# a Coriolis tutorial

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September 15, 2003

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**Summary:**

This essay is an introduction to the dynamics of Earth-attached reference frames, written especially for students beginning a study of geophysical fluid dynamics. It is accompanied by four Matlab scripts available from the Mathworks File Central archive.

An Earth-attached and thus rotating reference frame is almost always used for the analysis of the atmosphere and ocean. The equation of motion transformed into a general rotating frame includes two terms that involve the rotation rate — a centrifugal term and a Coriolis term. In the special case of an Earth-attached frame the centrifugal term is exactly balanced by a small tangential component of gravitational mass attraction and so drops out of the dynamical equations. The Coriolis term that remains is a part of the acceleration seen from an inertial frame, but is interpreted as a force (an apparent force) when we solve for the acceleration seen from the rotating frame. The rotating frame perspective gives up the properties of global momentum conservation and invariance to Galilean transformations, but leads to a greatly simplified analysis of geophysical flows.

The Coriolis force has a very simple mathematical form, $\propto 2\mathbf{\Omega} \times \mathbf{V}'$, where $\mathbf{\Omega}$ is the rotation vector and $\mathbf{V}'$ is the velocity observed from the rotating frame. The Coriolis force acts to deflect the velocity without changing the speed and gives rise to two important modes of motion or balances within the momentum equation. When the Coriolis force is balanced by the time rate of change of the velocity the result is a clockwise rotating velocity (northern hemisphere) having a constant speed. These free oscillations, usually called inertial oscillations, are commonly observed in the upper ocean following a wind event. When the Coriolis force is balanced by a persistent external force (e.g., a large-scale pressure gradient) the mean velocity is perpendicular to the external force. This geostrophic balance is characteristic of the large-scale circulation around pressure anomalies in the atmosphere and ocean.

# 1 Introduction to geophysical flows.

The large-scale, horizontal flows of Earth's atmosphere and ocean are characterized by circulation around centers of high or low pressure. Global-scale circulations of this sort include the atmospheric jet stream that encircles the mid-latitudes, and the Antarctic circumpolar current. On a smaller scale the weather is often dominated by storms, for example hurricanes, that have a nearly circular flow around a low pressure center, and many regions of the ocean are filled with slowly revolving eddies that can have a high or low pressure anomaly. The (hydrostatic) pressure anomaly at a given level is a direct consequence of mass excess or deficit (high pressure or low pressure) in the overlying fluid. What is at first surprising is that large scale mass and pressure anomalies often persist for many days or weeks; the flow of mass that would otherwise accelerate down a pressure gradient and disperse the mass anomaly is observed to be deflected to the right in the northern hemisphere and to the left in the southern hemisphere. On time mean, winds and currents tend toward a geostrophic (Earth-turning) momentum balance in which the pressure gradient force is closely balanced by the deflecting force, and the result is a mean flow that is nearly parallel to lines of constant pressure.[1]

We attribute profound physical consequences to the deflecting force, called the Coriolis force,[2] and yet we cannot point to a physical interaction or cause of the Coriolis force in the direct way that we can for a hydrostatic pressure anomaly. Instead, the origin of the Coriolis force is found in kinematics and in our common practice to use an Earth-attached and thus rotating and noninertial reference frame. This makes the Coriolis force distinct from other forces in ways and with consequences that will be a theme of this essay.

## 1.1 Inertial and rotating reference frames

For the purpose of studying the Coriolis force we can idealize the motion of a fluid continuum by the motion of a single particle. If the particle is observed from an inertial reference frame, then the classical (Newtonian) equation of motion for the particle is just

$$\frac{d\mathbf{V}}{dt} = \mathbf{F}/M. \tag{1}$$

$\mathbf{V}$ is the velocity in a three-dimensional space, and $M$ is the particle's mass and $\mathbf{F}$ is the sum of the forces that we can specify *a priori* given the complete knowledge of the environment, e.g., a pressure gradient, frictional drag with the ground or sea floor, or gravitation. These are all central forces.[3]

