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Supports of irreducible spherical representations of rational Cherednik algebras of finite Coxeter groups

SUPPORTS OF IRREDUCIBLE SPHERICAL REPRESENTATIONS OF RATIONAL CHEREDNIK ALGEBRAS OF FINITE COXETER GROUPS

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To my father Ilya Etingof on his 80-th birthday, with admiration

1. INTRODUCTION

In this paper we determine the support of the irreducible spherical representation (i.e., the irreducible quotient of the polynomial representation) of the rational Cherednik algebra of a finite Coxeter group for any value of the parameter c. In particular, we determine for which values of c this representation is finite dimensional. This generalizes a result of Varagnolo and Vasserot, [VV], who classified finite dimensional spherical representations in the case of Weyl groups and equal parameters (i.e., when c is a constant function).

Our proof is based on the Macdonald-Mehta integral and the elementary theory of distributions.

The organization of the paper is as follows. Section 2 contains preliminaries on Coxeter groups and Cherednik algebras. In Section 3 we state and prove the main result in the case of equal parameters. In Section 4 we deal with the remaining case of irreducible Coxeter groups with two conjugacy classes of reflections. In Section 5, as an application, we compute the zero set of the kernel of the renormalized Macdonald pairing in the trigonometric setting (in the equal parameter case). Finally, in the appendix, written by Stephen Griffeth, it is shown by a uniform argument (using only the theory of finite reflection groups) that our classification of finite dimensional spherical representations of rational Cherednik algebras with equal parameters coincides with that of Varagnolo and Vasserot.

2. Preliminaries

2.1. Coxeter groups. Let W be a finite Coxeter group of rank r with reflection representation $\mathfrak{h}_\mathbb{R}$ equipped with a Euclidean W-invariant inner product $(,)$. [1] Denote by $\mathfrak{h}$ the complexification of $\mathfrak{h}_\mathbb{R}$.

For $a \in \mathfrak{h}$, let W_a be the stabilizer of a in W. It is well known that W_a is also a Coxeter group, with reflection representation $\mathfrak{h}/\mathfrak{h}^{W_a} \cong (\mathfrak{h}^{W_a})^\perp$. The group W_a is called a *parabolic subgroup* of W. It is well known that the subgroup generated by a subset of the set of simple reflections of W (which corresponds to a subset of nodes of the Dynkin diagram) is parabolic, and any parabolic subgroup of W is conjugate to one of this type.

Denote by S the set of reflections of W. For each reflection s, pick a vector $\alpha_s \in \mathfrak{h}_\mathbb{R}$ such that $s\alpha_s = -\alpha_s$ and $(\alpha_s, \alpha_s) = 2$. Let

$$\Delta_W(\mathbf{x}) = \prod_{s \in S}(\alpha_s, \mathbf{x})$$

be the corresponding *discriminant polynomial* (it is uniquely determined up to a sign).

Let $d_i = d_i(W), i = 1, ..., r$, be the degrees of the generators of the algebra $\mathbb{C}[\mathfrak{h}]^W$. Let $\ell(w)$ be the length of $w \in W$. Let

$$P_W(q) = \sum_{w \in W} q^{\ell(w)}$$

be the Poincaré polynomial of W. Then we have the following well-known identity of Bott and Solomon:

$$(1) \qquad\qquad P_W(q) = \prod_{i=1}^{r} \frac{1 - q^{d_i}}{1 - q}.$$

If W is an irreducible Coxeter group which contains two conjugacy classes of reflections (i.e. an even dihedral group $I_2(2m)$ or a Weyl group of type B_n, $n \geq 2$, or F_4), then it is useful to consider the 2-variable Poincaré polynomial

$$P_W(q_1, q_2) := \sum_{w \in W} q_1^{\ell_1(w)} q_2^{\ell_2(w)},$$

where $\ell_i(w)$ is the number of simple reflections of i-th type occurring in a reduced decomposition of w.

In the Weyl group case, for a positive root α, denote by $\mathrm{ht}_i(\alpha)$, $i = 1, 2$, the number of simple roots of i-th type occuring in the decomposition of α.

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[1] As a basic reference on finite Coxeter groups, we use the book [Hu].

