18.03 Recitation Problems 21

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Vocabulary: System of ODEs; Linear: time-independent, homogeneous; Matrix, matrix multiplication; Solutions, initial conditions; Autonomous equations; Vector field, phase plane, trajectory; Equilibrium solutions; Companion matrix.

Notation. We will often write $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$ for a vector in the plane. \mathbf{i} is the unit vector pointing east, and \mathbf{j} is the unit vector pointing north. If $\mathbf{u}(t)$ is a parametrized curve, $\dot{\mathbf{u}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ is the velocity vector, tangent to the curve.

1. Define a vector field in the plane by putting the vector $y\mathbf{i}-x\mathbf{j}$ at the position (x,y). Sketch enough values of this vector field to visualize it and describe it in words. Then sketch a curve which is everywhere tangent to it and passes though the point (1,0).

The curve you drew is the trajectory of a solution of the system of ODEs $\dot{x} = y, \dot{y} = -x$. Solve this equation in the following way: substitute $\dot{y} = -x$ into the equation you get for \ddot{x} by differentiating $\dot{x} = y$. This gives you a second order LTI ODE. The initial conditions for x and y give initial conditions for this new ODE. Solve it.

Then graph x against t, graph y against t, and plot the path of the curve $\mathbf{u}(t)$ in the plane. Does it look right?

2. Now reverse engineer this, starting with the second order IVP

$$\ddot{x} + (1/2)\dot{x} + (17/16)x = 0,$$

with initial condition x(0) = 1, $\dot{x}(0) = 0$. Use $y = \dot{x}$ for one of the pair of equations. So: write an equation for \dot{y} in terms of x and y. Together this pair of equations determines $\dot{\mathbf{u}}$ in terms of \mathbf{u} . Solve the original second order ODE, and reinterpret your solution as a solution of the system you produced. Sketch graphs of x and of y as functions of t, and sketch the path of the curve $\mathbf{u}(t)$ (its "trajectory"). If you think of the variable x in the original equation as position, how is velocity, \dot{x} , represented in the picture of the trajectory?

3. Practice in matrix multiplication: Compute the following products:

$$\left[\begin{array}{cc}1&2\end{array}\right]\left[\begin{array}{c}x\\y\end{array}\right],\left[\begin{array}{c}1\\2\end{array}\right]\left[\begin{array}{c}x&y\end{array}\right],\left[\begin{array}{c}a&b\\c&d\end{array}\right]\left[\begin{array}{c}x\\y\end{array}\right],\left[\begin{array}{c}a&b\\c&d\end{array}\right]\left[\begin{array}{c}x&u\\y&v\end{array}\right].$$