This classical momentum equation has two fundamental properties that we remark upon here because we are about to give them up. **Global conservation:** For each of the central forces there will be a corresponding reaction force, $-\mathbf{F}$, acting on the part of environment that sets up the force $\mathbf{F}$. Thus the global change of momentum (particle plus the environment) due to the sum of all of the forces $\mathbf{F}$ is zero. Usually our attention is focused on the local problem, i.e., the particle only, with the global balance taken for granted and left unanalyzed. **Invariance to Galilean transformation:** A second fundamental property that is usually imposed upon Eq. (1) is that it should be invariant to Galilean transformations since there is no way to define or measure an absolute (as opposed to a relative) velocity. Thus a constant velocity added to $\mathbf{V}$ and to the environment should cause no change in the forces $\mathbf{F}$. Like the global balance just noted, this property is not invoked frequently, but is a very powerful guide to the possible appropriate forms of the forces $\mathbf{F}$. For example, a frictional force that satisfies Galilean invariance should depend upon differences of the velocity (say with respect to a surface or adjacent particles) and not the particle velocity only.

In practice we almost always observe and analyze the motion of the atmosphere and ocean from a reference frame that is attached to the rotating Earth. The equation of motion transformed into an Earth-attached, rotating reference frame has just one additional term,

$$\frac{d\mathbf{V}'}{dt} = -\ 2\mathbf{\Omega}\times\mathbf{V}' + \mathbf{F}'/M, \tag{2}$$

the Coriolis force (this transformation is done in detail below). The prime on a vector indicates that it is observed from the rotating frame, and $\mathbf{\Omega}$ is Earth's rotation vector. The Coriolis force has a very simple, linear form; it is perpendicular to the particle velocity and can do no work. The Coriolis force will, however, change the direction of the velocity unless balanced by another force, often a pressure gradient as noted in the opening paragraph. Given Eq. (2) we can begin to construct analyses and models that are suitable for an Earth-attached reference frame. A practical person would get on with the task, and so should you!

## 1.2 About this essay

But in this essay we take the time to indulge our curiosity — what *is* the Coriolis force? From the literature we might find any one of several plausible answers, that it is an acceleration, a pseudo force, a virtual force, an apparent force (our choice), and most equivocal of all, a fictitious correction force.[4] This is not the clear answer we had hoped for, and now we begin to wonder — is the Coriolis force real or is it a mathematical device to make things come out right? As we will see it is both, and to understand how that comes about we will address a sequence of more or less naive questions that probe the origin and some of the consequences of the Coriolis force, including:

1) Does the equation of motion (2) preserve the global conservation and Galilean transformation properties of Eq. (1)?

2) Does the factor 2 in the Coriolis force have any parallel or a larger significance?

3) What happened to the centrifugal force that also arises in a rotating frame? Are the Coriolis and centrifugal forces related?

4) Given that the Coriolis force causes only a deflection of velocity, does it affect energy balance?

5) Finally and most importantly for this essay, what is the Coriolis force and what should we call it?

A preview to this last question will show us where to begin. We have already indicated that the Coriolis and centrifugal forces are the result of the rotation of an Earth-attached reference frame and that the explanation of these forces is going to be largely kinematic (i.e., more mathematical than physical). Accordingly, we begin with the transformation of the inertial frame equation of motion into a rotating frame, developed in detail in Section 2. This important section is intended to be largely self-contained, however, readers are expected to come equipped with the basics of classical mechanics and with elementary vector and matrix algebra to include coordinate transformation. Two simple applications of the rotating frame momentum equations are considered in Section 3. These illustrate the marked difference between inertial and rotating frame descriptions of the same phenomenon. The special but important case of an Earth-attached reference frame suitable for analysis of atmospheric and oceanic motions is discussed in Section 4 and a problem that illustrates some aspects of adjustment toward a geostrophic balance is treated in Section 5. Closing remarks are in the final section, 6.

