# Combinatorics of Permutation Patterns, Interlacing Networks, and Schur Functions



by

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in partial fulfillment of the requirements for the degree of

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## Abstract

In the first part, we study pattern avoidance and permutation statistics. For a set of patterns $\Pi$ and a permutation statistic st, let $F_n^{\mathrm{st}}(\Pi; q)$ be the polynomial that counts st on the permutations avoiding all patterns in $\Pi$. Suppose $\Pi$ contains the pattern 312. For a class of permutation statistics (including inversion and descent statistics), we give a formula that expresses $F_n^{\mathrm{st}}(\Pi; q)$ in terms of these st-polynomials for some subblocks of the patterns in $\Pi$. Using this recursive formula, we construct examples of nontrivial st-Wilf equivalences. In particular, this disproves a conjecture by Dokos, Dwyer, Johnson, Sagan, and Selsor that all inv-Wilf equivalences are trivial.

The second part is motivated by the problem of giving a bijective proof of the fact that the birational RSK correspondence satisfies the octahedron recurrence. We define interlacing networks to be certain planar directed networks with a rigid structure of sources and sinks. We describe an involution that swaps paths in these networks and leads to a three-term relations among path weights, which immediately implies the octahedron recurrences. Furthermore, this involution gives some interesting identities of Schur functions generalizing identities by Fulmek-Kleber. Then we study the balanced swap graphs, which encode a class of Schur function identities obtained this way.

Thesis Supervisor: Alexander Postnikov
Title: Professor

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# Notation

In this thesis, we will use the following list of basic notations.

- $\mathbb{C}$ denotes the set of complex numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{N}$ denotes the set of nonnegative integers $\{0, 1, 2, ...\}$.
- $\mathbb{P}$ denotes the set of positive integers $\{1, 2, 3, ...\}$.
- $[n]$ denotes the set $\{1, 2, ..., n\}$.
- $[a, b]$ denoted the set $\{a, a + 1, ..., b\}$.
- $\binom{X}{k}$ denotes the set of $k$ element subsets of $X$.
- $2^X$ denotes the power set of $X$.
- $\overline{S}$ denotes the complement of the set $S$ with respect to its superset $X$.
- $[x^\alpha]f(x)$ denotes the coefficient of the $x^\alpha$ term of $f(x)$.

# Chapter 1

# Permutation patterns and statistics

This chapter is based on [35].

## 1.1 Introduction

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, 2, ..., n\}$ and let $\mathfrak{S} = \bigcup_{n \geq 0} \mathfrak{S}_n$, where $\mathfrak{S}_0$ contains only one element $\epsilon$ – the empty permutation. For $\pi \in \mathfrak{S}_p$ and $\sigma \in \mathfrak{S}$ we say that the permutation $\sigma$ *contains* $\pi$ if there is a subsequence $i_1 < ... < i_p$ such that $(\sigma(i_1), ..., \sigma(i_p))$ is *order isomorphic* to $\pi$, i.e. $\sigma(i_k) < \sigma(i_l)$ if and only if $\pi(k) < \pi(l)$ for all $k, l \in [p]$. In particular, every permutation contains $\epsilon$, and every permutation except $\epsilon$ contains $1 \in \mathfrak{S}_1$. For consistency, we will use the letter $\sigma$ to represent a permutation and $\pi$ to represent a pattern. We say that $\sigma$ *avoids* $\pi$ (or $\sigma$ is $\pi$-*avoiding*) if $\sigma$ does not contain $\pi$. For example, the permutation 46127538 contains 3142 (see Figure 1-1) while the permutation 46123578 avoids 3142. We denote by $\mathfrak{S}_n(\pi)$, where $\pi \in \mathfrak{S}$, the set of permutations $\sigma \in \mathfrak{S}_n$ avoiding $\pi$. More generally we denote by $\mathfrak{S}_n(\Pi)$, where $\Pi \subseteq \mathfrak{S}$, the set of permutations avoiding each pattern $\pi \in \Pi$ simultaneously, i.e. $\mathfrak{S}_n(\Pi) = \bigcap_{\pi \in \Pi} \mathfrak{S}_n(\pi)$. Two sets of patterns $\Pi$ and $\Pi'$ are called *Wilf equivalent*, written $\Pi \equiv \Pi'$, if $|\mathfrak{S}_n(\Pi)| = |\mathfrak{S}_n(\Pi')|$ for all integers $n \geq 0$.

Now we define $q$-analogues of pattern avoidance using permutation statistics. A *permutation statistic* (or sometimes just *statistic*) is a function $\mathrm{st} : \mathfrak{S} \to \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers. Given a permutation statistic st, we define the *st-polynomial* of $\Pi$-avoiding permutations to be

$$F_n^{\mathrm{st}}(\Pi) = F_n^{\mathrm{st}}(\Pi; q) := \sum_{\sigma \in \mathfrak{S}_n(\Pi)} q^{\mathrm{st}(\sigma)}.$$

We may drop the $q$ if it is clear from the context. The set of patterns $\Pi$ and $\Pi'$ are said to be st-*Wilf equivalent*, written $\Pi \stackrel{\mathrm{st}}{\equiv} \Pi'$, if $F_n^{\mathrm{st}}(\Pi; q) = F_n^{\mathrm{st}}(\Pi'; q)$ for all $n \geq 0$.

The study of pattern avoidance dates back to the beginning of the twentieth century. In 1915, MacMahon [25] proved that the number of 123-avoiding permutations in $\mathfrak{S}_n$ is the $n^{\text{th}}$ Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. This marks the first study of pattern avoidance. Decades later, Knuth [22] proved that the number of 132-avoiding permu-

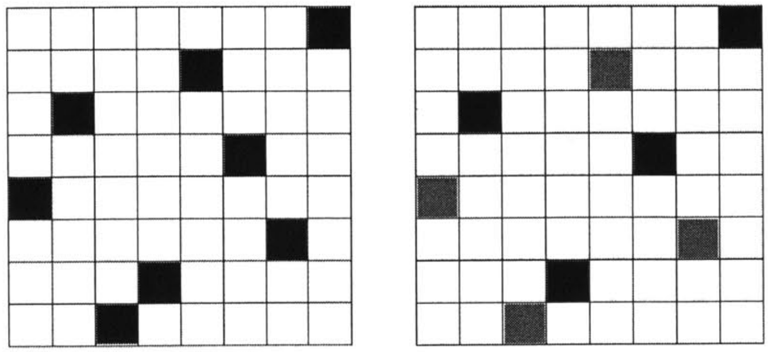

Figure 1-1: The permutation matrix of 46127538 (left) with an occurrence of 3142 colored (right)

tations in $\mathfrak{S}_n$ is also $C_n$. The systematical study of pattern avoidance eventually was begun in 1985 by Simion and Schmidt [30].

The study of $q$-analogues of pattern avoidance using permutation statistics and the st-Wilf equivalences began in 2002, initiated by Robertson, Saracino, and Zeilberger [28], with emphasis on the number of fixed points. Elizalde subsequently refined results of Robertson et al. by considering the excedance statistic [8] and later extended the study to cases of multiple patterns [9]. A bijective proof was later given by Elizalde and Pak [11]. Dokos et al. [4] studied pattern avoidance on the inversion and major statistics, as remarked by Savage and Sagan in their study of Mahonian pairs [29].

In [4], Dokos et al. conjectured that there are only essentially trivial inv-Wilf equivalences, obtained by rotations and reflections of permutation matrices. Let us describe these more precisely. The notations used below are mostly taken from [4].

Given a permutation $\sigma \in \mathfrak{S}_n$, we represent it geometrically using the squares $(1, \sigma(1)), (2, \sigma(2)), \ldots, (n, \sigma(n))$ of the $n$-by-$n$ grid, which is coordinatized according to the $xy$-plane. This will be referred as the *permutation matrix* of $\sigma$ (despite not using matrix coordinates). The diagram to the left in Figure 1 is the permutation matrix of 46127538. In the diagram to the right, the red squares correspond to the subsequence 4173, which is an occurrence of the pattern 3142.

By representing each $\sigma \in \mathfrak{S}$ as a permutation matrix, we have an action of the dihedral group of square $D_4$ on $\mathfrak{S}$ by the corresponding action on the permutation matrices. We denote the elements of $D_4$ by

$$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, r_{-1}, r_0, r_1, r_\infty\},$$

where $R_\theta$ is the counterclockwise rotation by $\theta$ degrees and $r_m$ is the reflection in a line of slope $m$. We will sometimes write $\Pi^t$ for $r_{-1}(\Pi)$. Note that $R_0, R_{180}, r_{-1}$, and $r_1$ each preserves the inversion statistic while the others reverse it, i.e.

$$\operatorname{inv}(f(\sigma)) = \begin{cases} \operatorname{inv}(\sigma) & \text{if } f \in \{R_0, R_{180}, r_{-1}, r_1\}, \\ \binom{n}{2} - \operatorname{inv}(\sigma) & \text{if } f \in \{R_{90}, R_{270}, r_0, r_\infty\}. \end{cases}$$

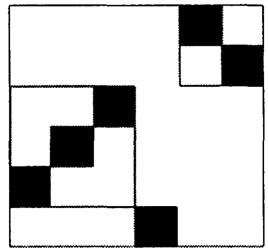

Figure 1-2: The permutation 213[123,1,21]

It follows that $\Pi$ and $f(\Pi)$ are inv-Wilf equivalent for all $\Pi \in \mathfrak{S}$ and $f \in \{R_0, R_{180}, r_{-1}, r_1\}$. We call these equivalences *trivial.* With these notations, the conjecture by Dokos et al. can be stated as follows.

**Conjecture 1.1.1** ([4], Conjecture 2.4). *$\Pi$ and $\Pi'$ are inv-Wilf equivalent iff $\Pi = f(\Pi')$ for some $f \in \{R_0, R_{180}, r_{-1}, r_1\}$.*

Given permutations $\pi = a_1 a_2 ... a_k \in \mathfrak{S}_k$ and $\sigma_1, ..., \sigma_k \in \mathfrak{S}$, the *inflation* $\pi[\sigma_1, ..., \sigma_k]$ of $\pi$ by the $\sigma_i$ is the permutation whose permutation matrix is obtained by putting the permutation matrices of $\sigma_i$ in the relative order of $\pi$; for instance, 213[123,1,21]=234165, as illustrated in Figure 2.

For convenience, we write

$$\pi_* := 21[\pi, 1].$$

In other words, $\pi_*$ is the permutation whose permutation matrix is obtained by adding a box to the lower right corner of the permutation matrix of $\pi$.

The next proposition is one of the main results of this paper, which disproves the conjecture above. This is a special case of the corollary of Theorem 1.2.4 in the next section.

**Proposition 1.1.2.** *Let $\iota_r$ be the permutation $12...r \in \mathfrak{S}_r$. Let $\pi_1, ..., \pi_r, \pi'_1, ..., \pi'_r$ be permutations such that $\{312, \pi_i\} \overset{inv}{\equiv} \{312, \pi'_i\}$ for all $i$. Set $\pi = \iota_r[\pi_{1*}, ..., \pi_{r*}]$ and $\pi' = \iota_r[\pi'_{1*}, ..., \pi'_{r*}]$. Then $\{312, \pi\}$ and $\{312, \pi'\}$ are inv-Wilf equivalent, i.e. $F_n^{inv}(312, \pi) = F_n^{inv}(312, \pi')$ for all $n$.*

In particular, if we set each $\pi'_i$ to be either $\pi_i$ or $\pi_i^t$, then the conditions $\{312, \pi_i\} \overset{\text{inv}}{\equiv} \{312, \pi'_i\}$ are satisfied. By this construction $\{312, \pi'\}$ is generally not of the form $f(\{312, \pi\})$ for any $f \in \{R_0, R_{180}, r_{-1}, r_1\}$. The pair $\Pi = \{312, 32415\}$ and $\Pi' = \{312, 24315\}$ is a minimal example of nontrivial inv-Wilf equivalences constructed this way. To clarify this, we note that $32415 = 12[213_*, \epsilon_*]$ and $24315 = 12[132_*, \epsilon_*]$ and that $213^t = 132$.

## 1.2 Avoiding two patterns

In this section, we study the st-polynomials in the case when $\Pi$ consists of 312 and another permutation $\pi$. For this set of patterns $\Pi$, Mansour and Vainshtein [26] gave a recursive formula for $|\mathfrak{S}_n(\Pi)|$. Here, we give a recursive formula for the st-polynomials $F_n^{\mathrm{st}}(\Pi)$, which generalizes the result of Mansour and Vainshtein. Then we present its corollary, which gives a construction of nontrivial st-Wilf equivalences. We note that Proposition 1.2.1 and Lemma 1.2.2 appear in [26] as small observations.

Suppose $\sigma \in \mathfrak{S}_{n+1}(312)$ with $\sigma(k+1) = 1$. Then, for every pair of indices $(i, j)$ with $i < k+1 < j$, we must have $\sigma(i) < \sigma(j)$; otherwise $\sigma(i)\sigma(k+1)\sigma(j)$ is an occurrence of the pattern 312 in $\sigma$. So $\sigma$ can be written as $\sigma = 213[\sigma_1, 1, \sigma_2]$ with $\sigma_1 \in \mathfrak{S}_k$ and $\sigma_2 \in \mathfrak{S}_{n-k}$. For the rest of the paper, we will always consider $\sigma$ in this inflation form.

We also assume that the permutation statistic $\mathrm{st} : \mathfrak{S}_n \to \mathbb{N}$ satisfies

$$\mathrm{st}(\sigma) = f(k, n-k) + \mathrm{st}(\sigma_1) + \mathrm{st}(\sigma_2) \tag{$\dagger$}$$

for some function $f : \mathbb{N}^2 \to \mathbb{N}$ which does not depend on $\sigma$. Some examples of such statistics are the inversion number, the descent number, and the number of occurrences of the consecutive pattern 213:

$$\underline{213}(\sigma) = \#\{i \in [n-2] : \sigma(i+1) < \sigma(i) < \sigma(i+2)\}.$$

For these statistics, we have

$$\begin{aligned}
\mathrm{inv}(\sigma) &= k + \mathrm{inv}(\sigma_1) + \mathrm{inv}(\sigma_2),\\
\mathrm{des}(\sigma) &= 1 - \delta_{0,k} + \mathrm{des}(\sigma_1) + \mathrm{des}(\sigma_2),\\
\underline{213}(\sigma) &= (1 - \delta_{0,k})(1 - \delta_{k,n}) + \underline{213}(\sigma_1) + \underline{213}(\sigma_2),
\end{aligned}$$

where $\delta$ is the Kronecker delta function.

It will be more beneficial to consider the permutation patterns in their *block decomposition* form as in the following proposition.

**Proposition 1.2.1.** *Every 312-avoiding permutation $\pi \in \mathfrak{S}_n(312)$ can be written uniquely as*

$$\pi = \iota_r[\pi_{1*}, \ldots, \pi_{r*}]$$

*where $r \geq 0$ and $\pi_i \in \mathfrak{S}(312)$.*

*Proof.* The uniqueness part is trivial. The proof of existence of $\pi_1, \ldots, \pi_r$ is by induction on $n$. If $n = 0$, there is nothing to prove. Suppose the result holds for $n$. Suppose that $\pi(k+1) = 1$. Then $\pi = 213[\pi_1, 1, \pi'] = 12[\pi_{1*}, \pi']$ where $\pi_1 \in \mathfrak{S}_k(312)$ and $\pi' \in \mathfrak{S}_{n-k}(312)$. Applying the inductive hypothesis on $\pi'$, we are done. $\square$

Suppose that $\pi \in \mathfrak{S}_n(312)$ has the block decomposition $\pi = \iota_r[\pi_{1*}, \ldots, \pi_{r*}]$. For $1 \leq i \leq r$, we define $\underline{\pi}(i)$ and $\overline{\pi}(i)$ to be

$$\underline{\pi}(i) = \begin{cases} \pi_1 & \text{if } i = 1,\\ \iota_i[\pi_{1*}, \ldots, \pi_{i*}] & \text{otherwise,} \end{cases}$$

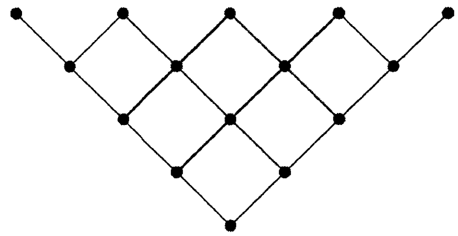

Figure 1-3: The poset $L_5$

and

$$\overline{\pi}(i) = \iota_{r-i+1}[\pi_{i*}, ..., \pi_{r*}].$$

Let $\Pi$ be a set of patterns containing 312. If $\pi \in \Pi \setminus \{312\}$ contains the pattern 312, then every permutation avoiding 312 will automatically avoid $\pi$, which means $F_n^{\mathrm{st}}(\Pi) = F_n^{\mathrm{st}}(\Pi \setminus \{\pi\})$. So we may assume that each pattern besides 312 in $\Pi$ avoids 312. The following lemma gives a recursive condition for a permutation $\sigma = 213[\sigma_1, 1, \sigma_2] \in \mathfrak{S}(312)$ to avoid $\pi$, in terms of $\sigma_1, \sigma_2$, and the blocks $\pi_{i*}$ of $\pi$.

**Lemma 1.2.2.** *Let $\sigma = 213[\sigma_1, 1, \sigma_2], \pi = \iota_r[\pi_{1*}, ..., \pi_{r*}] \in \mathfrak{S}(312)$. Then $\sigma$ avoids $\pi$ if and only if the condition*

$$(C_i) : \sigma_1 \textit{ avoids } \underline{\pi}(i) \textit{ and } \sigma_2 \textit{ avoids } \overline{\pi}(i).$$

*holds for some $i \in [r]$.*

*Proof.* First, suppose that $\sigma$ contains $\pi$. Let $j$ be the largest number for which $\sigma_1$ contains $\underline{\pi}(j)$. Then $\sigma_2$ must contain $\overline{\pi}(j+1)$. So $\sigma_1$ contains $\underline{\pi}(i)$ for all $i \le j$, and $\sigma_2$ contains $\overline{\pi}(i)$ for all $i > j$. Thus none of the $C_i$ holds.

On the other hand, suppose that there is a permutation $\sigma \in \mathfrak{S}(312)$ that avoids $\pi$ but does not satisfy any $C_i$. This means, for every $i$, either $\sigma_1$ contains $\underline{\pi}(i)$ or $\sigma_2$ contains $\overline{\pi}(i)$. Let $j$ be the smallest number such that $\sigma_1$ does not contain $\underline{\pi}(j)$. Note that $j$ exists and $j > 1$ since $j = 1$ implies $\sigma_2$ contains $\overline{\pi}(1) = \pi$, a contradiction. Since $\sigma_1$ does not contain $\underline{\pi}(j)$, $\sigma_2$ must contain $\overline{\pi}(j)$ (by $C_j$). But since $\sigma_1$ contains $\underline{\pi}(j-1)$ by minimality of $j$, we have found a copy of $\pi$ in $\sigma$ with $\underline{\pi}(j-1)$ from $\sigma_1$ and $\overline{\pi}(j)$ from $\sigma_2$, a contradiction. (For $j = 2$, the number 1 in $\sigma$ together with $\pi_1$ in $\sigma_1$ give $\pi_{1*}$.) □

Before presenting the main result, we state a technical lemma regarding the Möbius function of certain posets. See, for example, Chapter 3 of [32] for definitions and terminology about posets and the general treatment of the subject.

Let $\mathbf{r}$ be the chain of $r$ elements $0 < 1 < ... < r-1$. Let $L_r$ be the poset obtained by taking the elements of $\mathbf{r} \times \mathbf{r}$ of rank 0 to $r-1$, i.e. the elements of $L_r$ are the lattice points $(a, b)$ where $a, b \ge 0$ and $a + b < r$. For instance, $L_5$ is the poset shown in Figure 3. We denote its unique minimal element $(0, 0)$ by $\hat{0}$. Let $\hat{L}_r$ be the poset $L_r$ with the unique maximum element $\hat{1}$ adjoined.

For a poset $P$, we denote the Möbius function of $P$ by $\mu_P$. Note that for every element $a \in \hat{L}_r$ the up-set $U(a) := \{x \in \hat{L}_r : x \ge a\}$ of $a$ is isomorphic to $\hat{L}_{r-l(a)}$

where $l(a)$ is the rank of $a$ in $L_r$. Therefore, the problem of computing $\mu_{\hat{L}_r}(x, \hat{1})$ for every $r$ is equivalent to computing $\mu_{\hat{L}_r}(\hat{0}, \hat{1})$ for every $r$, which is given by the following lemma. The proof is omitted since it is by a straightforward calculation.

**Lemma 1.2.3.** *We have*

$$\mu_{\hat{L}_r}(\hat{0}, \hat{1}) = \begin{cases} (-1)^r, & \text{if } r = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

We now present the main theorem of this section.

**Theorem 1.2.4.** *Let $\Pi = \{312, \pi\}$, where $\pi = \iota_r[\pi_{1*}, ..., \pi_{r*}]$ Suppose that the statistic $st : \mathfrak{S} \to \mathbb{N}$ satisfies the condition (†). Then $F_n^{st}(\Pi; q)$ satisfies*

$$F_{n+1}^{st}(\Pi; q) = \sum_{k=0}^{n} q^{f(k,n-k)} \left[ \sum_{i=1}^{r} F_k^{st}(312, \underline{\pi}(i)) \cdot F_{n-k}^{st}(312, \overline{\pi}(i)) \right.$$
$$\left. - \sum_{i=1}^{r-1} F_k^{st}(312, \underline{\pi}(i)) \cdot F_{n-k}^{st}(312, \overline{\pi}(i+1)) \right], \qquad (*)$$

*for all $n \geq 0$, where $F_0^{st}(\Pi; q) = 0$ if $\pi = \epsilon$, and $1$ otherwise.*

*Proof.* For $k \in \{0, 1, ..., n\}$ and $\Sigma \subset \mathfrak{S}$, we write $\mathfrak{S}_{n+1}^k(\Sigma)$ to denote the set of permutations $\sigma \in \mathfrak{S}_{n+1}(\Sigma)$ such that $\sigma(k+1) = 1$. In particular,

$$\mathfrak{S}_{n+1}^k(312) = \{\sigma = 213[\sigma_1, 1, \sigma_2] : \sigma_1 \in \mathfrak{S}_k(312) \text{ and } \sigma_2 \in \mathfrak{S}_{n-k}(312)\}.$$

Fix $k$, and let $A_i (i \in [r])$ be the set of permutations in $\mathfrak{S}_{n+1}^k(312)$ satisfying the condition $C_i$. So $\mathfrak{S}_{n+1}^k(\Pi) = A_1 \cup A_2 \cup \cdots \cup A_r =: A$ by Lemma 1.2.2. Observe that if $i_1 < ... < i_k$ then

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} = A_{i_1} \cap A_{i_k} =: A_{i_1, i_k}$$

since satisfying the conditions $C_{i_1}, ..., C_{i_k}$ is equivalent to satisfying the conditions $C_{i_1}$ and $C_{i_k}$.

Let $P$ be the intersection poset of $A_1, ..., A_r$, where the order is given by $A \leq B$ if $A \subseteq B$. The elements of $P$ are $A$, $A_i$ $(1 \leq i \leq r)$, and $A_{i,j}$ $(1 \leq i < j \leq r)$. We see that $P$ is isomorphic to the set $\hat{L}_r$, so the Möbius function $\mu_P(T, A)$ for $T \in P$ is given by

$$\mu_P(T, A) = \begin{cases} 1 & \text{if } T = A \text{ or } A_{i,i+1} \text{ for some } i, \\ -1 & \text{if } T = A_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

For $T \in P$, we define $g : P \to \mathbb{C}(x : x \in A)$ by

$$g(T) = \sum_{x \in T} x.$$

The Möbius inversion formula ([32], Section 3.7) implies that

$$\begin{aligned} g(A) &= -\sum_{T<A} \mu_P(T,A) g(T) \\ &= \sum_{i=1}^{r} g(A_i) - \sum_{i=1}^{r-1} g(A_i \cap A_{i+1}). \end{aligned}$$

By mapping $\sigma \mapsto q^{\mathrm{st}(\sigma)}$ for all $\sigma \in A$, $g(A)$ is sent to $F^{\mathrm{st}}_{n+1,k}(\Pi; q) := \sum_{\sigma \in \mathfrak{S}^k_{n+1}(\Pi)} q^{\mathrm{st}(\sigma)}$. Hence,

$$\begin{aligned} F^{\mathrm{st}}_{n+1,k}(\Pi; q) &= \sum_{i=1}^{r} \sum_{\sigma \in A_i} q^{\mathrm{st}(\sigma)} - \sum_{i=1}^{r-1} \sum_{\sigma \in A_i \cap A_{i+1}} q^{\mathrm{st}(\sigma)} \\ &= q^{f(k,n-k)} \left[ \sum_{i=1}^{r} \sum_{\sigma \in A_i} q^{\mathrm{st}(\sigma_1)+\mathrm{st}(\sigma_2)} - \sum_{i=1}^{r-1} \sum_{\sigma \in A_i \cap A_{i+1}} q^{\mathrm{st}(\sigma_1)+\mathrm{st}(\sigma_2)} \right], \end{aligned}$$

where the second equality is obtained from the condition (†).
Note that $\sigma \in A_i$ iff $\sigma_1$ avoids $\underline{\pi}(i)$ and $\sigma_2$ avoids $\overline{\pi}(i)$, and $\sigma \in A_i \cap A_{i+1}$ iff $\sigma_1$ avoids $\underline{\pi}(i)$ and $\sigma_2$ avoids $\overline{\pi}(i+1)$. Thus

$$\sum_{\sigma \in A_i} q^{\mathrm{st}(\sigma_1)+\mathrm{st}(\sigma_2)} = F^{\mathrm{st}}_k(312, \underline{\pi}(i)) \cdot F^{\mathrm{st}}_{n-k}(312, \overline{\pi}(i))$$

and

$$\sum_{\sigma \in A_i \cap A_{i+1}} q^{\mathrm{st}(\sigma_1)+\mathrm{st}(\sigma_2)} = F^{\mathrm{st}}_k(312, \underline{\pi}(i)) \cdot F^{\mathrm{st}}_{n-k}(312, \overline{\pi}(i+1)).$$

Therefore

$$\begin{aligned} F^{\mathrm{st}}_{n+1,k}(\Pi; q) = q^{f(k,n-k)} \Bigg[ &\sum_{i=1}^{r} F^{\mathrm{st}}_k(312, \underline{\pi}(i)) \cdot F^{\mathrm{st}}_{n-k}(312, \overline{\pi}(i)) \\ &- \sum_{i=1}^{r-1} F^{\mathrm{st}}_k(312, \underline{\pi}(i)) \cdot F^{\mathrm{st}}_{n-k}(312, \overline{\pi}(i+1)) \Bigg]. \end{aligned}$$

We get the stated result by summing the preceding equation from $k = 0$ to $n$. □

**Example 1.2.5.** ($q$-analogues of odd Fibonacci numbers) It is well known that the permutations in $\mathfrak{S}_n$ avoiding 312 and 1432 are counted by the Fibonacci numbers $f_{2n-1}$, assuming $f_1 = f_2 = 1$ (see [36], for example). Let $a_n = f_{2n-1}$. It can be shown that the $a_n$ satisfy

$$a_{n+1} = a_n + \sum_{k=0}^{n-1} 2^{n-k-1} a_k.$$

Theorem 1.2.4 gives $q$-analogues of this relation. Here, we consider the inversion statistic.