Proposition 2.1. ([Ma]) (i) One has

$$P_{I_2(2m)}(q_1, q_2) = \frac{1 - q_1^2}{1 - q_1} \frac{1 - q_2^2}{1 - q_2} \frac{1 - (q_1 q_2)^m}{1 - q_1 q_2}.$$

(ii) For Weyl groups one has

$$P_W(q_1, q_2) = \prod_{\alpha > 0} \frac{1 - q_\alpha q_1^{\mathrm{ht}_1(\alpha)} q_2^{\mathrm{ht}_2(\alpha)}}{1 - q_1^{\mathrm{ht}_1(\alpha)} q_2^{\mathrm{ht}_2(\alpha)}},$$

where $q_\alpha = q_i$ if α is a root of i-th type, $i = 1, 2$.

From this proposition one can obtain the following more explicit formulas for the 2-variable Poincaré polynomials of $I_2(2m)$, B_n and F_4.

Proposition 2.2. ([Ma]) One has

$$P_{I_2(2m)}(q_1, q_2) = (1 + q_1)(1 + q_2)(1 + q_1 q_2 + ... + q_1^{m-1} q_2^{m-1}),$$

$$P_{B_n}(q_1, q_2) = \prod_{j=0}^{n-1} (1 + q_1 + ... + q_1^j) \prod_{j=1}^{n-1} (1 + q_1^j q_2),$$

and

$$P_{F_4}(q_1, q_2) =$$
$$(1+q_1)(1+q_1+q_1^2)(1+q_2)(1+q_2+q_2^2)(1+q_1^2 q_2)(1+q_1 q_2^2)(1+q_1 q_2)(1+q_1^2 q_2^2)(1+q_1^3 q_2^3).$$

2.2. **Cherednik algebras.** Let c be a W-invariant function on S. Let $H_c(W, \mathfrak{h})$ be the corresponding rational Cherednik algebra (see e.g. [E1]). Namely, $H_c(W, \mathfrak{h})$ is the quotient of $\mathbb{C}[W] \ltimes T(\mathfrak{h} \oplus \mathfrak{h})$ (with the two generating copies of $\mathfrak{h}$ spanned by x_a, y_a, $a \in \mathfrak{h}$), by the defining relations

$$[x_a, x_b] = [y_a, y_b] = 0, [y_a, x_b] = (a, b) - \sum_{s \in S} c_s(\alpha_s, a)(\alpha_s, b)s.$$

Let $M_c = H_c(W, \mathfrak{h}) \otimes_{\mathbb{C}W \ltimes \mathbb{C}[y_a]} \mathbb{C}$, where y_a act in $\mathbb{C}$ by 0 and $w \in W$ by 1. Then we have a natural vector space isomorphism $M_c \cong \mathbb{C}[\mathfrak{h}]$. For this reason M_c is called *the polynomial representation* of $H_c(W, \mathfrak{h})$. The elements y_a act in this representation by Dunkl operators (see [E1]).

The following proposition is standard, see e.g. [E1].

Proposition 2.3. There exists a unique W-invariant symmetric bilinear form β_c on M_c such that $\beta_c(1, 1) = 1$, which satisfies the contravariance condition

$$\beta_c(y_a v, v') = \beta_c(v, x_a v'), \ v, v' \in M_c, a \in \mathfrak{h}.$$

Polynomials of different degrees are orthogonal under β_c. Moreover, the kernel of β_c is the maximal proper submodule J_c of M_c, so M_c is reducible iff β_c is degenerate.

Let $L_c = M_c/J_c$ be the irreducible quotient of M_c. The module L_c is called *the irreducible spherical representation* of $H_c(W, \mathfrak{h})$.

3. THE MAIN THEOREM - THE CASE OF EQUAL PARAMETERS

3.1. **Statement of the theorem.** The goal of this paper is to determine the support of L_c as a $\mathbb{C}[\mathfrak{h}]$-module. We start with the case when c is a constant function. We will assume that $c \in (\mathbb{Q} \setminus \mathbb{Z})_{>0}$; otherwise, it is known from [DJO] that M_c is irreducible, so $L_c = M_c$, and the support of L_c is the whole space $\mathfrak{h}$. Let m be the denominator of c (written in lowest terms).

Theorem 3.1. A point $a \in \mathfrak{h}$ belongs to the support of L_c if and only if
$$\frac{P_W}{P_{W_a}}(e^{2\pi i c}) \neq 0,$$
i.e., if and only if
$$\#\{i | m \text{ divides } d_i(W)\} = \#\{i | m \text{ divides } d_i(W_a)\}.$$

Remark 3.2. The equivalence of the two conditions in Theorem 3.1 follows from the Bott-Solomon formula (1) for P_W.