Rotating reference frames and the Coriolis force have been textbook fare for many years and there is nothing new said here. There are useful discussions in many intermediate-level classical mechanics texts[5] and in most fluid mechanics textbooks that treat geophysical flows.[6] This essay aims to supplement those sources by providing greater (and if it succeeds, clearer) mathematical detail than do most fluid dynamics texts. Compared with most physics texts, this essay emphasizes geophysical phenomenon.[7] A notable source is the monograph by Stommel and Moore[8] which examines the Coriolis force at roughly the level of this essay but at considerably greater length; the present Section 4.1 has been especially strongly influenced by their work.

The aim (and style) of this essay is purely pedagogical. It is written at an introductory level for students beginning a study of Earth science and especially geophysical fluid dynamics. It may be freely copied and distributed for personal, educational purposes and it may be cited as an unpublished manuscript available from the web address on the title page. There are four companion Matlab scripts that can be downloaded from an anonymous ftp site.[9] Comments and questions are encouraged and may be addressed to the author at jprice@whoi.edu.

# 2 Transformation of vectors into a rotating reference frame.

The analysis in this section shows how to relate a vector seen in a steadily rotating reference frame to the same vector seen in a stationary reference frame. The stationary reference frame, also dubbed the 'master' reference frame, is defined by a triad of orthogonal unit vectors, $\mathbf{e_1}, \ \mathbf{e_2}, \ \mathbf{e_3}$, that are time-independent (Fig. 1). The rotating reference frame is defined by a triad of unit vectors that are (not surprisingly) rotating with respect to the master frame. The presumption is that position, velocity and acceleration vectors are given in the master frame and that our task is to find out how the same vectors will appear when seen from the rotating frame.

## 2.1 Reference frames, unit vectors and vector components; a brief review

We briefly review how the components of a vector depend upon a reference frame. For example, a point P can be located by the tip of a position vector $\mathbf{X}$ that has Cartesian components $x_i$ in the master reference frame as

$$\mathbf{X} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3}. \tag{3}$$

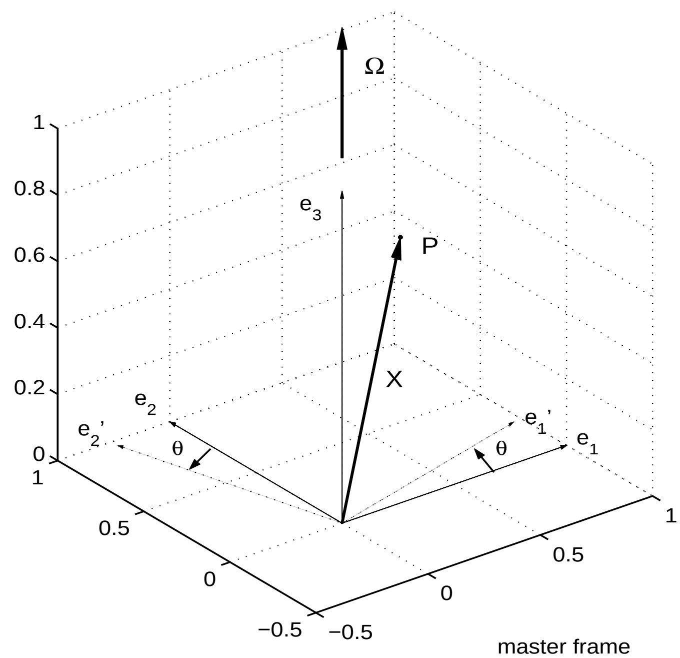


Figure 1: A point P is located by the tip of the vector **X**. The master reference frame has solid unit vectors that are presumed to be time-independent, and a rotating reference frame has dashed unit vectors. The reference frames have a common origin, and rotation is about the $\mathbf{e_3}$ axis at a rate $\Omega$ which is positive so that the unit vectors of the rotating frame turn counterclockwise.

