Then the mean-field Hamiltonian becomes:

\[
H_{\text{mean}} = J \sum_{<i,j>} [\Delta \psi_i^A \psi_j^B + h.c.] + a_0 \sum_i (\psi_i^A \psi_i^A - \psi_i^B \psi_i^B), \tag{4.17}
\]

which is just the simple tight-binding model in 2D and could be easily written in a block-diagonal form in \(k\)-space:

\[
H_{\text{mean}} = \sum_k \begin{pmatrix}
  \psi_k^A \\
  \psi_k^B 
\end{pmatrix} \begin{pmatrix}
  a_0 & J \Delta F(k) \\
  J \Delta^* F(k)^* & -a_0 
\end{pmatrix} \begin{pmatrix}
  \psi_k^A \\
  \psi_k^B 
\end{pmatrix}, \tag{4.18}
\]

where \( F(k) = 1 + e^{ik_1a} + e^{ik_2a} \), with \( k_1 \) and \( k_2 \) being the two wave numbers in the basis of the reciprocal lattice vectors and \( a \) being the lattice constant.

Note that the \( a_0 \) term acts like a “staggered chemical potential” term, which opens up a gap in the mean-field spectrum. We will take the ground state of the above Hamiltonian at half-filling as our first trial wave function, and we will call it the “pairing state”.

A second trial ground state is called the “hopping state”, which is the ground state
of a different mean-field Hamiltonian involving the hoping terms:

$$ H_{\text{mean}} = -J \sum_{<i,j>} [\chi f_{i\sigma}^\dagger f_{j\sigma} + h.c.] + a_0 \sum_i f_{i\sigma}^\dagger f_{i\sigma}, $$

(4.19)

which is a very natural metallic competing order. Writing this Hamiltonian in $k$-space, we can directly obtain the block-diagonalized Hamiltonian:

$$ H_{\text{mean}} = -\sum_k (f_{k}^{A\dagger}, f_{k}^{B\dagger}) \begin{pmatrix} -a_0 & J \Delta F(k) \\ J \Delta^* F(k)^* & -a_0 \end{pmatrix} \begin{pmatrix} f_{k}^{A} \\ f_{k}^{B} \end{pmatrix}, $$

(4.20)

where again $F(k) = 1 + e^{ik_1a} + e^{ik_2a}$. Note here the $a_0$ term acts like a uniform chemical potential term, and the spinon spectrum still has Dirac points. Upon doping, spinons will form fermi surfaces. We will take the ground state of the above Hamiltonian with all negative energy levels filled as our second trial wave function, and we will call it the “hopping state”.

A third trial ground state is obtained by giving the previous hopping Hamiltonian two different chemical potentials depending on $A$-$B$ sub-lattices and their spins:

$$ H_{\text{mean}} = -J \sum_{<i,j>} [\chi f_{i\sigma}^\dagger f_{j\sigma} + h.c.] $$

$$ + (a_0 + \delta) \sum_i (f_{i\uparrow}^{A\dagger} f_{i\uparrow}^{A} + f_{i\downarrow}^{B\dagger} f_{i\downarrow}^{B}) $$

$$ + (a_0 - \delta) \sum_i (f_{i\downarrow}^{A\dagger} f_{i\downarrow}^{A} + f_{i\uparrow}^{B\dagger} f_{i\uparrow}^{B}). $$

(4.21)

This assignment of “staggered chemical potential” effectively gives the ground state a staggered spin polarization, thus resembles the anti-ferromagnetic competing order in this formalism. Again by writing the Hamiltonian in $k$-space, we can obtain the
Figure 4-7: Energy plot of the $t_2 - J_1 - J_2$ model, using three different mean-field ansatz. It is clear that in the majority range of doping, the pairing state has the lowest energy, whereas the the other two states have very close energies.

$$H_{\text{mean}} = -\sum_{k} (f_{k\uparrow}^A, f_{k\uparrow}^B) \begin{pmatrix} -a_0 - \delta & J \Delta F(k) \\ J \Delta^* F(k)^* - a_0 - \delta \end{pmatrix} \begin{pmatrix} f_{k\uparrow}^A \\ f_{k\uparrow}^B \end{pmatrix} - \sum_{k} (f_{k\downarrow}^A, f_{k\downarrow}^B) \begin{pmatrix} -a_0 + \delta & J \Delta F(k) \\ J \Delta^* F(k)^* - a_0 + \delta \end{pmatrix} \begin{pmatrix} f_{k\downarrow}^A \\ f_{k\downarrow}^B \end{pmatrix}$$

(4.22)

where $F(k) = 1 + e^{i k_1 a} + e^{i k_2 a}$. We will take the ground state of the above Hamiltonian with all negative energies filled as our third trial wave function and call it the “staggered potential state”.