Let $\pi = 1432 = 12[\epsilon_*, 21_*]$ and $\Pi = \{312, \pi\}$. Since $\underline{\pi}(1) = \epsilon$ and $F_n^{\text{inv}}(312, \epsilon) = 0$ for all $n$, Theorem 1.2.4 implies

$$
\begin{aligned}
F_{n+1}^{\text{inv}}(\Pi) &= \sum_{k=0}^{n} q^k F_k^{\text{inv}}(\Pi) F_{n-k}^{\text{inv}}(312, 321) \\
&= q^n F_n^{\text{inv}}(\Pi) + \sum_{k=0}^{n-1} q^k (1+q)^{n-k-1} F_k^{\text{inv}}(\Pi),
\end{aligned}
$$

where the last equality is by [4], Proposition 4.2.

**Corollary 1.2.6.** *Let st be a statistic satisfying* (†). *Let* $\pi_1, \ldots, \pi_r, \pi'_1, \ldots, \pi'_r$ *be permutations such that* $\{312, \pi_i\} \overset{st}{\equiv} \{312, \pi'_i\}$ *for all* $i$. *Set* $\pi = \iota_r[\pi_{1*}, \ldots, \pi_{r*}]$ *and* $\pi' = \iota_r[\pi'_{1*}, \ldots, \pi'_{r*}]$. *Then* $\{312, \pi\}$ *and* $\{312, \pi'\}$ *are also st-Wilf equivalent, i.e.* $F_n^{st}(312, \pi) = F_n^{st}(312, \pi')$ *for all* $n$.

*Proof.* The proof is by induction on $n$. If $n = 0$, then the statement trivially holds. Now suppose the statement holds up to $n$. Then for $0 \le k \le n$ and $1 \le i \le r$, we have $F_k^{\text{st}}(312, \underline{\pi}(i)) = F_k^{\text{st}}(312, \underline{\pi'}(i))$ and $F_{n-k}^{\text{st}}(312, \overline{\pi}(i)) = F_{n-k}^{\text{st}}(312, \overline{\pi'}(i))$. Hence $F_{n+1}^{\text{st}}(312, \pi) = F_{n+1}^{\text{st}}(312, \pi')$ by comparing the terms on the right-hand side of (*). □

In particular, if we set each $\pi'_i$ to be either $\pi_i$ or $\pi_i^t$, then the conditions $\{312, \pi_i\} \overset{\text{inv}}{\equiv} \{312, \pi'_i\}$ are satisfied. By this construction $\Pi'$ is generally not of the form $f(\Pi)$ for any $f \in \{R_0, R_{180}, r_{-1}, r_1\}$. For example, the pair $\Pi = \{312, 32415\}$ and $\Pi' = \{312, 24315\}$ is an example of smallest size of nontrivial inv-Wilf equivalences constructed this way.

Of course, this construction works for every statistic st satisfying (†) and that $\text{st}(\sigma) = \text{st}(\sigma^t)$ for all $\sigma \in \mathfrak{S}(312)$. Besides the inversion statistic, the descent statistic for example also possesses this property. To justify this fact, we write $\sigma = 213[\sigma_1, 1, \sigma_2] \in \mathfrak{S}(312)$ where $\sigma_1, \sigma_2 \in \mathfrak{S}$. Observe that $\sigma^t = 132[\sigma_2^t, \sigma_1^t, 1]$ and

$$
\text{des}(\sigma^t) = \text{des}(\sigma_2^t) + \text{des}(\sigma_1^t) + (1 - \delta_{0,k})
$$

where $k = |\sigma_1^t| = |\sigma_1|$. The proof then proceeds by induction on $n = |\sigma|$. It is, however, not true in general that the matrix transposition preserves the descent number. For instance, if $\sigma = 2413$, then $\text{des}(\sigma) = 1$ while $\text{des}(\sigma^t) = 2$.

## 1.3 Avoiding multiple patterns

In this section, we generalize the results from Section 2 to the case when $\Pi$ consists of 312 and other patterns. We again begin with a lemma regarding the Möbius function.

**Lemma 1.3.1.** *Let* $L$ *be the poset* $L_{r_1} \times \cdots \times L_{r_m}$ *and* $\hat{L}$ *the poset* $L \cup \{\hat{1}\}$. *Let* $\mu = \mu_{\hat{L}}$ *be the Möbius function on* $\hat{L}$. *Then* $\mu(\hat{0}, \hat{1}) = 0$ *unless each* $r_i \in \{1, 2\}$, *in which case* $\mu(\hat{0}, \hat{1}) = (-1)^{|S|+1}$, *where* $S = \{i : r_i = 2\}$.

*Proof.* Let $a = (a_1, ..., a_m) \in L$. Then $\mu(\hat{0}, a) = \prod_{i=1}^{m} \mu_i(\hat{0}, a_i)$, where $\mu_i$ is the Möbius function on $L_{r_i}$. So

$$\mu(\hat{0}, \hat{1}) = -\sum_{a \in L} \mu(\hat{0}, a) = -\prod_{i=1}^{m} \left( \sum_{a_i \in L_{r_i}} \mu_i(\hat{0}, a_i) \right).$$

Note that if $r \geq 3$, the Möbius function $\mu_{L_r}(\hat{0}, a)$ vanishes unless $a \in \{(0,0), (1,0), (0,1), (1,1)\}$, in which cases the value of $\mu_{L_r}(\hat{0}, a)$ is 1,-1,-1,1, respectively. So $\sum_{a \in L_r} \mu_{L_r}(\hat{0}, a) = 0$ unless $r = 1, 2$. For $r = 1, 2$, it can easily be checked that $\sum_{a \in L_r} \mu_{L_r}(\hat{0}, a) = 1$ if $r = 1$ and -1 if $r = 2$. So if $r_i \geq 3$ for some $i$, then $\mu(\hat{0}, \hat{1}) = 0$. If each $r_i \in \{1, 2\}$, then each index $i$ for which $r_i = 2$ contributes a -1 to the product on the right-hand side of the previous equation. Thus $\mu(\hat{0}, \hat{1}) = (-1)^{|S|+1}$. □

For convenience, we introduce the following notations. Let $\Pi = \{312, \pi^{(1)}, ..., \pi^{(m)}\}$ where $\pi^{(j)} = \iota_{r_j}[(\pi_1^{(j)})_*, ..., (\pi_{r_j}^{(j)})_*]$. For $I = (i_1, ..., i_m)$, we define

$$\underline{\Pi}_I = \{312, \underline{\pi^{(1)}}(i_1), ..., \underline{\pi^{(m)}}(i_m)\}$$

and

$$\overline{\Pi}_I = \{312, \overline{\pi^{(1)}}(i_1), ..., \overline{\pi^{(m)}}(i_m)\}.$$

A generalization of Theorem 1.2.4 can be stated as the following.

**Theorem 1.3.2.** *Suppose that the statistic $st : \mathfrak{S} \to \mathbb{N}$ satisfies the condition (†). Let $\Pi = \{312, \pi^{(1)}, ..., \pi^{(m)}\}$ where $\pi^{(i)} = \iota_{r_i}[(\pi_1^{(i)})_*, ..., (\pi_{r_i}^{(i)})_*]$. Then $F_0^{st}(\Pi) = 0$ if $\pi_i = \epsilon$ for some $i$ and 1 otherwise, and for $n \geq 1$ the st-polynomial $F_n^{st}(\Pi; q)$ satisfies*

$$F_{n+1}^{st}(\Pi; q) = \sum_{k=0}^{n} q^{f(k,n-k)} \left[ \sum_{S \subseteq [m]} (-1)^{|S|} \sum_{\substack{I=(i_1,...,i_m): \\ 1 \leq i_j \leq r_j - \delta_j}} F_k^{st}(\underline{\Pi}_I) \cdot F_{n-k}^{st}(\overline{\Pi}_{I+\delta}) \right],$$

*where $\delta = (\delta_1, ..., \delta_m)$ with $\delta_j = 1$ if $j \in S$ and 0 if $j \notin S$.*

*Proof.* Recall that by Lemma 1.2.2 a permutation $\sigma = 213[\sigma_1, 1, \sigma_2] \in \mathfrak{S}(312)$ avoids $\pi^{(j)}$ iff $\sigma$ satisfies the condition

$(C_i^j)$: $\sigma_1$ avoids $\underline{\pi^{(j)}}(i)$ and $\sigma_2$ avoids $\overline{\pi^{(j)}}(i)$

for some $i \in [r_j]$. So $\sigma \in \mathfrak{S}(312)$ belongs to $\mathfrak{S}(\Pi)$ iff, for every $j$, there is an $i \in [r_j]$ for which $\sigma$ satisfies $(C_i^j)$. Fix $k$ and let $\mathfrak{S}_{n+1}^k(312)$ be as in the proof of Theorem 1.2.4. Let $A_i^j$ be the set of $\pi^{(j)}$-avoiding permutations in $\mathfrak{S}_{n+1}^k(312)$ satisfying the condition $(C_i^j)$. For $I = (i_1, ..., i_m) \in [r_1] \times [r_2] \times \cdots \times [r_m]$, we define the set $A_I$ to be

$$A_I = A_{i_1, i_2, ..., i_m} := A_{i_1}^1 \cap A_{i_2}^2 \cap A_{i_m}^m.$$

So $\mathfrak{S}_{n+1}^k(\Pi)$ is the union

$$\mathfrak{S}_{n+1}^k(\Pi) = \bigcup_{i_1, ..., i_m} A_{i_1, i_2, ..., i_m},$$

where the union is taken over all $m$-tuples $I = (i_1, ..., i_m)$ in $[r_1] \times [r_2] \times \cdots \times [r_m]$. Let $\hat{P}_j$ be the intersection poset of $A^j_1, ..., A^j_{r_j}$, and let $P_j$ be the poset $\hat{P}_j \setminus \{A^j\}$, where $A^j = A^j_1 \cup \cdots \cup A^j_{r_j}$ is the unique maximum element of $\hat{P}_j$. Recall that $P_j$ is isomorphic to $L_{r_j}$. Let $P$ be the intersection poset of the $A_I$. The elements of $P$ are the unique maximal element $A = \mathfrak{S}^k_{n+1}(\Pi)$ and

$$T = T^1 \cap T^2 \cap \cdots \cap T^m,$$

where each $T^j$ is an element of $P_j$. Thus $P$ is isomorphic to $L_{r_1} \times \cdots \times L_{r_m} \cup \{\hat{1}\}$. For $S \subseteq [n]$, we say that an element $T \in P$ has type $S$ if $T^j = A^j_i$ for some $i$ when $j \notin S$ and $T^j = A^j_i \cap A^j_{i+1}$ for some $i$ when $j \in S$. Using Lemma 1.3.1, we know that the value of $\mu_P(T, A)$ where $T = T^1 \cap T^2 \cap \cdots \cap T^m \neq A$ is

$$\mu_P(T, A) = \begin{cases} (-1)^{|S|+1}, & \text{if } T \text{ has type } S, \\ 0, & \text{otherwise.} \end{cases}$$

For $T \in P$, we define $g : P \to \mathbb{C}(x : x \in A)$ by $g(T) = \sum_{x \in T} x$, so that

$$g(A) = \sum_{S \subseteq [n]} (-1)^{|S|} \sum_{T \text{ has type } S} g(T)$$

by the Möbius inversion formula.

Now, by the definition of type $S$, we have

$$\sum_{T \text{ has type } S} g(T) = \sum_{\substack{i_1, \ldots, i_m: \\ 1 \le i_j \le r_j - \delta_j}} g\left( \bigcap_{j \notin S} A^j_{i_j} \cap \bigcap_{j \in S} (A^j_{i_j} \cap A^j_{i_j+1}) \right),$$

where $\delta_j = 1$ if $j \in S$ and $0$ if $j \notin S$. Recall that $\sigma \in A^j_{i_j}$ iff $\sigma_1$ avoids $\underline{\pi^{(j)}}(i_j)$ and $\sigma_2$ avoids $\overline{\pi^{(j)}}(i_j)$, and $\sigma \in A^j_{i_j} \cap A^j_{i_j+1}$ iff $\sigma_1$ avoids $\underline{\pi^{(j)}}(i_j)$ and $\sigma_2$ avoids $\overline{\pi^{(j)}}(i_j + 1)$. Therefore, by mapping $\sigma \mapsto q^{\mathrm{st}(\sigma)}$, we have

$$g\left( \bigcap_{j \notin S} A^j_{i_j} \cap \bigcap_{j \in S} (A^j_{i_j} \cap A^j_{i_j+1}) \right) \mapsto q^{f(k,n-k)} F^{\mathrm{st}}_k(312, \underline{\pi^{(1)}}(i_1), \ldots, \underline{\pi^{(m)}}(i_m)) \\ \cdot F^{\mathrm{st}}_{n-k}(312, \overline{\pi^{(1)}}(i_1 + \delta_1), \ldots, \overline{\pi^{(m)}}(i_m + \delta_m)).$$

Therefore,

$$F^{\mathrm{st}}_{n+1,k}(\Pi; q) = q^{f(k,n-k)} \left[ \sum_{S \subseteq [m]} (-1)^{|S|} \sum_{\substack{i_1, \ldots, i_m: \\ 1 \le i_j \le r_j - \delta_j}} F^{\mathrm{st}}_k(\underline{\Pi}_I) \cdot F^{\mathrm{st}}_{n-k}(\overline{\Pi}_{I+\delta}) \right],$$

and we are done. $\square$

**Example 1.3.3.** Let $\Pi = \{312, \pi^{(1)}, \pi^{(2)}\}$ where $\pi^{(1)} = 2314 = 12[12_*, \epsilon_*]$ and $\pi^{(2)} = 2143 = 12[1_*, 1_*]$. We want to compute $a_n = F_n^{\mathrm{inv}}(\Pi)$ by using Theorem 1.3.2. There are four possibilities of $S \subseteq \{1,2\}$, and for each possibility the following table shows the appearing terms, where $\delta$ is again the Kronecker delta function.

| | | |
|---|---|---|
| $S = \emptyset$: | $F_k^{\mathrm{inv}}(312, 12, 1) \cdot F_{n-k}^{\mathrm{inv}}(\Pi)$ | $= \delta_{0,k} \cdot a_{n-k}$ |
| | $F_k^{\mathrm{inv}}(312, 2314, 1) \cdot F_{n-k}^{\mathrm{inv}}(312, 1, 2143)$ | $= \delta_{0,k} \cdot \delta_{0,n-k}$ |
| | $F_k^{\mathrm{inv}}(312, 12, 2143) \cdot F_{n-k}^{\mathrm{inv}}(312, 2314, 21)$ | $= 1$ |
| | $F_k^{\mathrm{inv}}(\Pi) \cdot F_{n-k}^{\mathrm{inv}}(312, 1, 21)$ | $= \delta_{0,n-k} \cdot a_k$ |
| $S = \{1\}$: | $F_k^{\mathrm{inv}}(312, 12, 1) \cdot F_{n-k}^{\mathrm{inv}}(312, 1, 2143)$ | $= \delta_{0,k} \cdot \delta_{0,n-k}$ |
| | $F_k^{\mathrm{inv}}(312, 12, 2143) \cdot F_{n-k}^{\mathrm{inv}}(312, 1, 21)$ | $= \delta_{0,n-k}$ |
| $S = \{2\}$: | $F_k^{\mathrm{inv}}(312, 12, 1) \cdot F_{n-k}^{\mathrm{inv}}(312, 2314, 21)$ | $= \delta_{0,k}$ |
| | $F_k^{\mathrm{inv}}(312, 2314, 1) \cdot F_{n-k}^{\mathrm{inv}}(312, 1, 21)$ | $= \delta_{0,k} \cdot \delta_{0,n-k}$ |
| $S = \{1,2\}$: | $F_k^{\mathrm{inv}}(312, 21, 1) \cdot F_{n-k}^{\mathrm{inv}}(312, 1, 21)$ | $= \delta_{0,k} \cdot \delta_{0,n-k}$ |

Hence the $a_n$ satisfy

$$\begin{aligned} a_{n+1} &= \sum_{q=0}^{n} q^k \left[\delta_{0,k} a_{n-k} + \delta_{0,n-k} \cdot a_k + 1 - \delta_{0,k} - \delta_{0,n-k}\right] \\ &= (1+q^n)a_n + \frac{1-q^{n+1}}{1-q} - (1+q^n) \\ &= (1+q^n)a_n + q\left(\frac{1-q^{n-1}}{1-q}\right). \end{aligned}$$

In particular, by setting $q = 1$ we get $a_{n+1} = 2a_n + n - 1$ with $a_0 = a_1 = 1$. Thus

$$|\mathfrak{S}_n(312, 2314, 2143)| = 2^n - n.$$

The following construction of st-Wilf equivalences can be extracted from Theorem 1.3.2. A proof of this corollary uses a similar argument to that of Corollary 1.2.6 and is omitted here.

**Corollary 1.3.4.** *Let st be a statistic satisfying* (†). *Let* $\pi_i^{(j)}, \pi_i'^{(j)}, 1 \leq j \leq m, 1 \leq i \leq r_m$, *be permutations such that*

$$\{312, \pi_{i_1}^{(1)}, ..., \pi_{i_m}^{(m)}\} \overset{st}{\equiv} \{312, \pi_{i_1}'^{(1)}, ..., \pi_{i_m}'^{(m)}\}$$

*for all m-tuples* $I = (i_1, ..., i_m) \in [r_1] \times ... \times [r_m]$. *Set* $\pi^{(j)} = \iota_r[\pi_{1*}^{(j)}, ..., \pi_{r_j*}^{(j)}]$ *and* $\pi'^{(j)} = \iota_r[\pi_{1*}'^{(j)}, ..., \pi_{r_j*}'^{(j)}]$. *Then* $\Pi = \{312, \pi^{(1)}, ..., \pi^{(m)}\}$ *and* $\Pi' = \{312,\ \pi'^{(1)}, ..., \pi'^{(m)}\}$ *are st-Wilf equivalent.*

## 1.4 Consecutive patterns

In this section we redirect our attention to study the number of occurrences of some consecutive patterns in 312-avoiding permutations. Benabei et al. [2] show that the distributions of the consecutive patterns 132 and 231 in 312-avoiding permutations are identical. Here we give a generalization to this result, where we consider certain inflations of 132 and 231.

For permutations $\tau$ and $\sigma$ we define $(\tau)\sigma$ to be the number of occurrences of the consecutive pattern $\tau$ in $\sigma$. More precisely, if $\tau \in \mathfrak{S}_k$ and $\sigma \in \mathfrak{S}_n$, then

$$(\tau)\sigma := \#\{i \in [n-k+1] : (\sigma(i), ..., \sigma(i+k-1)) \text{ is order isomorphic to } \tau\}.$$

We define $[\tau)\sigma$ to be 1 if $(\sigma(1), ..., \sigma(k))$ is order isomorphic to $\tau$ and 0 otherwise. Similarly, we define $(\tau]\sigma$ to be 1 if $(\sigma(n-k+1), ..., \sigma(n))$ is order isomorphic to $\tau$ and 0 otherwise. In other words, the "[" and "]" indicate that the occurrence of the consecutive pattern must be the leftmost and the rightmost position, respectively. Finally, we define $[\tau]\sigma = \delta_{\tau,\sigma}$. For example, let $\tau = 213$ and $\sigma = 562831749$. We have $(\tau)\sigma = 3$ since the consecutive subsequences 628, 317, 749 are order isomorphic to $\tau$. Furthermore, we have $(\tau]\sigma = 1$ since 749 is order isomorphic to $\tau$ and $[\tau)\sigma = 0$ since 572 is not order isomorphic to $\tau$.

For a permutation $\tau$, we set

$$F_n^\tau(q) = \sum_{\sigma \in \mathfrak{S}_n(312)} q^{(\tau)\sigma}$$

and

$$F^\tau(q,x) = \sum_{n \geq 0} F_n(q) x^n = \sum_{n,r \geq 0} a_{n,r}^\tau x^n q^r,$$

where $a_{n,r}^\tau$ is the number of permutations $\sigma \in \mathfrak{S}_n(312)$ containing $r$ occurrences of the consecutive patterns $\tau$.

The following theorem is the main result of this section.

**Theorem 1.4.1.** *For permutations $\pi \in \mathfrak{S}$ and $l \in \mathbb{N}$, define the permutations $\alpha_l(\pi)$ and $\beta_l(\pi)$ by*

$$\alpha_l(\pi) = 132[\iota_l, \pi, 1], \ \beta_l(\pi) = 231[1, \pi, \delta_l].$$

*Then for any two permutations $\pi, \pi' \in \mathfrak{S}_p$ of the same size, we have*

$$F^{\alpha_l(\pi)} = F^{\alpha_l(\pi')} = F^{\beta_l(\pi)} = F^{\beta_l(\pi')}.$$

*Proof.* For $\pi \in S_n$ and $0 \leq k \leq l$, we define the functions $f(k,l,\pi), g(k,l,\pi) : \mathfrak{S} \to \mathbb{N}$ by

$$f(k,l,\pi)(\sigma) = (\alpha_l(\pi))\sigma + [\alpha_{l-1}(\pi))\sigma + ... + [\alpha_{l-k}(\pi))\sigma$$

and

$$g(k,l,\pi)(\sigma) = (\beta_l(\pi))\sigma + (\beta_{l-1}(\pi)]\sigma + ... + (\beta_{l-k}(\pi)]\sigma.$$

We will show that

$$F_n^{f(k,l,\pi)} = F_n^{g(k,l,\pi)}$$

and are independent of $\pi$. In particular for $k = 0$ we have $F^{\alpha_l(\pi)} = F^{\beta_l(\pi)}$, which proves the theorem.

**Lemma 1.4.2.** *Let $\sigma = 213[\sigma_1, 1, \sigma_2] \in \mathfrak{S}_{n+1}$ with $\sigma_1 \in \mathfrak{S}_r$. Then*

$$(\alpha_l(\pi))\sigma = \begin{cases} (\alpha_l(\pi))\sigma_1 + (\alpha_l(\pi))\sigma_2 + [\alpha_{l-1}(\pi))\sigma_2 & \text{if } l > 0, \\ (\alpha_l(\pi))\sigma_1 + (\alpha_l(\pi))\sigma_2 + (\pi]\sigma_1 & \text{if } l = 0, \end{cases}$$

*and*

$$(\beta_l(\pi))\sigma = \begin{cases} (\beta_l(\pi))\sigma_1 + (\beta_l(\pi))\sigma_2 + (\beta_{l-1}(\pi)]\sigma_1 & \text{if } l > 0, \\ (\beta_l(\pi))\sigma_1 + (\beta_l(\pi))\sigma_2 + [\pi)\sigma_2 & \text{if } l = 0. \end{cases}$$

*Proof.* We will only prove the assertion for $\alpha_l(\pi)$. The statement about $\beta_l(\pi)$ can be proved similarly. First suppose $l > 0$. The terms $(\alpha_l(\pi))\sigma_1$ and $(\alpha_l(\pi))\sigma_2$ are from the case when the pattern $\alpha_l(\pi)$ appears inside $\sigma_1$ or $\sigma_2$. The other case can $\alpha_l(\pi)$ occur in $\sigma$ is when $1 = \sigma(r+1)$ involves. This can only happen when $\alpha_l(\pi)$ starts at the position $r+1$. That is precisely when $(\sigma_2(1), ..., \sigma_2(k))$ is order isomorphic to $\alpha_{l-1}(\pi)$, where $k$ is the size of $\alpha_{l-1}(\pi)$, hence the term $[\alpha_{l-1}(\pi))\sigma_2$. Suppose $l = 0$. Other than the occurrences of $\alpha_l(\pi)$ in either $\sigma_1$ or $\sigma_2$, $\sigma(r+1)$ must be the end of the pattern. This means $(\sigma_1(r-k+1), ..., \sigma_1(r))$ is order isomorphic to $\pi$, hence the term $(\pi]\sigma_1$ □

We also need to compute $[\alpha_l(\pi))\sigma$ in terms of $\sigma_1$ and $\sigma_2$. It can be proved similarly to the lemma above.

$$[\alpha_l(\pi))\sigma = \begin{cases} [\alpha_l(\pi))\sigma_1 & \text{if } r > 0, l > 0, \\ [\alpha_0(\pi))\sigma_1 + [\pi]\sigma_1 & \text{if } r > 0, l = 0, \\ [\alpha_{l-1}(\pi))\sigma_2 & \text{if } r = 0, l > 0, \\ 0 & \text{if } r = 0, l = 0. \end{cases}$$

With these tools, we can compute $f(k,l,\pi)(\sigma)$ in terms of $\sigma_1$ and $\sigma_2$. There are four cases to be considered.