The same vector can also be represented in a rotated or rotating reference frame by summing a different set of components and the unit vectors that define that frame as

$$\mathbf{X} = x'_1\mathbf{e'_1} + x'_2\mathbf{e'_2} + x'_3\mathbf{e'_3}. \tag{4}$$

The superscript prime $(\ \ )'$ is now all important as the indicator of the components and unit vectors of the rotated reference frame. The position vector **X** has a physical or objective existence, while the components $x'_i$ will necessarily vary according to the accidental orientation of the rotated reference frame. Thus we do not have a vector until we sum the product of the appropriate components and unit vectors.

To relate the components and unit vectors in the rotated frame to those of the master frame it is helpful to rewrite Eqs. (3) and (4) using a consistent matrix notation. The unit vectors are concatenated into a single 3x3 matrix

$$\mathbb{E} = \begin{bmatrix} \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{e_3} \end{bmatrix}, \tag{5}$$

where the first row of $\mathbb{E}$ is the (1x3) row matrix $\mathbf{e_1} = [\ 1\ \ 0\ \ 0\ ]$, etc. Assuming that the components are $x_j$ in a (3x1) column matrix, then Eqs. (3) and (4) may be written in a way that conforms with the usual matrix multiplication rules as

$$\mathbf{X} = \sum_i E_{ij}x_j = \sum_i E'_{ij}x'_j \tag{6}$$

where $j$ and then $i$ are summed from 1 to 3.

The unit vectors of the rotating frame are shown in Fig. (1) having been rotated an angle $\theta$ with respect to the master reference frame. The unit vectors that define the rotated frame, $\mathbf{e_i'}$, are related to the unit vectors that define the master frame, $\mathbf{e_i}$, by

$$E'_{ij} = E_{in} D^t_{nj} \tag{7}$$

where $\mathbb{D}^t$ is the transpose of the rotation matrix,[10] $\mathbb{D}$,

$$\mathbb{D} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8}$$

and thus

$$\mathbb{D}^t = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This particular rotation matrix corresponds to a rotation about the $\mathbf{e_3}$ axis and so leaves the unit vector in that direction unchanged.

The components of $\mathbf{X}$ in the rotated frame, $x'_j$, are then related to the components in the master reference frame by

$$x'_j = D_{jn} x_n, \tag{9}$$

and where the $(\ )'$ is used to indicate a rotated component. Note that $\mathbb{D}^t = \mathbb{D}(-\theta)$ so that the sense of rotation is reversed in going from $\mathbb{D}$ to $\mathbb{D}^t$ and thus

$$\mathbb{D}\mathbb{D}^t = \mathbb{D}^t\mathbb{D} = \mathbb{I}, \tag{10}$$

where $\mathbb{I}$ is the identity matrix that leaves vectors unchanged. To recall the important transformation rule, Eqs. (7) and (9), we need only remember the orthogonality property of $\mathbb{D}$ shown by Eq. (10) and notice that we can insert an identity matrix into Eq. (6) as

$$\mathbf{X} = \sum_i E_{ij} x_j = \sum_i E_{ij} I_{jn} x_n = \sum_i (E_{in} D^t_{nj})(D_{jn} x_n) = \sum_i E'_{ij} x'_j = \mathbf{X}, \tag{11}$$

given the associative law for matrix multiplication.

## 2.2 When seen from the rotating frame

For our purpose it is essential to know how a position vector and its time derivatives will appear when seen from the rotating frame.