After obtaining the above “pairing state”, “hopping state” and “staggered potential state”, we now treat them as variational trial wave functions and apply energy minimization on the exact Hamiltonian (4.11). Then we compare the three energies obtained by plotting the three energies at different doping levels, which is shown in Fig. 4-7. Here, doping level is defined as $x \equiv < b_i^+ b_i > = 1 - < f_{1\sigma}^j f_{i\sigma} >$, according to eqn (4.14). As can be seen in Fig. 4-7, at typical values of $J_2 = t_2 = \frac{1}{2} J_1$, the “pairing
Figure 4-8: A zoomed-in version of Fig. 4-7. The three solid lines still represent the mean-field energies of the corresponding ansatz as Fig. 4-7. We see that when doping is very small (< 6%), the "staggered potential state" starts to deviate from the "hopping state" and eventually dominates at very low doping. Also plotted in the graph is the staggered chemical potential $\delta$ (see eqn (4.21)) obtained from energy minimization. We can see that $\delta$ becomes non-zero only at very small doping, confirming the anti-ferromagnetic order.
state” has lowest energy in most of the doping range, indicating our mean-field ground state is a superconducting state. When doping is close to zero (< 2%), see Fig. 4-8, the “staggered potential state” is favored, representing the anti-ferromagnetic order as ground state.

Now to go beyond mean-field theory and to understand some special features of our superconducting state, we include extra gauge degree of freedom, which will promote our mean-field theory to a \( U(1) \) gauge theory [5, 43]. Including the gauge fields, we have:

\[
H_{\text{gauge}} = \sum_{<i,j>} J[\Delta_{ij} \epsilon^{iaij} \psi_i^A \psi_j^B + h.c.] \\
+ \sum_i a_0(i)(\psi_i^A \psi_i^A - \psi_i^B \psi_i^B) \\
- t_2 \sum_{\langle i,j \rangle} (\eta_{ij} \epsilon^{iaij} b_i^A b_j^A - \eta_{ij} e^{-iaij} b_i^B b_j^B + h.c.). \tag{4.23}
\]

where \( \eta_{ij} = \sum_\sigma \langle \psi_{i\sigma}^\dagger \psi_{j\sigma} \rangle \). This Hamiltonian (and the corresponding Lagrangian) is invariant under the \( U(1) \) gauge transformation of matter fields:

\[
\begin{pmatrix}
\psi_i^A \\
\psi_i^B
\end{pmatrix} \rightarrow 
\begin{pmatrix}
\epsilon^{i\phi_i} \psi_i^A \\
\epsilon^{i\phi_i} \psi_i^B
\end{pmatrix}, \quad 
\begin{pmatrix}
b_i^A \\
b_i^B
\end{pmatrix} \rightarrow 
\begin{pmatrix}
e^{i\phi_i} b_i^A \\
e^{-i\phi_i} b_i^B
\end{pmatrix} \tag{4.24}
\]

together with that of gauge fields:

\[
a_{ij} \rightarrow a_{ij} + \phi_i - \phi_j, \\
a_0(i) \rightarrow a_0(i) + \frac{\partial \phi_i(\tau)}{\partial \tau}. \tag{4.25}
\]

Now if we further include external electro-magnetic field \( A \) and consider the situation when holons on both \( A \) and \( B \) sub-lattice independently Bose-Einstein condense, we obtain:

\[
L \sim (A + a)^2 + (A - a)^2 + F_{\mu\nu} F^{\mu\nu} \tag{4.26}
\]
where due to the famous Anderson-Higgs mechanism[, $U(1)$-gauge field is gapped out, and we readily have the Meissner effect. This confirms that our mean-field pairing state is indeed a stable superconducting state. Here, it is worthwhile to emphasize that holon condensations on $A, B$ sub-lattice are completely independent (since holons only hop within the same sub-lattice), and both condensations are needed for superconductivity. This is a special feature of this superconducting state.

