NUMERICAL STUDY OF TIME DEPENDENT PERTURBATION OF
A QUANTUM MECHANICAL HARMONIC OSCILLATOR

by

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Signature of Author.

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Accepted by . . . .

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Chairman, Departmental Committee on Theses
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I  Abstract

This paper deals with the problem of a quantum mechanical harmonic oscillator with a harmonic perturbation. The problem is solved by numerical integration of the time dependent Schroedinger equation on an IBM 7094 and the results are displayed as graphs of the state populations for various perturbation frequencies. These results are in agreement with time dependent perturbation theory where that theory is valid and with a classical calculation of the energy of the oscillator.
II Summary

The problem at hand is that of a quantum mechanical harmonic oscillator with no damping acted upon by a harmonic driving force turned on at time equal to zero. The classical differential equation is

\[ m \frac{d^2x}{dt^2} = -kx + F(t) \]

where \( m \) is the mass, \( k \) is the spring constant and \( F(t) \) is the driving force. In this case \( F(t) = -B \phi(t) \sin \omega t \) where

\[ \begin{align*}
\phi(t) &= 0 & t < 0 \\
\phi(t) &= 1 & t > 0
\end{align*} \]

Quantum mechanically, the problem is formulated in terms of the time dependent Schroedinger equation:

\[ \frac{-\hbar^2}{2m} \frac{\delta^2 \psi}{\delta x^2} + V \psi = i \hbar \frac{\delta \psi}{\delta t} \]

where

\[ V = \frac{kx^2}{2} + \phi(t) Bx \sin \omega t \]

Classically we solve for the position as a function of time and determine the energy of the oscillator. Quantum mechanically we solve for the wavefunction, \( \psi(x,t) \), from which the expectation value of the energy can be calculated. An exact quantum mechanical solution is determined numerically and an approximate solution by perturbation theory. These two solutions are shown to be in agreement with the classical solution.

The results of the numerical solution are shown in Appendix #1 as graphs of the state populations, \( |a_n|^2 \), for various values of time.
and of the driving frequency where

\[ a_n = \int_{-\infty}^{+\infty} u_n(x) \psi(x,t) \, dx \]

are the expansion coefficients. In this expression \( u_n(x) \) are the harmonic oscillator eigenfunctions. These graphs will eventually be presented in the form of a computer generated motion picture showing the development of the state populations in time.
III Numerical Integration of the Time Dependent Schroedinger Equation\(^*\)

In order to calculate an exact solution to this problem, it is necessary to use numerical integration of the time dependent Schroedinger equation or of the differential equations for the expansion coefficients. We choose to integrate the Schroedinger equation:

\[
\frac{\hbar^2}{2m} \frac{\delta^2 \psi}{\delta x^2} + V(x,t) = i\hbar \frac{\delta \psi}{\delta t}.
\]

At this point we adopt a system of units, suitable for machine computation, in which \(\hbar = 1\) and \(m = 1/2\). In this system the above equation becomes

\[
-H\psi = \frac{\delta^2 \psi}{\delta x^2} - \psi = -i \frac{\delta \psi}{\delta t}.
\]

(\#1)

In order to derive a finite difference equation, time is denoted by a superscript, \(n\), and spatial position by a subscript, \(j\). \(\psi^n_j\) replaces \(\psi(j\epsilon, n\delta)\) where \(\epsilon\) is the spatial mesh width and \(\delta\) is the time step.

We start by deriving an expression for \(\frac{\delta^2 \psi^n_j}{\delta x^2}\). Here we can ignore the time superscript since it will remain constant for the spatial derivative. We write Taylor expansion series for \(\psi_{j+1}\) and \(\psi_{j-1}\) as follows:

\[
\begin{align*}
\psi_{j+1} &= \psi_j + \epsilon \psi'_j + \frac{1}{2} \epsilon^2 \psi''_j + \frac{1}{6} \epsilon^3 \psi'''_j + \ldots; \\
\psi_{j-1} &= \psi_j - \epsilon \psi'_j + \frac{1}{2} \epsilon^2 \psi''_j - \frac{1}{6} \epsilon^3 \psi'''_j + \ldots.
\end{align*}
\]