Corollary 3.3. L_c is finite dimensional if and only if $\dfrac{P_W}{P_{W'}}(e^{2\pi i c}) = 0$, i.e., if and only if
$$\#\{i | m \text{ divides } d_i(W)\} > \#\{i | m \text{ divides } d_i(W')\},$$
for any maximal parabolic subgroup $W' \subset W$.

We note that Varagnolo and Vasserot [VV] proved that if W is a Weyl group then L_c is finite dimensional if and only if there exists a regular elliptic element in W of order m (i.e. an element with no eigenvalue 1 in $\mathfrak{h}$ and an eigenvector v not fixed by any reflection, see [Sp]). A direct uniform proof of the equivalence of this condition to the condition of Corollary 3.3, based solely on the theory of finite reflection groups, is given in the appendix to this paper, written by S. Griffeth.

Remark 3.4. If W is a Weyl group, then the values of the denominator m of $c > 0$ for which L_c is finite dimensional are listed in [VV]. Let us list these values in the noncrystallographic cases.

For dihedral groups $I_2(p)$: $m \geq 2$ is any number dividing p (this follows from the paper [Chm]).

For H_3: $m = 2, 6, 10$ (this is due to M. Balagovic and A. Puranik).

For H_4: m is any divisor of a degree of H_4, i.e. $m = 2, 3, 4, 5, 6, 10, 12, 15, 20, 30$.

3.2. **Proof of Theorem 3.1.**

3.2.1. *Tempered distributions.* Let $\mathcal{S}(\mathbb{R}^n)$ be the set of Schwartz functions on $\mathbb{R}^n$, i.e.

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | \forall \alpha, \beta, \sup |\mathbf{x}^\alpha \partial^\beta f(\mathbf{x})| < \infty\}.$$

This space has a natural topology.

A tempered distribution on $\mathbb{R}^n$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$. Let $\mathcal{S}'(\mathbb{R}^n)$ denote the space of tempered distributions.

We will use the following well known lemma (see [H]).

Lemma 3.5. (i) $\mathbb{C}[\mathbf{x}]e^{-\mathbf{x}^2/2} \subset \mathcal{S}(\mathbb{R}^n)$ is a dense subspace.

(ii) Any tempered distribution ξ has finite order, i.e., $\exists N = N(\xi)$ such that if $f \in \mathcal{S}(\mathbb{R}^n)$ satisfies $f = \mathrm{d}f = \cdots = \mathrm{d}^{N-1}f = 0$ on $\mathrm{supp}\xi$, then $\langle \xi, f \rangle = 0$.

3.2.2. *The Macdonald-Mehta integral.* The Macdonald-Mehta integral is the integral

$$F_W(c) := (2\pi)^{-r/2} \int_{\mathfrak{h}_\mathbb{R}} e^{-\mathbf{x}^2/2} |\Delta_W(\mathbf{x})|^{-2c} \mathrm{d}\mathbf{x}.$$

It is convergent for $\mathrm{Re}(c) \leq 0$.

The following theorem gives the value of the Macdonald-Mehta integral.

Theorem 3.6. One has

$$F_W(c) = \prod_{i=1}^r \frac{\Gamma(1 - d_i c)}{\Gamma(1 - c)}.$$

This theorem was conjectured by Macdonald and proved by Opdam [O1] for Weyl groups and by F. Garvan (using a computer) for H_3 and H_4 (for dihedral groups, the formula follows from Euler's beta integral). Later, a uniform and computer-free proof for all Coxeter groups was given in [E2].

3.2.3. *The Gaussian inner product.* Let a_i be an orthonormal basis of $\mathfrak{h}$, and $\mathbf{f} = \frac{1}{2} \sum y_{a_i}^2$. Introduce the *Gaussian inner product* on M_c as follows:

Definition 3.7. The Gaussian inner product γ_c on M_c is given by the formula

$$\gamma_c(v, v') = \beta_c(\exp(\mathbf{f})v, \exp(\mathbf{f})v').$$

This makes sense because the operator $\mathbf{f}$ is locally nilpotent on M_c.

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Proposition 3.8. ([Du], Theorem 3.10) [2] For $\mathrm{Re}(c) \leq 0$, one has

$$\gamma_c(P, Q) = \frac{(2\pi)^{-r/2}}{F_W(c)} \int_{\mathfrak{h}_{\mathbb{R}}} e^{-\mathbf{x}^2/2} |\Delta_W(\mathbf{x})|^{-2c} P(\mathbf{x}) Q(\mathbf{x}) \mathrm{d}\mathbf{x},$$

where P, Q are polynomials.