4.5 Summary

In this chapter, we proposed a non-BCS mechanism for superconductivity through entanglement. In a highly entangled system, charge carriers may have statistical confusions and behave like bosons, thus directly Bose-Einstein condense, forming superconducting state.

To make the above idea more concrete, we first considered a $J_1-J_2$ model on the honeycomb lattice. Through a tensor-network calculation, we demonstrated that the ground state is highly entangled, and NN RVB state is a good competing order which could potentially be a parent state for superconductivity upon doping.

We then presented a simple single/double hole picture, where one/two hole(s) was introduced into the NN RVB background. By applying hopping operators in particular orders, we demonstrated that holes may indeed behave like bosons when $J_2$ is large enough and/or in the large $N$ limit of $SO(N)$ spin systems.

To make the argument more quantitative, we detailed a slave-boson calculation where we assumed spinons were fermions and holons were bosons. Through a mean-field calculation, we demonstrated that a superconducting state could indeed be the ground state for a $t_2-J_1-J_2$ model, confirming our proposal.

This proposal opens up new doors in approaching the problem of superconductivity. By focusing on entanglement-driven rather than BCS pair-attraction-driven superconductivity, we hope to bring a new perspective to the study of this old phenomena, which could potentially lead to construction and discovery for completely new superconducting states.
Chapter 5

Concluding Remarks

In this thesis, we proposed a new theory of topological orders in $2 + 1D$. This is achieved through introducing the non-Abelian geometric phases represented by $T$- and $S$-matrices, which generate a representation of the mapping class group of the torus $SL(2, \mathbb{Z})$ for exact topological orders. We conjectured that all the $(S, T)$-matrices which satisfy a series of conditions could form a complete and one-to-one description of exact topological orders in $2 + 1D$, thus giving rise to a new characterization of topological orders. This new characterization acts like a “non-local order parameter”, from which many topological properties could be obtained, such as number of quasiparticle types (from the dimension of the $T$ or $S$), the quasiparticle statistics (from the diagonal elements of $T$), the quantum dimensions of quasiparticles, etc..

After the proposal of this new characterization, we then detailed how the labeling $(S, T)$-matrices could be obtained from the ideal fixed-point wave functions, which were previously defined in the string-net approach by Levin and Wen and/or the local unitary transformation approach by Chen, Gu and Wen. This is achieved through applying two discrete deformations, “$T$-transformation” and “$S$-transformation”, on these fixed-point ground states on a torus. We presented the resulting $(S, T)$-matrices label for a large list of topological orders in much details.

We then addressed the question of how to label different topological orders for non-ideal wave functions, which are not fixed-point wave functions. This is achieved through the concept of the so called “environment matrix”, which is capable of char-
acterizing any topological orders described by gauge theories as well as 1D symmetry protected topological (SPT) orders. With the help of this “environment matrix”, a very efficient tensor-network method was introduced, which could even describe phase transitions and calculate phase boundaries between the aforementioned topological phases. These phase transitions are marked by a sudden change in the symmetry structure of the “environment matrix”, making the environment matrix another “order-parameter-like” description that captures patterns of entanglement in topological phases for a generic wave function.

Following that, as an application of the concept of many-body entanglement and the tensor-network method introduced in this thesis, we proposed a new mechanism for superconductivity. In this proposal, charge carriers are “dressed” by the strong entanglement in the system, causing them to be statistically confused. When they behave like bosons, they directly Bose-Einstein condense, without the need of BCS-like pair-attraction. To make this idea concrete, we detailed a slave-boson calculation, where we compared the mean-field energies of several competing orders, and demonstrated that such a superconducting state is indeed possible. This could potentially lead to construction of completely new superconductors.

Finally, in this thesis, we hope that we have demonstrated the power and richness of quantum many-body entanglement. Much work is of course still needed to fully understand the beauty of many-body entanglement, which will certainly enhance our understanding of many more new and old phenomena across the entire field of physics.
Appendix A

Local unitary transformation and fixed-point tensors

In this appendix, we will briefly summarize the local-unitary transformation approach in [11], and review how to obtain the associated fixed-point tensors for different topologically ordered states. We want to remind the readers that this approach is equivalent to the “string-net” approach, and the resulting fixed-point tensors describe the “string-net” states.