(i) $r > 0, k < l$:

$$\begin{aligned} f(k,l,\pi)(\sigma) &= \Big((\alpha_l(\pi))\sigma_1 + (\alpha_l(\pi))\sigma_2 + [\alpha_{l-1}(\pi))\sigma_2\Big) + \sum_{i=1}^{k}[\alpha_{l-i}(\pi))\sigma_1 \\ &= f(k,l,\pi)(\sigma_1) + f(1,l,\pi)(\sigma_2). \end{aligned}$$

(ii) $r > 0, k = l$:

$$f(k,l,\pi)(\sigma) = \Big( (\alpha_l(\pi))\sigma_1 + (\alpha_l(\pi))\sigma_2 + [\alpha_{l-1}(\pi))\sigma_2 \Big) + \sum_{i=1}^{k} [\alpha_{l-i}(\pi))\sigma_1 + [\pi]\sigma_1$$
$$= f(k,l,\pi)(\sigma_1) + f(1,l,\pi)(\sigma_2) + [\pi]\sigma_1.$$

(iii) $r = 0, k < l$:

$$f(k,l,\pi)(\sigma) = \Big( (\alpha_l(\pi))\sigma_2 + [\alpha_{l-1}(\pi))\sigma_2 \Big) + \sum_{i=1}^{k} [\alpha_{l-i-1}(\pi))\sigma_2$$
$$= f(k+1,l,\pi)(\sigma_2).$$

(iv) $r = 0, k = l$:

$$f(k,l,\pi)(\sigma) = \Big( (\alpha_l(\pi))\sigma_2 + [\alpha_{l-1}(\pi))\sigma_2 \Big) + \sum_{i=1}^{l-1} [\alpha_{l-i-1}(\pi))\sigma_2$$
$$= f(l,l,\pi)(\sigma_2).$$

Recall that every $\sigma \in \mathfrak{S}_{n+1}(312)$ can be written as $213[\sigma_1, 1, \sigma_2]$ with $\sigma_1 \in \mathfrak{S}_r$ and $\sigma_2 \in \mathfrak{S}_{n-r}$ for some $r$. Therefore

$$F_{n+1}^{f(k,l,\pi)} = \sum_{\sigma \in \mathfrak{S}_{n+1}(312)} q^{f(k,l,\pi)(\sigma)} = \sum_{r=0}^{n} \sum_{\sigma_1 \in \mathfrak{S}_r, \sigma_2 \in \mathfrak{S}_{n-r}} q^{f(k,l,\pi)(\sigma)}$$
$$= \sum_{\sigma_2 \in \mathfrak{S}_n} q^{f(k,l,\pi)(\sigma)} + \sum_{r=1}^{n} \sum_{\sigma_1 \in \mathfrak{S}_r, \sigma_2 \in \mathfrak{S}_{n-r}} q^{f(k,l,\pi)(\sigma)}.$$

So if $k < l$ we have

$$F_{n+1}^{f(k,l,\pi)} = \sum_{\sigma_2 \in \mathfrak{S}_n} q^{f(k+1,l,\pi)(\sigma_2)} + \sum_{r=1}^{n} \sum_{\sigma_1 \in \mathfrak{S}_r, \sigma_2 \in \mathfrak{S}_{n-r}} q^{f(k,l,\pi)(\sigma_1) + f(1,l,\pi)(\sigma_2)}$$
$$= F_n^{f(k+1,l,\pi)} + \sum_{r=1}^{n} F_r^{f(k,l,\pi)} \cdot F_{n-r}^{f(1,l,\pi)}.$$

If $k = l$ we have

$$F_{n+1}^{f(l,l,\pi)} = \sum_{\sigma_2 \in \mathfrak{S}_n} q^{f(l,l,\pi)(\sigma_2)} + \sum_{r=1}^{n} \sum_{\sigma_1 \in \mathfrak{S}_r, \sigma_2 \in \mathfrak{S}_{n-r}} q^{f(l,l,\pi)(\sigma_1) + [\pi]\sigma_1 + f(1,l,\pi)(\sigma_2)}$$
$$= F_n^{f(l,l,\pi)} + \sum_{r=1}^{n} F_r^{f(l,l,\pi) + [\pi]} \cdot F_{n-r}^{f(1,l,\pi)}.$$

Note that $[\pi]\sigma = 1$ only when $\sigma = \pi$ and 0 otherwise, and in the case $\sigma = \pi$, $f(l,l,\pi)(\sigma) = 0$. So

$$F_n^{f(l,l,\pi)+[\pi]} = \begin{cases} F_n^{f(l,l,\pi)} + (q-1) & \text{if } n = p, \\ F_n^{f(l,l,\pi)} & \text{otherwise.} \end{cases}$$

Thus in this case $F^{f(l,l,\pi)}$ satisfies

$$\begin{aligned} F_{n+1}^{f(l,l,\pi)} &= F_n^{f(l,l,\pi)} + \sum_{r=1}^{n} \left(F_r^{f(l,l,\pi)} + (q-1)\delta_{r,p}\right) \cdot F_{n-r}^{f(1,l,\pi)} \\ &= F_n^{f(l,l,\pi)} + (q-1)F_{n-p}^{f(1,l,\pi)} + \sum_{r=1}^{n} F_r^{f(l,l,\pi)} \cdot F_{n-r}^{f(1,l,\pi)}. \end{aligned}$$

To summarize, we have

$$F_{n+1}^{f(k,l,\pi)} = \begin{cases} F_n^{f(k+1,l,\pi)} + \sum_{r=1}^{n} F_r^{f(k,l,\pi)} \cdot F_{n-r}^{f(1,l,\pi)} & \text{if } k < l \\ F_n^{f(l,l,\pi)} + (q-1)F_{n-p}^{f(1,l,\pi)} + \sum_{r=1}^{n} F_r^{f(l,l,\pi)} \cdot F_{n-r}^{f(1,l,\pi)} & \text{if } k = l. \end{cases} \tag{1.1}$$

Next we perform a similar computation for $g(k,l,\pi)$. Here, instead, we let $r = |\sigma_2|$. First we start with $(\beta_l(\pi)]\sigma$:

$$(\beta_l(\pi)]\sigma = \begin{cases} (\beta_l(\pi)]\sigma_2 & \text{if } r > 0, l > 0, \\ (\beta_l(\pi)]\sigma_2 + [\pi]\sigma_2 & \text{if } r > 0, l = 0, \\ (\beta_{l-1}(\pi)]\sigma_1 & \text{if } r = 0, l > 0, \\ 0 & \text{if } r = 0, l = 0. \end{cases}$$

So $g(k,l,\pi)(\sigma)$ can be expressed as

$$g(k,l,\pi)(\sigma) = \begin{cases} g(1,l,\pi)(\sigma_1) + g(k,l,\pi)(\sigma_2) & \text{if } r > 0, k < l, \\ g(1,l,\pi)(\sigma_1) + g(k,l,\pi)(\sigma_2) + [\pi]\sigma_2 & \text{if } r > 0, k = l, \\ g(k+1,l,\pi)(\sigma_1) & \text{if } r = 0, k < l, \\ g(l,l,\pi)(\sigma_1) & \text{if } r = 0, k = l. \end{cases}$$

Therefore the polynomials $F_n^{g(k,l,\pi)}$ satisfy

$$F_{n+1}^{g(k,l,\pi)} = \begin{cases} F_n^{g(k+1,l,\pi)} + \sum_{r=1}^{n} F_{n-r}^{g(1,l,\pi)} \cdot F_r^{g(k,l,\pi)} & \text{if } k < l \\ F_n^{g(l,l,\pi)} + (q-1)F_{n-p}^{g(1,l,\pi)} + \sum_{r=1}^{n} F_{n-r}^{g(1,l,\pi)} \cdot F_r^{g(l,l,\pi)} & \text{if } k = l. \end{cases} \tag{1.2}$$

Comparing the equations 1.1 and 1.2, we see that both $F^{f(k,l,\pi)}$ and $F^{g(k,l,\pi)}$ satisfy the same recurrence relations. They also satisfy the same initial conditions: $F_n^{f(k,l,\pi)} = F_n^{g(k,l,\pi)} = C_n$ for all $n < p$ since $f(k,l,\pi)(\sigma) = g(k,l,\pi)(\sigma) = 0$. Therefore $F^{f(k,l,\pi)} = F^{g(k,l,\pi)}$ for all $0 \le k \le l$ and $\pi \in \mathfrak{S}$, and is independent of $\pi$. Thus the Theorem 1.4.1 is proved. $\square$

For the rest of this Section, we show how to calculate $F^\tau$, where $\tau$ is either $132[1, \pi, 1]$ or $132[12, \pi, 1]$.

**Proposition 1.4.3.** *Let $\pi \in \mathfrak{S}_p$ be any permutation and $\tau = 132[1, \pi, 1]$. Then*

$$F^\tau(q,x) = 2\left[1 + (q-1)x^p - \sqrt{(q-1)^2x^{2p} - 2(q-1)x^p - 4x + 1}\right]^{-1}.$$

*Proof.* Let $F_n = F_n^{f(0,1,\pi)}(q) =$ and $G_n = F_n^{f(1,1,\pi)}(q)$. Then we have

$$F_{n+1} = G_n + \sum_{r=1}^{n} F_r \cdot G_{n-r} = \sum_{r=0}^{n} F_r \cdot G_{n-r},$$

$$G_{n+1} = (q-1)G_{n-p} + \sum_{r=0}^{n} G_r \cdot G_{n-r}.$$

From the second equation, we get

$$0 = xG(x)^2 + ((q-1)x^p - 1)\, G(x) + 1.$$

Since, if $q = 1$, $G(x) = \sum_{n\geq 0} C_n x^n = frac12x(1 - \sqrt{1-4x})$,

$$G(x) = \frac{1}{2x}\left[1 - (q-1)x^p - \sqrt{(q-1)^2x^{2p} - 2(q-1)x^p - 4x + 1}\right],$$

and by the first equation, $F(x)$ has the closed form

$$F(x) = \frac{1}{1 - xG(x)} = 2\left[1 + (q-1)x^p + \sqrt{(q-1)^2x^{2p} - 2(q-1)x^p - 4x + 1}\right]^{-1}.$$

□

**Proposition 1.4.4.** *Let $\pi \in \mathfrak{S}_p$ be any permutation and $\tau = 132[12, \pi, 1]$. Then $F^\tau$ satisfies*

$$(q-1)x^{p+1}F^3 + \left(x - (q-1)x^{p+1}\right)F^2 - F + 1 = 0.$$

*Proof.* Set $F_n^{(k)} = F_n^{f(k,2,\pi)}$. Then we have

$$F_{n+1}^{(0)} = \sum_{r=0}^{n} F_r^{(0)} \cdot F_{n-r}^{(1)}$$

$$F_{n+1}^{(1)} = F_n^{(2)} + \sum_{r=1}^{n} F_r^{(1)} \cdot F_{n-r}^{(1)}$$

$$F_{n+1}^{(2)} = F_n^{(2)} + \sum_{r=1}^{n} F_r^{f(2,2,\pi)+[\pi]} \cdot F_{n-r}^{(1)}.$$

From these relations, respectively, we get

$$F^{(0)} = \left(1 - xF^{(1)}\right)^{-1},$$

$$F^{(1)} - 1 = xF^{(2)} - xF^{(1)} + x(F^{(1)})^2,$$
$$F^{(2)} - 1 = xF^{(2)} + xF^{(1)} \cdot F^{(2)} - xF^{(1)} + (q-1)x^p F^{(1)}.$$

Manipulating the second and third equations, we get

$$xF^{(2)} = F^{(1)} - 1 + xF^{(1)} - x(F^{(1)})^2,$$
$$(1 - x - xF^{(1)})F^{(2)} = 1 - xF^{(1)} + (q-1)x^p F^{(1)}.$$

So we obtain

$$x\left(1 - xF^{(1)} + (q-1)x^p F^{(1)}\right) = \left(1 - x - xF^{(1)}\right)\left((1+x)F^{(1)} - x(F^{(1)})^2 - 1\right),$$

or, equivalently,

$$x^2(F^{(1)})^3 - 2x(F^{(1)})^2 + \left(1 + x - (q-1)x^{p+1}\right)F^{(1)} - 1 = 0.$$

Substituting $xF^{(1)} = 1 - F^{-1}$, where $F = F^{(0)}$, we get

$$\left(1 - \frac{1}{F}\right)^3 - 2\left(1 - \frac{1}{F}\right)^2 + (1 + x - (q-1)x^{p+1})\left(1 - \frac{1}{F}\right) - 1 = 0,$$

or

$$(q-1)x^{p+1}F^3 + \left(x - (q-1)x^{p+1}\right)F^2 - F + 1 = 0.$$

□

# Chapter 2

# Interlacing networks

The chapter is based on joint works with Miriam Farber and Sam Hopkins [13] [14].

## 2.1 Introduction

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ be a partition of $n$. A *Young diagram* of shape $\lambda$ is an array of boxes with $\lambda_i$ boxes in the $i^{\text{th}}$ row, adjusted to the top left. A *standard Young tableaux* (SYT) of shape $\lambda$ is a filling of the Young diagram by the numbers 1 to $n$ without repetition, so that the numbers in each row and column are strictly increasing. A *semistandard Young tableau* (or SSYT) is a filling of the Young diagram by positive integers (repetitions allowed) so that the numbers in each row are weakly increasing and the numbers in each columns are strictly increasing. An $n$-semistandard Young tableau is a SSYT with content from $[n]$.

The *Robinson-Schensted correspondence* (RS) is a bijective map which sends a permutation $\sigma$ in the symmetric group $\mathfrak{S}_n$ to a pair $(P, Q)$ of standard Young tableau of the same shape $\lambda \vdash n$, defined via a row-insertion algorithm. The *Robinson-Schensted-Knuth correspondence* (RSK) is a generalization of the RS correspondence which sends an arbitrary $n \times n$ $\mathbb{N}$-matrix $A$ to a pair $(P, Q)$ of $n$-semistandard Young tableaux of the same shape $\lambda$. For the precise definition of RSK and its properties see [33, Chapter 7.11]. Often, we regard RSK as a map which sends an $n \times n$ $\mathbb{N}$-matrix $A$ to an $\mathbb{N}$-matrix of the same size for which the rows and columns are weakly increasing, as described below.

For an $n$-SSTY $T$ of shape $\lambda$, let $g_{ij}$ with $1 \leq i \leq j \leq n$ be the number of entries in row $j - i + 1$ of $T$ which are $\leq n + 1 - i$. Then the array $G = (g_{ij})_{1 \leq i \leq j \leq n}$ is a

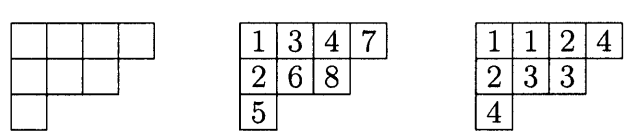


Figure 2-1: From left to right: Young diagram of $\lambda = (4, 3, 1)$, a standard Young tableau of shape $\lambda$, and a semistandard Young diagram of shape $\lambda$.

$$A=\begin{bmatrix}1&0&2&1\\0&3&0&1\\2&1&0&0\\1&1&1&0\end{bmatrix}\longmapsto\quad\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3\\\hline 2&2&2&4\\\cline{1-4} 3&3&4\\\cline{1-3}\end{array}\qquad\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&4&4\\\hline 2&2&2&3\\\cline{1-4} 3&3&4\\\cline{1-3}\end{array}$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\widehat{A}=\begin{bmatrix}0&2&3&4\\2&3&4&5\\3&3&4&5\\4&6&7&7\end{bmatrix}\longleftarrow\quad\begin{array}{ccccccc}7&&4&&3&&0\\&7&&3&&2&\\&&6&&3&&\\&&&4&&&\end{array}\qquad\begin{array}{ccccccc}7&&4&&3&&0\\&5&&4&&2&\\&&5&&3&&\\&&&4&&&\end{array}$$

Figure 2-2: An example of RSK.

*Gelfand-Tsetlin pattern*, i.e., $G$ satisfies $g_{i,j} \geq g_{i+1,j+1} \geq g_{i,j+1}$ for all $1 \leq i \leq j \leq n-1$. We denote this resulting array by $GT(T)$. Given an $n \times n$ $\mathbb{N}$-matrix $A$, let $G = (g_{ij}) = GT(P)$ and $H = (h_{ij}) = GT(Q)$, where $(P, Q)$. Since $P$ and $Q$ have the same shape, we see that $g_{1j} = h_{1j}$ for all $j$. We obtain the matrix $\widehat{A}$ by gluing along the first row of $G$ and $H$ and reflecting along in the anti-diagonal. More precisely, the matrix $\widehat{A} = (\widehat{a}_{ij})$ is given by

$$\widehat{a}_{ij} := \begin{cases} g_{i-j+1,n+1-j} & \text{if } i \geq j; \\ h_{j-i+1,n+1-i} & \text{otherwise.} \end{cases}$$

Since $G$ and $H$ are Gelfand-Tsetlin patterns, the matrix $\widehat{A}$ has weakly increasing rows and columns. We may also regard the composition $A \mapsto (P, Q) \mapsto \widehat{A}$ as RSK.

The RSK enjoys the symmetry: if $A \mapsto \widehat{A}$, then $A^t \mapsto \widehat{A^t}$. This, however, is not transparent in the original definition of RSK via row-insertions. Another equivalent definition of RSK via octahedron recurrences makes this symmetry transparent and allows the entries of $A$ to be any real numbers. Before we explain this construction, we introduce the Greene-Kleitman invariant, which will play an important role in verification of this definition of RSK.

Let $A$ be an $n \times n$ $\mathbb{N}$-matrix. Suppose the RSK gives $A \mapsto (P, Q) \mapsto \widehat{A}$ where $\lambda = (\lambda_1, \ldots \lambda_l)$ is the shape of a $P$ (and $Q$). Then $\lambda_1 + \ldots + \lambda_k$ is equal to the maximum sum of the weights of $k$ non-crossing lattice paths from $(1,1), \ldots, (1,k)$ to $(n, n-k+1), \ldots, (n,n)$. Here the weight of a path is the sum of $a_{ij}$ for all points $(i,j)$ along that path. If $A$ is a permutation matrix corresponding to a permutation $\pi$ (i.e., the Robinson-Schensted case), then this phenomenon says that $\lambda_1 + \ldots + \lambda_k$ is the maximum size of $k$ disjoint increasing subsequences of $\pi$. This is known as Greene's theorem [18]. The generalization to RSK of this theorem is widely accepted as "well-known." The best reference of this theorem is probably [23].

Passing this to $(P, Q) \mapsto \widehat{A}$, we see that $\lambda_1 + \ldots + \lambda_k = \sum_{r=0}^{k-1} \widehat{a}_{n-r,n-r}$. Indeed the maximum sum of the weights of $k$ non-crossing lattice paths from $(1,1), \ldots, (1,k)$

to $(i, n-k+1), ..., (i, n)$ (resp. $(n, j-k+1), ..., (n, j)$) to is equal to $\sum_{r=0}^{k-1} \widehat{a}_{i-r,n-r}$ (resp. $\sum_{r=0}^{k-1} \widehat{a}_{n-r,j-r}$). Note that this property uniquely determines $\widehat{A}$ from $A$.

Now we describe the RSK via octahedron recurrences (see [5] and [6]). Given an $n \times n$ $\mathbb{N}$-matrix $A$, we let $\text{rect}(i,j) := \sum_{1\le r\le i, 1\le s\le j} a_{ij}$. We construct a three-dimensional array $\widetilde{Y} = (\widetilde{y}_{ijk})$ with indices $i, j, k \in [0, n]$ such that $k \le \min(i, j)$ by the following recurrence relations, which we refer as *octahedron recurrence*:

$$\widetilde{y}_{i,j,k} + \widetilde{y}_{i-1,j-1,k-1} = \max\left(\widetilde{y}_{i-1,j,k} + \widetilde{y}_{i,j-1,k-1}, \widetilde{y}_{i-1,j,k-1} + \widetilde{y}_{i,j-1,k}\right) \tag{2.1}$$

for all $i, j, k \ge 1$ with initial conditions:

$$\widetilde{y}_{ikk} = \widetilde{y}_{kjk} = 0 \ \text{ and } \widetilde{y}_{ij0} = -\text{rect}(i,j).$$

Finally, let

$$\widehat{a}_{ij} = \begin{cases} \widetilde{y}_{n,n-i+j,n-i} - \widetilde{y}_{n,n-i+j,n-i+1} & \text{if } i \ge j \\ \widetilde{y}_{n-j+i,n,n-j} - \widetilde{y}_{n-j+i,n,n-j+1} & \text{if } i < j. \end{cases}$$

**Example 2.1.1.** In this example, we take the same matrix $A$ from Figure 2-2. We construct the array $\widetilde{Y} = (\widetilde{y}_{ijk})$ as follows. The first $(k = 0)$ level $(\widetilde{y}_{ij0})_{i,j\in[0,n]}$ is given by $\widetilde{y}_{i00} = \widetilde{y}_{0j0} = 0$ for $i, j \in [0, n]$ and, for $i, j \in [n]$, $\widetilde{y}_{ij0}$ is the negative of the sum of the elements $a_{rs}$ of $A$ where $r \in [i]$ and $s \in [j]$. Then for each $k \in [n]$ we build $(\widetilde{y}_{ijk})$, $i, j \in [k, n]$ successively by $\widetilde{y}_{ikk} = \widetilde{y}_{kjk} = 0$ and for $i, j \in [k+1, n]$ we compute $\widetilde{y}_{ijk}$ from the octahedron recurrence. We obtain the following array $\widetilde{Y}$. Here we present the array $\widetilde{Y} = (\widetilde{y}_{ijk})$ by the index $k$ as an $(n+1-k) \times (n+1-k)$ matrix for each $k = 0, 1, ..., n$.

$$\widetilde{Y} = \begin{array}{|ccccc|cccc|ccc|cc|c|} \hline 0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\ 0 & -1 & -1 & -3 & -4 & 0 & 0 & 0 & 0 & & & & & & \\ 0 & -1 & -4 & -6 & -8 & 0 & 0 & -2 & -3 & 0 & 0 & 0 & & & \\ 0 & -3 & -7 & -9 & -11 & 0 & -2 & -4 & -6 & 0 & -2 & -2 & 0 & 0 & \\ 0 & -4 & -9 & -12 & -14 & 0 & -3 & -5 & -7 & 0 & -2 & -3 & 0 & 0 & 0 \\ \multicolumn{5}{|c|}{k=0} & \multicolumn{4}{c|}{k=1} & \multicolumn{3}{c|}{k=2} & \multicolumn{2}{c|}{k=3} & \text{k=4} \\ \hline \end{array}$$

To obtain $\widehat{A} = (\widehat{a}_{ij})$, we first form an intermediate matrix $B = (b_{ij})_{i,j\in[0,n]}$ by putting together the last row and the last column of each $k$-level of $\widetilde{Y}$. In this example, we have

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -3 & -4 \\ 0 & -2 & -3 & -6 & -8 \\ 0 & -3 & -5 & -7 & -11 \\ 0 & -4 & -9 & -12 & -14 \end{bmatrix}$$

Finally, the elements $\widehat{a}_{ij}$ of $\widehat{A}$ are obtained by taking $\widehat{a}_{ij} = b_{i-1,j-1} - b_{ij}$. In this case, we see that the matrix $\widehat{A}$ is indeed the same as the matrix $\widehat{A}$ from the Figure 2-2.

To see that, in general, the map $A = (a_{ij}) \mapsto \widehat{A} = (\widehat{a_{ij}})$ is indeed equivalent to RSK, we let $\overline{y}_{ijk}$ to be the maximum sum of the weights of $k$ non-crossing lattice paths from $(1,1),...,(1,k)$ to $(i,j-k+1),...,(i,j)$. The following theorem is one of the main results in this chapter.

**Theorem 2.1.1.** *Let $Y' = (y'_{ijk})$ be the array defined by $(y'_{ijk}) := (y_{ijk} - rect(i,j))$. Then $Y'$ satisfy the octahedron recurrence 2.1.*

Assuming this result, we see that the $(\widehat{a}_{ij})$ satisfy

$$\sum_{r=0}^{k-1} \widehat{a}_{i-r,n-r} = \widetilde{y}_{i,n,k} - \widetilde{y}_{i,n,0} = \overline{y}_{i,n,k}$$

and

$$\sum_{r=0}^{k-1} \widehat{a}_{n-r,j-r} = \widetilde{y}_{n,j,k} - \widetilde{y}_{n,j,0} = \overline{y}_{n,j,k}.$$

So this $\widehat{A} = (\widehat{a}_{ij})$ is the same as the result of RSK:$A \mapsto \widehat{A}$.

The main motivation of this chapter is to give a combinatorial prove a birational version of this theorem, where maximums and sums are replaced by sums and products, respectively (Theorem 2.2.22). In Section 2.2, we introduce the notion of an interlacing network, a planar directed network with a rigid sources and sinks structure, and an $n$-bottlenecked network, which is a less restricted version of interlacing networks. We construct an involution that swaps pairs of tuples of noncrossing paths connecting sources and sinks. This involution on interlacing networks leads to three-term relations, which imply the octahedron recurrences. In Section 2.3, we give examples of Schur function identities obtained from the involution on the $n$-bottlenecked networks. Then we explain how these identities prove some special cases of a conjectured by Lam-Postnikov-Pylyavskyy. In Section 2.4, we define the balanced swap graphs, which the graphs defined by the end-patterns and the changes under the involution. These graphs encode a class of Schur function identities. In the case $m - n = 1$, we classify the connected components of these graphs and characterize the components which are acyclic.

## 2.2 Interlacing networks

### 2.2.1 Terminology

Let $G = (V, E, \omega)$ be a graph with vertex set $V$, edge set $E \subseteq V \times V$, and edge-weight function $\omega : E \to \mathbb{R}_{\geq 0}$. Unless noted otherwise, we assume that all the graphs are finite, directed, acyclic, and planar. A *path* in $G$ is a sequence $\pi = (v_i)_{i=0}^{n}$ of distinct elements of $V$ with $(v_{i-1}, v_i) \in E$ for all $i \in [n]$. We say that such a path $\pi$ *connects* $v_0$ and $v_n$, and that $v_0$ is the *start point* of $\pi$ and $v_n$ is its *end point.* We use $\mathrm{Vert}(\pi)$ to denote the set of vertices in $\pi$. The *weight* of $\pi$ is $\mathrm{wt}(\pi) := \prod_{i=1}^{n} \omega(v_{i-1}, v_i)$. A *subpath* of $\pi$ is a subsequence of consecutive vertices. Sometimes we view paths as simple curves embedded in the plane in the obvious way. Two paths $\pi$ and $\sigma$ are *noncrossing* if $\mathrm{Vert}(\pi) \cap \mathrm{Vert}(\sigma) = \emptyset$. Let $\Pi = (\pi_1, \dots, \pi_n)$ be a tuple of paths. We say $\Pi$ is *noncrossing* if $\pi_i$ and $\pi_j$ are noncrossing for all $1 \leq i \neq j \leq n$. Suppose that $\mathcal{X} = (x_1, ..., x_n)$ and $\mathcal{Y} = (y_1, ..., y_n)$ are two $n$-tuples of vertices in $V$ of the same length $n$. We say that an $n$-tuple $(\pi_1, ..., \pi_n)$ of paths in $G$ *connects* $\mathcal{X}$ and $\mathcal{Y}$ if $\pi_i$ connects $x_i$ and $y_i$ for each $i \in [n]$. We denote the set of all $n$-tuples of noncrossing paths connecting $\mathcal{X}$ and $\mathcal{Y}$ by $\mathrm{NCPath}_G(\mathcal{X}, \mathcal{Y})$. We omit the subscript $G$ when the network is clear from context. We define the *weight* of the tuple $\Pi$ to be $\mathrm{wt}(\Pi) := \prod_{i=1}^n \mathrm{wt}(\pi)$. For a pair $(\Pi, \Sigma)$ of tuples of paths we define $\mathrm{wt}(\Pi, \Sigma) := \mathrm{wt}(\Pi) \cdot \mathrm{wt}(\Sigma)$.

**Remark 2.2.1.** Given a vertex-weighted graph $G = (V, E, \omega')$ where $\omega' : V \to \mathbb{R}_{>0}$, we can convert $\omega$ into an edge-weight $\omega : E \to \mathbb{R}_{>0}$ of $G$ by defining

$$\omega(u,v) = \sqrt{\omega'(u) \cdot \omega'(v)} \qquad\qquad \forall (u,v) \in E.$$

Let us define the vertex-weight $\mathrm{wt}'(P)$ of a path $P$ in $G$ to be product of the vertex-weights $\omega(v)$ of all vertices that $P$ visits. If $P$ is a path connecting vertices $u$ and $v$, then

$$\mathrm{wt}'(P) = \mathrm{wt}(P)\sqrt{\omega'(u) \cdot \omega'(v)}.$$

So we can interchange between vertex-weighted graphs and edge-weighted graphs.

Let $m > n \geq 1$. A *network* is a triple $(G, S, T)$, where

- $G = (V, E, \omega)$ is a graph;
- $S = (s_1, \dots, s_{m+n}) \in V^{m+n}$ is a tuple of *source* vertices;
- $T = (t_1, \dots, t_{m+n}) \in V^{m+n}$ is a tuple of *sink* vertices,

such that $G$ is embedded inside a planar disc with $s_1, \dots, s_{m+n}, t_{m+n}, \dots, t_1$ arranged in clockwise order on the boundary of this disc. Note that both $S$ and $T$ are allowed to have repeated vertices. Needless to say, such networks are considered up to homeomorphism. We assume the edges of $G$ intersect the boundary of the disc into which $G$ is embedded only at vertices. In this section and the next we will work with a fixed network $(G, S, T)$; we will refer to this network from now on as simply $G$ with the sources and sinks implicit. A *pattern* on $G$ is just a pair $(I, J)$ with $I, J \in \binom{[m+n]}{n}$, where we think of $s_i$ and $t_j$ being colored red for all $i \in I$ and $j \in J$, and the other source and sink vertices being colored blue. We call $I$ the *source pattern* of $(I, J)$, and $J$ its *sink pattern.* We will use $\mathrm{Pat}(G)$ to denote the set of patterns on $G$.