3.2.4. *Proof of Theorem 3.1.* Consider the distribution

$$\xi_c^W = \frac{(2\pi)^{-r/2}}{F_W(c)} |\Delta_W(\mathbf{x})|^{-2c}.$$

It is well-known that this distribution extends to a meromorphic distribution in c (Bernstein's theorem). Moreover, since $\gamma_c(P, Q)$ is a polynomial in c for any P and Q, this distribution is in fact holomorphic in $c \in \mathbb{C}$.

Proposition 3.9.

$$\begin{aligned}
\mathrm{supp}(\xi_c^W) &= \{a \in \mathfrak{h}_{\mathbb{R}} | \frac{F_{W_a}}{F_W}(c) \neq 0\} = \{a \in \mathfrak{h}_{\mathbb{R}} | \frac{P_W}{P_{W_a}}(e^{2\pi i c}) \neq 0\} \\
&= \{a \in \mathfrak{h}_{\mathbb{R}} | \#\{i | \text{denominator of } c \text{ divides } d_i(W)\} \\
&\qquad = \#\{i | \text{denominator of } c \text{ divides } d_i(W_a)\}\}.
\end{aligned}$$

Proof. First note that the last equality follows from the Bott-Solomon formula (1) for the Poincaré polynomial, and the second equality from Theorem 3.6. Now let us prove the first equality.

Look at ξ_c^W near $a \in \mathfrak{h}$. Equivalently, we can consider

$$\xi_c^W(\mathbf{x} + a) = \frac{(2\pi)^{-r/2}}{F_W(c)} |\Delta(\mathbf{x} + a)|^{-2c}$$

with $\mathbf{x}$ near 0. We have

$$\begin{aligned}
\Delta_W(\mathbf{x} + a) &= \prod_{s \in S} \alpha_s(\mathbf{x} + a) = \prod_{s \in S} (\alpha_s(\mathbf{x}) + \alpha_s(a)) \\
&= \prod_{s \in S \cap W_a} \alpha_s(\mathbf{x}) \cdot \prod_{s \in S \setminus S \cap W_a} (\alpha_s(\mathbf{x}) + \alpha_s(a)) \\
&= \Delta_{W_a}(\mathbf{x}) \cdot G(\mathbf{x}),
\end{aligned}$$

where G is a nonvanishing function near 0 (since $\alpha_s(a) \neq 0$ if $s \notin S \cap W_a$).

So near 0, we have

$$\xi_c^W(\mathbf{x} + a) = \frac{F_{W_a}}{F_W}(c) \cdot \xi_c^{W_a}(\mathbf{x}) \cdot |G(\mathbf{x})|^{-2c},$$

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[2] A proof of this theorem can also be found in [ESG] (Proposition 4.9).

and the last factor is well defined since G is nonvanishing. Thus $\xi_c^W(\mathbf{x})$ is nonzero near a if and only if $\frac{F_{W_a}}{F_W}(c) \neq 0$, which finishes the proof. $\square$

Now consider the support of L_c. Note that $\mathfrak{h}$ has a stratification by stabilizers of points in W, and by the results of [Gi] (see also [BE]), the support of L_c is a union of strata of this stratification.

Proposition 3.10. For any $c \in \mathbb{C}$,
$$\text{supp}(\xi_c^W) = (\text{supp}L_c)_\mathbb{R},$$
where the right hand side denotes the set of real points of the support.

Proof. Let $a \notin \text{supp}L_c$ and assume $a \in \text{supp}\xi_c^W$. Then we can find a $P \in J_c = \ker\gamma_c$ such that $P(a) \neq 0$. Pick a compactly supported test function $\phi \in C_c^\infty(\mathfrak{h}_\mathbb{R})$ such that P does not vanish anywhere on $\text{supp}\phi$, and $\langle \xi_c^W, \phi \rangle \neq 0$ (this can be done since $P(a) \neq 0$ and ξ_c^W is nonzero near a). Then we have $\phi/P \in \mathcal{S}(\mathfrak{h}_\mathbb{R})$. Thus from Lemma 3.5(i) it follows that there exists a sequence of polynomials P_n such that
$$P_n(\mathbf{x})e^{-\mathbf{x}^2/2} \to \frac{\phi}{P} \text{ in } \mathcal{S}(\mathfrak{h}_\mathbb{R}), \text{ when } n \to \infty.$$
So $PP_n e^{-\mathbf{x}^2/2} \to \phi$ in $\mathcal{S}(\mathfrak{h}_\mathbb{R})$, when $n \to \infty$.