The key in the local unitary transformation approach is a new definition of equivalence class on many-body ground states. Two gapped many-body ground states were defined as belonging to the same equivalence class (i.e. the same “phase”) if and only if they are connected by a local unitary (LU) transformation defined as follows:

\[ |\Phi(1) > \sim |\Phi(0) > \text{ iff } |\Phi(1) > = T |e^{-i \int_0^1 dg \tilde{H}(g)}| \Phi(0) > \]  

(A.1)

where \( T \) is the path-ordering operator and \( \tilde{H}(g) = \sum_i O_i(g) \) is a sum of local Hermitian operators. It can be proved that the above definition is equivalent to the standard definition through “adiabatic evolution” (i.e. two states belong to the same phase if and only if they can be connected through a gapped adiabatic evolution).[11]

The new definition of equivalence class is advantageous because it provides a very operational way to determine if two states belong to the same phase, thus provides a
very natural way for renormalization. Under this definition, “long-range entangled” states are those that are not in the same phase as direct-product states. Different “long-range entanglements” are then called “topological orders”.

Following the new definition of equivalence class, a wave function renormalization scheme can be introduced. The wave function renormalization contains two basic moves called “F-move” and “P-move”, both of which are the generalized local unitary (gLU) transformations. Under this renormalization scheme, gapped ground states will only flow within the same equivalence class, i.e. renormalization does not change its topological order. Thus a fixed point in this renormalization scheme can be used to represent a whole equivalence class.

Now, we will briefly review the renormalization scheme mentioned above; all details could be found in [11]. Following the convention in [11], we will represent a quantum state by a graph (see Fig. 2-1). Degrees of freedom live on both edges and vertices of the graph. Each edge has $N + 1$ states, labeled by $i = 0, ..., N$, and the edges have orientations, with $i^*$ representing the inverse of $i$, and we have trivially that $(i^*)^* = i$. Each vertex has $N_{ijk}$ states, labeled by $\alpha = 1, ..., N_{ijk}$. The first renormalization move corresponds to locally deforming the graph (here only relevant part of the graph is drawn), and is referred to as the “F-move”. Note that this is a unitary transformation, as the support dimension of both graphs fixing $i, j, k, l$ are the same, with the saturation assumption. The second renormalization move corresponds to locally reducing the graph (here only relevant part of the graph is drawn), and is referred to as the “P-move”. Note that this move only removes local entanglement, and does not change the topological order of the state. We could represent the two transformations using tensors:

$$
\Phi_{\text{fix}} \left( \begin{pmatrix} i & \tilde{\alpha} & i^* \end{pmatrix} \right) \simeq \sum_{n=0}^{N} \sum_{\chi=1}^{N_{kn}} \sum_{\delta=1}^{N_{il}} F_{\delta, \chi, n}^{i, j, m, \alpha, \beta} \Phi_{\text{fix}} \left( \begin{pmatrix} \tilde{\delta} \end{pmatrix} \right)
$$

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we would like to remind the reader that using the two renormalization moves above, we could reduce any graph to the following graph:

\[ \Phi_{\text{fix}} \left( \begin{array}{c}
  \circ \\
  j \to i \\
  k \\
  i \\
  \end{array} \right) \approx P_{\alpha}^{kj,\alpha\beta} \Phi_{\text{fix}} \left( \begin{array}{c}
  i \\
  \end{array} \right), \quad (A.2) \]

thus once we know the above, the full fixed-point wave function could be reconstructed.

The above-mentioned fixed point tensors can be obtained through solving a set of self-consistency equations. Since the fixed-point state will not change under renormalization, we can apply renormalization moves in arbitrary orders. This arbitrariness gives us a set of self-consistency conditions that fixed point tensors must satisfy. The self-consistency conditions are summarized below:[11]

\[ N_{ikj} = N_{kji} = N_{i^*j^*k^*} \geq 0, \quad \sum_{j,k} N_{i^*k} N_{jk^*j^*} \geq 1, \]

\[ \sum_{m=0}^{N} N_{jm^*} N_{kml^*} = \sum_{n=0}^{N} N_{kn^*} N_{l^*ni}, \quad \text{(Saturation assumption)} \]

\[ (F_{kln,\chi}^{ijm,\alpha\beta})^* = F_{l^*m^*,\beta\alpha}^{kn,\chi \delta} \]