Let $I, J \in \binom{[m+n]}{k}$ where the elements of $I$ are $i_1 < \cdots < i_k$ and the elements of $J$ are $j_1 < \cdots < j_k$. Define the set of *tuples of noncrossing paths of type* $(I, J)$ to be $\mathrm{NCPath}_G(I, J) := \mathrm{NCPath}_G((s_{i_l})_{l=1}^k, (t_{j_l})_{l=1}^k)$. Fix a pattern $(I, J) \in \mathrm{Pat}(G)$. Define the set of *pairs of tuples of noncrossing paths of type* $(I, J)$ to be

$$\mathrm{PNCPath}_G(I, J) := \mathrm{NCPath}_G(I, J) \times \mathrm{NCPath}_G(\overline{I}, \overline{J})$$

where for a subset $K \subseteq [m+n]$ we set $\overline{K} := [m+n] \setminus K$. Again, we omit the subscripts of $\mathrm{NCPath}_G(I, J)$ and $\mathrm{PNCPath}_G(I, J)$ when the network is clear from context. We then define the *weight* of a pattern $(I, J) \in \mathrm{Pat}(G)$ to be

$$\mathrm{wt}(I, J) := \sum_{(R,B) \in \mathrm{PNCPath}(I,J)} \mathrm{wt}(R, B).$$

Finally, set

$$\mathrm{PNCPath}(G) := \bigcup_{(I,J) \in \mathrm{Pat}(G)} \mathrm{PNCPath}(I, J).$$

We now define what it means for a network to be interlacing, the key property that will allow us to find three-term relations among the pattern weights. This condition may at first appear ad-hoc, but the later algebraic treatment of these networks will show that this definition suffices for the corresponding matrix to have a certain easily-stated rank property. Let us call $U \subseteq V$ *non-returning* if for all $u_1, u_2 \in U$ and paths $\pi$ connecting $u_1$ and $u_2$, we have $\mathrm{Vert}(\pi) \subseteq U$ (this is a technical condition required for our sink-swapping algorithm to work). Then we say $G$ is $k$*-bottlenecked* if there exists a non-returning subset $N \subseteq V$ with $|N| \leq k$ so that for all $i, j \in [m+n]$ and paths $\pi$ connecting $s_i$ to $t_j$, there is $v \in \mathrm{Vert}(\pi)$ for some $v \in N$.

For the most part of this section, we will only require our graph to be $m$-bottlenecked. However, to get the three-term relations, we will need our graph to be interlacing. We assume now, for the sake of define the interlacing property, that $m = n + 1$. Let us call $U \subseteq V$ *sink-branching* if for all $u \in U$, $i \in [2, 2n]$, $j \in \{1, 2n+1\}$ and paths $\pi'$ connecting $u$ and $t_i$ and $\pi''$ connecting $u$ and $t_j$, we have that $\mathrm{Vert}(\pi') \cap \mathrm{Vert}(\pi'') = \{u\}$ (this is another technical condition). Then we say $G$ is $k$*-sink-bottlenecked* if there exists a non-returning and sink-branching $N_T \subseteq V$ with $|N_T| \leq k$ so that for all $i \in [2n+1], j \in [2, 2n]$ and paths $\pi$ connecting $s_i$ to $t_j$, there is $v \in \mathrm{Vert}(\pi)$ for some $v \in N_T$. We say $G$ is *interlacing* if it is both $(n+1)$-bottlenecked and $n$-sink-bottlenecked.

**Example 2.2.2.** The following interlacing network, the rectangular grid $\Gamma^n_{r,s}$, will serve as our running example and will also be key for the motivating problem concerning birational RSK. Let $r, s \geq 3$ and $1 \leq n < \min(r, s)$. The graph $\Gamma_{r,s}$ has vertex set $V := \{(i, j) \in \mathbb{Z}^2 \colon i \in [r], j \in [s]\}$ and edge set $E := E_1 \cup E_2$ where

$$E_1 := \{((i, j), (i + 1, j)) \colon i \in [r - 1], j \in [s]\};$$
$$E_2 := \{((i, j), (i, j + 1)) \colon i \in [r], j \in [s - 1]\}.$$

We allow the weight function $\omega$ of the graph $\Gamma_{r,s}$ to be arbitrary. The network $\Gamma^n_{r,s}$ has underlying graph $\Gamma_{r,s}$ with sources and sinks

$$S := ((n + 1, 1), (n, 1), (n, 2), (n - 1, 2), \dots, (1, n), (1, n + 1))$$
$$T := ((r, s - n), (r - 1, s - n), (r - 1, s - n + 1), \cdots, (r - n, s - 1), (r - n, s)).$$

Our term "interlacing network" derives from the fact that these sinks and sources

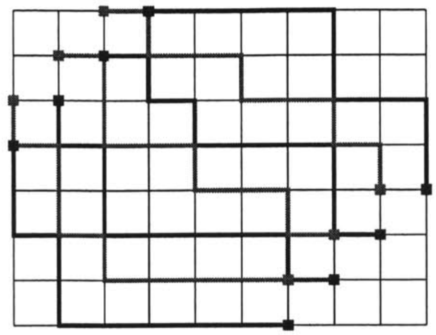

Figure 2-3: $\Gamma^3_{8,10}$ and an element of $\mathrm{PNCPath}(\{2,4,6\},\{2,4,6\})$.

are arranged in a zig-zag. Strictly speaking, in order to satisfy the network condition requiring our graph to lie inside a disc with the source and sink vertices on the boundary, we should restrict the vertex set of $\Gamma^n_{r,s}$ to $V' \subseteq V$ where $V' := \{v \in V : s_i \leq v \leq t_j \text{ for some } i,j \in [2n+1]\}$. However, vertices in $V \setminus V'$ will never be used in a path connecting a source to a sink, so this technicality will not concern us from now on.

Observe that $\Gamma^n_{r,s}$ is interlacing: we may take $N = \{s_1, s_3, \ldots, s_{2n+1}\}$ to satisfy the $(n+1)$-bottlenecked condition, and $N_T = \{t_2, t_4, \ldots, t_{2n}\}$ to satisfy the $n$-sink-bottlenecked condition. Figure 2-3 depicts $\Gamma^3_{8,10}$ along with an element of $\mathrm{PNCPath}(\{2,4,6\},\{2,4,6\})$. Note that vertex $(1,1)$ is the top-leftmost vertex in this picture, $(8,1)$ is the bottom-leftmost vertex, and $(1,10)$ is the top-rightmost: we use "matrix coordinates" with edges directed downwards and rightwards in $\Gamma^n_{r,s}$. We warn the reader that there are various conventions for orientation of such a grid, and we will at different times use several of them.

### 2.2.2 The sink-swapping involution

In this section we obtain three-term Plücker-like relations between pattern weights of an interlacing network $G$ via an algorithmically-defined involution on $\mathrm{PNCPath}(G)$ that swaps sink patterns. As previously mentioned, we only need the $m$-bottlenecked property for the algorithm to make sense. The interlacing property will ensure that we get three-term relation.

**Definition 2.2.3.** Let $m$ and $n$ be positive integers with $m > n$. For $J, J' \in \binom{[m+n]}{n}$, we say that $J'$ is a *swap* of $J$ if $J \cap J' = \emptyset$. Clearly the relation of being a swap is symmetric. If $J$ and $J'$ are swaps of one another, we call the set $P(J,J') := [m+n] \setminus (J \cup J')$ their *pivot set*. We say $J'$ is a *balanced swap* of $J$ if it is a swap of $J$ and that $|J \cap [j]| = |J' \cap [j]|$ for all $p \in P(J,J')$.

This condition is equivalent to that $|J \cap [j,j']| = |J' \cap [j,j']|$ for all $j, j' \in P(J,J') \cup \{0, m+n+1\}$. This means, for two consecutive pivots $p, p'$ in $P(J,J') \cup \{0, m+n-1\}$, the interval $[p+1, p'-1]$ is balanced in the sense that there are equally numbers of elements from $J$ and elements from $J'$. For instance, if $m = 6$ and $n = 3$, $J = \{1,2,8\}$ is a balanced swap of $J' = \{3,4,7\}$ with the pivot set $P(J,J') = \{5,6,9\}$.

Define

$$\mathrm{bswap}(J) := \left\{ J' \in \binom{[m+n]}{n} : J' \text{ is a balanced swap of } J \right\}.$$

If $m = n+1$, we say that $J'$ is a *end swap* of $J$ if it is a swap of $J$ and their pivot $j^*$ is either 1 or $2n+1$. Observe that $J'$ being an end swap of $J$ implies it is a balanced swap of $J$. Define

$$\mathrm{eswap}(J) := \left\{ J' \in \binom{[2n+1]}{n} : J' \text{ is an end swap of } J \right\}.$$

Our goal in this section is to prove the following theorem and corollaries:

**Theorem 2.2.4.** *Suppose $G$ is $n$-bottlenecked. Then there is a weight-preserving involution $\tau\colon \mathrm{PNCPath}(G) \to \mathrm{PNCPath}(G)$ with*

$$\tau(\mathrm{PNCPath}(I,J)) \subseteq \bigcup_{J' \in \mathrm{bswap}(J)} \mathrm{PNCPath}(I,J'),$$

*for all $(I,J) \in \mathrm{Pat}(G)$.*

*Suppose further that $m = n+1$ and $G$ is interlacing. Then for all $(I,J) \in \mathrm{Pat}(G)$ we have*

$$\tau(\mathrm{PNCPath}(I,J)) \subseteq \bigcup_{J' \in \mathrm{eswap}(J)} \mathrm{PNCPath}(I,J').$$

**Corollary 2.2.5.** *Suppose $m = n+1$ and $G$ is $n$-bottlenecked. Fix a source pattern $I \in \binom{[2n+1]}{n}$. Fix some $K \subseteq [2n+1]_{\mathrm{even}}$ and set $K' := [2n+1]_{\mathrm{even}} \setminus K$. Then*

$$\tau\left( \bigcup_{\substack{(I,J)\in \mathrm{Pat}(G)\\ J_{\mathrm{even}}=K}} \mathrm{PNCPath}(I,J) \right) = \bigcup_{\substack{(I,J')\in \mathrm{Pat}(G)\\ J'_{\mathrm{even}}=K'}} \mathrm{PNCPath}(I,J'),$$

*and thus*

$$\sum_{\substack{(I,J)\in \mathrm{Pat}(G)\\ J_{\mathrm{even}}=K}} \mathrm{wt}(I,J) = \sum_{\substack{(I,J')\in \mathrm{Pat}(G)\\ J'_{\mathrm{even}}=K'}} \mathrm{wt}(I,J').$$

**Corollary 2.2.6.** *Suppose $G$ is interlacing. Fix a source pattern $I \in \binom{[2n+1]}{n}$. Suppose that the sink pattern $J \in \binom{[2n+1]}{n}$ is such that $\{1, 2n+1\} \cap J = \emptyset$. Define $J' := [2, 2n+1] \setminus J$ and $J'' := [1, 2n] \setminus J$. Then*

$$\tau(\mathrm{PNCPath}(I,J)) = \mathrm{PNCPath}(I,J') \cup \mathrm{PNCPath}(I,J'')$$

*and thus*

$$\mathrm{wt}(I,J) = \mathrm{wt}(I,J') + \mathrm{wt}(I,J'').$$

**Remark 2.2.7.** If $J \in \binom{[2n+1]}{n}$ with $\{1, 2n+1\} \subseteq J$, then $\mathrm{eswap}(J) = \emptyset$. So if $G$ is interlacing, Theorem 2.2.4 implies $\mathrm{PNCPath}(I,J) = \emptyset$ for any $I \in \binom{[2n+1]}{n}$.

But we can see without Theorem 2.2.4 as well. Let $N_T \subseteq V$ be the set of vertices guaranteed by the $n$-sink-interlacing property of $G$. Suppose that $i'_1 < \cdots < i'_{n+1}$ are all the elements of $[2n+1] \setminus I$ and $j'_1 < \cdots < j'_{n+1}$ are those of $[2n+1] \setminus J$. Then let $B = (b_1, \ldots, b_{n+1}) \in \mathrm{PNCPath}(\mathcal{Y}, \mathcal{Z})$ be a tuple of noncrossing paths, where $\mathcal{Y} := (s_{i'_l})_{l=1}^{n+1}$ and $\mathcal{Z} := (t_{j'_l})_{l=1}^{n+1}$. Note that $[2n+1] \setminus J \subseteq [2, 2n]$ and thus by the sink-interlacing property of $G$, for each $l \in [n+1]$ we must have that $v \in \mathrm{Vert}(b_l)$ for some $v \in N_T$. But since $|N_T| \leq n$, by the pigeonhole principle we get $1 \leq k \neq l \leq n+1$ and $v \in N_T$ such that $v \in \mathrm{Vert}(b_k) \cap \mathrm{Vert}(b_l)$. This contradicts that $B$ was noncrossing, so there can be no such $B$. Therefore $\mathrm{PNCPath}(I, J) = \emptyset$.

Before we can describe the bijection $\tau$ we need a technical result about posets. First we recall some poset terminology. Let $(P, \leq)$ be a finite poset. For $x, y \in P$, we write $x < y$ to denote $x \leq y$ and $x \neq y$, as is standard. At some point we will require the notion of a downset; for a subset $P' \subseteq P$ the *downset* of $P'$ is the set of all $x \in P$ with $x \leq y$ for some $y \in P'$. Recall that a *chain* in $P$ is a subset $C \subseteq P$ such that any two elements of $C$ are related, and an *antichain* in $P$ is a subset $A \subseteq P$ such that no two elements of $A$ are related. If $|A| = k$, then we say $A$ is a *$k$-antichain.* Suppose that $m$ is the maximal size of an antichain in $P$. In this case, we can find a partition $\{C_1, \ldots, C_m\}$ of $P$ into chains: that is, each $C_i$ is a chain and we have $\cup_{i=1}^m C_i = P$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. (This result is known as Dilworth's theorem; see for example Freese [12], who proves not only that the poset of maximal size antichains has a minimum, as we show below, but also that this poset is in fact a lattice.) Let $\mathcal{A}_m(P)$ denote the set of $m$-antichains of $P$. Note that for any $A \in \mathcal{A}_m(P)$, we must have that $|A \cap C_i| = 1$ for all $i \in [m]$. Thus we can define the following partial order on $\mathcal{A}_m(P)$: for two $m$-antichains $X = \{x_i\}_{i=1}^m$ and $Y = \{y_i\}_{i=1}^m$ such that $X \cap C_i = \{x_i\}$ and $Y \cap C_i = \{y_i\}$ for all $i \in [m]$, we say that $X \leq Y$ if and only if $x_i \leq y_i$ for all $i \in [m]$.

**Proposition 2.2.8.** *The poset $\mathcal{A}_m(P)$ has a minimum.*

*Proof.* Given any $X, Y \in \mathcal{A}_m(P)$ which are incomparable, we claim that there is $Z \in \mathcal{A}_m(P)$ so that $Z \leq X$ and $Z \leq Y$. Define $z_i := \min(x_i, y_i)$ for all $i \in [m]$ and set $Z := \{z_1, \ldots, z_m\}$. Note that $Z$ is still an antichain: if $z_i < z_j$, then $\min(x_i, y_i) < \min(x_j, y_j)$, which means $\min(x_i, y_i) < x_j$ and $\min(x_i, y_i) < y_j$, which forces a relation in $X$ or in $Y$. Because $\mathcal{A}_m(P)$ is evidently finite, and by definition nonempty, it has a minimum. □

The order we defined on $\mathcal{A}_m(P)$ above in principle depended on the choice of chains $\{C_1, \ldots, C_m\}$; but in fact we can give a description of this order which does not depend on such a choice. Namely, for $X, Y \in \mathcal{A}_m(P)$, let us say $X \leq' Y$ if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$. Then $X \leq' Y$ if and only if $X \leq Y$. The implication $X \leq Y \Rightarrow X \leq' Y$ is trivial. To see $X \leq' Y \Rightarrow X \leq Y$, write $X = \{x_i\}_{i=1}^m$ and $Y = \{y_i\}_{i=1}^m$ with $X \cap C_i = \{x_i\}$ and $Y \cap C_i = \{y_i\}$ for all $i \in [m]$. Then for $i \in [m]$, we have that there is some $y_j$ such that $x_i \leq y_j$. If $i = j$, then we are okay. So suppose $i \neq j$. It cannot be that $y_i \leq x_i$ as then $y_i \leq y_j$ and $Y$ would fail to be an antichain. But $C_i$ is a chain, so this means $x_i < y_i$. Therefore we have $X \leq Y$.

We now define the poset of intersections of a tuple of paths in $G$, which will be key in defining the bijection $\tau$ of Theorem 2.2.4. Let $\Pi = (\pi_1, \ldots, \pi_m)$ be a tuple of paths in $G$. Then define $\mathrm{Int}^{\Pi} := \cup_{i \neq j} \mathrm{Vert}(\pi_i) \cap \mathrm{Vert}(\pi_j)$ to be the set of all intersections between paths in $\Pi$. First of all, we give $\mathrm{Int}^{\Pi}$ a labeling function $\ell^{\Pi} \colon \mathrm{Int}^{\Pi} \to \mathcal{P}(\{\pi_1, \ldots, \pi_m\})$, whereby $\ell^{\Pi}(u) := \{\pi_i \colon u \in \mathrm{Vert}(\pi_i)\}$. Secondly, we give $\mathrm{Int}^{\Pi}$ a partial order $\leq$ as follows. For a path $\pi_i = \{v_j\}_{j=0}^{n}$, if $v_i, v_j \in \mathrm{Int}^{\Pi}$ for $0 \leq i \leq j \leq n$ we declare $v_j \preceq v_i$. We then define $\leq$ to be the transitive closure of $\preceq$. It is routine to verify that $\leq$ indeed defines a partial order on $\mathrm{Int}^{\Pi}$ (but note that here we use the acyclicity of $G$ in an essential way).

We need just a little more terminology related to paths in order to define $\tau$. For a tuple $\Pi = (\pi_1, \ldots, \pi_n)$ of paths, let us say a vertex $v \in V$ is a *2-crossing* of $\Pi$ if $|\{i \colon v \in \mathrm{Vert}(\pi_i)\}| = 2$. Let $v$ be a 2-crossing of $\Pi$ and suppose that $v \in \pi_i \cap \pi_j$ for $i \neq j$. Say $\pi_i = \{u_1, \ldots, u_a, v, u_{a+1}, \ldots, u_b\}$ and $\pi_j = \{w_1, \ldots, w_c, v, w_{c+1}, \ldots, w_d\}$. Then define the *flip of* $\Pi$ *at* $v$ to be $\mathrm{flip}_v(\Pi) := (\pi'_1, \ldots, \pi'_n)$ where

$$\pi'_k := \begin{cases} \{u_1, \ldots, u_a, v, w_{c+1}, \ldots, w_d\} & \text{if } k = i; \\ \{w_1, \ldots, w_c, v, u_{a+1}, \ldots, u_b\} & \text{if } k = j; \\ \pi_k, & \text{otherwise.} \end{cases}$$

For any two 2-crossings $u, v$ of $\Pi$, we have $\mathrm{flip}_u(\mathrm{flip}_v(\Pi)) = \mathrm{flip}_v(\mathrm{flip}_u(\Pi))$. Thus for a set $U = \{u_1, \ldots, u_p\} \subseteq V$ of 2-crossings of $\Pi$, let us define the *flip of* $\Pi$ *at* $U$ to be $\mathrm{flip}_U(\Pi) = \mathrm{flip}_{u_1}(\mathrm{flip}_{u_2}(\cdots \mathrm{flip}_{u_p}(\Pi) \cdots))$, where the composition may be taken in any order. Finally, for two tuples of paths $\Pi = (\pi_1, \ldots, \pi_m)$ and $\Sigma = (\sigma_1, \ldots, \sigma_n)$, set $\Pi + \Theta := (\pi_1, \ldots, \pi_m, \sigma_1, \ldots, \sigma_n)$.

We proceed to define the involution $\tau$. So we assume from now on that $G$ is $m$-bottlenecked. Let $N \subseteq V$ be the subset guaranteed by the $m$-bottlenecked property of $G$. If $|N| < m$, then $\mathrm{PNCPath}(G) = \emptyset$; so we may assume $|N| = m$. Let $(I, J) \in \mathrm{Pat}(G)$ and let $(R, B) \in \mathrm{PNCPath}(I, J)$. Here we use $R$ for "red" and $B$ for "blue" as the example below will make clear. Say $R = (r_1, \ldots, r_n)$ and $B = (b_1, \ldots, b_m)$ and set $\Pi := R + B$. Because $N$ is non-returning there is a subset of $N$ of size $n$ consisting of 2-crossings of $\Pi$ which in fact is a $n$-antichain of $\mathrm{Int}^{\Pi}$. It is also clear that there is no antichain of size greater than $n$: indeed, given an antichain $U$ of $\mathrm{Int}^{\Pi}$ and any two elements $u, v \in U$, for each $r \in R$ we have that $r \in \ell^{\Pi}(u) \Rightarrow r \notin \ell^{\Pi}(v)$; but on the other hand, for any $u \in U$, there must be some $r \in R$ with $r \in \ell^{\Pi}(u)$. So by Proposition 2.2.8, we conclude that $\mathcal{A}_n(\mathrm{Int}^{\Pi})$ has a minimum. Starting with this minimum antichain, we define $\tau(R, B)$ by the following algorithm.

**Algorithm defining $\tau$**

Let $U \subseteq \mathrm{Int}^{\Pi}$ be the minimum of $\mathcal{A}_n(\mathrm{Int}^{\Pi})$.
Let FLIP $:= U$.
Let $A = \{\alpha \in [m] : b_\alpha \notin \ell^{\Pi}(u) \text{ for all } u \in U\}$.
FOR EACH $\alpha \in A$: DO

Initialize the counter $c$ to 0 and $n_c$ to $\alpha$.
IF there is no $w$ in the downset of $U$ with $b_\alpha \in \ell^\Pi(w)$:
Continue to DO for the next $\alpha \in A$.
Let $w_{c+1}$ be maximal in the downset of $U$ with $b_\alpha \in \ell^\Pi(w_{c+1})$.
Increment the counter $c$ by 1.
LOOP:
Let $r_{m_c}$ be the unique $r_i$ with $r_{m_c} \in \ell^\Pi(w_c)$.
Let $v_c$ be minimal in $\mathrm{Int}^\Pi$ with $w_c < v_c$ and $r_{m_c} \in \ell^\Pi(v_c)$.
Let FLIP := FLIP $\Delta\{v_c, w_c\}$ (with $\Delta =$ "symmetric difference").
Let $b_{n_c}$ be the unique $b_i$ with $b_{n_c} \in \ell^\Pi(v_c)$.
If there is $w$ with $b_{n_c} \in \ell^\Pi(w)$ and $w < v_c$:
Let $w_{c+1}$ be maximal in $\mathrm{Int}^\Pi$ with $w_{c+1} < v_c$ and $b_{n_c} \in \ell^\Pi(w_{c+1})$.
Increment the counter $c$ by 1.
Return to loop.
Else:
Exit the loop.
OUTPUT:
Define $\tau(R, B) := (R', B')$ where $R' + B' := \mathrm{flip}_{\mathrm{FLIP}}(\Pi)$.

**Example 2.2.9.** Before we prove the correctness of this algorithm, we give an example run of it. Let our network be $\Gamma^3_{9,9}$ and consider the pair of tuples of non-crossing paths $(R, B) \in \mathrm{PNCPath}(\{2,4,6\},\{2,4,6\})$ depicted in Figure 2-4. Suppose $R = (r_1, r_2, r_3)$ and $B = (b_1, b_2, b_3, b_4)$ so the paths are labeled in left-to-right order in the figure. To apply $\tau$ to $(R, B)$, first we find the minimum 3-antichain $U$ in $\mathrm{Int}^\Pi$ where $\Pi := R + B$. This vertices in this antichain are circled by small olive-colored circles in Figure 2-4, and the poset $\mathrm{Int}^\Pi$ is depicted to the right in the figure. In this case it turns out that $U = \{(7,3),(2,3),(1,4)\}$. We initialize $\mathrm{FLIP}_0 := \{(7,3),(2,3),(1,4)\}$ and find that $n_0 = 2$. There is some $w$ in the downset of $U$ with $\theta_2 \in \ell^\Pi(w)$, so we set $w_1 := (5,5)$ and enter the loop.

1. We find $m_1 = 2$ and $v_1 = (3,4)$, and we set

$$\mathrm{FLIP}_1 := \{(7,3),(2,3),(1,4),(5,5),(3,4)\}.$$

We find $n_1 = 3$ and there is $w$ with $\theta_3 \in \ell^\Pi(w)$ and $w < v_1$, so we set $w_2 := (5,6)$ and enter the loop again.

2. We find $m_2 = 2$ and $v_2 = (5,5)$, and we set

$$\mathrm{FLIP}_2 := \{(7,3),(2,3),(1,4),(3,4),(5,6)\}.$$

We find $n_2 = 2$ and there is $w$ with $\theta_2 \in \ell^\Pi(w)$ and $w < v_2$, so we set $w_3 := (7,5)$ and enter the loop again.

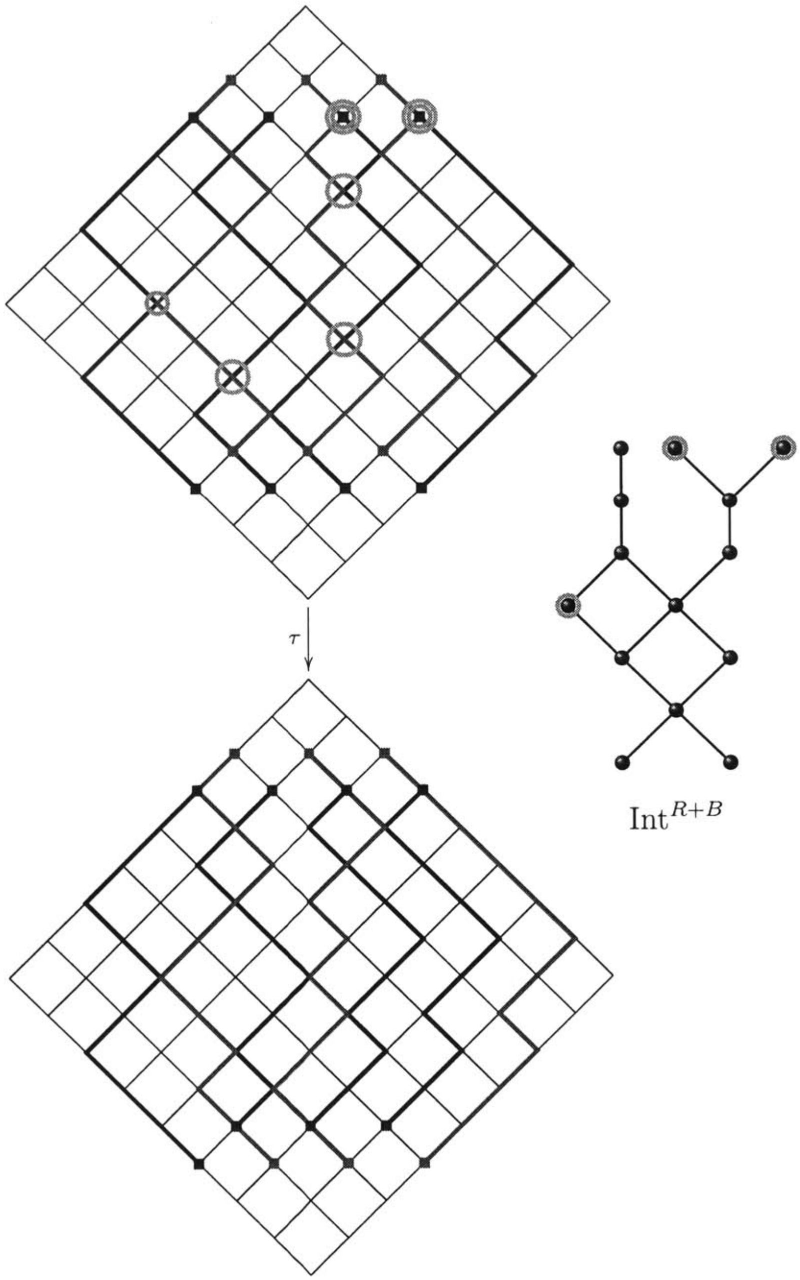


Figure 2-4: Example 2.2.9: the interlacing network is $\Gamma^3_{9,9}$; above we depict $(R, B) \in \mathrm{PNCPath}(\{2,4,6\},\{2,4,6\})$ and below we depict $\tau(R, B) \in \mathrm{PNCPath}(\{2,4,6\},\{3,5,7\})$. The intersection poset $\mathrm{Int}^{R+B}$ is depicted to the right.