But by Proposition 3.8, we have $\langle \xi_c^W, PP_n e^{-\mathbf{x}^2/2} \rangle = \gamma_c(P, P_n)$. Hence, $\langle \xi_c^W, PP_n e^{-\mathbf{x}^2/2} \rangle = 0$, which is a contradiction. This implies that $\text{supp}\xi_c^W \subset (\text{supp}L_c)_\mathbb{R}$.

To establish the opposite inclusion, let P be a polynomial on $\mathfrak{h}$ which vanishes identically on $\text{supp}\xi_c^W$. By Lemma 3.5(ii), there exists N such that $\langle \xi_c^W, P^N(\mathbf{x})Q(\mathbf{x})e^{-\mathbf{x}^2/2} \rangle = 0$. Thus, using Proposition 3.8, we see that for any polynomial Q, $\gamma_c(P^N, Q) = 0$, i.e. $P^N \in \text{Ker}\gamma_c$. Thus, $P|_{\text{supp}L_c} = 0$. This implies the required inclusion, since $\text{supp}\xi_c^W$ is a union of strata. $\square$

Theorem 3.1 follows from Proposition 3.9 and Proposition 3.10.

4. The main theorem - the case of non-equal parameters

4.1. **Statement of the theorem.**
Consider now the case when W is an irreducible Coxeter group with two conjugacy classes of reflections. In this case, $c = (c_1, c_2)$, and by $e^{2\pi i c}$ we will mean the pair (q_1, q_2), where $q_j = e^{2\pi i c_j}$, $j = 1, 2$.

Define a *positive* line in the plane with coordinates (c_1, c_2) to be any line of the form $a_1 c_1 + a_2 c_2 = b$, where $a_1, a_2 \geq 0, b > 0$.

Theorem 4.1. A point $a \in \mathfrak{h}$ belongs to the support of L_c if and only if there is no positive line passing through c on which the function $z \mapsto \frac{P_W}{P_{W_a}}(e^{2\pi i z})$ identically vanishes.

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Corollary 4.2. L_c is finite dimensional if and only if for every maximal parabolic subgroup $W' \subset W$, there exists a positive line ℓ passing through c such that the function $\frac{P_W}{P_{W'}}(e^{2\pi i z})$ vanishes on ℓ.

4.2. **Computation of points c for which L_c is finite dimensional.** Let us use Corollary 4.2 to compute explicitly the set Σ_c of points c for which L_c is finite dimensional. The computation is straightforward using Propositions 2.1 and 2.2 (although somewhat tedious), so we will only give the result.

1. The dihedral case, $I_2(2m)$. In this case, the set Σ_c is the union of the following lines and isolated points.

1) The lines are $c_1 + c_2 = \frac{r}{m}$, where $r \in \mathbb{N}$ and r is not divisible by m.

2) The isolated points are $(\frac{p_1}{2}, \frac{p_2}{2})$, where p_j are odd positive integers. This description coincides with the one of [Chm].

2. The case F_4. In this case, the set Σ_c is the union of the following lines and isolated points.

1) The lines are $c_1 + c_2 = \frac{p}{4}$ and $c_1 + c_2 = \frac{p}{6}$, where p is an odd positive integer.

2) The isolated points are:

2a: $(\frac{p_1}{2}, \frac{p_2}{2})$, where p_1, p_2 are odd positive integers;

2b: $(\frac{p_1}{3}, \frac{p_2}{3})$, where p_1, p_2 are positive integers not divisible by 3;

2c: $(\frac{p_1}{3}, \frac{p_2}{4} - \frac{p_1}{6})$ and $(\frac{p_2}{4} - \frac{p_1}{6}, \frac{p_1}{3})$, where p_2 is an odd positive integer and p_1 is a positive integer not divisible by 3;

2d: $(\frac{2p_2 - p_1}{6}, \frac{2p_1 - p_2}{6})$, where p_1, p_2 are odd positive integers such that $p_1 + p_2$ is not divisible by 3.

3. The case B_n, $n \geq 2$. In this case, let c_1 correspond to long roots. Then the set Σ_c is the union of the following lines and isolated points.

1) The lines are $(n-1)c_1 + c_2 = \frac{p}{2}$, where p is an odd positive integer.