\[ \sum_{n,\chi,\delta} F_{kln,\chi}^{ijm,\alpha\beta'} (F_{kln,\chi}^{ijm,\alpha\beta})^* = \delta_{m\alpha, m'\alpha'}, \quad \text{(F-move being unitary)} \quad (A.4) \]
\[
\sum_{t} \sum_{\eta=1}^{N_{k\ell t}} \sum_{\varphi=1}^{N_{m\ell t}} \sum_{\kappa=1}^{N_{n\ell t}} F_{k\ell t, \eta \varphi}^{\alpha \beta} F_{m\ell t, \varphi \kappa}^{\alpha \gamma} F_{n\ell t, \kappa \ell}^{\gamma \delta} = e^{i \theta \rho} \sum_{\epsilon=1}^{N_{q\ell m}} F_{q\ell m, \beta \varphi}^{\alpha \gamma} F_{l\ell q, \varphi \delta}^{\alpha \gamma} \quad (\text{Pentagon identity})
\]

\[
e^{i \theta \rho_1} P_{i}^{kj, \alpha \beta} = \sum_{m, \lambda, \gamma, \nu, \mu} F_{m j, \beta \alpha}^{k, \gamma \lambda} F_{i \alpha m, \gamma \lambda}^{j, \nu \gamma} P_{l m, \mu \nu}^{l, \nu \gamma} P_{k \mu, \nu \gamma},
\]

\[
e^{i \theta \rho_2} P_{i}^{kj, \alpha \beta} = \sum_{\nu=1}^{N_{k j k}} F_{k l, \alpha \beta}^{j, \mu \nu} P_{l \mu, \nu \gamma}^{l, \nu \gamma},
\]

for all \( k, i, l \) satisfying \( N_{kl} > 0 \),

\[
\Phi_{\text{fix}} \left( \begin{array}{c}
i \\
j
\end{array} \right) \equiv A^i = A^i \neq 0, \quad \sum_{i} A^i (A^i)^* = 1,
\]

\[
P_{i}^{m j, \gamma \lambda} A^i = e^{i \theta \lambda_1} P_{j}^{i \alpha \gamma, \lambda \gamma} A^i,
\]

\[
\Phi_{i k j, \alpha \beta} = e^{i \theta \rho} \sum_{m, \lambda, \gamma} F_{j i k, \alpha \beta}^{m \gamma \lambda} P_{m i, \gamma \lambda}^{j \gamma \lambda} A^i,
\]

\[
\Phi_{i k j, \alpha \beta} = e^{i \theta \lambda_2} \Phi_{k, j i, \alpha \beta},
\]

\[
\Phi_{i k j, \alpha \beta} = 0 \text{ if } N_{ikj} = 0,
\]

\[
\det[\Phi_{i k j, \alpha \beta}] \neq 0. \quad (A.5)
\]
Conditions (A.5) form a set of non-linear equations with variables $N_{ijk}$, $F^{ijm,\alpha\beta}_{kl\eta,\gamma\lambda}$, $P^{k,ij,\alpha\beta}_i$, $A^i$ and $(\theta_F, \theta_P, \theta_P', \theta_A, \theta_A')$, which are tensor labels for fixed points. Following the previous discussion, the solutions $(N_{ijk}, F^{ijm,\alpha\beta}_{kl\eta,\gamma\lambda}, P^{k,ij,\alpha\beta}_i, A^i)$ will then give us a labeling of different topological orders.
Ideal Hamiltonian for fixed-point states on a torus

The Hamiltonian construction is similar to the one in appendix of [25]. The basic idea is to construct a Hamiltonian which is a sum of commuting projectors and is thus exactly solvable. On specific lattices for example the honeycomb lattice, it can be shown that all fixed-point wave functions \((N_{ijk}, F_{kl,n}^{ijm,\alpha\beta}, P_{i}^{kj,\alpha\beta}, A^{i})\) that we obtained from solving (A.5) are exact gapped ground states of such a local Hamiltonian.[25]

The Hamiltonian is of the following form:

\[
\hat{H} = \sum_{v} (1 - \hat{Q}_{v}) + \sum_{p} (1 - \hat{B}_{p})
\]  

(B.1)

where \(\sum_{v}\) sums over all vertices and \(\sum_{p}\) sums over all plaquettes. The Hamiltonian \(\hat{H}\) acts on the Hilbert space \(V_{G}\) formed by all the graph states. Operator \(\hat{Q}_{v}\) in \(\hat{H}\) acts on the states of the 3 links that connect to the vertex \(v\):

\[
\hat{Q}_{v} \left| i^{\alpha_{k}}_{i_{j}} \right> = \left| i^{\alpha_{k}}_{i_{j}} \right> \quad \text{if} \quad N_{ijk} > 0,
\]

\[
\hat{Q}_{v} \left| i^{\alpha_{k}}_{i_{j}} \right> = 0 \quad \text{otherwise.}
\]  

(B.2)

Clearly, \(\hat{Q}_{v}\) is a projector \(\hat{Q}_{v}^{2} = \hat{Q}_{v}\). Operator \(\hat{B}_{p}\) in \(\hat{H}\) acts on the states of all the
links and vertices belonging to the same plaquette \( p \).

