# 18.03 Recitation Problems 21 <br> April 27, 2004 

Vocabulary: System of ODEs; Linear: time-independent, homogeneous; Matrix, matrix multiplication; Solutions, initial conditions; Autonomous equations; Vector field, phase plane, trajectory; Equilibrium solutions; Companion matrix.

Notation. We will often write $\mathbf{u}=x \mathbf{i}+y \mathbf{j}$ for a vector in the plane. $\mathbf{i}$ is the unit vector pointing east, and $\mathbf{j}$ is the unit vector pointing north. If $\mathbf{u}(t)$ is a parametrized curve, $\dot{\mathbf{u}}=\dot{x} \mathbf{i}+\dot{y} \mathbf{j}$ is the velocity vector, tangent to the curve.

1. Define a vector field in the plane by putting the vector $y \mathbf{i}-x \mathbf{j}$ at the position $(x, y)$. Sketch enough values of this vector field to visualize it and describe it in words. Then sketch a curve which is everywhere tangent to it and passes though the point $(1,0)$.
The curve you drew is the trajectory of a solution of the system of ODEs $\dot{x}=y, \dot{y}=-x$. Solve this equation in the following way: substitute $\dot{y}=-x$ into the equation you get for $\ddot{x}$ by differentiating $\dot{x}=y$. This gives you a second order LTI ODE. The initial conditions for $x$ and $y$ give initial conditions for this new ODE. Solve it.
Then graph $x$ against $t$, graph $y$ against $t$, and plot the path of the curve $\mathbf{u}(t)$ in the plane. Does it look right?
2. Now reverse engineer this, starting with the second order IVP

$$
\ddot{x}+(1 / 2) \dot{x}+(17 / 16) x=0,
$$

with initial condition $x(0)=1, \dot{x}(0)=0$. Use $y=\dot{x}$ for one of the pair of equations. So: write an equation for $\dot{y}$ in terms of $x$ and $y$. Together this pair of equations determines $\dot{\mathbf{u}}$ in terms of $\mathbf{u}$. Solve the original second order ODE, and reinterpret your solution as a solution of the system you produced. Sketch graphs of $x$ and of $y$ as functions of $t$, and sketch the path of the curve $\mathbf{u}(t)$ (its "trajectory"). If you think of the variable $x$ in the original equation as position, how is velocity, $\dot{x}$, represented in the picture of the trajectory?
3. Practice in matrix multiplication: Compute the following products:

$$
\left[\begin{array}{ll}
1 & 2
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right],\left[\begin{array}{l}
1 \\
2
\end{array}\right]\left[\begin{array}{ll}
x & y
\end{array}\right],\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right],\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]\left[\begin{array}{ll}
x & u \\
y & v
\end{array}\right] .
$$