Adding and solving for \(\psi''_j\) we find

\[
\psi''_j = \frac{1}{\epsilon^2}(\psi_{j+1} - 2\psi_j + \psi_{j-1}).
\]

* The material in this and the following section has been developed from the following paper which should be consulted for details on the numerical integration procedure:
This is correct to order $\epsilon^2$. It replaces the second derivative of $\psi_j$ in the Hamiltonian, thus

$$H\psi = -\left[\frac{\delta^2}{\delta x^2} - V\right]_{j} = \frac{1}{\epsilon^2}[\psi_{j+1} - 2\psi_j + \psi_{j-1}] + V_j \psi_j \quad (#2)$$

To treat the time development of $\psi_j^n$, we consider the formal solution to equation #1,

$$\psi(x,t) = e^{-i(t-t_0)H} \psi(x,t_0)$$

or,

$$\psi_{j}^{n+1} = e^{-i\delta H} \psi_{j}^{n}$$

Approximating $e^{-i\delta H}$ by the Cayley form (see reference 1)

$$e^{-i\delta H} = (1 - \frac{i}{2}\delta H) / (1 + \frac{i}{2}\delta H)$$

we have

$$(1 + \frac{i}{2}\delta H) \psi_{j}^{n+1} = (1 - \frac{i}{2}\delta H) \psi_{j}^{n}.$$  

Carrying out the necessary algebra and using equation #2 for the Hamiltonian, we get the following:

$$\psi_{j+1}^{n+1} + (i\lambda - \epsilon^2 V_{j}^{n+1} - 2) \psi_{j}^{n+1} + \psi_{j-1}^{n+1} = -\psi_{j+1}^{n} + (i\lambda + \epsilon^2 V_{j}^{n} + 2) \psi_{j}^{n} - \psi_{j-1}^{n} \quad (#3)$$

where $\lambda=2\epsilon^2/\delta$.

This implicit equation for $\psi_{j}^{n}$ is solved in the manner discussed in reference 1. The equation is integrated in a restricted region of the variable $x$ outside of which the wavefunction is required to be zero at all times, i.e. $\psi_{0}^{n}=\psi_{J}^{n}=0$ where $J$ is the maximum value of the spatial index. In order to carry out this integration, we must also specify the initial values of the wavefunction, $\psi_{j}^{0}$ for all $j$.

Once $\psi_{j}^{n}$ is calculated for all $j$ at a single time, the expansion coefficients are calculated as
\[ a_n(t) = \int_{-\infty}^{+\infty} u_n(x) \psi(x,t) \, dx. \]

This integral was approximated by the following sum for computational purposes:

\[ a_k(t) = \epsilon \sum_{j=1}^{\infty} (u_k)_j \psi^n_j. \]
IV Criteria for Choice of Parameters for Numerical Integration

One of the assumptions made in solving the finite difference equation was that the wavefunction always be equal to zero at the limits of the interval over which the integration was performed. That is, \( \psi(-A,t)=\psi(A,t)=0 \), where \( \pm A \) are the ends of the interval. This may be insured by limiting the amplitude and/or the duration of the perturbation so that only those states whose eigenfunctions approximately satisfy these conditions will be populated. In our numerical calculation, we choose to limit the perturbation so that the ninth state would be the highest state populated. For a spring constant \( k=2 \), the case that we considered, the value of \( A \) was set at 20 which is four times the turning radius for a classical oscillator with energy equal to the ninth energy eigenvalue of the quantum mechanical oscillator.

In order that the two approximations made in the derivation of the finite difference equation (#3) be valid, we must make sure that the spatial mesh width and the time step are sufficiently small. Considering first the errors introduced by replacing the continuous variable \( x \) by the index \( j \), we note that the minimum distance that can be resolved be the approximation for \( \psi''_j \) is \( \varepsilon \). We can approximate the range of \( x \) over which the wavefunction is non-negligible by the range of the classical oscillation with energy equal to the energy eigenvalue of the highest eigenfunction appearing to a significant amount in the eigenfunction expansion of the wavefunction. Using the fact that the number of nodes in an eigenfunction is equal to the quantum number, \( n \), of the state, and assuming these nodes to be equally spaced, we find the distance between them is of the order of

\[
\frac{2\sqrt{2n+1}}{n+1} \frac{\sqrt{\omega_0/k}}{
\]
This quantity must be large compared with \( \varepsilon \). In this expression \( n \) is
the largest oscillator quantum number included in the eigenfunction
expansion.