3. We find $m_3 = 1$ and $v_3 = (7, 3)$, and we set

$$\mathrm{FLIP}_3 := \{(2, 3), (1, 4), (3, 4), (5, 6), (7, 5)\}.$$

We find $n_3 = 1$ and there is no $w$ with $\theta_1 \in \ell^{\Pi}(w)$ and $w < v_3$, so we exit the loop.

Finally, we define $\tau(R, B) := (R', B')$ where $R' + B' := \mathrm{flip}_{\mathrm{FLIP}_3}(\Pi)$. The elements of $\mathrm{FLIP}_3$ are circled by large light green circles in Figure 2-4 and $\tau(R, B)$ is shown below $(R, B)$. Note that $\tau(R, B) \in \mathrm{PNCPath}(\{2, 4, 6\}, \{3, 5, 7\})$ and this is consistent with Theorem 2.2.4 because $\{3, 5, 7\} \in \mathrm{eswap}(\{2, 4, 6\})$.

We proceed to verify the correctness of the algorithm defining $\tau$. In the following series of claims we refer to the variables defined above in the description of the algorithm. First, we note that the algorithm is independent of the choice of the ordering of $A$ since we loop for each $\alpha \in A$ independently. Let $k = m - n = |A|$. For the sake of the argument, we fix the order of elements of $A$ by $A = \{\alpha_1 < \alpha_2 < \ldots < \alpha_k\}$. For $i \in [k]$, let $c_i$ be the the value of the counter $c$ at the end of the $i^{\text{th}}$ loop. We let $v_{i,j}$, $w_{i,j}$, and $r_{m_{i,j}}$ to be the $v_j$, $w_j$, and $r_{m_j}$, respectively, in the loop corresponding to the element $\alpha_i \in A$. Also, say $R' = (r'_1, \ldots, r'_n)$ and $B' = (b'_1, \ldots, b'_m)$ and set $\Pi' := R' + B'$. Some of the analysis that follows is tedious but it is all necessary.

**Claim 2.2.10.** *For $i \in [k]$ and $j \in [c_i]$, there exists a unique $u_{i,j}$ with $w_{i,j} < u_{i,j}$ and $r_{m_{i,j}} \in \ell^{\Pi}(u_{i,j})$ such that $u_{i,j} \in U$.*

*Proof.* This claim is required for the algorithm to make sense because it shows that $v_{i,j}$ is always well-defined. This claim also shows that $v_{i,j}$ always belongs to the downset of $U$. Observe that the uniqueness is trivial because for any $r \in R$ there is a unique $u \in U$ with $r \in \ell^{\Pi}(u)$. So existence is what is an issue. Fix $i \in [k]$, we prove this claim by induction on $j$. For $j = 1$ it is clear because $w_{i,1} \le u_{i,1}$ by definition, where $u_{i,1}$ is the unique element of $U$ with $r_{m_{i,1}} \in \ell^{\Pi}(u_{i,1})$, and $w_{i,1} \ne u_{i,1}$ because $b_{n_{i,0}} \in \ell^{\Pi}(w_{i,1})$ but $b_{n_{i,0}} \notin \ell^{\Pi}(u_{i,1})$. So suppose $j > 1$ and the claim holds for smaller $j$. Then assume $w_{i,j} \ge u_{i,j}$ where $u_{i,j}$ is the unique element of $U$ with $r_{m_{i,j}} \in \ell^{\Pi}(u_{i,j})$. Note that $w_{i,j} < v_{i,j-1}$ and $v_{i,j-1} \le u_{i,j-1}$ by our inductive assumption. Thus we conclude $u_{i,j} < u_{i,j-1}$. But this contradicts the fact that $U$ is an antichain. So in fact $w_{i,j} < u_{i,j}$. The claim follows by induction. $\square$

**Claim 2.2.11.** *The algorithm terminates.*

*Proof.* We claim it is impossible that $w_{i,j} = w_{i,j'}$ for $i \in [k]$ and $j < j'$. If $j > 1$, then $w_{i,j} = w_{i,j'}$ implies $w_{i,j-1} = w_{i,j'-1}$. So suppose $j = 1$. Then $w_{i,1} = w_{i,j'}$ for some $j' > 1$ implies that $v_{i,j'-1} > w_{i,1}$ with $b_{n_{i,0}} \in \ell^{\Pi}(v_{i,j'-1})$. But $v_{i,j'-1}$ is in the downset of $U$ and $w_{i,1}$ was chosen to be maximal in the downset of $U$ such that $b_{n_{i,0}} \in \ell^{\Pi}(w_{i,1})$, which is a contradiction. So indeed $w_{i,j} \ne w_{i,j'}$ for all $j \ne j'$. Therefore, the algorithm terminates for each $i \in [k]$ since $\mathrm{Int}^{\Pi}$ is finite. $\square$

**Lemma 2.2.12.** *Let $(R, B) \in \mathrm{PNCPath}(I, J)$ be a pair of tuples of noncrossing paths. For any $p, q \in [m + n]$, and a (directed) path $P$ connecting $s_p$ and $t_q$ such that every* *step of $P$ is a step in $(R, B)$. Then the parity of $|\mathit{Vert}(P) \cap \mathrm{FLIP}|$ only depends on $q$. Furthermore, $|\mathit{Vert}(P) \cap \mathrm{FLIP}|$ is even if $t_q$ is the end point of $b_{n_{i,c_i}}$ for some $i \in [k]$ and is odd otherwise*

*Proof.* We think of FLIP as a multiset with the line $\mathrm{FLIP} := \mathrm{FLIP}\Delta\{v_i, w_i\}$ in the algorithm replaced by $\mathrm{FLIP} := \mathrm{FLIP} \sqcup \{v_i, w_i\}$. This operation does not change the parity of $|\mathrm{Vert}(P) \cap \mathrm{FLIP}|$. Let $V = \{v_{i,j} : i \in [k], j \in [0, c_i]\}$ and $W = \{w_{i,j} : i \in [k], j \in [0, c_i]\}$. So $\mathrm{FLIP} = U \sqcup V \sqcup W$.

Let $P$ be a path connecting a source $s_p$ and a sink $t_q$, considered as a path from the sink $t_q$ to the source $s_p$. We consider the sequence of points in FLIP that $P$ visits (from sink to source). If $P$ visits a vertex $v$ which is both $v_{i,j}$ and $w_{i',j'}$ (resp. $u \in U$) at the same time, we say that $P$ visits $v_{i,j}$ first then $w_{i',j'}$ (resp. $u$). Consider the following observations.

1. The last point in FLIP that $P$ visits must be either $u \in U$ or $w_{i,1}$ for some $i$.
2. If $P$ visits any point in $U$, then $P$ cannot visit other points in FLIP.
3. If $P$ visits $w_{i,j}$, then this is either that this is the last point in FLIP that $P$ visits (only if $j = 0$) or the next point $P$ visits must be $v_{i,j-1}$ or $v_{i,j}$.
4. Unless $j = c_i$, if $P$ visits $v_{i,j}$, then the previous point $P$ visits must be $w_{i,j}$ or $w_{i,j+1}$.

So once $P$ visits a point $v_1 \in V$, the rest of the sequence (including this $v_1$) must be of the form $(v_1, w_1, v_2, w_2, \ldots, v_l, w_l)$, where $v_i \in V$ $(i \in [l])$, $w_i \in W$ $(i \in [l-1])$, and $w_l \in W \cup U$. If $t_q$ is the end point of $b_{n_{i,c_i}}$, then the first point $P$ visits is $v_{i,c_i}$. Thus $|\mathrm{Vert}(P) \cap \mathrm{FLIP}|$ is even in this case. If $t_q$ is not the end point of any $b_{n_{i,c_i}}$, then the first point $P$ visits cannot be a point in $V$. If the first point comes from $U$, this is also the last point. Thus $|\mathrm{Vert}(P) \cap \mathrm{FLIP}|$ is odd in this case. If the first point comes from $W$, this is either the last point or the next point must be from $V$. In either case, $|\mathrm{Vert}(P) \cap \mathrm{FLIP}|$ is odd. $\square$

**Claim 2.2.13.** *The tuples of paths $R'$ and $B'$ are noncrossing.*

*Proof.* Suppose otherwise that $(R', B')$ is not noncrossing, and a bad crossing occurs at a vertex $v$. Without losing of generality, assume that two blue lines cross at $v$, say $\ell^{\Pi'}(v) = \{b_i, b_j\}$ for some $i, j \in [m]$. Let $v'$ (resp. $v''$) be the minimal vertex such that $v' > v$ and $b_i \in \ell^{\Pi'}(v')$ (resp. $v'' > v$ and $b_j \in \ell^{\Pi'}(v'')$). Since $\Pi$ is noncrossing, one of the paths $(v', v)$ and $(v'', v)$ is red in $\Pi$, and the other is blue in $\Pi$. WLOG, assume that $(v', v)$ is red in $\Pi$ and $(v'', v)$ is blue in $\Pi$. This means $(v', v)$ has been flipped odd number of times in $\mathrm{flip}_{\mathrm{FLIP}}$. So there is a path $P'$ from a source $s_{i'}$ to $v'$ such that $|\mathrm{Vert}(P') \cap \mathrm{FLIP}|$ is odd. (In fact, this is true for any path from any source $s_{i'}$ to $v'$.) Similarly, there is a path $P''$ from a source $s_{i''}$ to $v'$ such that $|\mathrm{Vert}(P'') \cap \mathrm{FLIP}|$ is even. However, by extending these paths $P'$ and $P''$ by $(v', v)$ and $(v'', v)$, respectively, and then extend both paths by any path from $v$ to a sink $t_q$, we have found two paths $\widetilde{P}'$ and $\widetilde{P}''$ to the same end point $t_q$ for which the intersection with FLIP have different parity. This contradicts the previous lemma that the parity of their intersections with FLIP is independent of the path. Therefore, the pair $(R', B')$ is noncrossing. $\square$

**Claim 2.2.14.** *The map $\tau$ is an involution.*

*Proof.* Suppose we run the algorithm again on $(R', B')$. Let us use primes to denote the variables for this run of the algorithm; so we have $A'$, $\mathrm{FLIP}'_{i,j}$, $n'_{i,j}$, $w'_{i,j}$, $v'_{i,j}$, $c'_i$ and so on. Observe that $A = A'$ since the intersection poset $\mathrm{Int}^{\Pi}$ is unchanged. So $|A'| = |A| = k$. To prove $\tau(R', B') = (R, B)$ it suffices to show $\mathrm{FLIP}'_{k,c'_k} = \mathrm{FLIP}_{k,c_k}$. We claim that in fact $c_i = c'_i$ for all $i \in [k]$ and $\mathrm{FLIP}'_{i,j} = \mathrm{FLIP}_{i,j}$ for all $j \in [0, c_i]$. Note that $\mathrm{FLIP}'_{0,0} = \mathrm{FLIP}_{0,0}$ because both of these equal the minimum $n$-antichain of $\mathrm{Int}^{\Pi}$ and we gave a characterization earlier of this antichain just in terms of $\mathrm{Int}^{\Pi}$ as an abstract poset, independent of how it is labeled. Fix an $i \in [k]$. It is clear that $n'_{i,0} = n_{i,0}$. If $c_i = 0$, then no vertex on $b_{n_{i,0}}$ ever flips, so no vertex in this path belongs to the downset of $U$ and thus we get $c'_i = 0$ as well. If $c_i > 0$ we get $w_{i,1} = w'_{i,1}$ as these are both equal to the first place where $b_{n_0}$ intersects the downset of $U$. But then if $w_{i,j} = w'_{i,j}$, we get $v'_{i,j} = v_{i,j}$. This is because each element of $\mathrm{Int}^{\Pi'}$ has two paths in $\Pi'$ coming into it, and since $R'$ and $B'$ are noncrossing these paths must be colored differently (where by the color of the path we mean in the sense of Figure 2-4). We followed one of these paths in to arrive at $w'_{i,j}$ and thus we must follow the other out to arrive at $v'_{i,j}$. And if $v'_{i,j} = v_{i,j}$ and $j < c_i$, then similarly we have $w'_{i,j+1} = w_{i,j+1}$. If $j = c_i$ then both algorithms terminate on this step and so $c_i = c'_i$. The result follows by induction. □

**Claim 2.2.15.** *There exists $J' \in \binom{[m+n]}{n}$ which is a swap of $J$ so that for all paths $\pi' \in \Pi'$ there is some $j' \in J'$ with $t_{j'}$ an end point of $\pi'$.*

*Proof.* This is an immediate consequence of 2.2.12. □

Let $D$ denote the disc into which $G$ is embedded, and let $\partial D$ denote its boundary. Let $b \in B$ with start point $s$ and end point $t$. Denote by $\mathrm{rt}(b)$ (respectively, $\mathrm{lt}(b)$) the compact subset of the plane whose boundary is the closed curve obtained by adjoining $b$ with the arc on $\partial D$ that connects $s$ to $t$ clockwise (respectively, counter-clockwise). It is easy to see $\mathrm{rt}(b), \mathrm{lt}(b) \subseteq D$ and $\mathrm{rt}(b) \cap \mathrm{lt}(b) = b$. Also, we have $b_j \in \mathrm{rt}(b_i)$ if and only if $j \geq i$, and similarly $b_j \in \mathrm{lt}(b_i)$ if and only if $j \leq i$. For $r \in R$ we define $\mathrm{rt}(r)$ and $\mathrm{lt}(r)$ analogously, and have the similar result that $r_j \in \mathrm{rt}(r_i)$ if and only if $j \geq i$, and $r_j \in \mathrm{lt}(r_i)$ if and only if $j \leq i$.

**Lemma 2.2.16.** *For $x_1, x_2 \in \mathrm{Int}^{\Pi}$ which are not related, if $\ell^{\Pi}(x_1) = \{r_{i_1}, b_{j_1}\}$ and $\ell^{\Pi}(x_2) = \{r_{i_2}, b_{j_2}\}$ then $i_1 \leq i_2$ if and only if $j_1 \leq j_2$.*

*Proof.* We may assume the inequalities of the indices are strict because otherwise $x_1$ and $x_2$ would certainly be related. So let $x_1, x_2 \in \mathrm{Int}^{\Pi}$ be such that $\ell^{\Pi}(x_1) = \{r_{i_1}, b_{j_1}\}$ and $\ell^{\Pi}(x_2) = \{r_{i_2}, b_{j_2}\}$ where $i_1 < i_2$ but $j_1 > j_2$. Let $s_{p_1}$ be the start point of $r_{i_1}$ and $t_{q_1}$ its end point, and let $s_{p_2}$ be the start point of $b_{j_1}$ and $t_{q_2}$ its end point. Assume by symmetry that $p_1 \leq p_2$, so $q_1 \leq q_2$. Let $Y_1$ (respectively, $Y_2$) denote the compact subset of the plane that is bounded by the closed curve obtained by adjoining the subpath of $r_{i_1}$ connecting $s_{p_1}$ to $x_1$ (resp., the subpath of the reverse of $r_{i_1}$ connecting $t_{q_1}$ to $x_1$), the subpath of the reverse of $b_{j_1}$ connecting $x_1$ to $s_{p_2}$ (resp, the subpath of $b_{j_1}$ connecting $x_1$ to $t_{q_2}$), and the arc on $\partial D$ connecting $s_{p_2}$ to $s_{p_1}$

counter-clockwise (resp., the arc on $\partial D$ connecting $t_{q_2}$ to $t_{q_1}$ clockwise). Because $x_2$ lies in $\mathrm{rt}(r_{i_1}) \cap \mathrm{lt}(b_{j_1})$, it must lie in one of $Y_1$ or $Y_2$. Assume by symmetry that it lies in $Y_1$. We claim that the subpath of $r_{i_2}$ below $x_2$ cannot lie inside $Y_1$: if it did, its end point would lie clockwise between $s_{p_1}$ and $s_{p_2}$ on $\partial D$, contradicting our assumption about how sources and sinks of $G$ are arranged on this boundary. So it must exit $Y_1$. When it does so, it crosses $b_{j_1}$ above $x_1$. Thus $x_2 > x_1$. $\square$

**Lemma 2.2.17.** *For $r \in R$, if $x_1 < \cdots < x_l$ are the elements of $\mathrm{Int}^\Pi \cap \mathrm{Vert}(r)$ and $b_{p_i} \in \ell^\Pi(x_i)$ for all $i \in [l]$, then $|p_i - p_{i-1}| \le 1$ for all $i > 1$.*

*Proof.* This is an immediate consequence of the facts that $G$ is planar and $B$ is noncrossing. $\square$

**Lemma 2.2.18.** *For $\alpha \in A$, let $L_\alpha$ be the (undirected) path $(v_{\alpha,0}, w_{\alpha,1}, v_{\alpha,1}, \dots, v_{\alpha,c_\alpha}, w_{\alpha,c_\alpha})$. Then for $\alpha < \beta \in A$, the line $L_\alpha$ lies to the left of the line $L_\beta$. That is the intersection of the interior of $\mathrm{lt}(L_\alpha)$ and the interior of $\mathrm{rt}(L_\beta)$ is empty. In particular, if $\alpha < \beta \in A$, then $n_{\alpha,c_\alpha} < n_{\beta,c_\beta}$.*

*Proof.* Since we only flip at the points in the downset of $U$, we may assume that $R$ and $B$ intersect only at $U$ or below $U$. Let $\alpha < \beta \in A$, but suppose that $L_\beta$ crosses into $\mathrm{lt}(L_\alpha)$. Let $u \in \mathrm{Vert}(L_\beta)$ be the first vertex inside the interior of $\mathrm{lt}(L_\alpha)$. First suppose that $u = v_{\beta,j}$, then $w_{\beta,j}$ must be a vertex in $\mathrm{Vert}(L_\alpha)$ since otherwise there is another vertex in between $v_{\beta,j}$ and $w_{\beta,j}$, which contradicts the way we choose $v_{i,j}$ and $w_{i,j}$. If $w_{\beta,j} = w_{\alpha,j'}$ for any $j'$, then by tracing back the points on $L_\alpha$ and $L_\beta$ we arrive at the same starting points of $L_\alpha$ and $L_\beta$. In particular, the segments $(v_{\alpha,0}, w_{\alpha,1})$ and $(v_{\beta,0}, w_{\beta,1})$ are identical, violating the noncrossing assumption. If $w_{\beta,j} = v_{\alpha,j'}$ and that $w_{\beta,j} \neq v_{\alpha,j''}$ for any $j''$, then we see that $v_{\beta,j-1}$ is also in the interior of $\mathrm{lt}(L_\alpha)$, contradicting our choice of $u$. Now suppose that $u = w_{\beta,j}$ for some $j$. Then $v_{\beta,j-1} \in \mathrm{Vert}(L_\alpha)$. If $v_{\beta,j-1} = v_{\alpha,j'}$ for some $j'$, then $w_{\beta,j} = w_{\alpha,j'+1}$ which is not in the interior of $\mathrm{lt}(L_\alpha)$. So $v_{\beta,j-1} = w_{\alpha,j'}$ for some $j'$. In this case $w_{\beta,j-1}$ is also in the interior of $\mathrm{lt}(L_\alpha)$, which contradicts our choice of $u$. From these cases, we see that the point $u$ does not exists, i.e. $L_\beta$ does not cross $L_\alpha$. $\square$

**Claim 2.2.19.** *The sink pattern $J'$ is a balanced swap of $J$.*

*Proof.* Claim 2.2.15 tells us that $J'$ and $J$ are swaps of one another and their pivot set $\{p_1^*, \dots, p_k^*\}$ is such that $t_{p_i^*}$ is the endpoint of $b_{n_{i,c_i}}$. Define $w_{i,c_i+1}$ to be $t_{p_i^*}$. Define $v_{i,0}$ to be first element that comes strictly before $w_{i,1}$ in $b_{n_{i,0}}$ and belongs to $\mathrm{Int}^\Pi$, or to be the start point of $b_{n_{i,0}}$ if there is no such element. Then for $i \in [k], j \in [0, c_i]$, let $\widetilde{b}_{i,j}$ be the subpath of $b_{n_{i,j}}$ connecting $v_{i,j}$ to $w_{i,j+1}$. Also, for $i \in [n+1]$ define closed subsets $X_i$ of $D$ by $X_1 := \mathrm{lt}(r_1)$, $X_i := \mathrm{rt}(r_{i-1}) \cap \mathrm{lt}(r_i)$ if $i \in [2, n]$, and $X_{n+1} := \mathrm{rt}(r_n)$. Again, we assume that $R$ and $B$ intersect only at $U$ or below $U$.

Our key subclaim is that for $i \in [k]$ and $j \in [0, c_i]$, the curve $\widetilde{b}_{i,j}$ lies in $X_{n_{i,j}-(i-1)}$. Fix $i \in [k]$. We prove this by induction on $j$. First of all, for any $i$ it is clear that each $\widetilde{b}_{i,j}$ must lie in one of the $X_l$ because if it did not it would have to intersect too many paths in $R$ (only the start point and end point of $\widetilde{b}_i$ can belong to $\mathrm{Int}^\Pi$). So each $\widetilde{b}_{i,j}$ intersects the interior of at most one of the $X_l$. The case $j = 0$ is clear since no intersections are allowed in $v_{i,0}$ and $w_{i,1}$. Note that there are $i - 1$ blue sources inside $\mathrm{lt}(b_{\alpha_i})$.

Now assume $j > 0$ and the key subclaim holds for smaller values of $j$. We know that $\widetilde{b}_{i,j-1}$ lies in $X_{n_{i-1}-(i-1)}$, so either $r_{n_{i,j-1}-i}$ or $r_{n_{i,j-1}-(i-1)}$ is in $\ell^{\Pi}(w_i)$; let us assume by symmetry that it is $r_{n_{i,j-1}-(i-1)}$. Lemma 2.2.17 gives $n_{i,j} = n_{i,j-1} + \delta$ for $\delta \in \{-1, 0, 1\}$. First suppose that $\delta = 0$. Then we claim that $\widetilde{b}_{i,j}$ cannot enter the interior of $X_{n_{i,j}-i+2}$; in particular, the subpath of $b_{n_{i,j}}$ connecting $v_{i,j}$ to $w_{i,j}$ cannot enter the interior of $X_{n_{i,j}-i+2}$. Suppose that it did. This subpath must eventually enter the interior of $X_{n_{i,j}-(i-1)}$ by our inductive supposition and so it would have to cross $r_{n_{i,j}-(i-1)}$ at some point to do so. However, if it crossed $r_{n_{i,j}-(i-1)}$ above $v_{i,j}$ this would cause a cycle in $G$, and if it crossed below $w_{i,j}$ this would also cause a cycle. So it would have to cross between $v_{i,j}$ and $w_{i,j}$; but this is also impossible because there are no elements of $\mathrm{Int}^{\Pi}$ that lie on $r_{n_{i,j}-(i-1)}$ between $v_{i,j}$ and $w_{i,j}$. So indeed $\widetilde{b}_{i,j}$ lies in $X_{n_{i,j}-(i-1)}$.

Now suppose that $\delta \neq 0$. Let $s_{l_1}$ be the start point of $r_{n_{i,j-1}-(i-1)}$ and $s_{l_2}$ be the start point of $b_{n_{i,j-1}}$. Assume $l_1 \leq l_2$ for simplicity of the following exposition; the other case is symmetric. Let $Y$ denote the compact subset of the plane that is bounded by the closed curve obtained by adjoining the subpath of $b_{n_{i,j-1}}$ connecting $s_{l_1}$ to $w_{i,j}$, the subpath of the reverse of $r_{n_{i,j-1}-(i-1)}$ connecting $w_{i,j}$ to $s_{l_2}$, and the arc on $\partial D$ connecting $s_{l_2}$ to $s_{l_1}$ counter-clockwise. We claim that the subpath of $b_{n_{i,j}}$ below $v_{i,j}$ cannot enter the interior of $Y$. Suppose it did. Then it could not exit $Y$ because it cannot intersect $b_{n_{i,j-1}}$ at all, and it cannot intersect $r_{n_{i,j-1}-(i-1)}$ above $w_{i,j}$ without creating a cycle. Thus the end point of $b_{n_{i,j}}$ lies on $\partial D$ clockwise between $s_{l_1}$ and $s_{l_2}$. But because $l_1 < l_2$, this contradicts our assumption of how the sources and sinks of $G$ are arranged on this boundary. (Note $l_1 = l_2$ is impossible in this case because that would force the end point of $b_{n_{i,j}}$ to be $s_{l_1}$ as well, creating a cycle.) So $\widetilde{b}_{i,j}$ does not enter the interior of $Y$. Thus $\widetilde{b}_{i,j}$ lies in $X_{n_{i,j-1}-i+2}$.

To finish the proof of the key subclaim, we need to show that $\delta \neq -1$. First consider the case $j = 1$. Let $u$ be the unique element of $U$ such that $r_{n_{i,0}-(i-1)} \in \ell^{\Pi}(u)$. Note that $b_{n_{i,0}+1} \in \ell^{\Pi}(u)$ by Lemma 2.2.16. Also note that $w_{i,1}$ is the maximal element below $u$ with $\ell^{\Pi}(w_{i,1}) = \{r_{n_{i,0}-(i-1)}, b_{n_{i,0}}\}$. So by Lemma 2.2.17, as we look at the $b_p$ intersecting the vertices $r_{n_{i,0}-(i-1)}$ we encounter below $u$ but above $w_{i,1}$ we can never see $b_{n_{i,0}-1}$. Thus $\delta = 1$ as claimed. Now consider the case $j > 1$. Then either $r_{n_{i,j-1}-i}$ or $r_{n_{i,j-1}-(i-1)}$ is in $\ell^{\Pi}(v_{i,j-1})$. Suppose first that $r_{n_{i,j-1}-i} \in \ell^{\Pi}(v_{i,j-1})$. Then $v_{i,j-1}$ and $v_{i,j}$ are unrelated and so by Lemma 2.2.16 we get that $\delta \neq -1$. Suppose next that $r_{n_{i,j-1}-(i-1)} \in \ell^{\Pi}(v_{i,j-1})$. Then note that $w_{i,j}$ is the maximal element below $v_{i,j-1}$ with $\ell^{\Pi}(w_{i,j}) = \{r_{n_{i,j-1}-(i-1)}, b_{n_{i,j-1}}\}$. Also, the element of $\mathrm{Int}^{\Pi}$ below $v_{i,j-1}$ on $r_{n_{i,j-1}-(i-1)}$ is $w_{i,j-1}$ and has $b_{n_{i,j-2}} \in \ell^{\Pi}(w_{i,j-1})$, so again by Lemma 2.2.17 we get $b_{n_{i,j-2}} \in \ell^{\Pi}(v_{i,j})$. So $n_{i,j} = n_{i,j-2}$ and by induction we obtain $\delta = 1$ again. The key subclaim is thus proved by induction.

