A note on light geometric graphs

Eyal Ackerman∗ Jacob Fox† Rom Pinchasi‡

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Abstract

Let $G$ be a geometric graph on $n$ vertices in general position in the plane. We say that $G$ is $k$-light if no edge $e$ of $G$ has the property that each of the two open half-planes bounded by the line through $e$ contains more than $k$ edges of $G$. We extend the previous result in [1] and with a shorter argument show that every $k$-light geometric graph on $n$ vertices has at most $O(n\sqrt{k})$ edges. This bound is best possible.

Keywords: Geometric graphs, $k$-near bipartite.

1 Introduction

Let $G$ be an $n$-vertex geometric graph. That is, $G$ is a graph drawn in the plane such that its vertices are distinct points and its edges are straight-line segments connecting corresponding vertices. It is usually assumed, as we will assume in this paper, that the set of vertices of $G$ is in general position in the sense that no three of them lie on a line.

A typical question in geometric graph theory asks for the maximum number of edges that a geometric graph on $n$ vertices can have assuming a forbidden configuration in that graph. This is a popular area of study extending classical extremal graph theory, utilizing diverse tools from both geometry and combinatorics. For example, an old result of Hopf and Pannwitz [3] and independently Sutherland [7] states that any geometric graph on $n$ vertices with no pair of disjoint edges has at most $n$ edges. This is a special case of Conway’s thrackle conjecture.

Let $e$ be an edge of $G$. We say that $G$ has a $k$-light side with respect to $e$, if one of the two open half-planes bounded by the line through $e$ contains at most $k$ edges of $G$. If $G$ has a $k$-light side with respect to every edge $e$, then we say that $G$ is $k$-light. In other words, $G$ is $k$-light if no edge of $G$ has the property that each of the two open half-planes bounded by the line through $e$ contains more than $k$ edges of $G$.

The notion of a $k$-light graph is a weakening of the notion of a $k$-near bipartite graph defined in [1]. A graph $G$ is $k$-near bipartite if every line in the plane bounds an open half plane containing at most $k$ edges of $G$. Therefore, every $k$-near bipartite graph is also a $k$-light graph. It is shown in [1] that $k$-near bipartite graphs on $n$ vertices contain $O(\sqrt{k}n)$ edges. In this paper we prove the same result for $k$-light graphs, thus strengthening the result in [1]. Moreover, our proof is much shorter but on the other hand relies on other results about geometric graphs.

∗Department of Mathematics, Physics, and Computer Science, University of Haifa at Oranim, Tivon 36006, Israel. ackerman@sci.haifa.ac.il.
†Department of Mathematics, MIT, Cambridge, MA 02139-4307. Research supported by a Simons Fellowship and NSF grant DMS-1069197. fox@math.mit.edu.
‡Mathematics Department, Technion—Israel Institute of Technology, Haifa 32000, Israel. room@math.technion.ac.il. Supported by BSF grant (grant No. 2008290).
2 The maximum number of edges in $k$-light geometric graphs

We are interested in the maximum number of edges of an $n$-vertex $k$-light geometric graph. A simple construction from [1] shows an $n\sqrt{k}$ lower bound for $k \leq (\frac{9}{4} - 1)^2$, even for $k$-near bipartite graphs. In this construction every line contains at most $k$ edges of $G$ in one of the two open half-planes bounded by it. Another construction of a $k$-light graph with $n\sqrt{k}$ edges is obtained by taking the vertices of a regular $n$-gon and connecting by edges vertices whose cyclic distance is at most $\sqrt{k}$. In this construction, however, it is no longer true that every line bounds an open half-plane containing at most $k$ edges of $G$.

Our main result shows that these constructions are essentially best possible.

**Theorem 1.** Let $n$ and $k$ be positive integers. Every $n$-vertex $k$-light geometric graph has at most $O(n\sqrt{k})$ edges.

**Proof.** Let $G$ be an $n$-vertex $k$-light geometric graph with $m$ edges. We orient every edge $e$ of $G$ in such a way that the open half-plane bounded to the left of $e$ contains at most $k$ edges of $G$. Because $G$ is $k$-light such an orientation exists.

We will need the following two lemmas.

**Lemma 2.1.** Let $G$ be an oriented geometric graph on $n$ vertices. There exists an absolute constant $c_3$ such that if $G$ has more than $c_3n$ edges, then it contains an edge $e$ such that the open half-plane bounded to the left of $e$ contains an edge of $G$.