We now approximate the eigenfunction by a sine having the proper
number of nodes between the two classical turning points. Hence

\[ u_n \approx \sin \frac{\pi (x + C)(n+1)}{C} \]

where \( C \) gives the range of motion of the corresponding classical oscillator.

From this we have

\[ \frac{d^2 u_n}{dx^2} = \frac{\pi^2 (n+1)^2}{C^2} u_n . \]

The difference approximation to this derivative is

\[ \frac{\Delta^2 u_n}{\Delta x^2} = - \frac{u_n(x+\varepsilon) - 2u_n(x) + u_n(x-\varepsilon)}{\varepsilon^2} = - \frac{2\pi(1 - \cos \frac{\pi(n+1)\varepsilon}{C})}{\varepsilon^2} u_n . \]

For \( \frac{\pi(n+1)\varepsilon}{C} \) sufficiently small, the cosine may be approximated by the
first two terms of a Taylor series, yielding

\[ \frac{\Delta^2 u_n}{\Delta x^2} \approx \frac{\pi^2 (n+1)^2}{C^2} (1 - \frac{\pi^2 (n+1)^2 \varepsilon^2}{12C^2}) u_n . \]

In order that this be a good approximation to the second derivative, we
must have

\[ \frac{\pi^2 (n+1)^2 \varepsilon^2}{12C^2} \ll 1 . \]

We now consider the limitations on the size of the time step imposed
by the finite difference approximation. We can write the time development
of an eigenfunction of the unperturbed potential as

\[ u_n(x,t) = e^{-iE_nt} u_n(x,0) . \]
For a single time step this is

$$u_n(x,t+\delta) = e^{-iE_n\delta} u_n(x,t) .$$  (#4)

The finite difference approximation to this time development is

$$u_n(x,t+\delta) = \left[1 - i\delta E_n/2\right] u_n(x,t) = e^{-2i \tan^{-1}(E_n\delta/2)} u_n(x,t) .$$

We expand the inverse tangent in a Taylor series to obtain

$$u_n(x,t+\delta) = \left[e^{-iE_n\delta} e^{iE_n^3\delta^3/12}\right] u_n(x,t) .$$  (#5)

Comparing equations #4 and #5 we find that the effect of the finite difference is to introduce a phase error approximately equal to \(E_n^3\delta^3/12\) for each time step. After \(N\) time steps, the cumulative phase error is thus \(NE_n^3\delta^3/12\). In order to minimize this error we require

$$NE_n^3\delta^3/12 \ll 1 .$$

In summary the inequalities involving the time step and the mesh width are

$$\varepsilon < 2\sqrt{2n+1} \sqrt{\omega_0/k}$$

$$\frac{k\pi^2(n+1)^2\varepsilon^2}{24E_n} \ll 1 \quad \quad \quad E_n = (n+1/2) \hbar \omega_0 .$$  (#6)

and \(NE_n^3\delta^3/12 \ll 1 .\)

Since these are inequalities, they provide only upper limits on the values used for \(\varepsilon\) and \(\delta\). The values of \(\varepsilon\) and \(\delta\) used to produce the graphs shown in Appendix 2 were obtained by assuming reasonable values for \(\varepsilon\) and \(\delta\). These values were then decreased until there was no significant further change in the calculated values of \(a_n(t)\) for further
reductions in the time step. These results were also checked against
time dependent perturbation theory (see Section VI) where that theory
was valid and against a classical calculation discussed below in Section
V. The values of $\delta$ and $\epsilon$ used were

$$\delta = 0.031$$

and $$\epsilon = 0.04$$.

If $n$ is set equal to 6, the highest state significantly populated at
$N=400$ with $B$ (the strength of the perturbation) equal to 0.25, the three
inequalities (equation #6) become

$$\frac{2\sqrt{2n+1}}{n+1} \frac{\sqrt{\omega_o/k}} = 1.03 >> 0.04$$

$$k \pi^2 (n+1)^2 \epsilon^2 = 0.0076 << 1$$

$$NE_n^3 \delta^3/12 = 0.218 << 1$$.