To conclude, note that $\widetilde{b}_{i,c_i}$ is in $X_{n_{i,c_i}-(i-1)}$ which means $t_{p_i^*}$ is clockwise between the end point of $r_{n_{i,c_i}-i}$ and $r_{n_{i,c_i}-(i-1)}$ on $\partial D$. This means there are $n_{i,c_i} - i$ red end points inside $\mathrm{lt}(b_{n_{i,c_i}})$. On the other hand, by 2.2.18, there are $n_{i,c_i} - 1 - (i - 1) =$

$n_{i,c_i} - i$ blue end points inside $\mathrm{lt}(b_{n_{i,c_i}})$ which are not fixed by $\mathrm{flip}_{\mathrm{FLIP}}$ Together with Claim 2.2.15, this means exactly that $|J \cap [p_i^*]| = |J' \cap [p_i^*]|$. Thus $J'$ is a balanced swap of $J$. $\square$

**Claim 2.2.20.** *If $G$ is interlacing then $J'$ is an end swap of $J$.*

*Proof.* Let $\Pi_T$ be the subtuple of $\Pi$ consisting of paths whose end points are among $t_j$ for $j \in [2, 2n]$. Let $N_T$ be the subset of $V$ guaranteed by the $n$-sink-bottlenecked property of $G$. There is a subset of $N_T$ of size $n-1$ consisting of 2-crossings of $\Pi_T$. This subset is a $(n-1)$ antichain of $\mathrm{Int}^{\Pi_T}$ because $N_T$ is non-returning. There are no antichains of $\mathrm{Int}^{\Pi_T}$ of greater cardinality. So $\mathcal{A}_{n-1}(\mathrm{Int}^{\Pi_T})$ has a minimum; call that minimum $U_T$. We claim that $U_T$ belongs to the downset of $U$. To see this, let $U^T \subseteq U$ be the set of those $u \in U$ for which $\ell^{\Pi}(u) \subseteq \Pi_T$. Note that because $N_T$ is sink-branching, it also must be that $U_T$ is sink-branching. So no element of $U_T$ is greater than an element of $U \setminus U^T$; but also, every element of $U_T$ is comparable to some element of $U$. Thus if we let $U_{\min}$ be the set of minimal elements of $U \cup U_T$, there is a subset of $U_{\min}$ that belongs to $\mathcal{A}_{n-1}(\mathrm{Int}^{\Pi_T})$ and is in the downset of $U$. But $U_T$ is minimal among all such antichains; so indeed $U_T$ must be in the downset of $U$.

Let $j^*$ be the pivot of $J$ and $J'$. We want to show that $j^* \notin [2, 2n]$. Suppose to the contrary. Recall the paths $\widetilde{b}_i$ and regions $X_i$ defined in the proof of Claim 2.2.19. Let $\pi^*$ be the unique element of $\Pi_T$ not among the labels of elements of $U_T$. If $j^* \in [2, 2n]$, it must be that there is $i$ such that $\widetilde{b}_i = \pi^*$ and either $w_{i+1}$ is in he downset of $U_T$ or $w_{i+1} = t_{j*}$. This is because if $\widetilde{b}_i$ passes through some $u \in U_T$, that $u$ must either be $v_i$ or $w_{i+1}$. If that $u$ is a $w_{i+1}$, then $v_{i+1}$ will not belong to the downset of $U_T$ so we will have to pass through $U_T$ again at some later step. On the other hand, if that $u$ is a $v_i$, then we must have $i > 0$ and $w_{i-1}$ already belongs to the downset of $U_T$ and is strictly below an element of $U_T$. Thus indeed there exists $i$ such that $\widetilde{b}_i = \pi^*$ and with $w_{i+1}$ as described above. But then by the same logic as the second paragraph of the proof of Claim 2.2.19, we conclude that either $\widetilde{b}_i$ lies in $X_{n_i+1}$ (if $1 \in J$) or $\widetilde{b}_i$ lies in $X_{n_i-1}$ (if $1 \notin J$). At any rate, we get that $\widetilde{b}_i$ does not lie in $X_{n_i}$ which is a contradiction with the key subclaim in the proof of Claim 2.2.19. So indeed $j^* \notin [2, 2n]$. $\square$

Having established the above facts about the behavior of the algorithm defining $\tau$, the proofs of Theorem 2.2.4 and its corollaries are easy.

*Proof of Theorem 2.2.4.* Claims 2.2.10 and 2.2.11 establish that $\tau$ is well-defined, and Claim 2.2.13 shows that $\tau$ maps into $\mathrm{PNCPath}(G)$. The map $\tau$ is weight-preserving because the multisets of edges visited by paths in $\Pi$ and in $\Pi'$ are identical. Claim 2.2.14 shows $\tau$ is an involution. Claim 2.2.19 gives us an estimate of the image $\tau(\mathrm{PNCPath}(I, J))$, and Claim 2.2.20 gives a more refined estimate on $\tau(\mathrm{PNCPath}(I, J))$ when $G$ is interlacing. $\square$

*Proof of Corollary 2.2.5.* For any $J, J' \in \binom{[2n+1]}{n}$ with $J'$ a balanced swap of $J$, their pivot $j^*$ cannot be even, so we have $J'_{\mathrm{even}} = [2n+1]_{\mathrm{even}} \setminus J_{\mathrm{even}}$. Thus with $K$ as in the statement of the corollary, by Theorem 2.2.4 we have

$$\tau\left(\bigcup_{\substack{(I,J)\in \mathrm{Pat}(G)\\ J_{\mathrm{even}}=K}} \mathrm{PNCPath}(I,J)\right) \subseteq \bigcup_{\substack{(I,J')\in \mathrm{Pat}(G)\\ J'_{\mathrm{even}}=K'}} \mathrm{PNCPath}(I,J').$$

But the reverse inclusion follows for the same reason. ☐

*Proof of Corollary 2.2.6.* For $J, J', J'' \in \binom{[2n+1]}{n}$ as in the statement of the corollary, we have $\mathrm{eswap}(J) \subseteq J' \cup J''$, but we also have $\mathrm{eswap}(J') \subseteq J$ and $\mathrm{eswap}(J'') \subseteq J$. Then Theorem 2.2.4 tells us that

$$\tau(\mathrm{PNCPath}(I,J)) \subseteq \mathrm{PNCPath}(I,J') \cup \mathrm{PNCPath}(I,J'')$$

and also the reverse inclusion. ☐

**Remark 2.2.21.** The definition of $k$-bottlenecked is symmetric with respect to sources and sinks, so we can easily obtain from $\tau$ a source-swapping involution as well. We define $(G^{\mathrm{op}}, S^{\mathrm{op}}, T^{\mathrm{op}})$, the opposite network of $G$, as follows: $G^{\mathrm{op}}$ is the same graph as $G$ but with edge directions reversed, $S^{\mathrm{op}} := (t_{2k-1}, \ldots, t_1)$, and $T^{\mathrm{op}} := (s_{2k-1}, \ldots, s_1)$. For $I \subseteq [2k-1]$ let us set $I^\circ := \{2k-i \colon i \in I\}$. There is a weight-preserving bijection

$$\Psi\colon \mathrm{PNCPath}(G) \to \mathrm{PNCPath}(G^{\mathrm{op}})$$

with $\Psi(\mathrm{PNCPath}(I,J)) = \mathrm{PNCPath}(I^\circ, J^\circ)$ for $(I,J) \in \mathrm{Pat}(G)$ whereby $\Psi$ just reverses all paths. Suppose $G$ is $k$-bottlenecked. Then so is $G^{\mathrm{op}}$. So we may define the source-swapping involution $\sigma\colon \mathrm{PNCPath}(G) \to \mathrm{PNCPath}(G)$ by $\sigma := \Psi^{-1} \circ \tau_{G^{\mathrm{op}}} \circ \Psi$ and it will satisfy

$$\sigma(\mathrm{PNCPath}(I,J)) \subseteq \bigcup_{I'\in \mathrm{bswap}(I)} \mathrm{PNCPath}(I',J),$$

for all $(I,J) \in \mathrm{Pat}(G)$. Here $\tau_{G^{\mathrm{op}}}$ denotes the involution $\tau$ defined above in this section but applied to the opposite network. The involution $\sigma$ leads to a source-swapping analogue of Corollary 2.2.5. However, $G$ being interlacing does not in general imply that $G^{\mathrm{op}}$ is interlacing, so we do not in general get a source-swapping analogue of Corollary 2.2.6.

### 2.2.3 Proof of octahedron recurrence

Instead of proving Theorem 2.1.1, we will give a proof of the birational version of this octahedron recurrence. Roughly speaking, birationalization is a process in turning a piecewise linear function to a subtraction-free expression by the process of replacing $a+b \mapsto a\cdot b$, $a-b \mapsto a/b$, and $\max\{a,b\} \mapsto a+b$. For example, $\max\{x+y, y-z\}$ becomes $xy + y/z$. The reverse of this process is called tropicalization.

Let us define this process more formally. We follow the notations used in [27]. Let $x = \{x_i\}_{i\in I}$ be a set of formal variables. In our purpose, we assume that $x$ is finite.

We express a subtraction-free rational function $f$ as a quotient $f(x) = a(x)/b(x)$, where $a(x) = \sum_{\alpha \in A} a_\alpha x^\alpha$ and $b(x) = \sum_{\beta \in B} b_\beta x^\beta$ are polynomials with coefficients in $\mathbb{R}_{>0}$. We define the tropicalization of $f$ to be the piecewise linear function $M(f)$ on $x$ given by

$$M(f) := \max\{\langle \alpha, x\rangle : \alpha \in A\} - \max\{\langle \beta, x\rangle : \beta \in B\}.$$

Note that $M(f)$ is independent of the choice of the expression $f = a/b$. Also it is easy to see that, under this map, we have

$$M(fg) = M(f) + M(g), \qquad M(\frac{f}{g}) = M(f) - M(g),$$

$$M(f+g) = \max\{M(f) + M(g)\},$$

for all subtraction-free rational functions $f$ and $g$.

With this tool, we can restate Theorem 2.1.1 in the birational setting, which implies Theorem 2.1.1 by tropicalization.

Let $X = (x_{i,j})_{1 \le i,j \le n}$ be a positive real matrix. Define the rectangular product $\mathrm{rect}(i,j) = \mathrm{rect}_X(i,j)$ at $(i,j)$ by

$$\mathrm{rect}(i,j) = \mathrm{rect}_X(i,j) := \prod_{1 \le k \le i, 1 \le l \le j} x_{ij}.$$

Now we define a three-dimensional array $\overline{Y} = (\overline{y})_{i,j,k}$ with $i,j,k \in \mathbb{N}$, $k \le \min(i,j)$ by

$$\overline{y}_{i,j,k} := \sum_{\Pi \in \mathrm{RSKPath}(i,j,k)} \mathrm{wt}(\Pi),$$

where $\mathrm{RSKPath}(i,j,k) := \mathrm{NCPath}_{\Gamma_{m,n}}(S,T)$ with sources and sinks: $S = \{(1,1), ..., (1,k)\}$ and $T = \{(i, j-k+1), ..., (i,j)\}$. Then we define a normalized array $\widetilde{Y} = (\widetilde{y})_{i,j,k}$ with the same set of indices of $Y$ by $\widetilde{y}_{i,j,k} = \overline{y}_{i,j,k}/\mathrm{rect}(i,j)$.

**Theorem 2.2.22.** *The three-dimensional array $\widetilde{Y} = (\widetilde{y}_{i,j,k})$ can be computed as follows: the boundary conditions are $\widetilde{y}_{i,j,0} = 1/\mathrm{rect}(i,j)$ and $\widetilde{y}_{i,j,\min(i,j)} = 1$, and for $1 \le k \le \min(i,j) - 1$ we have the recursive formula*

$$\widetilde{y}_{i,j,k}\widetilde{y}_{i-1,j-1,k-1} = \widetilde{y}_{i-1,j,k}\widetilde{y}_{i,j-1,k-1} + \widetilde{y}_{i-1,j,k-1}\widetilde{y}_{i,j-1,k}$$

*In other words, $\widetilde{Y}$ satisfies the (bounded) octahedron recurrence.*

*Proof.* Of course $\overline{y}_{i,j,0} = 1$ and $\overline{y}_{i,j,\min(i,j)} = \mathrm{rect}(i,j)$ are equivalent boundary conditions. We have $\overline{y}_{i,j,0} = 1$ by definition. We have $\overline{y}_{i,j,\min(i,j)} = \mathrm{rect}(i,j)$ because there is a single tuple of paths in $\mathrm{RSKPath}(i,j,,\min(i,j))$ and it covers exactly those vertices in $\Gamma_{m,n}$ that are less than or equal to $(i,j)$.

Now let $1 \le k \le \min(i,j) - 1$. The key to proving the recursive condition is to show that

$$\overline{y}_{ijk}\overline{y}_{i-1,j-1,k-1} = (\overline{y}_{i-1,j,k}\overline{y}_{i,j-1,k-1} + \overline{y}_{i-1,j,k-1}\overline{y}_{i,j-1,k})x_{ij}. \qquad (*)$$

For $k = 1$, we have $\overline{y}_{i-1,j-1,k-1} = \overline{y}_{i,j-1,k-1} = \overline{y}_{i-1,j,k-1} = 1$ and (*) follows from the fact that every path connecting $(1,1)$ to $(i,j)$ goes through exactly one of $(i-1,j)$ or $(i,j-1)$, and conversely any path to either $(i-1,j)$ or $(i,j-1)$ can be uniquely extended to a path to $(i,j)$. Assume $k \geq 2$. For $(i,j) \in \Gamma_{m,n}$, define the *increasing* and *decreasing triangular products* of length $l$ at $(i,j)$ as

$$\operatorname{tri}^+(i,j,l) := \prod_{r=i}^{i+l-1} \prod_{s=j}^{j+i+l-r-1} x_{rs} \text{ and } \operatorname{tri}^-(i,j,l) := \prod_{r=i-l+1}^{i} \prod_{s=j+i-l-r+1}^{j} x_{rs}.$$

The first equation makes sense for $1 \leq l \leq \min(m-i+1, n-j+1)$, and the second equation makes sense for $1 \leq l \leq \min(i,j)$. Consider the vertex-weighted network (see 2.2.1) $\Gamma = \Gamma_{i,j}^{k-1}$ with weight function $\omega : (i,j) \mapsto x_{ij}$ . Set $I, J := \{2, 4, \ldots, 2k-2\}$ and $\kappa := \operatorname{tri}^+(1,1,k-2) \cdot \operatorname{tri}^+(1,1,k-1)$. Then there is a bijection

$$\varphi\colon \operatorname{PNCPath}_\Gamma(I,J) \to \operatorname{RSKPath}(i-1,j-1,k-1) \times \operatorname{RSKPath}(i,j,k)$$

such that

$$\operatorname{wt}(R,B) \cdot \kappa \cdot \operatorname{tri}^-(i-1,j-1,k-2) \cdot \operatorname{tri}^-(i,j,k-1) = \operatorname{wt}(\varphi(R,B)).$$

Specifically, if $(R,B) = ((r_1,\ldots,r_{k-1}),(b_1,\ldots,b_k)) \in \operatorname{PNCPath}_\Gamma(I,J)$ then we define $\varphi(R,B) := ((\pi_1,\ldots,\pi_{k-1}),(\sigma_1,\ldots,\sigma_k))$ where

$$\begin{aligned} \pi_s &:= \{(t,s)\}_{t=1}^{k-1-s} \cdot r_s \cdot \{(i-s+t, j-k+s)\}_{t=1}^{s-1} \\ \sigma_s &:= \{(t,s)\}_{t=1}^{k-s} \cdot b_s \cdot \{(i-s+t, j-k+s)\}_{t=2}^{s}. \end{aligned}$$

(Here $\cdot$ denotes concatenation of sequences.) In other words, $\varphi$ extends the paths vertically to connect to the appropriate start and end points for paths in $\operatorname{RSKPath}(i-1,j-1,k-1)$ and $\operatorname{RSKPath}(i,j,k)$; there is a unique way to do this. Similarly, if we set $J' := \{1,3,\ldots,2k-3\}$ then there is a bijection

$$\varphi'\colon \operatorname{PNCPath}_\Gamma(I,J') \to \operatorname{RSKPath}(i,j-1,k-1) \times \operatorname{RSKPath}(i-1,j,k)$$

such that

$$\frac{\operatorname{wt}(R',B')}{x_{i,j-k+1}} \cdot \kappa \cdot \operatorname{tri}^-(i,j-1,k-2) \cdot \operatorname{tri}^-(i-1,j,k-1) = \operatorname{wt}(\varphi'(R',B')).$$

Here for $(R',B') = ((r'_1,\ldots,r'_{k-1}),(b'_1,\ldots,b'_k)) \in \operatorname{PNCPath}_{\Gamma_{i,j}^k}(I,J')$ we define $\varphi'(R',B') := ((\pi'_1,\ldots,\pi'_{k-1}),(\sigma'_1,\ldots,\sigma'_k))$ where

$$\begin{aligned} \pi'_s &:= \{(t,s)\}_{t=1}^{k-1-s} \cdot r'_s \cdot \{(i-s+t, j-k+s)\}_{t=2}^{s} \\ \sigma'_s &:= \{(t,s)\}_{t=1}^{k-s} \cdot b'_s \cdot \{(i-s+t, j-k+s)\}_{t=1}^{s-1}. \end{aligned}$$

Again, $\varphi'$ just extends paths vertically. And if we set $J'' := \{3,5,\ldots,2k-1\}$ then there is a bijection

$$\varphi'' \colon \mathrm{PNCPath}_\Gamma(I, J'') \to \mathrm{RSKPath}(i-1, j, k-1) \times \mathrm{RSKPath}(i, j-1, k)$$

such that

$$\frac{\mathrm{wt}(R'', B'')}{x_{i-k+1,j}} \cdot \kappa \cdot \mathrm{tri}^-(i-1, j, k-2) \cdot \mathrm{tri}^-(i, j-1, k-1) = \mathrm{wt}(\varphi''(R'', B'')).$$

Here for $(R'', B'') = ((r''_1, \ldots, r''_{k-1}), (b''_1, \ldots, b''_k)) \in \mathrm{PNCPath}_\Gamma(I, J'')$ such that $b''_1 = \{v_t\}_{t=0}^{l}$ we define $\varphi''(R'', B'') := ((\pi''_1, \ldots, \pi''_{k-1}), (\sigma''_1, \ldots, \sigma''_k))$ where

$$\pi_s := \{(t, s)\}_{t=1}^{k-1-s} \cdot r_s \cdot \{(i-s+t, j-k+s+1)\}_{t=1}^{s-1}$$
$$\sigma_s := \begin{cases} \{(t, 1)\}_{t=1}^{k-1} \cdot \{v_t\}_{t=0}^{l-1} & \text{if } s = 1 \\ \{(t, s)\}_{t=1}^{k-s} \cdot b_s \cdot \{(i-s+t, j-k+s-1)\}_{t=2}^{s} & \text{otherwise.} \end{cases}$$

Now $\varphi''$ has to slide the end point of $b_1$ to the left, but the other paths it again just extends vertically. Corollary 2.2.6 tells us that

$$\sum_{(R,B) \in \mathrm{PNCPath}(I,J)} \mathrm{wt}(R, B) = \sum_{(R',B') \in \mathrm{PNCPath}(I,J')} \mathrm{wt}(R', B') + \sum_{(R'',B'') \in \mathrm{PNCPath}(I,J'')} \mathrm{wt}(R'', B'')$$

and together with

$$\begin{aligned} x_{ij} &= \frac{\mathrm{tri}^-(i-1, j-1, k-2) \cdot \mathrm{tri}^-(i, j, k-1) \cdot x_{i,j-k+1}}{\mathrm{tri}^-(i-1, j, k-2) \cdot \mathrm{tri}^-(i, j-1, k-1)} \\ &= \frac{\mathrm{tri}^-(i-1, j-1, k-2) \cdot \mathrm{tri}^-(i, j, k-1) \cdot x_{i-k+1,j}}{\mathrm{tri}^-(i, j-1, k-2) \cdot \mathrm{tri}^-(i-1, j, k-1)} \end{aligned}$$

we conclude that indeed equation (*) holds. To finish, we compute

$$\begin{aligned} \widetilde{y}_{ijk} = \frac{\overline{y}_{ijk}}{\mathrm{rect}(i,j)} &= \frac{x_{ij}(\overline{y}_{i-1,j,k}\overline{y}_{i,j-1,k-1} + \overline{y}_{i,j-1,k}\overline{y}_{i-1,j,k-1})}{\mathrm{rect}(i,j) \cdot \overline{y}_{i-1,j-1,k-1}} \\ &= \frac{\mathrm{rect}(i-1, j-1) \cdot (\overline{y}_{i-1,j,k}\overline{y}_{i,j-1,k-1} + \overline{y}_{i,j-1,k}\overline{y}_{i-1,j,k-1})}{\mathrm{rect}(i-1, j) \cdot \mathrm{rect}(i, j-1) \cdot \overline{y}_{i-1,j-1,k-1}} \\ &= \frac{\widetilde{y}_{i-1,j,k}\widetilde{y}_{i,j-1,k-1} + \widetilde{y}_{i,j-1,k}\widetilde{y}_{i-1,j,k-1}}{\widetilde{y}_{i-1,j-1,k-1}}. \end{aligned}$$

Thus, $\widetilde{Y}$ satisfies the octahedron recurrence [31] [20]. □

## 2.3 Schur functions

Given a SSYT $T$ of a fixed shape $\lambda$, we associate it with a monomial $x^T$ defined by the product $\prod_{i \geq 0} x_i^{c_T(i)}$, where $c_T(i)$ is the numbers of $i$'s in $T$. Then the *Schur function*

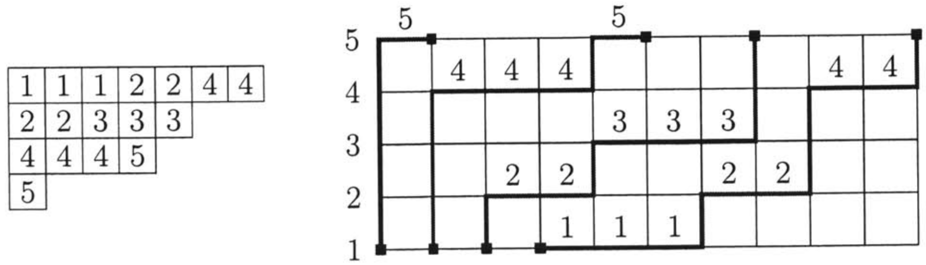

Figure 2-5: A semistandard Young tableau of shape $\lambda = (7, 5, 4, 1)$ and its corresponding paths.

$s_\lambda$ is defined by

$$s_\lambda(x_1, x_2, \ldots) := \sum_{T \in \mathrm{SSYT}(\lambda)} x^T.$$

Furthermore, we define $s_\lambda^X$, where $X \subset \mathbb{Z}_{>0}$ by $s_\lambda(x_1, x_2, \ldots)$ with specialization $x_i = 0$ for $i \notin X$.

The set $\{s_\lambda\}$ forms a linear basis of the ring of symmetric functions $\Lambda$ over $\mathbb{Z}$. So the product $s_\lambda s_\mu$ can be written as $s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$. We call the $c_{\lambda\mu}^\nu$ the *Littlewood-Richardson coefficients* (or *LR-coefficients*). It follows form representation theory of symmetric groups that the LR-coefficients $c_{\lambda\mu}^\nu$ are always nonnegative.

We now recall an equivalent definition of Schur functions in terms of nonintersecting paths. Already Gessel and Viennot [17] were aware of the connection between tableaux and nonintersecting lattice paths in $\mathbb{Z}^2$. Let us make $\mathbb{Z}^2$ into a graph with horizontal edges $((i,j),(i-1,j))$ and vertical edges $((i,j),(i,j-1))$. We now use Cartesian coordinates for $\mathbb{Z}^2$ so $(-\infty,-\infty)$ will be in the bottom-left corner. Although $\mathbb{Z}^2$ is infinite, this is no problem for us as we will only ever use a finite portion of it. We set the edge-weight function $\omega$ of $\mathbb{Z}^2$ to be $\omega((i,j),(i-1,j)) := x_j$ for horizontal edges and $\omega((i,j),(i,j-1)) := 1$ for vertical edges. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition. For $n \geq 1$ let

$$\mathrm{SPath}(\lambda, n) := \mathrm{NCPath}_{\mathbb{Z}^2}(\{(\lambda_{k+1-i} + i, n)\}_{i=1}^k, \{(i,1)\}_{i=1}^k).$$

Then $s_\lambda^{[n]} = \sum_{\Pi \in \mathrm{SPath}(\lambda,n)} \mathrm{wt}(\Pi)$, which follows from a simple bijection between $n$-tableaux of shape $\lambda$ and paths in $\mathrm{SPath}(\lambda, n)$ (see [33, Theorem 7.16.1]). In fact, we obtain the following by translation:

**Proposition 2.3.1.** *For $a, b, c \in \mathbb{Z}$ with $1 \leq a \leq b$ let*

$$\mathrm{SPath}^c(\lambda, a, b) := \mathrm{NCPath}_{\mathbb{Z}^2}(\{(\lambda_{k+1-i} + i + c, b)\}_{i=1}^k, \{(i + c, a)\}_{i=1}^k).$$

*Then $s_\lambda^{[a,b]} = \sum_{\Pi \in \mathrm{SPath}^c(\lambda,a,b)} \mathrm{wt}(\Pi)$.*

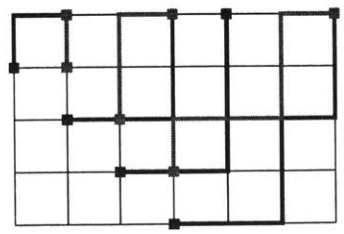

Figure 2-6: For $\lambda = (3,2,2,1)$, $t = 1$, and $n = 5$: the network $G$ and an element of $\mathrm{PNCPath}_G(\{2,4,6\},\{2,4,6\})$.

### 2.3.1 Schur function identities

First we show two examples of Schur function identities obtained from the theory of interlacing networks (or rather $m$-bottlenecked networks). In the first example, we think of the network that flows from top to bottom while in the second example, the network flows from bottom to top. That is, in the second example we consider the opposite graph $G^{\mathrm{op}}$ of what we introduced earlier.

**Theorem 2.3.2.** *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $t \in [0, k-1]$. Then*

$$\begin{aligned} s_\lambda s^{[2,\infty)}_{(\lambda_1,\ldots,\lambda_t,\lambda_{t+2}-1,\ldots,\lambda_k-1)} =& s^{[2,\infty)}_\lambda s_{(\lambda_1,\ldots,\lambda_t,\lambda_{t+2}-1,\ldots,\lambda_k-1)} \\ &+ x_1 s_{(\lambda_1-1,\ldots,\lambda_k-1)} s^{[2,\infty)}_{(\lambda_1+1,\ldots,\lambda_t+1,\lambda_{t+2},\ldots,\lambda_k)}. \end{aligned}$$

*Proof.* In order to prove this identity we use an interlacing network $G$. Fix some $n \geq k$. For $i \in [k]$ define $v_i := (\lambda_{k+1-i} + i, n) \in \mathbb{Z}^2$. Define $G$ to be the network whose underlying graph is the subgraph of $\mathbb{Z}^2$ with vertices in the rectangle between $(1,1)$ and $(\lambda_1 + k, n)$ and with sources

$$S = (s_1, \ldots, s_{2k-1}) := (v_1, v_1, v_2, v_2, \ldots, v_{k-t}, \overline{v_{k-t}}, \ldots, v_k, v_k)$$

(where the overline denotes omission) and sinks

$$T = (t_1, \ldots, t_{2k-1}) := ((1,k), (2,k), (2,k-1), (3,k-1), \ldots, (k,2), (k,1)).$$

To witness that $G$ is interlacing we may take $N = \{s_1, s_3, \ldots, s_{2k-1}\}$ as a $k$-bottleneck and $N_T = \{t_2, t_4, \ldots, t_{2k-2}\}$ as a $(k-1)$-sink bottleneck. Figure 2-6 illustrates $G$ together with an element of $\mathrm{PNCPath}(G)$ for some specific parameters $\lambda$, $t$ and $n$.