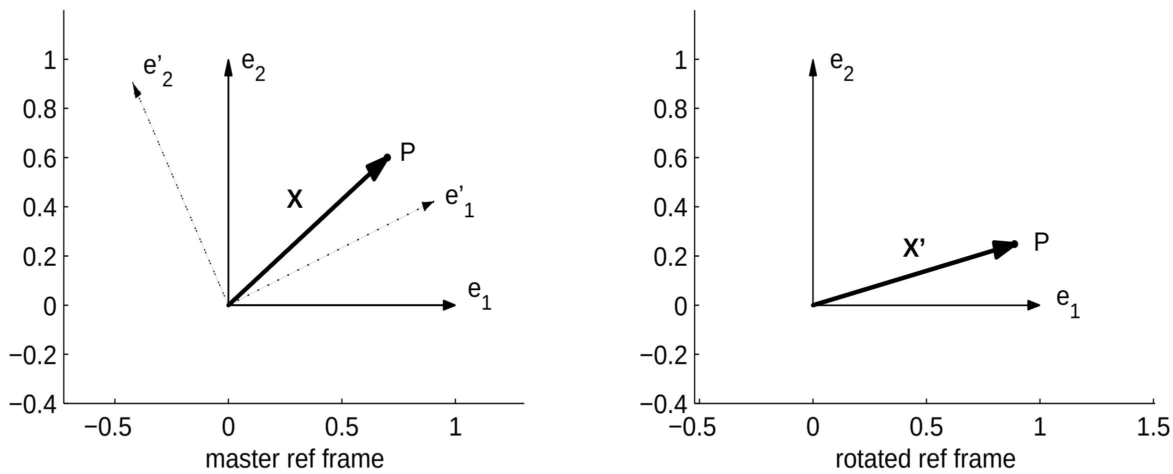


Figure 2: (left) A point P is located by the tip of the solid vector $\mathbf{X}$ in the master reference frame. (right) The point P as seen from the rotated frame, and the associated position vector $\mathbf{X}' = \sum_i E_{ij} x'_j$. By indicating that this is the 'rotated ref frame' we mean that the unit vectors used to construct the vector $\mathbf{X}'$ are fixed; that they are aligned with the page is not relevant. The meaning of $\mathbf{X} = \sum_i E_{ij} x_j = \sum_i E'_{ij} x'_j$ is that these two representations of the vector $\mathbf{X}$ are identical when the rotated reference frame is plotted on the master reference frame, as at left.

### 2.2.1 Position

To construct the position vector as seen from the rotating frame, $\mathbf{X}'$, we use the components $x'_i$ and a set of unit vectors that are fixed; we may as well use $\mathbb{E}$ since we already have it, and so

$$\mathbf{X}' = \sum_i E_{ij} x'_j. \tag{12}$$

The prime symbol on a vector thus means the primed (rotated) components summed with fixed unit vectors. For example, in Fig. (2, left), a position vector $\mathbf{X}$ is shown making an angle of about 50 degrees counterclockwise from the $\mathbf{e_1}$ axis. The rotated reference frame is presumed to be rotated about 30 degrees counterclockwise from the master frame (these angles are arbitrary). That being so, the position vector of point P viewed from the rotated reference frame makes an angle of 50 - 30 = 20 degrees to the $\mathbf{e'_1}$ axis, Fig. (2, right).

### 2.2.2 Velocity

The velocity of point P seen in the master frame is just the time rate of change of the position vector seen in that frame,

$$\frac{d\mathbf{X}}{dt} = \frac{d}{dt} \sum_i E_{ij} x_j = \sum_i E_{ij} \frac{dx_i}{dt},$$

since $E_{ij}$ is time-independent. The velocity of point P as seen from the rotating reference frame is similarly

$$\frac{d\mathbf{X}'}{dt} = \frac{d}{dt}\sum_i E_{ij}x'_j = \sum_i E_{ij}\frac{dx'_i}{dt}.$$

This last expression indicates that the time derivatives of the rotated components are going to be very important in what follows. For the first derivative we find

$$\frac{dx'_i}{dt} = \frac{d(D_{in}x_n)}{dt} = \frac{dD_{in}}{dt}x_n + D_{in}\frac{dx_n}{dt}. \tag{13}$$

The second term on the right side of Eq. (13) represents the velocity components from the master frame that have been rotated into the rotated frame. If the rotation angle $\theta$ is constant so that $\mathbb{D}$ is independent of time, then the velocity vector would transform exactly as does the position vector, cf. Eq. (9).