In spite of the third inequality being rather poorly satisfied, the
calculations were considered satisfactory since further reduction of the
time step produced little change in the results and since the energy
calculated at $N=400$ was within .25% of the energy calculated classically.
This error was of course considerably smaller for many of the smaller
$N$ and was considered quite acceptable.
V Classical Solution

For this section and the section on time-dependent perturbation theory, we return to natural units.

Classically we solve for the position of a harmonic oscillator by using the Green's function for the oscillator:

\[ g(t,t_0) = \frac{1}{m\omega_0} \sin\omega_0 t . \]

This is the response at time \( t \) to a unit impulse at time \( t_0 \). In this expression \( \omega_0 \) is the classical angular frequency, \( \omega_0 = \sqrt{k/m} \). If the oscillator is initially \( t=0 \) at rest, the position at a time \( t \) is

\[ x(t) = \int_0^t g(t,t_0) F(t_0) \, dt_0 . \]

Substituting \( F(t_0) = -B \sin \omega_0 t_0 \) as the driving force and evaluating the integral for the case \( \omega = \omega_0 \) (resonance) we get

\[ x(t) = -\frac{B}{2m\omega_0^2} (\sin \omega_0 t - \omega_0 t \cos \omega_0 t) . \]

We would like this classical oscillator to be equivalent to the quantum mechanical oscillator considered previously. Since the quantum mechanical oscillator was originally in the ground state with energy \( \frac{\hbar \omega_0}{2} \), we initialize the motion of the classical oscillator so that at \( t=0 \) it has energy \( \frac{\hbar \omega_0}{2} \). The position for times less than zero becomes:

\[ x(t) = \sqrt{\frac{\hbar \omega_0}{k}} \sin \omega_0 (t-t') . \]

In this equation \( t' \) is a constant of integration representing a phase difference between the driving force and the initial motion.

Thus we have
\[ x(t) = -\frac{B}{2m\omega_o^2} \left[ \sin\omega_o t - \omega_o t \cos\omega_o t \right] + \sqrt{\hbar\omega_o/k} \sin\omega_o (t-t'). \] (#7)

Differentiating we find

\[ \frac{dx(t)}{dt} = -\frac{B}{2m\omega_o} \left( \omega_o t \sin\omega_o t + \omega_o \sqrt{\hbar\omega_o/k} \cos\omega_o (t-t') \right). \] (#8)

The energy is

\[ E(t) = \frac{kx^2(t)}{2} + \frac{m\dot{x}(t)^2}{2}. \]

Substituting equations #7 and #8 into this equation for energy we get:

\[ E(t) = \frac{\hbar\omega_o}{2} + \frac{B^2 t^2}{8m} + \frac{B^2 \sin^2\omega_o t}{8k} - \frac{B^2 t}{4m\omega_o} (\sin\omega_o t \cos\omega_o t) \]

\[ -\frac{B\sqrt{\hbar\omega_o/k}}{2} \sin\omega_o t \sin\omega_o (t-t') - \frac{Bt\sqrt{\hbar\omega_o/k}}{2} \sin\omega_o t'. \]

Since the quantum mechanical problem is phrased in terms of probabilities which are meaningful for an ensemble of oscillators, it seems most reasonable to assume that all possible starting phases will be represented equally. Alternatively this may be considered in terms of the uncertainty principle: initially the energy is known exactly, therefore there must be an infinite uncertainty in the time. Thus we average the above expression for \( E(t) \) over all values of \( t' \) and get

\[ E(t) = \frac{\hbar\omega_o}{2} + \frac{B^2 t^2}{8m} + \frac{B^2 \sin^2\omega_o t}{8k} - \frac{B^2 t}{4m\omega_o} \sin\omega_o t \cos\omega_o t \quad (\omega=\omega_o) \] (#9)

Carrying out a similar derivation for the off resonance case we get:

\[ E(t) = \frac{\hbar\omega_o}{2} + \frac{B}{2m(\omega^2 - \omega_o^2)^2} [(\omega \sin\omega_o t - \omega_o \sin\omega t)^2 + \omega^2 (\cos\omega_o t - \cos\omega t)^2] \]

\[ (\omega \neq \omega_o) \] (#10)