For our purpose in this paper, we only need to consider the Hamiltonian acting on the subspace of all non-contractable graphs on a torus (See Fig. 2-2). The \( \hat{B}_p \) operator on such a torus can be constructed from a combination of the \( F \)-moves and \( P \)-moves as follows:

\[
\Phi \left( \begin{array}{c}
\includegraphics{example1}
\end{array} \right) = \sum_{t,s} (P_{i,s}^{t,s}) \Phi \left( \begin{array}{c}
\includegraphics{example2}
\end{array} \right)
\]

\[
= \sum_{t,s} \sum_{i'} (P_{i,s}^{t,s}) F^{i'i'i'}_{s} \Phi \left( \begin{array}{c}
\includegraphics{example3}
\end{array} \right)
\]

\[
= \sum_{t,s} \sum_{i'} \sum_{k'} (P_{i,s}^{t,s}) F^{i'i'i'}_{s} F_{k'k'k'}^{i'i'i'} \Phi \left( \begin{array}{c}
\includegraphics{example4}
\end{array} \right)
\]

\[
= \sum_{t,s} \sum_{i'} \sum_{k'} \sum_{j'^*} \sum_{j'^*} (P_{i,s}^{t,s}) F^{i'i'i'}_{s} F_{k'k'k'}^{i'i'i'} F_{j'j'j'^*}^{i'i'i'} \Phi \left( \begin{array}{c}
\includegraphics{example5}
\end{array} \right)
\]

\[
= \sum_{t,s} \sum_{i'} \sum_{k'} \sum_{j'^*} \sum_{j'^*} (P_{i,s}^{t,s}) F^{i'i'i'}_{s} F_{k'k'k'}^{i'i'i'} F_{j'j'j'^*}^{i'i'i'} F_{j'j'j'^*}^{i'i'i'} \Phi \left( \begin{array}{c}
\includegraphics{example6}
\end{array} \right)
\]
\[= \sum_{l,s} \sum_{i} \sum_{i'} \sum_{i''} \sum_{j} \sum_{j'} \sum_{j''} \sum_{k} \sum_{k'} \sum_{k''} (P^{*})^{l,s} F^{s'*it*}_{k'_j l} F^{j'*s^*i'}_{k''_k} F^{j'*s^*i'}_{k''_k} F^{j'*s^*i'}_{k''_k} \times \]

\[F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} \Phi \]

(B.3)

thus we have the matrix \( B \) given by

\[B = \sum_{l,s} \sum_{i} \sum_{i'} \sum_{i''} \sum_{j} \sum_{j'} \sum_{j''} \sum_{k} \sum_{k'} \sum_{k''} (P^{*})^{l,s} F^{s'*it*}_{k'_j l} F^{j'*s^*i'}_{k''_k} F^{j'*s^*i'}_{k''_k} F^{j'*s^*i'}_{k''_k} \times \]

\[F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} F^{k''_s^*i'}_{j'_{i''}a} \Phi \]

(B.4)

which is the matrix form of operator \( \hat{B}_p \).

Recall that all fixed-point states are exact ground states of the Hamiltonian \( \hat{H} \) defined in (B.2). Thus by finding the exact ground states of \( \hat{H} \) among all the non-contractable graph states, we can find the subspace of all the fixed-point states.

In sections 2.4.2 and 2.4.3, we are required to reduce the modular transformations to only within the ground-state subspace of the above Hamiltonian. This can now be easily achieved. We can first define \( T \) and \( S \) on all the non-contractable graphs on a torus, then find the ground-state subspace by diagonalizing the Hamiltonian on these non-contractable graphic states. We can then project \( T \) and \( S \) only onto the ground-state subspace of the Hamiltonian, thus obtain the required \( T \) and \( S \).
Bibliography