When the rotation angle is time-varying, as we presume it is here, then the first term on the right side of Eq. (13) is non-zero and represents the effect of the rotation. Specifically, we are going to take the angle $\theta$ to be

$$\theta = \theta_0 + \Omega t$$

with the rotation rate $\Omega$ a constant, an excellent approximation for the rotating Earth. The $\mathbf{e_3}$ unit vector is presumed to be aligned with the axis of rotation and thus $\mathbf{\Omega} = \Omega\mathbf{e_3}$. The rotation matrix is then time-dependent, and its time derivative is just

$$\frac{d\mathbb{D}}{dt} = \Omega\begin{bmatrix} -\sin\theta(t) & \cos\theta(t) & 0 \\ -\cos\theta(t) & -\sin\theta(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

Notice that this has all the elements of $\mathbb{D}(\theta(t))$, but shuffled around. By inspection this matrix can be factored into the product of $\mathbb{D}(\pi/4)$ and $\mathbb{D}$, where

$$\mathbb{D}(\tfrac{\pi}{4}) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

represents a rotation of $\pi/4$ in the direction opposite $\Omega$ (when left multiplied onto a column matrix of components). Thus the time derivative operating on this rotation matrix brings out the rotation frequency and shifts the phase by $\pi/4$,

$$\frac{d\mathbb{D}}{dt} = \Omega\ \mathbb{D}(\pi/4)\mathbb{D}(\theta(t)), \tag{16}$$

just like the time derivative operating on a scalar sine function. Substitution into Eq. (13) gives the velocity components in the rotating frame:

$$\frac{dx'_i}{dt} = \Omega D(\pi/4)_{ij}D(\theta)_{jn}x_n + D(\theta)_{in}\frac{dx_n}{dt}. \tag{17}$$

This can be simplified a little by using the prime notation to indicate rotated components:

$$\frac{dx'_i}{dt} = \Omega D(\pi/4)_{ij}x'_j + \left(\frac{dx_i}{dt}\right)'. \tag{18}$$

An interpretation using vectors can clarify the first term on the right hand side of Eq. (17). Assume that $\Omega > 0$ so that the rotating frame is turning counterclockwise, and assume that the point P is stationary in the master reference frame so that $dx_n/dt = 0$. Point P as viewed from the rotating frame will then appear to move clockwise at a rate that can be calculated from the geometry. Let the rotation in a time interval $\delta t$ be given by $\delta\theta = \Omega\delta t$; in that time interval the tip of the vector will move a distance given by the magnitude of the vector times $\delta\theta$, i.e., $\delta\mathbf{X}' = |\mathbf{X}'|\delta\theta$ and in a direction that is perpendicular to $\mathbf{X}'$ (Fig. 3). The velocity of point P seen from the rotating frame and due solely to the coordinate system rotation is thus

$$\lim_{\delta t \to 0} \frac{\delta\mathbf{X}'}{\delta t} = -\mathbf{\Omega}\times\mathbf{X}',$$

which is the vector equivalent of the first term of Eq. (17). If there is a velocity of P relative to the master frame, then that velocity (suitably rotated) will add to this rotation-induced velocity to give the velocity of P as seen from the rotating frame and thus

$$\frac{d\mathbf{X}'}{dt} = -\mathbf{\Omega}\times\mathbf{X}' + \left(\frac{d\mathbf{X}}{dt}\right)', \tag{19}$$

which is the vector version of Eq. (17). Notice that we could just as well construct this vector from the components of Eq. (18) and the $\mathbb{E}$ unit vectors.

The relation Eq. (19) between the time rate of change of a vector as seen from the rotating frame and as seen from the master frame holds for all vectors, as the matrix form Eq. (17) shows most clearly.[11] Thus the relationship between the time derivatives may be written as the operator equation

$$\frac{d(\ )'}{dt} = -\mathbf{\Omega}\times(\ )' + \left(\frac{d(\ )}{dt}\right)'. \tag{20}$$

From left to right the terms of this important equation are 1) the time rate of change of a vector as seen in the rotating reference frame, 2) the cross product of the rotation vector with the vector, and 3) the time rate change of the vector as seen in the master frame and then rotated into the rotating frame. If we are observing or analyzing from a rotating frame, then term 1) is the time rate of change that we observe directly or that we seek to solve.