As we would expect, the energy increases indefinitely in the resonant case while it oscillates, returning to \( \frac{\hbar\omega_o}{2} \) periodically in the off-resonance case.
From time-dependent perturbation theory, we get the following formula for the expansion coefficients:

\[ a_k = (\hbar i)^{-1} \sum_n H'_{kn} a_n e^{i\omega_{kn}t} \]

where \( H'_{kn} \) is the matrix element of the perturbation

\[ H'_{kn} = \int_{-\infty}^{+\infty} \bar{u}_k H' u_n \, dx \]

\( H' \) is the perturbation of the Hamiltonian, and \( \omega_{kn} \) is the Bohr angular frequency

\[ \omega_{kn} = \frac{E_k - E_n}{\hbar} \]

\( E_k \) and \( E_n \) are the energy eigenvalues of the kth and nth states respectively.

Integrating the expression for \( \dot{a}_k \) and assuming that \( a_n = \delta_{mn} \), we find

\[ a_k(t) = (\hbar i)^{-1} \int_0^t H'_{km}(t') e^{i\omega_{km}t'} \, dt'. \]

If we let \( H'_{km}(t') = H'_{km} \sin \omega t' \) and integrate, we find

\[ a_k(t) = -\frac{H'_{km}}{2i\hbar} \left[ \frac{e^{i(\omega_{km} + \omega)t} - 1}{\omega_{km} + \omega} - \frac{e^{i(\omega_{km} - \omega)t} - 1}{\omega_{km} - \omega} \right] \quad (\omega \neq \omega_{km}) \]

\[ a_k(t) = \frac{H'_{km}}{2\hbar} \left[ \frac{\sin \omega t \, e^{-i\omega t} - \omega t}{\omega} \right] \]
\( (\omega = \omega_{km}) \)

Taking the absolute squares we get

\[ |a_k(t)|^2 = \frac{|H'_{km}|^2}{\hbar^2 (\omega^2 - \omega_{km}^2)^2} \left[ \omega^2 (\cos \omega_{km} t - \cos \omega t)^2 + (\omega \sin \omega_{km} t - \omega k \sin \omega t)^2 \right] \]
\( (\omega \neq \omega_{km}) \) (#11)

and
\[
|a_k(t)|^2 = \left| \frac{H_{km}^0}{\omega} \right|^2 \left( \sin^2 \omega t - 2\omega t \cos \omega t \sin \omega t + \omega^2 t^2 \right) (\omega=\omega_0). (#12)
\]

For the harmonic oscillator with \( h' = Bx \sin \omega t \), we can calculate \( H_{km}^0 \) very easily. We know the value of the integral
\[
\int_{-\infty}^{\infty} \bar{u}_n(x) \cdot u_m(x) \, dx = \begin{cases} \frac{1}{\alpha} \left( \frac{n+1}{2} \right)^{1/2} & \text{for } m = n+1 \\ \frac{1}{\alpha} \left( \frac{n}{2} \right)^{1/2} & \text{for } m = n \\ 0 & \text{otherwise} \end{cases}
\]

where \( \alpha = \frac{\sqrt{mk}}{\hbar^2} \). Since \( H_{km}^0 = Bx \), \( H_{km}^0 \) is equal to \( B \) times the above integral. Thus according to first order time dependent perturbation theory this perturbation will cause transitions only to the two states adjacent to the initial state. If the initial state is the ground state, \( n=0 \), the only state to which transitions can occur will be the first excited state \( n=1 \). We note that first order perturbation theory gives \( a_0 = 1 \) with \( a_1 \neq 0 \). It is obvious that this approximation is non-unitary, i.e. it destroys the normalization of the wavefunction. However, by slightly modifying the theory we can obtain an approximation for \( |a_0|^2 \) and \( |a_1|^2 \) which will be correct in second order and will be unitary. We let \( a_1 \) have the value predicted by first order perturbation theory, but set \( |a_0|^2 = 1 - |a_1|^2 \). This approximation obviously insures the normalization of the wavefunction and can be shown to have an error term of third order. Using this approximation we can calculate the time average expectation value of the energy as
\[
E(t) = \int \bar{\psi} H_0 \psi = |a_0|^2 E_0 + |a_1|^2 E_1 = (1-|a_1|^2)E_0 + |a_1|^2 E_1.
\]
Substituting $E_n = \frac{\hbar \omega}{2} (2n+1)$ in the above, we find

$$E(t) = \frac{\hbar \omega}{2} (1 + 2 |a_1|^2).$$

The value of $|a_1|^2$ is given by equations #11 and #12 for the on and off resonance cases respectively. Using these equations with the above equation for $E(t)$, results in expressions for energy identical to equations #7 & #8 (the equations for the energy calculated classically.)