2) The isolated points are $(\frac{r}{n}, \frac{p}{2} - r + \frac{rs}{n})$, where r is a positive integer not divisible by n, p is an odd positive integer, and $2 \leq s \leq \frac{n}{\gcd(r,n)}$ is an integer.

Remark 4.3. Note that in the case $c_1 = c_2$, we recover precisely the result of Varagnolo and Vasserot, [VV] for W of types B_n, F_4 and G_2, while setting $c_2 = 0$, we recover their result for W of type D_n.

4.3. **Proof of Theorem 4.1.** First we need to formulate the appropriate generalization of the Macdonald-Mehta integral. Let S_1, S_2 be the sets of reflections in W of the first and second kind, and let

$$\Delta_{W,j}(\mathbf{x}) = \prod_{s \in S_j} (\alpha_s, \mathbf{x}).$$

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Define the Macdonald-Mehta integral with two parameters:

$$F_W(c_1, c_2) := (2\pi)^{-r/2} \int_{\mathfrak{h}_\mathbb{R}} e^{-\mathbf{x}^2/2} |\Delta_{W,1}(\mathbf{x})|^{-2c_1} |\Delta_{W,2}(\mathbf{x})|^{-2c_2} d\mathbf{x}.$$

As before, it is convergent for $\operatorname{Re}(c_j) \leq 0$.

The following theorem gives the value of the two-parameter Macdonald-Mehta integral.

Theorem 4.4. (i) For dihedral groups $I_2(2m)$, one has

$$F_W(c_1, c_2) = \frac{\Gamma(1 - 2c_1)}{\Gamma(1 - c_1)} \frac{\Gamma(1 - 2c_2)}{\Gamma(1 - c_2)} \frac{\Gamma(1 - m(c_1 + c_2))}{\Gamma(1 - (c_1 + c_2))}.$$

(ii) For Weyl groups, one has

$$F_W(c_1, c_2) = \prod_{\alpha > 0} \frac{\Gamma(1 - c_\alpha - c_1 \mathrm{ht}_1(\alpha) - c_2 \mathrm{ht}_2(\alpha))}{\Gamma(1 - c_1 \mathrm{ht}_1(\alpha) - c_2 \mathrm{ht}_2(\alpha))}.$$

Proof. (i) follows from Euler's beta integral, and (ii) is proved in [O1]. $\square$

Also, we need an analog of the integral formula for the Gaussian form γ_c. This analog is given by the following proposition, whose proof is a straightforward generalization of the proof of Proposition 3.8.

Proposition 4.5. For $\operatorname{Re}(c_j) \leq 0$, one has

$$\gamma_c(P, Q) = \frac{(2\pi)^{-r/2}}{F_W(c_1, c_2)} \int_{\mathfrak{h}_\mathbb{R}} e^{-\mathbf{x}^2/2} |\Delta_{W,1}(\mathbf{x})|^{-2c_1} |\Delta_{W,2}(\mathbf{x})|^{-2c_2} P(\mathbf{x}) Q(\mathbf{x}) d\mathbf{x},$$

where P, Q are polynomials.

Now we are ready to prove Theorem 4.1. Consider the distribution

$$\xi_c^W = \frac{(2\pi)^{-r/2}}{F_W(c_1, c_2)} |\Delta_{W,1}(\mathbf{x})|^{-2c_1} |\Delta_{W,2}(\mathbf{x})|^{-2c_2}.$$

As before, this distribution extends to a meromorphic distribution in c (by Bernstein's theorem), and since $\gamma_c(P, Q)$ is a polynomial in c for any P and Q, this distribution is in fact holomorphic in c.

Proposition 4.6. One has

$$\mathrm{supp}(\xi_c^W) = \{a \in \mathfrak{h}_\mathbb{R} | \frac{F_{W_a}}{F_W}(c) \neq 0\}.$$

Proof. The proof is parallel to the proof of Proposition 3.9. $\square$

Corollary 4.7. A point $a \in \mathfrak{h}_{\mathbb{R}}$ belongs to the support of ξ_c^W if and only if there is no positive line passing through c on which the function $z \mapsto \frac{P_W}{P_{W_a}}(e^{2\pi i z})$ identically vanishes.

Proof. The Corollary follows from Propositions 4.6 and 2.1 and Theorem 4.4, using the bijective correspondence between the factors in the product formulas for $P_W(q_1, q_2)$ in Proposition 2.1 and the Γ-factors in the product formulas for $F_W(c_1, c_2)$ in Theorem 4.4. $\square$

Proposition 4.8. For any $c \in \mathbb{C}^2$,
$$\mathrm{supp}(\xi_c^W) = (\mathrm{supp} L_c)_{\mathbb{R}}.$$

Proof. Parallel to the proof of Proposition 3.10, using Proposition 4.5. $\square$

Proposition 4.6 and 4.8 imply Theorem 4.1.