### 2.2.3 Acceleration

Our goal is to relate the accelerations seen in the two frames and so we differentiate Eq. (17) once more and after rearrangement of the kind used above we find that

$$\frac{d^2x_i'}{dt^2} = 2\Omega D(\pi/4)_{ij}\frac{dx_j'}{dt} + \Omega^2 D(\pi/2)_{ij}x_j' + \left(\frac{d^2x_i}{dt^2}\right)'. \tag{21}$$

To calculate the vector version we apply the left and right sides of Eq. (20) to the respective sides of Eq. (19). Noting that $d(\mathbf{A}\times\mathbf{B})/dt = (d\mathbf{A}/dt)\times\mathbf{B} + \mathbf{A}\times(d\mathbf{B}/dt)$ and after collecting terms the result is

$$\frac{d^2\mathbf{X}'}{dt^2} = -2\mathbf{\Omega}\times\frac{d\mathbf{X}'}{dt} - \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{X}' + \left(\frac{d^2\mathbf{X}}{dt^2}\right)', \tag{22}$$

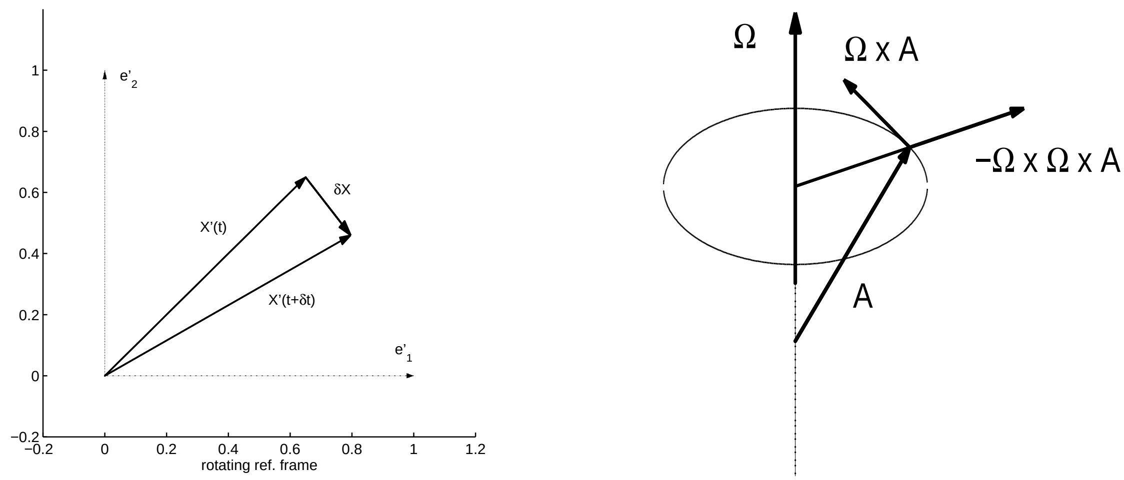


Figure 3: (left) A view of point P as seen from the rotating reference frame. In this frame the basis vectors $\mathbf{e}'$ appear to be stationary. (When we write an equation for a vector as seen in the rotating frame, as Eq. (12), we use $\mathbb{E}$ to indicate unit vectors that are time-independent.) Assume that the point P is stationary in the master frame; in this reference frame P appears to be moving in a clockwise direction. Over a time interval $\delta t$ P moves a distance $\delta\mathbf{X}'$ in a direction that is perpendicular to its position vector $\mathbf{X}'$ (imagine that $\delta\mathbf{X}'$ goes to zero). (right) A schematic showing the relationship of a vector $\mathbf{A}$, and various cross products with a second vector $\boldsymbol{\Omega}$ (note the signs). There is no attempt to show a scale since these vectors may have different units. The vector $\mathbf{A}$ is shown with its tail along the axis of the vector $\boldsymbol{\Omega}$. This helps when we use the right hand rule to visualize the direction of the resulting cross-product, but is irrelevant mathematically; the vector $\mathbf{A}$ could be defined at any point in space and the cross-product with $\boldsymbol{\Omega}$ would be exactly the same. That this is so is, one might say, the trouble with vectors — they are almost too visual. This is the main reason that the present analysis relies mostly upon matrix methods. Another way to understand this is to note that a vector in three space is defined by just three numbers (three components) but when we draw a vector in a three-dimensional diagram like this one we have to specify six numbers, a start and an end.