Since the time dependent perturbation theory is valid for small perturbations, we may conclude that the classical calculations give the correct result for such perturbations. It is unlikely that classical calculations would be correct near the ground state energy and become incorrect between this quantum mechanical region and the classical region, thus we may tentatively conclude that the classical calculations are correct for all amplitudes and durations of the perturbation. Comparison with the results obtained by numerical integration will show this to be true.
VIII Results and Discussion

The results of these numerical calculations are presented graphically in Appendix #1 for three different perturbation frequencies; \( \omega = \frac{3}{4} \omega_0 \), \( \omega = \omega_0 \), and \( \omega = \frac{5}{4} \omega_0 \). The calculations were also performed for \( \omega = \frac{1}{2} \omega_0 \) and \( \omega = 2 \omega_0 \), but the resulting state populations were less than 2% for the first state and less than 0.02% for the second state and were therefore not suitable for graphic display. Appendix #2 contains a brief summary of the results of the calculations for perturbations at one-half and twice the resonant frequency.

In the graphs of \( |q_n|^2 \) with the perturbation at the resonant frequency transitions to higher states are apparent. Quantum mechanically, the oscillator will accept energy indefinitely at the classical resonant frequency. In the off resonance cases we find that the oscillator accepts energy for a time, but eventually starts to return the energy to the potential, returning to the ground state for times \( t = \frac{2m\pi}{|\omega - \omega_0|} \) where \( m \) is an integer. In both the resonance and off resonance cases, the agreement with classical theory and with time-dependent perturbation theory is quite good. (see Tables #1 and #2 for numerical examples.)

By comparing the state populations predicted by time dependent perturbation theory with the state populations calculated by numerical integration, we can come to some significant conclusions concerning the region of validity of time dependent perturbation theory (eg. see Table #3). Generally the condition for the applicability of time dependent perturbation theory is that the perturbation be "small", but this criterion of smallness is not very well defined. As it turns out, the effect of a perturbation is determined by its frequency
as well as its amplitude. Judging by the numerical comparison shown in Table #3, we can conclude that time dependent perturbation theory will be accurate if the total depletion of the initial state as calculated by that theory is less than about 2%. When the initial state is depleted by more than this amount there will be significant population of the two states immediately adjacent to the initial state. Transitions from these newly populated states will then occur resulting in population of states not immediately adjacent to the initial state. Population of such states is not possible according to first order perturbation theory.

The criteria for the validity of time dependent perturbation theory are probably more stringent in the harmonic oscillator problem than in most other physical systems. In other systems, the energy levels are generally not equally spaced. If transitions to an excited state are caused by a certain frequency, there is little probability of transition from this excited state to any state other than the initial state, since a different frequency would be required to cause these secondary transitions.

In conclusion, we feel that the graphs presented in this thesis provide a useful insight into the behaviour of the harmonic oscillator and other quantum mechanical systems with harmonic time dependent perturbations. The three derivations of the average energy as a function of time (exact quantum mechanical, time dependent perturbation theory, and classical) elucidate the correspondence between the quantum mechanical and the classical harmonic oscillator. Although the present numerical integration is too slow for the production of movies, we hope to be able to reformulate the problem in terms of the differential equations
for the expansion coefficients, thereby allowing the production of movies for pedagogic purposes.
Table #1

<table>
<thead>
<tr>
<th>N</th>
<th>t</th>
<th>classical E(t)</th>
<th>quantum E(t)</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0.00</td>
<td>1.0000</td>
<td>1.0000</td>
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<tr>
<td>40</td>
<td>1.24</td>
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<td>400</td>
<td>12.40</td>
<td>3.4625</td>
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</table>

This table compares the energy calculated classically for $\omega=\omega_0=2$ with the time average expectation value of the quantum mechanical energy (i.e., $\sum |a_n|^2 E_n$). $N$ is the number of time steps, $t$ is the time, and $E(t)$ is the energy.