5. APPLICATION: THE ZERO SET OF THE KERNEL OF THE RENORMALIZED MACDONALD PAIRING

As an application of the above technique, let us compute the zero set of the kernel of the renormalized Macdonald pairing in the trigonometric setting (in the equal parameter case).

Let R be a reduced irreducible root system of rank r, W be the Weyl group of R, R_+ a system of positive roots, P the weight lattice of R, and $H = \mathrm{Hom}(P, \mathbb{C}^*)$ be the complex torus attached to R. Let $\mathfrak{h} = \mathrm{Lie}(H)$. For $a \in H$, the stabilizer W_a is a reflection subgroup of W which is not necessarily parabolic; such a subgroup is called a quasiparabolic subgroup.

The following lemma is well known.

Lemma 5.1. If $W' \subset W$ is any reflection subgroup, then $P_W(q)$ is divisible by $P_{W'}(q)$.

Proof. $P_W/P_{W'}$ is the character of the generators of the module $\mathbb{C}[\mathfrak{h}]^{W'}$ over $\mathbb{C}[\mathfrak{h}]^W$. $\square$

Let
$$D_R = \prod_{\alpha \in R_+} (e^\alpha - 1)$$
be the Weyl denominator of R. Let $\mathbb{C}[H]$ denote the algebra of regular functions on H. Let $c \in \mathbb{C}$. Recall that the *Macdonald pairing* on the space $\mathbb{C}[H]$ (or $\mathbb{C}[H]^W$) is defined by the formula
$$\langle P, Q \rangle_c := \int_{H_{\mathbb{R}}} |D_R(\mathbf{t})|^{-2c} P(\mathbf{t}) Q(\mathbf{t}) d\mathbf{t},$$

where $H_{\mathbb{R}}$ is the maximal compact subgroup of H. It is known that this pairing is well defined when $\mathrm{Re}(c) < 0$, and develops poles when $c > 0$ and $e^{2\pi i c}$ is a root of P_W.

For $c \in \mathbb{C}$, ler d be the order of the pole of the Macdonald pairing at c. Define the *renormalized Macdonald pairing* by the formula

$$(P, Q)_c = \lim_{k \to c} (k - c)^d \langle P, Q \rangle_k.$$

This pairing is well defined and nonzero for any c.

Moreover, it is easy to see that the kernel I_c of this pairing is an ideal in $\mathbb{C}[H]$; in fact, this ideal is invariant under the trigonometric Dunkl operators, so the quotient ring $V_c := \mathbb{C}[H]/I_c$ is a representation of the trigonometric Cherednik algebra $H_c^{\mathrm{trig}}(W, H)$ (i.e., degenerate double affine Hecke algebra, see [Ch], [EM]). Our goal is to find the support $\mathrm{supp}(V_c)$ as a $\mathbb{C}[H]$-module, i.e. the zero set of I_c.

The main result of this section is the following theorem.

Theorem 5.2. Suppose that $c \in \mathbb{Q}_{>0}$. A point $a \in H$ belongs to $\mathrm{supp}(V_c)$ if and only if the polynomial P_W/P_{W_a} takes a nonzero value at $e^{2\pi i c}$.

Proof. We have a stratification of H by stabilizers of points in W. By the results of [BE], $\mathrm{supp}(V_c)$ is a union of strata of this stratification. Thus, it suffices to prove the result for $a \in H_{\mathbb{R}}$.

Consider the distribution on $H_{\mathbb{R}}$ defined by the formula

$$\xi_c^R := \frac{(2\pi)^{-r/2}}{F_W(c)} |D_R|^{-2c}.$$

This distribution is defined for all c, and up to a scalar,

$$(P, Q)_c = \langle \xi_c^R, PQ \rangle.$$

Let $\mathrm{supp}(V_c)_{\mathbb{R}}$ be the intersection of $\mathrm{supp}(V_c)$ with $H_{\mathbb{R}}$.

Proposition 5.3. A point $a \in H_{\mathbb{R}}$ belongs to $\mathrm{supp}(\xi_c^R)$ if and only if the polynomial P_W/P_{W_a} takes a nonzero value at $e^{2\pi i c}$.