**Proof.** It is enough to show that in any (unoriented) geometric graph $G$ with $n$ vertices and sufficiently many (that is, at least $c_3n$) edges there is an edge $e$ such that each of the two open half-planes bounded by the line through $e$ contains an edge of $G$. This is in fact the case $k = 1$ in Theorem 1 that we wish to prove. The reader is encouraged to find a simple proof of this fact. Here we will rely on a rather elaborate argument of Valtr [8] that proves a much stronger statement than what we need.

We refer the reader to [5, 4, 8]. Two edges of a geometric graph are called *avoiding* or sometimes *parallel* if no line passing through one edge meets the other edge. Equivalently, two edges are avoiding if they are opposite edges in a convex quadrilateral.

The notion of avoiding edges was first defined by Kupitz [5], who conjectured that any geometric graph on $n$ vertices with more than $2n - 2$ edges must contain a pair of avoiding edges. In [4] it is shown that if a graph $G$ on $n$ vertices does not contain a pair of avoiding edges, then the number of edges in $G$ is at most $2n - 1$. In [8] Valtr improved this bound by one, completing the proof of Kupitz’ conjecture. He further generalized this result, showing that for any fixed $k$, every geometric graph with more than $c_3n$ edges contains $k$ pairwise avoiding edges. Here $c_3$ is an absolute constant that depends only on $k$.

In fact, Valtr’s result is a bit stronger. Looking into the proof in [8] reveals that he actually shows that a geometric graph with more than $c_3n$ edges contains $k$ edges $e_1, \ldots, e_k$ that are pairwise avoiding, but what is more important to our needs is that the line through $e_i$ separates $e_1, \ldots, e_{i-1}$ from $e_{i+1}, \ldots, e_k$. More specifically, Valtr defines three partial orders on a set of edges in $G$ and any chain with respect to any of the partial orders is a collection of such edges. It is then shown that if the number of edges in $G$ is large enough, then there exists a chain of length $k$ in one of the partial orders.

Thus, for the case $k = 3$ it follows that if $G$ contains more than $c_3n$ edges, then there are three pairwise avoiding edges $e, f, g$ such that the line through $f$ separates $e$ and $g$. This immediately implies Lemma 2.1 as in any orientation of $f$ the half-plane bounded to the left of $f$ will contain an edge of $G$.

**Lemma 2.2.** Let $G$ be an oriented geometric graph on $n$ vertices with $m$ edges. There exists a positive absolute constant $d$ with the following property. If the number of edges in $G$ is greater than $2c_3n$ (where $c_3$ is the constant from Lemma 2.1), then $G$ contains at least $dn^3/n^2$ pairs of edges $(e, f)$ such that the open half-plane bounded to the left of $e$ contains $f$.
Proof. This is by now a quite standard consequence of the result in Lemma 2.1 and is carried out by a similar probabilistic technique used to derive a similar bound for the number of pairs of crossing edges in a geometric graph (see p. 55 in [6], also p. 45 in [2]).

Denote by \( x(G) \) the number of pairs of edges \((e, f)\) in \( G \) such that the open half-plane bounded to the left of \( e \) contains \( f \). Pick every vertex of \( G \) independently with probability \( p \), and denote by \( G' = (V', E') \) the subgraph of \( G \) that is induced by the chosen vertices. Clearly, \( E[|V'|] = pn \), \( E[|E'|] = p^2m \), and \( E[x(G')] = p^4x(G) \). On the other hand, it follows from Lemma 2.1 that \( x(G') \geq |E'| - c_3|V'| \), and this holds also for the expected values: \( E[x(G')] \geq E[|E'|] - c_3E[|V'|] \). Plugging in the expected values and setting \( p = \frac{2c_3n}{m} < 1 \) we get that \( x(G) \geq \frac{1}{8c_3^2} \frac{m}{n^2} \).

Let \( c_3 \) and \( d \) be the constants from Lemmas 2.1 and 2.2. Clearly we may assume that \( G \) contains at least \( 2c_3n \) edges or else we are done. By Lemma 2.2, \( G \) contains at least \( dm^3/n^2 \) pairs \((e, f)\) of edges such that the open half-plane bounded to the left of \( e \) contains \( f \). However, by the choice of orientation of the edges in \( G \), an edge \( e \) can belong to at most \( k \) such pairs \((e, f)\). We conclude that \( dm^3/n^2 \leq km \). This now easily implies that \( m \leq \frac{1}{\sqrt{d}}n\sqrt{k} \) as desired.

References