Table #2

<table>
<thead>
<tr>
<th>N</th>
<th>t</th>
<th>classical E(t)</th>
<th>quantum E(t)</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>160</td>
<td>4.96</td>
<td>1.2075</td>
<td>1.2052</td>
</tr>
<tr>
<td>320</td>
<td>9.92</td>
<td>1.0699</td>
<td>1.0749</td>
</tr>
</tbody>
</table>

This table is the equivalent of table #1 with $\omega=1.5$ and $\omega_0=2.0$.

Table #3

| N | t  | perturbation theory $|a_1(t)|^2$ | numerical integration $|a_1(t)|^2$ |
|---|----|----------------|----------------|
| 0 | 0.00 | 0.00000 | 0.00000 |
| 40 | 1.24 | 0.01715 | 0.01688 |
| 80 | 2.48 | 0.05320 | 0.05046 |
| 120 | 3.72 | 0.09730 | 0.07127 |
| 160 | 4.96 | 0.17555 | 0.14737 |

Calculations with $\omega=\omega_0=2.0$. 


Appendix #1

The following six pages show computer generated graphs of the state populations for various perturbing frequencies and times. These were generated with B=0.25, \( \omega_0 = 2.0 \) and the values of \( \delta \) and \( \epsilon \) given in Section IV. The parabola in the graphs is the potential at each time, \( t \). The bar graphs represent the state populations and are placed along the \( y \) (energy) axis at a position corresponding to each state's energy eigenvalue. The \( x \) axis represents position for the parabola graphs with the end of the axis representing approximately 4.6 and represents state population for the bar graphs with the end of the axis representing a population of 1.0.
Appendix #2

Values of $|a_n|^2$ for $\omega = 2\omega_0$

| N  | t     | $|a_0|^2$ | $|a_1|^2$ | $|a_2|^2$ |
|-----|-------|-----------|-----------|-----------|
| 0   | 0.00  | 1.000000  | 0.000000  | 0.000000  |
| 40  | 1.24  | 0.992470  | 0.007496  | 0.000027  |
| 80  | 2.48  | 0.993661  | 0.006307  | 0.00019   |
| 120 | 3.72  | 0.995149  | 0.004820  | 0.00012   |
| 160 | 4.96  | 0.989851  | 0.010074  | 0.00074   |
| 200 | 6.20  | 0.999960  | 0.000010  | 0.000000  |
| 240 | 7.44  | 0.993640  | 0.006305  | 0.00019   |
| 280 | 8.68  | 0.993250  | 0.006687  | 0.00021   |
| 320 | 9.92  | 0.996549  | 0.003298  | 0.00006   |
| 360 | 11.16 | 0.987897  | 0.011979  | 0.00071   |
| 400 | 12.40 | 0.999827  | 0.000115  | 0.000000  |

for $\omega = \omega_0/2$

| N  | t     | $|a_0|^2$ | $|a_1|^2$ | $|a_2|^2$ |
|-----|-------|-----------|-----------|-----------|
| 0   | 0.00  | 1.000000  | 0.000000  | 0.000000  |
| 40  | 1.24  | 0.990423  | 0.009524  | 0.000046  |
| 80  | 2.48  | 0.979463  | 0.020315  | 0.000208  |
| 120 | 3.72  | 0.980954  | 0.018846  | 0.000180  |
| 160 | 4.96  | 0.987681  | 0.012218  | 0.000076  |
| 200 | 6.20  | 0.999969  | 0.000001  | 0.000000  |
| 240 | 7.44  | 0.992290  | 0.007644  | 0.000030  |
| 280 | 8.68  | 0.978525  | 0.021207  | 0.000225  |
| 320 | 9.92  | 0.982073  | 0.017110  | 0.000159  |
| 360 | 11.16 | 0.985560  | 0.014280  | 0.000106  |
| 400 | 12.40 | 0.999702  | 0.000014  | 0.000000  |
Bibliography