To simplify notation, set

$$\begin{aligned} \mu &:= (\lambda_1, \ldots, \lambda_t, \lambda_{t+2} - 1, \ldots, \lambda_k - 1) \\ \nu &:= (\lambda_1 + 1, \ldots, \lambda_t + 1, \lambda_{t+2}, \ldots, \lambda_k) \\ \rho &:= (\lambda_1 - 1, \ldots, \lambda_k - 1). \end{aligned}$$

Let $I, J := \{2, 4, \ldots, 2k-2\}$ and $J' := [1, 2k-2] \setminus J$ and $J'' := [2, 2k-1] \setminus J$. Then there are bijections

$$\begin{aligned}
\varphi\colon \mathrm{PNCPath}_G(I,J) &\to \mathrm{SPath}^1(\mu,2,n)\times \mathrm{SPath}^0(\lambda,1,n)\\
\varphi'\colon \mathrm{PNCPath}_G(I,J') &\to \mathrm{SPath}^0(\nu,2,n)\times \mathrm{SPath}^1(\rho,1,n)\\
\varphi''\colon \mathrm{PNCPath}_G(I,J'') &\to \mathrm{SPath}^0(\mu,1,n)\times \mathrm{SPath}^0(\lambda,2,n)
\end{aligned}$$

such that

$$\begin{aligned}
\mathrm{wt}(R,B) &= \mathrm{wt}(\varphi(R,B))\\
\mathrm{wt}(R',B') &= x_1\cdot \mathrm{wt}(\varphi'(R',B'))\\
\mathrm{wt}(R'',B'') &= \mathrm{wt}(\varphi''(R'',B''))
\end{aligned}$$

for all appropriate $(R,B),(R',B'),(R'',B'')\in \mathrm{PNCPath}(G)$. These bijections have a very similar description to those in the proof of Theorem 2.2.22: the maps $\varphi$ and $\varphi''$ merely extend the paths vertically to reach the necessary start and end points; $\varphi'$ also just extends paths vertically, except for $b'_{2k-1}$ (the rightmost blue path) which it moves to the right, thus accounting for the factor of $x_1$. Corollary 2.2.6 tells us that

$$\sum_{(R,B)\in\mathrm{PNCPath}(I,J)} \mathrm{wt}(R,B) = \sum_{(R',B')\in\mathrm{PNCPath}(I,J')} \mathrm{wt}(R',B') + \sum_{(R'',B'')\in\mathrm{PNCPath}(I,J'')} \mathrm{wt}(R'',B'')$$

and together with Proposition 2.3.1 we conclude $s_\mu^{[2,n]}s_\lambda^{[n]} = x_1 s_\nu^{[2,n]}s_\rho^{[n]} + s_\mu^{[n]}s_\lambda^{[2,n]}$. Taking the limit $n\to\infty$ gives us the result. $\square$

Because the involution $\tau$ makes sense not just for interlacing networks, but also more generally for $m$-bottlenecked networks, it can actually be applied in a different way to obtain another result about Schur positivity. In fact, $\tau$ leads another (multi-term) Schur function identity. The following identity appeared earlier in [19] (Proposition 3.1 and Corollary 3.2). It is also a consequence of Lemma 16 in [16]. Our proof is independent of the above and uses the properties of our involution $\tau$.

**Theorem 2.3.3.** *Let $\lambda=(\lambda_1,\ldots,\lambda_k)$ and $\mu=(\mu_1,\ldots,\mu_{k-1})$ be partitions that interlace in the sense that $\lambda_i\ge\mu_i\ge\lambda_{i+1}$ for $i\in[k-1]$. For $1\le i\le k$, define $\lambda^i=(\lambda^i_1,\ldots,\lambda^i_k)$ and $\mu^i=(\mu^i_1,\ldots,\mu^i_{k-1})$ to be*

$$\lambda^i_j := \begin{cases}\mu_j-1 & \text{if } j<i\\ \lambda_i & \text{if } j=i\\ \mu_{j-1} & \text{if } j>i\end{cases} \quad\text{and}\quad \mu^i_j := \begin{cases}\lambda_j+1 & \text{if } j<i\\ \lambda_{j+1} & \text{if } j\ge i.\end{cases}$$

*Then we have $s_\lambda s_\mu = \sum_{i=1}^k s_{\lambda^i}s_{\mu^i}$ where $s_\nu$ is taken to be $0$ if $\nu$ is not a partition.*

*Proof.* In order to prove this identity we use a $k$-bottlenecked network $G$. Fix $n\ge k$. For $i\in[k]$ define $v_i := (\lambda_{k+1-i}+i,n)\in\mathbb{Z}^2$ and for $i\in[k-1]$ define $u_i := (\mu_{k-i}+i,n)$. Define $G$ to be the network whose underlying graph is the subgraph of $\mathbb{Z}^2$ with vertices in the rectangle between $(1,1)$ and $(\lambda_1+k,n)$ and with sources and sinks

$$S = (s_1, \ldots, s_{2k-1}) := (v_1, u_1, v_2, u_2, \ldots, v_{k-1}, u_{k-1}, v_k)$$
$$T = (t_1, \ldots, t_{2k-1}) := ((1,1),(1,1),(2,1),(2,1),\ldots,(k-1,1),(k-1,1),(k-1)).$$

To witness that $G$ is $k$-bottlenecked we may take $N = \{t_1, t_3, \ldots, t_{2k-1}\}$.

Let $I, J := \{2, 4, \ldots, 2k-2\}$ and define $I^i := [2k-1] \setminus (\{2i-1\} \cup I)$ for all $i \in [k]$. Then we have

$$\mathrm{PNCPath}_G(I,J) = \mathrm{SPath}(\mu, n) \times \mathrm{SPath}(\lambda, n)$$
$$\mathrm{PNCPath}_G(I^i,J) = \mathrm{SPath}(\mu^i, n) \times \mathrm{SPath}(\lambda^i, n)$$

for all $i \in [k]$. Also, Remark 2.2.21 tells us that $\mathrm{wt}(I,J) = \sum_{i=1}^{k} \mathrm{wt}(I^i, J)$. So we conclude $s_\lambda^{[n]} s_\mu^{[n]} = \sum_{i=1}^{k} s_{\lambda^i}^{[n]} s_{\mu^i}^{[n]}$. Taking $n \to \infty$ gives us the result. □

Setting $\lambda = (\nu_1, \ldots, \nu_k, 0)$ and $\mu = (\nu_2, \ldots, \nu_{k+1})$, we obtain the following identity.

**Corollary 2.3.4** (Fulmek and Kleber). *Let $\nu = (\nu_1, \nu_2, \ldots, \nu_{k+1})$ be a partition with $k \geq 1$. Then*

$$s_{(\nu_1,\ldots,\nu_k)} s_{(\nu_2,\ldots,\nu_{k+1})} = s_{(\nu_2,\ldots,\nu_k)} s_{(\nu_1,\ldots \nu_{k+1})} + s_{(\nu_2-1,\ldots,\nu_{k+1}-1)} s_{(\nu_1+1,\ldots,\nu_k+1)}.$$

Fulmek and Kleber [16] give a bijective proof of this identity. Their proof also goes through a certain algorithm that swaps pairs of tuples of nonintersecting paths. In fact, their notion of *changing tail* is quite similar to the path visiting the vertices $v_0, w_1, v_1, \ldots, v_c, w_c$ we build as part of the algorithm defining $\tau$ in §2.2.2. However, there are significant differences: for one, their networks are not interlacing (and so they never use bottlenecks); also, their procedure changes the size of each tuple, whereas ours does not. The result is that our identities oddly involve Schur functions in different sets of variables.

### 2.3.2 Schur positivity

We now explain how those Schur function identities from previous section lead to some results about Schur positivity. Recall that we say that a symmetric function is *Schur positive* if it has all nonnegative coefficients in the basis of Schur functions. For two symmetric functions $f$ and $g$, we write $f \geq_s g$ if the difference $f - g$ is Schur positive. There has been some interest in understanding when we have $s_\nu s_\rho \geq_s s_\lambda s_\mu$ for partitions $\nu, \rho, \lambda, \mu$. If we let $c^\alpha_{\lambda,\mu}$ be the Littlewood–Richardson coefficients given by $s_\lambda s_\mu = \sum_\alpha c^\alpha_{\lambda,\mu} s_\alpha$, this question is equivalent to the question of when we have $c^\alpha_{\nu,\rho} \geq c^\alpha_{\lambda,\mu}$ for all $\alpha$. Research on this problem has focused on the case where the partitions $\nu$ and $\rho$ are thought of as "functions" of $\lambda$ and $\mu$ as in [15] [24] [1] [34]. We will now state a Schur positivity conjecture of this form. This conjecture was communicated to us privately by Alex Postnikov, who discovered it in collaboration with Pavlo Pylyavskyy and Thomas Lam (see also the papers [7] [3] which investigate this conjecture).

**Conjecture 2.3.5** (Lam-Postnikov-Pylyavskyy). *Let $\nu = (\nu_1, \ldots, \nu_n)$ be a partition. For all $\sigma \in \mathfrak{S}_n$, $i \in [n]$, and choices $\pm \in \{+, -\}$, define a new sequence $\nu^{\pm}(\sigma, i) = (\nu^{\pm}(\sigma, i)_1, \ldots, \nu^{\pm}(\sigma, i)_n)$ by*

$$\nu^{\pm}(\sigma, i)_j := \begin{cases} \nu_j \pm 1 & \text{if } \sigma^{-1}(j) \in [i] \\ \nu_j & \text{otherwise.} \end{cases}$$

*Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be two partitions and let their difference vector be $\delta = (\delta_1, \ldots, \delta_n) := (\lambda_1 - \mu_1, \ldots, \lambda_n - \mu_n)$. Let $\sigma \in \mathfrak{S}_n$ be the unique permutation so that $\delta_{\sigma(1)} \geq \cdots \geq \delta_{\sigma(n)}$ and $\delta_{\sigma(i)} = \delta_{\sigma(j)}$ for $i < j$ implies that $\sigma(i) < \sigma(j)$. Set $\mathcal{D} := \{i \in [n] \colon \delta_{\sigma(i)} > 0$ and $(i = n$ or $\delta_{\sigma(i)} > \delta_{\sigma(i+1)})\}$. Then for all $i \in \mathcal{D}$ we have $s_{\lambda^-(\sigma,i)} s_{\mu^+(\sigma,i)} \geq_s s_\lambda s_\mu$.*

That $\lambda^-(\sigma, i)$ and $\mu^+(\sigma, i)$ remain partitions for all $i \in \mathcal{D}$ in Conjecture 2.3.5 just requires checking some cases. The Schur function identities we have been studying in previous section resolve some special cases of this conjecture. For instance, by applying $\frac{\partial}{\partial x_1}$ to both sides and setting $x_1 = 0$ in Theorem 2.3.2, we get the following identity of Schur functions that all use the same set of variables, albeit involving skew Schur functions.

**Corollary 2.3.6.** *For $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $0 \leq t \leq k-1$,*

$$\begin{aligned} s_{\lambda/1} s_{(\lambda_1, \ldots, \lambda_t, \lambda_{t+2}-1, \ldots, \lambda_k - 1)} = & s_{(\lambda_1, \ldots, \lambda_t, \lambda_{t+2}-1, \ldots, \lambda_k-1)/1} s_\lambda \\ & + s_{(\lambda_1 - 1, \ldots, \lambda_k - 1)} s_{(\lambda_1+1, \ldots, \lambda_t+1, \ldots, \lambda_k)}. \end{aligned}$$

*Here $\nu/1$ denotes the skew shape of $\nu$ minus its top-leftmost box.*

But we can in fact obtain the following Schur positivity result that concerns only regular Schur functions.

**Proposition 2.3.7.** *Let $c, r \geq 1$ and $0 \leq t \leq r-1$. Then*

$$s_{(c^{r-1}, c-1)} s_{(c^t, (c-1)^{r-t-1})} - s_{(c-1)^r} s_{((c+1)^t, c^{r-t-1})}$$

*is Schur positive.*

This proposition is a special case of Conjecture 2.3.5. In order to see why, let $\lambda = ((c+1)^t, c^{r-t-1})$ and $\mu = (c-1)^r$. Then $\delta = (2^t, 1^{r-t-1}, -(c-1))$ and so $\sigma$ is the identity permutation. Note $r - 1 \in \mathcal{D}$, so with $i = r-1$ we get $\lambda^-(\sigma, r-1) = (c^t, (c-1)^{r-t-1})$ and $\mu^+(\sigma, r-1) = (c^{r-1}, c-1)$. The conjecture says we should have $s_{\lambda^-(\sigma,r-1)} s_{\mu^+(\sigma,r-1)} \geq_s s_\lambda s_\mu$, which is exactly what Proposition 2.3.7 asserts.

*Proof of Proposition 2.3.7.* : Applying Corollary 2.3.6 to the case in which $\lambda$ is the rectangular partition $c^r$, and using the skew version of Pieri's rule [33, Corollary 7.5.19] leads us to the following three cases:

1. If $1 \leq t \leq r-2$ then $s_{(c^{r-1}, c-1)} s_{(c^t, (c-1)^{r-t-1})}$ is equal to

$$\left[ s_{(c^{t-1}, (c-1)^{r-t})} + s_{(c^t, (c-1)^{r-t-2}, c-2)} \right] s_{c^r} + s_{(c-1)^r} s_{((c+1)^t, c^{r-t-1})}.$$

2. If $t = 0$ then $s_{(c^{r-1},c-1)}s_{(c-1)^{r-1}} = s_{((c-1)^{r-2},c-2)}s_{c^r} + s_{(c-1)^r}s_{c^{r-1}}$.

3. If $t = r-1$ then $s_{(c^{r-1},c-1)}s_{c^{r-1}} = s_{(c^{r-2},c-1)}s_{c^r} + s_{(c-1)^r}s_{(c+1)^{r-1}}$.

Thus, because products of Schur functions are Schur positive (in other words, because Littlewood–Richardson coefficients are nonnegative) we are done. □

Another example of a special case of Conjecture 2.3.5 is obtained from Theorem 2.3.3.

**Corollary 2.3.8.** *Let $\nu = (\nu_1, \ldots, \nu_k)$ be a partition and let $1 \leq t \leq k$. Then*

$$s_\nu s_{(\nu_1,\ldots,\nu_{t-1},\nu_{t+1},\ldots,\nu_k)} - s_{(\nu_1-1,\ldots,\nu_{t-1}-1,\nu_t,\ldots,\nu_k)}s_{(\nu_1+1,\ldots,\nu_{t-1}+1,\nu_{t+1},\ldots,\nu_k)}$$

*is Schur positive.*

To see why Corollary 2.3.8 is a special case of Conjecture 2.3.5, we can take $\lambda = (\nu_1+1, \ldots, \nu_{t-1}+1, \nu_{t+1}, \ldots, \nu_k)$ and $\mu = (\nu_1 - 1, \ldots, \nu_{t-1}-1, \nu_t, \ldots, \nu_k)$. Let $\sigma$ be as in that conjecture. Note that $\delta_i = 2$ for $i \leq t-1$ and $\delta_i < 0$ for $i \geq t$, so $\sigma(i) = i$ for $i \leq t-1$ and $t-1 \in \mathcal{D}$. Then the conjecture predicts $s_{\lambda^-(\sigma,t-1)}s_{\mu^+(\sigma,t-l)} \geq_s s_\lambda s_\mu$. But $\lambda^-(\sigma, t-1) = (\nu_1, \ldots, \nu_{t-1}, \nu_{t+1}, \ldots, \nu_k)$ and $\mu^+(\sigma, t-1) = \nu$ so Corollary 2.3.8 indeed verifies this Schur inequality.

## 2.4 Balanced swap graphs

Recall the definition of balanced swap.

**Definition 2.4.1.** Let $m$ and $n$ be positive integers such that $m > n$. For $J, J' \in \binom{[m+n]}{n}$, we say that $J'$ is a *swap* of $J$ if $J \cap J' = \emptyset$. Clearly the relation of being a swap is symmetric. If $J$ and $J'$ are swaps of one another, we call the set $P(J, J') := [m+n] \setminus (J \cup J')$ their *pivot set.* We say $J'$ is a *balanced swap* of $J$ if it is a swap of $J$ and that $|J \cap [p]| = |J' \cap [p]|$ for all $p \in P(J, J')$.

Consider the undirected graph $B_{m,n}$ whose vertex set is $\binom{[m+n]}{n}$ and whose edges are all $\{J, J'\}$ such that $J'$ is a balanced swap of $J$. We will refer to $B_{m,n}$ as *balanced swap graph.* In the case $m - n = 1$, we will sometimes write $B_n$ instead of $B_{n+1,n}$.

Figure 2-7 and Figure 2-8 show the balanced swap graphs $B_{4,3}$ and $B_{4,2}$ respectively.

The following theorem shows how the balanced swap graphs represent Schur function identities.

Let $A = \{a_1 > ... > a_n \geq 0\}$. For $J = \{j_1 < ... < j_r\} \subset [n]$, we denote by $A(J)$ the partition $(a_{j_1} - (r-1), a_{j_2} - (r-2), ..., a_{j_r})$.

**Theorem 2.4.2.** *Let $B$ be a bipartite connected component of $B_{m,n}$ with bipartition $B = B_1 \sqcup B_2$. Then, for any set of nonnegative integers $A = \{a_1 > ... > a_{m+n} \geq 0\} \in \binom{\mathbb{N}}{m+n}$, we have*

$$\sum_{J \in B_1} s_{A(J)}s_{A(\bar{J})} = \sum_{J \in B_2} s_{A(J)}s_{A(\bar{J})}.$$

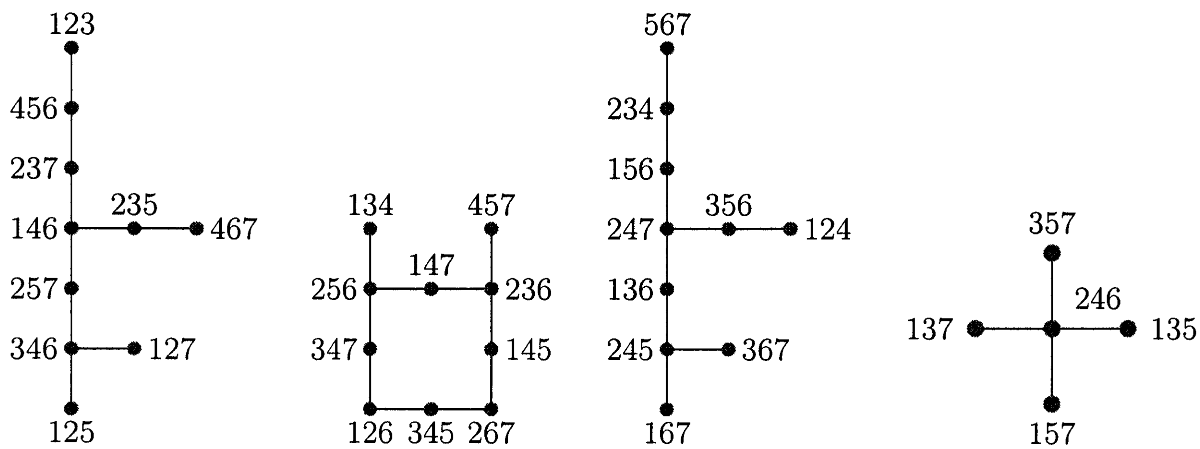


Figure 2-7: The graph $B_{4,3}$

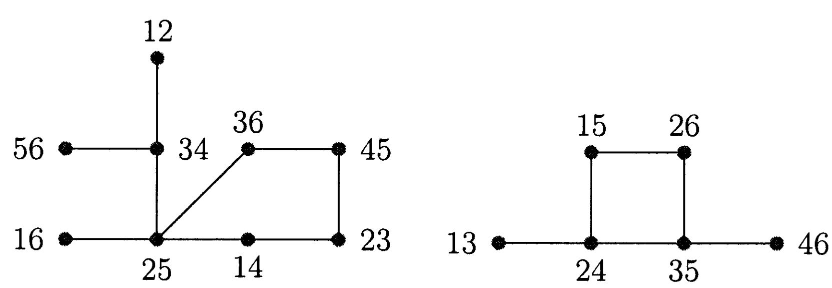


Figure 2-8: The graph $B_{4,2}$

*Proof.* Let $G$ be the grid graph $\mathbb{Z}^2$ with the source $S = \{s_1, ..., s_{m+n}\}$, where $s_{2i-1} = s_{2i} = (i-1, 0)$ for $i \in [n]$ and $s_i = (i-n, 0)$ for $i \in [2n+1, m+n]$, and the sink $T = \{t_1, ..., t_{m+n}\}$, where $t_i = (a_{m+n+1-i}, 0)$.

Fix $I = 2[n]$. By a similar argument to the proof of Corollary 2.2.5, we have

$$\sum_{J \in B_1} \mathrm{wt}(I, J) = \sum_{J \in B_2} \mathrm{wt}(I, J).$$

Then the result follows from Proposition 2.3.1. □

The goal of this section is to understand the structure of $B_{m,n}$. It turns out that the $B_{m,n}$ behave very differently when $m - n = 1$ and $m - n \geq 2$. First, the graphs $B_n$ are always bipartite. Indeed $(\{J \in \binom{[2n+1]}{n} : 2 \in J\}, \{J' \in \binom{[2n+1]}{n} : 2 \notin J'\})$ is a bipartition of $B_n$, because if $J$ and $J'$ are balanced swaps of one another then their pivot $p_0$ is odd, so $p_0 \neq 2$ and thus 2 must belong to exactly one of them. However $B_{m,n}$ are generally not; $B_{4,2}$ is already not bipartite. Nevertheless, the following proposition shows that there is at least one components of $B_{m,n}, m - n \geq 2$, that is bipartite.

We call an element $J = \{j_1 < ... < j_n\}$ *special* if $j_{i+1} - j_i$ is even for all $i = 1, ..., n-1$.

**Proposition 2.4.3.** *The special elements of $B_{m,n}$ form a connected components. This component is bipartite.*

*Proof.* Let $B$ be the set of special elements of $B_{m,n}$. Let $B'$ (resp. $B''$) be the set of $J = \{j_1 < ... < j_n\} \subset [m+n]$ such that all $j_i$ are odd (resp. even). Suppose $J''$ is a balanced swap of $J' \in B'$ with the pivot set $P = \{p_1 < ... < p_{m-n}\}$. For each $i \in [m-n-1]$, the interval $I = [p_i + 1, p_{i+1} - 1]$ must have equal number of elements from $J'$ and $J''$ and that $\overline{J'} \cap I = J'' \cap I$. So the number of elements of $I$ is even. Since $J'$ contains only odd numbers, we have $J' \cap I = I_{\text{odd}}$ and so $J'' \cap I = I_{\text{even}}$. Therefore $J''$ contains only even numbers. This shows that $B' \sqcup B''$ is a bipartition of $B$.

To show that $B$ is indeed connected, we will show that every element of $B$ is connected to the element $J_0 = \{1, 3, ..., 2n-1\}$. Let $J = \{j_1 < ... < j_n\} \neq J_0$ be an element in $B'$. Choose an index $i$ such that $j_i - 2 \in [m+n] \setminus J$. (If there is no such $i$, then $J = J_0$.) It is easy to see that $J' = \{j_1 + 1, ..., j_{i-1} + 1, j_i - 1, ..., j_n - 1\} \in B''$ is a balanced swap of $J$ and that $J'' = \{j_1, ..., j_{i-1}, j_i - 2, ..., j_n - 2\} \in B'$ is a balanced swap of $J'$. So we found that $J''$ is in the same connected component as $J$ and has the smaller sum than $J$. We can repeat this process until we get $J_0$. So the elements of $B'$ are in same connected component in $B$. Now for $J = \{j_1 < ... < j_n\} \in B''$, we have $J' = \{j_1 - 1, ..., j_n - 1\} \in B'$ is a balanced swap of $J$, so $J$ belongs to the same component as $B'$. Thus $B$ is a connected component of $B_{m,n}$. □

Next we partially give the number of connected components of $B_{m,n}$. The graphs $B_n$ decompose into many connected components, while the graphs $B_{m,n}$ appear to have much fewer connected components. The next proposition gives an explanation of what these connected components are.

**Proposition 2.4.4.** *Two vertices $J, J' \in \binom{[2n+1]}{n}$ are in the same connected component of $B_n$ iff $J_{\text{even}} = J'_{\text{even}}$ or $J_{\text{even}} = 2[n] \setminus J'_{\text{even}}$.*

*Proof of Proposition 2.4.4:* The "only if" direction of this proposition is clear since the pivot $j_0 = P(J, J')$ is always odd. So every swap from $J$ changes all $J_{\text{even}}$. What needs to be checked is the "if" direction. Thanks to Darij Grinberg and Alex Postnikov for showing us the proof of this implication.

Define $b\colon \binom{[2n+1]}{n} \to [0,n]$ by

$$b(J) := \max\{k \in [0,n] \colon |J \cap [2i]| = i \text{ for all } 1 \leq i \leq k\}.$$

Fix $K \subseteq [2n+1]_{\text{even}}$. Define $J(K) := K \cup \{i \in [2n-1]_{\text{odd}} \colon i+1 \notin K\}$. Note that $J(K)$ is the unique vertex in $B_n$ which satisfies $J(K)_{\text{even}} = K$ and $b(J(K)) = n$. Set $K' := [2n+1]_{\text{even}} \setminus K$. Then $J(K)$ and $J(K')$ are balanced swaps of one another with pivot $2n+1$; so $\{J(K), J(K')\}$ is an edge of $B_n$.

We claim that any $J$ with $J_{\text{even}} = K$ is in the same connected component as $J(K)$ in $B_n$. This proves the proposition because it follows that for any other $I$ with $I_{\text{even}} = K$ is also in this connected component, as in any $J'$ with $J'_{\text{even}} = K'$ by the observation that $J(K)$ and $J(K')$ are connected. If $b(J) = n$ then $J = J(K)$ and the claim is clear. So suppose $b(J) < n$. Then we claim there is some $J'$ with $b(J') > b(J)$ such that $J$ is in the same connected component as $J'$. To see this, observe that either $\{2b(J)+1, 2b(J)+2\} \cap J = \emptyset$ or $\{2b(J)+1, 2b(J)+2\} \subseteq J$ by the definition (in particular, maximality) of $b(J)$. In the first case, note that we can define $J'$ to be the swap of $J$ whose pivot is $2b(J)+1$; this swap is balanced precisely because $|J \cap [2b(J)]| = b(J)$. Then $b(J') > b(J)$ because $|J' \cap [2i]| = |J \cap [2i]| = i$ for all $1 \leq i \leq b(J)$, and also $|J' \cap [2b(J)+2]| = b(J)+1$. In the second case, we claim there is some $i$ with $i > b(J)$ and $2i+1 \notin J$ such that $|J \cap [2i]| = i$. Indeed, this has to be the case because $|J \cap [2b(J)+2]| = b(J)+2$ but $|J \cap [2n+1]| = n$. So then we may define $J'$ to be the swap of $J$ whose pivot is $2i+1$, and $b(J') \geq b(J)$. If $b(J') = b(J)$, then we find some $J''$ with $b(J'') > b(J')$ by applying the first case to $J'$, which has $\{2b(J)+1, 2b(J)+2\} \cap J' = \emptyset$. $\square$

One immediate corollary of Proposition 2.4.4 is that the number of connected components of $B_n$ is $2^{n-1}$. Another corollary is that the size of the connected component of $B_n$ containing $J \in \binom{[2n+1]}{n}$ is $\binom{n+1}{|J_{\text{even}}|} + \binom{n+1}{|J_{\text{even}}|+1}$. However, the structure of these connected components is in general quite complicated. For example, already for the graph $B_3$ has a cycle. In fact, we can exactly classify those connected components of $B_n$ which are acyclic. We will give a proof in the next section.

**Proposition 2.4.5.** *Let $J \in \binom{[2n+1]}{n}$ with $2 \in J$. Then the connected component of $B_n$ containing $J$ is acyclic iff $\max(J_{\text{even}}) < \min([2n+1]_{\text{even}} \setminus J_{\text{even}})$, i.e. $J_{\text{even}} = 2[r]$ for some $r \in [n]$.*

The number of the connected components of $B_{m,n}$ is, on the other hand, still not known. From experimental results, we have the following conjecture of the exact number of connected components of $B_{m,n}$.