Proof. The proof is parallel to the proof of Proposition 3.9. Namely, for small $\mathbf{x} \in \mathfrak{h}_{\mathbb{R}}$, we have

$$\xi_c^R(ae^{\mathbf{x}}) = \frac{F_{W_a}(c)}{F_W(c)} \xi_c^{W_a}(\mathbf{x}) f(\mathbf{x}),$$

where f is a nonvanishing smooth function. So the result follows. $\square$

Proposition 5.4. The set $\mathrm{supp}(V_c)_{\mathbb{R}}$ coincides with the support $\mathrm{supp}(\xi_c^R)$ of the distribution ξ_c^R.

Proof. The proof is parallel to the proof of Proposition 3.10. $\square$

Theorem 5.2 follows from Proposition 3.9 and Proposition 3.10. □

Example 5.5. Let R be the root system of type B_2. Then $P_W(q) = (1+q)^2(1+q^2)$, so setting $q = e^{2\pi i c}$, we see that at $c = 1/2$ (i.e. $q = -1$), the only points $a \in H = (\mathbb{C}^*)^2$ for which P_W/P_{W_a} does not vanish at q are $(1,1)$ ($W_a = W$) and $(-1,-1)$ ($W_a = \mathbb{Z}_2 \times \mathbb{Z}_2$, $P_{W_a} = (1+q)^2$). So the support of V_c is the set consisting of these two points.

This example shows that unlike the rational case, the module V_c is not necessarily irreducible. Namely, local analysis near the two points of the support using the results of [Chm] shows that V_c is the direct sum of a 1-dimensional irreducible representation supported at $(-1, -1)$ and a 4-dimensional irreducible representation supported at $(1, 1)$.

6. APPENDIX

by Stephen Griffeth

Let W be a finite real reflection group with reflection representation $\mathfrak{h}$. Recall that an *elliptic element* of W is an element not contained in any proper parabolic subgroup, or, equivalently, an element with fix space $\{0\}$ in $\mathfrak{h}$. Recall also that a positive integer m is a *regular number* for W if there is an element $g \in W$ that has a regular eigenvector (i.e., one not fixed by any reflection) with eigenvalue a primitive m-th root of 1. (By Theorem 4.2(i) of [Sp], in this case the order of g is m; such elements are called *regular*). If in addition this element g can be chosen to be elliptic, then m is called an *elliptic number* for W.

Let $d_1(W), \ldots, d_r(W)$ be the degrees of W, and let m be a positive integer. Denote by $a_W(m)$ the number of degrees divisible by m: $a_W(m) = \#\{1 \le i \le r \mid m \text{ divides } d_i(W)\}$.

The purpose of this appendix is to give a uniform proof of the following theorem.

Theorem 6.1. Let W be a finite real reflection group. Then m is an elliptic number for W if and only if for every maximal parabolic subgroup W' of W, one has $a_W(m) > a_{W'}(m)$.

Proof. First suppose m is an elliptic number for W. This means that there exists an elliptic element $b \in W$ and a regular vector $v \in \mathfrak{h}$ such that $bv = \zeta v$, where ζ is a primitive m-th root of unity. Assume towards a contradiction that $a_W(m) \le a_{W'}(m)$ for some maximal parabolic subgroup W'. Then by part (i) of Theorem 3.4 of [Sp], $a_W(m) = a_{W'}(m)$, and there is an element $g \in W'$ so that the ζ-eigenspace of g has dimension exactly equal to $a_W(m)$. By part (iv) of Theorem 4.2 of [Sp], the elements b and g are conjugate in W. This is a contradiction, since b is an elliptic elememt, and g is not.

Conversely, assume the inequalities in the statement of the theorem hold. These inequalities together with Part (i) of Theorem 3.4 of [Sp] imply that for any primitive m-th root of unity ζ there exists an element $g \in W$ with ζ-eigenspace of dimension $a_W(m)$, and the fix space of any such g in $\mathfrak{h}$ is zero (i.e., g is elliptic). Since W is a real reflection group, this implies that the determinant of g on $\mathfrak{h}$ is $(-1)^r$ (i.e., is independent of g). Examining the left hand side of the equation in Corollary 2.6 of [LM] shows that the term $(-T)^{a_W(m)}$ occurs with non-zero coefficient. Hence, looking at the right hand side of this equation, we see that the number of codegrees of W divisible by m is $a_W(m)$. Now part (ii) of Theorem 3.1 of [LM] implies that m is a regular number, and hence elliptic. $\qquad\square$

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