**Conjecture 2.4.6.** *Let $n \geq 2$ and $m \geq n+2$ be integers. Then $B_{m,n}$ has $\lfloor \frac{n}{2} \rfloor + 1$ connected components.*

We end this section with the following conjecture, which is a stronger version of 2.4.2.

**Conjecture 2.4.7.** *Let $A = \{a_1 > ... > a_{m+n} \geq 0\}$. There is a function $f : E(B_{m,n}) \to S$, where $S$ is the semiring of Schur-positive symmetric functions, such that*

$$s_{A(J)} s_{A(\bar{J})} = \sum_e f(e),$$

*where the sum is taken over the edges $e \in E(B_{m,n})$ of the form $(J, J')$, $J' \in \binom{[m+n]}{n}$.*

*In particular if $J$ is a leaf of $B_{m,n}$ and $(J, J')$ is the only edge adjacent to $J$, then $s_{A(J')} s_{A(\bar{J}')} - s_{A(J)} s_{A(\bar{J})}$ is Schur positive.*

### 2.4.1 Proof of Proposition 2.4.5

Recall from Proposition 2.4.4 that the connected components of $B_n$ correspond to the subsets of $2[n]$ which contain 2. For $X \subset [n]$ such that $1 \in X$, we let $B_n(X)$ be the connected component

$$\{J \in \binom{[2n+1]}{n} : J_{\text{even}} = 2X \text{ or } \overline{J}_{\text{even}} = 2X\}$$

of $B_n$. We know that the number of vertices of $B_n(X)$ is

$$|V(B_n(X))| = \binom{n+1}{k} + \binom{n+1}{k+1} = \binom{n+2}{k+1},$$

where $k = |X|$. So $B_n(X)$ is acyclic if and only if $|E(B_n(X))| = \binom{n+2}{k+1} - 1$, or equivalently $\sum_{v \in V(B_n(X))} \deg(v) = 2\binom{n+2}{k+1} - 2$. We will show that this is the case precisely when $X = [k]$.

Let $\mathcal{P}(n)$ be the set of lattice paths from $(0,0)$ to $(n+1, n)$ where each step is of the form $(i,j) \to (i+1, j)$ or $(i,j) \to (i, j+1)$. We call a step $(i,j) \to (i+1,j)$ (resp. $(i,j) \to (i,j+1)$) an *east step* (resp. a *north step*). For $P \in \mathcal{P}(n)$, we define the weight of $P$, written $wt(P)$, to be the number of steps of $P$ from $(i,i)$ to $(i+1,i)$. We define the polynomial $P_n(x_1, ..., x_n; q)$ by

$$P_n(x_1, ..., x_n; q) = \sum_{P \in \mathcal{P}(n)} q^{wt(P)} x_1^{\epsilon_1} x_2^{\epsilon 2} \cdots x_n^{\epsilon_n},$$

where $\epsilon_i$ is 1 if the $(2i)^{\text{th}}$ step is an east step and -1 otherwise. For example, $P_2(x, y; q) = q^3 x^{-1} y^{-1} + 2q^2(xy^{-1} + x^{-1}y) + q(3xy + xy^{-1} + x^{-1}y)$ (see Figure 2-9).

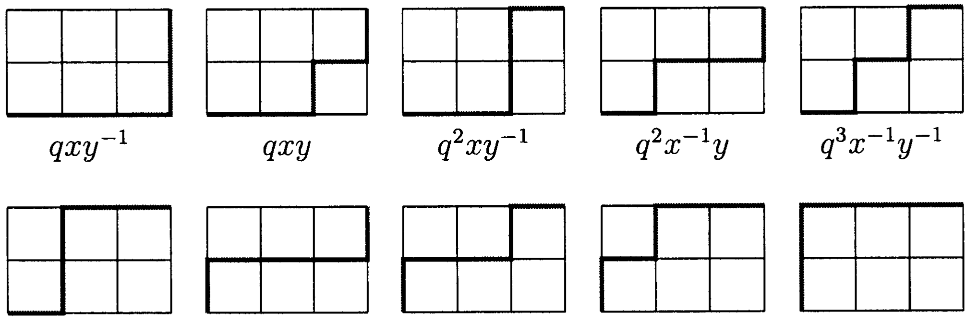


Figure 2-9: The paths in $\mathcal{P}(2)$ and their corresponding monomials.

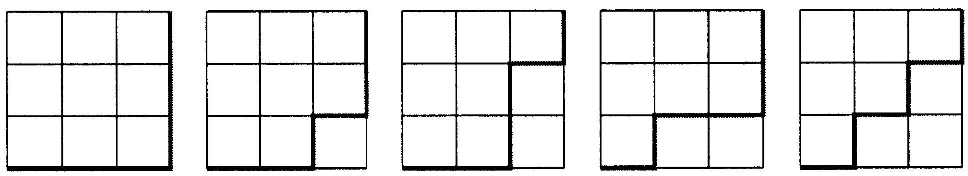


Figure 2-10: The paths in $\mathcal{C}(3)$ and their corresponding monomials.

Let $\mathcal{C}(n)$ be the set of lattice paths from $(0,0)$ to $(n,n)$ which stays weakly below the line $y = x$ (i.e. the set of Dyck paths of length $2n$). We define $C_n(x_1, ..., x_n)$ by

$$C_n(x_1, ..., x_n) = \sum_{P \in \mathcal{C}(n)} x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n},$$

where $\epsilon_i$ is 1 if the $(2i-1)^{\text{st}}$ step is an east step and -1 otherwise. For example $C_3(x, y, z) = x(yz + 2yz^{-1} + y^{-1}z + y^{-1}z^{-1})$ (see Figure 2-10).

For an $n$-tuple $u = (u_1, u_2, ..., u_n)$, we define $u^k$ and ${}^k u$ by

$$u^k := (u_1, u_2, ..., u_{k-1}), \quad \text{and} \quad {}^k u := (u_{k+1}, u_{k+2}, ..., u_n).$$

**Proposition 2.4.8.** *Let* $\mathbf{x} = (x_1, ..., x_n)$ *be an* $n$*-tuple of variables and* $\mathbf{x}^{-1} = (x_1^{-1}, ..., x_n^{-1})$. *Then* $P_n$ *and* $C_n$ *satisfy the recurrent relations*

$$C_n(\mathbf{x}) = \sum_{k=1}^{n} x_1 C_{k-1}(x_k^{-1}, ..., x_2^{-1}) C_{n-k}({}^k\mathbf{x})$$

*and*

$$P_n(\mathbf{x}; q) = qC_n(\mathbf{x}) + \sum_{k=1}^{n} \left[ qC_{k-1}(\mathbf{x}^k) x_k^{-1} + C_{k-1}((\mathbf{x}^{-1})^k) x_k \right] \cdot P_{n-k}({}^k\mathbf{x}; q).$$

*Proof.* We start with the first relation. For any path $P \in \mathcal{C}(n)$, we look at the first step of the form $(k, k-1) \to (k, k)$, i.e., the first north step which touches the line $y = x$. The paths from $(k, k) \to (n, n)$ give us $C_{n-k}({}^k\mathbf{x})$. Consider the part of $P$ from $(0, 0)$ to $(k, k)$. Clearly the first and the last steps must be $(0, 0) \to (1, 0)$ (hence the term $x_1$) and $(k-1, k) \to (k, k)$, respectively. The part of $P$ from $(1, 0)$ to $(k-1, k)$ lies strictly below the line $y = x$, so by moving it one unit to the left we have a path $P'$ in $\mathcal{C}(k-1)$. However, by doing this, the even steps of $P$ becomes odd steps of $P'$, and vise versa. This can be resolved by considering the path backward from $(k-1, k)$ to $(1, 0)$. So this part of $P$ gives us $C_{k-1}(x_k^{-1}, \ldots, x_2^{-1})$. The first identity is obtained by summing over $k$ from 1 to $n$.

Now we show the second relation. Let $P \in \mathcal{P}(n)$ be a lattice path. If the path does not touch the line $y = x$ again after the first step, we get $qC_n(\mathbf{x})$. Suppose that the first time the path $P$ touches the line $y = x$ is at $(k, k)$. This means the path from $(0, 0)$ to $(k, k)$ is either strictly below or strictly above the line $y = x$. In which cases, we get the factor $qC_{k-1}(\mathbf{x}^k)x_k^{-1}$ and $C_{k-1}((\mathbf{x}^{-1})^k)x_k$, respectively. The paths from $(k, k)$ to $(n+1, n)$ give us $P_{n-k}({}^k\mathbf{x}; q)$. $\square$

Define $h : \{1, -1\}^n \to \mathbb{Z}$ by

$$h(\epsilon) = ([\mathbf{x}^\epsilon] + [\mathbf{x}^{-\epsilon}]) \left.\frac{\partial P_n(\mathbf{x}; q)}{\partial q}\right|_{q=1}.$$

Since the set $\{1, -1\}^n$ is obviously in a bijection with $2^{[n]}$, by $\epsilon \mapsto X = X(\epsilon) := \{i : \epsilon_i = 1\}$, we will use $\epsilon$ and $X$ interchangeably under this bijection. We denote by $\epsilon(X) \in \{1, -1\}^n$, where $X \in 2^{[n]}$, the preimage of $X$ under this bijection. For $\epsilon \in \{1, -1\}^n$, we say that the length of $\epsilon$ is $\ell(\epsilon) = n$. We let $|\epsilon|$ denote the number of 1's in $\epsilon$, i.e. $|\epsilon| := |X(\epsilon)|$.

**Proposition 2.4.9.** *$h(\epsilon)$ is the sum of the degrees of the vertices of $B_n(X)$.*

*Proof.* For $v \in V(B_n(X))$, the degree of $v$ is the weight $wt(P)$ of the corresponding path $P = P(v)$ in $\mathcal{P}(n)$ since the steps which count toward $wt(P)$ are the places we could perform a balanced swap of $v$. Thus

$$\begin{aligned} h(\epsilon) &= ([\mathbf{x}^\epsilon] + [\mathbf{x}^{-\epsilon}]) \sum_{P \in \mathcal{P}(n)} wt(P)\mathbf{x}^\epsilon \\ &= \sum_{P:\ \epsilon(P)=\epsilon} wt(P) + \sum_{P:\ \epsilon(P)=-\epsilon} wt(P) \\ &= \sum_{v:\ v_{\text{even}}=2X} \deg(v) + \sum_{v:\ \overline{v}_{\text{even}}=2X} \deg(v) = \sum_{v \in V(B_n(X))} \deg(v). \end{aligned}$$

$\square$

So Proposition 2.4.5 is equivalent to:

**Proposition 2.4.10.** *Suppose $\epsilon_1 = 1$ and $|\epsilon| = r$. Then*

$$h(\epsilon) \geq 2\binom{n+2}{r+1} - 2,$$

*and the equality holds iff $\epsilon = (1^r, (-1)^{n-r})$.*

By taking the partial derivative $\frac{\partial}{\partial q}$ on $P_n$, we get

$$\begin{aligned}\frac{\partial P_n}{\partial q}(\mathbf{x}; q) = C_n(\mathbf{x}) + \sum_{k=1}^{n} C_{k-1}(\mathbf{x}^k) x_k^{-1} P_{n-k}({}^k\mathbf{x}; q)\\ + \left[qC_{k-1}(\mathbf{x}^k)x_k^{-1} + C_{k-1}\left((\mathbf{x}^{-1})^k\right) x_k\right] \cdot \frac{\partial P_{n-k}}{\partial q}({}^k\mathbf{x}; q).\end{aligned}$$

Let $f(\epsilon) := [\mathbf{x}^\epsilon]C_n(x_1, ..., x_n)$. Suppose $\epsilon_1 = 1$. Then

$$\begin{aligned}[\mathbf{x}^\epsilon] \left.\frac{\partial P_n}{\partial q}\right|_{q=1} &= f(\epsilon) + [{}^1(\mathbf{x}^\epsilon)]\frac{\partial P_n}{\partial q}({}^1\mathbf{x}; 1)\\ &\quad + \sum_{k:\ \epsilon_k=-1} [(\mathbf{x}^\epsilon)^k]C_{k-1}(\mathbf{x}^k) \cdot [{}^k(\mathbf{x}^\epsilon)] \left(\frac{\partial P_{n-k}}{\partial q}({}^k\mathbf{x}; 1) + P_{n-k}({}^k\mathbf{x}; 1)\right)\\ &= f(\epsilon) + [{}^1(\mathbf{x}^\epsilon)]\frac{\partial P_n}{\partial q}({}^1\mathbf{x}; 1)\\ &\quad + \sum_{k:\ \epsilon_k=-1} f(\epsilon^k) \cdot \left([{}^k(\mathbf{x}^\epsilon)]\frac{\partial P_{n-k}}{\partial q}({}^k\mathbf{x}; 1) + \binom{n-k+1}{|{}^k\epsilon|}\right),\end{aligned}$$

and

$$\begin{aligned}[\mathbf{x}^{-\epsilon}] \left.\frac{\partial P_n}{\partial q}\right|_{q=1} &= [{}^1(\mathbf{x}^{-\epsilon})] \left(\frac{\partial P_n}{\partial q}({}^1\mathbf{x}; 1) + P_{n-1}({}^1\mathbf{x}; 1)\right)\\ &\quad + \sum_{k:\ \epsilon_k=-1} [(\mathbf{x}^{-\epsilon})^k]C_{k-1}((\mathbf{x}^{-1})^k) \cdot [{}^k(\mathbf{x}^{-\epsilon})]\frac{\partial P_{n-k}}{\partial q}({}^k\mathbf{x}; 1)\\ &= [{}^1(\mathbf{x}^{-\epsilon})]\frac{\partial P_n}{\partial q}({}^1\mathbf{x}; 1) + \binom{n}{|\epsilon|} + \sum_{k:\ \epsilon_k=-1} f(\epsilon^k) \cdot [{}^k(\mathbf{x}^{-\epsilon})]\frac{\partial P_{n-k}}{\partial q}({}^k\mathbf{x}; 1).\end{aligned}$$

Therefore $h(\epsilon)$ satisfies

$$h(\epsilon) = \binom{n}{|\epsilon|} + h({}^1\epsilon) + f(\epsilon) + \sum_{k:\ \epsilon_k=-1} f(\epsilon^k) \left[h({}^k\epsilon) + \binom{n-k+1}{|{}^k\epsilon|}\right].$$

Every $\epsilon \in \{1, -1\}^n$ with $|\epsilon| = r$ corresponds to a lattice path $P = P(\epsilon)$ from $(0, 0)$ to $(r, n-r)$ such that the $i^{\text{th}}$ step of $P$ is an east step if $\epsilon_i = 1$ and a north step if $\epsilon_i = -1$. This path $P$, in turn, corresponds to a partition which fits inside the rectangle $(r^{n-r})$ by taking the part above $P$ but inside the rectangle with vertices

$(0,0),(0,n-r),(r,n-r)$, and $(r,0)$. We denote this partition by $p(\epsilon)$. For example, if $\epsilon=(1,-1,1,1,-1,1)$, then $p(\epsilon)$ is the partition $(3,1)$ inside the rectangle $(4^2)$.

**Lemma 2.4.11.** *Suppose $\epsilon_1=1$. Then $f(\epsilon)$ is the number of partitions $\lambda$ which are covered by $p({}^1\epsilon)$.*

*Proof.* Let $n=l(\epsilon)$. To prove this lemma, we first note that $f(\epsilon)$ is the number of lattice paths from $(0,0)$ to $(n,n)$ with odd steps $\epsilon$ that stays below the line $x=y$. Consider the even steps $\delta=(\delta_1,...,\delta_n)$. Then $f(\epsilon)$ is the number of $n$-tuples $\delta=(\delta_1,...,\delta_{n-1},-1)\in\{1,-1\}^n$ satisfying

$$1\le\epsilon_1+...+\epsilon_k+\delta_1+...+\delta_{k-1}\le 2n-2k+1 \tag{2.2}$$

for all $k$. On the other hand, the number of partitions covered by $p({}^1\epsilon)$ is the number of $n$-tuples $\alpha=(1,\alpha_2,...,\alpha_n)\in\{1,-1\}^n$ satisfying

$$(k-1)-2(n-r)\le\alpha_2+...+\alpha_k\le\epsilon_2+...\epsilon_k \tag{2.3}$$

for all $k$, where $r=|\epsilon|$. We claim that the map $\alpha\mapsto\delta=(-\alpha_2,...,-\alpha_n,-1)$ gives a bijection between the set of possible $\delta$'s and the set of possible $\alpha$'s. To verify this, first we note that $\epsilon_1+...+\epsilon_n=2r-n$, given $|\epsilon|=r$. Suppose $\delta=(\delta_1,...,\delta_{n-1},-1)$ satisfies (2.2). This is equivalent to

$$0\le\epsilon_2+...+\epsilon_k-(\alpha_2+...+\alpha_k)\le 2n-2k.$$

So the inequality to the right of (2.3) is satisfied. Furthermore we have

$$\begin{aligned}\alpha_2+...+\alpha_k&\ge\epsilon_1+...+\epsilon_k-(2n-2k+1)\\&\ge\epsilon_1+...+\epsilon_n-(n-k+1)\\&=2r-n-(n-k+1)=(k-1)-2(n-r),\end{aligned}$$

which is precisely the inequality to the left of (2.3). On the other hand, suppose $\alpha=(1,\alpha_2,...,\alpha_n)$ satisfies (2.3). We have

$$(k-1)-2(n-r)\le-(\delta_1+...+\delta_{k-1})\le\epsilon_2+...+\epsilon_k.$$

Thus the inequality to the left of (2.2) is satisfied. Also from

$$(k-1)-2(n-r)+(\delta_1+...+\delta_{k-1})\le 0,$$

we obtain

$$\delta_1+...+\delta_{k-1}+\epsilon_1+...+\epsilon_n\le n-k+1.$$

Hence

$$\begin{aligned}\delta_1+...+\delta_{k-1}+\epsilon_1+...+\epsilon_k&\le n-k+1-(\epsilon_{k+1}+...+\epsilon_n)\\&\le 2n-2k+1.\end{aligned}$$

Therefore, the number of possible $\delta$'s is equal to the number of possible $\alpha$'s, which finishes the proof. $\square$

**Lemma 2.4.12.** *Suppose $\epsilon_1 = 1$, $l(\epsilon) = n$, and $|\epsilon| = r$. Then*

$$f(\epsilon) + \sum_{k:\ \epsilon_k=-1} f(\epsilon^k)\binom{n+1-k}{|^k\epsilon|} = \binom{n}{r}.$$

*Proof.* Let $\mathcal{P}(m,n)$ denote the set of lattice paths from $(0,0)$ to $(m,n)$. Fix a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ with $\lambda_1 \leq n-r$, represented as a fixed lattice path in $\mathcal{P}(n-r,r)$. We will construct a bijection between the set $\mathcal{P}(n-r+1,r)$ and the set $\mathcal{A}$ of triples $(i,\alpha,\beta)$ where $i \in [0,r]$, $\alpha$ is a lattice path from $(0,0)$ to $(\lambda_{r+1-i}, i)$ which lies weakly above $\lambda$, and $\beta$ is a lattice path from $(\lambda_{r-i}, i)$ to $(n-r,r)$. Define $\Phi : \mathcal{A} \to \mathcal{P}(n-r+1,r)$ as follows. Given $(i,\alpha,\beta)$, we construct a lattice path $P = \Phi(i,\alpha,\beta)$ from $(0,0)$ to $(n-r+1,r)$ by concatenating $\alpha$, $x$, and $\beta$, where $x$ is the path from $(\lambda_{r+1-i}, i)$ to $(\lambda_{r-i}+1, i)$. Here we regard $\beta$ as a path from $(\lambda_{r-i}+1, i)$ to $(n-r+1, r)$.

For $P \in \mathcal{P}(m,n)$, we define the *height* of $P$ to be the sequence $ht(P) = a_1 \dots a_m$, where $a_j$ is the number of north steps before the $j^{\text{th}}$ east step. Now for $P \in \mathcal{P}(n-r+1,r)$, suppose that $ht(P) = a_1 \dots a_{n-r+1}$ and $ht(\lambda) = b_1 \dots b_{n-r}$. Let $k$ be the smallest index such that $a_k < b_k$, or $k = n-r+1$ if $a_j \geq b_j$ for all $j$. Removing the step corresponding to $a_k$ (i.e. the $(k+a_k)^{\text{th}}$-step) from $P$, we get a path $P' \in \mathcal{P}(n-r,r)$. Suppose $l$ is the largest index such that $b_l < a_k$ (if $b_1 = a_k$, then $l = 0$). Let $i = a_k$, and let $\alpha$ and $\beta$ be the paths with heights $a_1 \dots a_l$ and $a_{k+1} \dots a_{n-r+1}$, respectively. Since $a_i \geq b_i$ for $i \leq l$, $\alpha$ lies weakly above $\lambda$. By this choice of $l$, we have $b_l < b_{l+1} = \dots = b_{k-1} = a_{l+1} = \dots = a_k < b_k$. So the corresponding path to the subsequence $b_{l+1} \dots b_{k-1}$ is the path from $(\lambda_{r+1-i}, i)$ to $(\lambda_{r-i}+1, i)$, which are the ending point of $\alpha$ and the starting point of $\beta$, respectively. Hence, $(i,\alpha,\beta) \in \mathcal{A}$. We denote this map by $\Psi : \mathcal{P}(n-r+1,r) \to \mathcal{A}$. It remains to check that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identity on $\mathcal{A}$ and $\mathcal{P}(n-r+1,r)$, respectively. Let $P \in \mathcal{P}(n-r+1,r)$ and $(i,\alpha,\beta) = \Phi(P)$. It is easy to see that $\Phi(\Psi(P)) = P$ since we add an east step of height $i$ back when applying $\Phi$ to $(i,\alpha,\beta)$, where the an east step of height $i$ is removed from $P$ when applying $\Psi$. So $\Phi \circ \Psi = \mathrm{id}_{\mathcal{P}(n-r+1,r)}$. Suppose $(i,\alpha,\beta) \in \mathcal{A}$ and $P = \Phi(i,\alpha,\beta)$. Then the step that gets removed in the process of applying $\Psi$ to $P$ is the step from $(\lambda_{r-i}, i)$ to $(\lambda_{r-i}+1, i)$ since it is the first step that lies strictly below $\lambda$. Hence $\Psi \circ \Phi = \mathrm{id}_{\mathcal{A}}$.

Let $k_0 < \dots < k_{n-r-1}$ be the set $\{k : \epsilon_k = -1\}$. Let $\lambda = (\lambda_1, \dots, \lambda_{n-r})$, with $\lambda_1 \leq r$, be the partition traced by the lattice path $^1\epsilon$. From above, we know that the number of triples $(i,\alpha,\beta)$, where $i \in [0, n-r]$, $\alpha$ is a partition from which covered by $(\lambda_{n-r-i+1}, \dots, \lambda_{n-r})$, and $\beta$ is a lattice path from $(\lambda_{n-r-i}, i)$ to $(r-1, n-r)$, is $|\mathcal{P}(r, n-r)| = \binom{n}{r}$. For $i = n-r$, the number of such triples is just the number of partitions covered by $\lambda$, which is $p(^1\epsilon) = f(\epsilon)$. For $i < n-r$, the partition corresponding to the sequence $^1(\epsilon^{k_i})$ is $(\lambda_{n-r+1-i}, \dots, \lambda_{n-r})$ embedded inside the rectangle $(\lambda_{n-r-i})^i$. Thus the number of such triples $(i,\alpha,\beta)$ is $f(\epsilon^{k_i})\binom{n+1-k}{|^k\epsilon|}$. So we obtain the stated identity by summing over $i$. $\square$

We can rewrite the recurrence formula for $h$ as:

$$h(\epsilon) = 2\binom{n}{|\epsilon|} + h({}^1\epsilon) + \sum_{k:\ \epsilon_k=-1} f(\epsilon^k)h({}^k\epsilon).$$

**Lemma 2.4.13.** *Suppose $\epsilon_1 = 1, \ell(\epsilon) = n$, and $|\epsilon| = r$. Then*

$$f(\epsilon) + \sum_{k:\epsilon_k=-1} f(\epsilon^k) \leq \binom{n}{r},$$

*and the equality holds iff $\epsilon = (1^r, (-1)^{n-r})$.*

*Proof.* By induction on $n$. If $n = 0$, there is nothing to prove. Suppose the inequality holds for $n$. Let $\bar{\epsilon} = (\epsilon, a)$ where $\ell(\epsilon) = n$, $|\epsilon| = r$, and $a \in \{1, -1\}$. If $a = 1$, then

$$f(\bar{\epsilon}) + \sum_{k:\bar{\epsilon}_k=-1} f(\bar{\epsilon}^k) = f(\epsilon) + \sum_{k:\epsilon_k=-1} f(\epsilon^k) \leq \binom{n}{r} \leq \binom{n+1}{r+1}.$$

Here the equality holds only when $r = n$, which means $\bar{\epsilon} = (1^{n+1})$. Now suppose $a = -1$. Then

$$\sum_{k:\bar{\epsilon}_k=-1} f(\bar{\epsilon}^k) = \sum_{k:\epsilon_k=-1} f(\epsilon^k) + f(\epsilon) \leq \binom{n}{r}$$

So

$$f(\bar{\epsilon}) + \sum_{k:\bar{\epsilon}_k=-1} f(\bar{\epsilon}^k) \leq f(\bar{\epsilon}) + \binom{n}{r} \leq \binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}.$$

The equality holds iff $\epsilon = (1^r, (-1)^{n-r})$ and $f(\bar{\epsilon}) = \binom{n}{r-1}$, which is equivalent to the condition $\epsilon = (1^r, (-1)^{n+1-r})$. □

Now we prove our main claim alongside with another inequality.

**Lemma 2.4.14.** *Suppose $\epsilon_1 = 1$ and $|\epsilon| = r$. Then*

$$\sum_{k:\ \epsilon_k=-1} f(\epsilon^k)h({}^k\epsilon) \geq 2\binom{n}{r+1},$$

*and*

$$h(\epsilon) \geq 2\binom{n+2}{r+1} - 2.$$

*Each equality holds iff $\epsilon = (1^r, (-1)^{n-r})$.*

*Proof.* By induction on the length of $\epsilon$. Assume both inequalities hold for all $\epsilon$ of length up to $n-1$. Suppose now that $\epsilon$ has length $n$ and $|\epsilon| = r$. Then

$$\begin{aligned}\sum_{k:\ \epsilon_k=-1} f(\epsilon^k)h({}^k\epsilon) &\geq 2 \sum_{k:\ \epsilon_k=-1} f(\epsilon^k)\left[\binom{n-k+2}{|{}^k\epsilon|+1} - 1\right] \\ &= 2 \sum_{k:\ \epsilon_k=-1} f(\epsilon^k)\binom{n-k+2}{|{}^k\epsilon|+1} - 2 \sum_{k:\ \epsilon_k=-1} f(\epsilon^k).\end{aligned}$$

Using Lemma 2.4.12 on $\bar{\epsilon} := (\epsilon_1, ..., \epsilon_n, 1)$, we get

$$\sum_{k:\ \epsilon_k=-1} f(\epsilon^k)\binom{n-k+2}{|{}^k\epsilon|+1} = \binom{n+1}{r+1} - f(\epsilon).$$

So the desired inequality reduces to

$$f(\epsilon) + \sum_{k:\epsilon_k=-1} f(\epsilon^k) \leq \binom{n}{r},$$

which is precisely Lemma 2.4.13.

The other assertion is more transparent:

$$\begin{aligned}h(\epsilon) &= 2\binom{n+1}{r} + h({}^1\epsilon) + \sum_{k:\ \epsilon_k=-1} f(\epsilon^k)h({}^k\epsilon) \\ &\geq 2\binom{n+1}{r} + 2\binom{n+2}{r} - 2 + 2\binom{n+1}{r+1} \\ &= 2\binom{n+3}{r+1} - 2.\end{aligned}$$

□

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