which is the vector equivalent of Eq. (21). It is important that you verify the steps leading to Eqs. (21) and (22).[12] From left to right the terms of this equation are 1) the acceleration as seen in the rotating frame, 2) the Coriolis term, 3) the centrifugal[13] term, and 4) the acceleration as seen in the master frame and then rotated into the rotating frame. As before, term 1) is the acceleration that we directly observe or seek to analyze when we are working from the rotating reference frame.

### 2.3 Master ⇒ Inertial; Rotating ⇒ Earth-Attached

Now for the key physical step; we identify the master reference frame as an inertial reference frame within which the equation of motion is Eq. (1). To make this frame inertial we presume that the unit vectors $\mathbf{e_i}$ could in principle be aligned on the distant, fixed stars[14]. The rotating frame is presumed to be attached to the rotating Earth, whose rotation rate is then defined by the rate at which the same fixed stars rotate overhead,

$$\Omega = 7.2921 \times 10^{-5} \text{ rad sec}^{-1}.$$

Earth's rotation rate is approximately constant, and the axis of rotation maintains a nearly steady bearing on a point on the celestial sphere near the North Star, Polaris.[15]

Assuming that the inertial frame equation of motion is

$$\frac{d^2 x_i}{dt^2} = F_i/M, \tag{23}$$

then the rotated form is just

$$D_{ij}\frac{d^2 x_j}{dt^2} = D_{ij}F_j/M, \tag{24}$$

or using the prime and vector notation,

$$\left(\frac{d^2 x_i}{dt^2}\right)' = F_i'/M \quad \text{and} \quad \left(\frac{d^2 \mathbf{X}}{dt^2}\right)' = \mathbf{F}'/M. \tag{25}$$

The rotated equation preserves the global conservation and Galilean transformation properties of Eq. (23).

To find the rotating frame equation of motion we need only use Eqs. (21) and (22) to eliminate the rotated acceleration, i.e., $\left(\frac{d^2\mathbf{X}}{dt^2}\right)'$, and then solve for the acceleration in the rotating frame:

$$\boxed{\frac{d^2 x_i'}{dt^2} = 2\Omega D(\pi/4)_{ij}\frac{dx_j'}{dt} + \Omega^2 D(\pi/2)_{ij}x_j' + F_j'/M,} \tag{26}$$

and the vector equivalent

$$\boxed{\frac{d^2 \mathbf{X}'}{dt^2} = -\ 2\boldsymbol{\Omega}\times\frac{d\mathbf{X}'}{dt}\ -\ \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{X}' + \mathbf{F}'/M.} \tag{27}$$

Thus the transformation of the equation of motion from an inertial to a rotating frame requires two terms - the Coriolis term, $2\boldsymbol{\Omega}\times(d\mathbf{X}'/dt)$, and the centrifugal term, $\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{X}'$. Much of the rest of this essay aims to clarify what these terms are, and to understand what they contribute to dynamics seen in a rotating reference frame.

### 2.4 How we use and interpret the transformed equation of motion

The origin of the Coriolis and centrifugal terms is the transformation of the acceleration vector into a steadily rotating frame, and very often these terms are written on the left side of an equation of motion as if they were going to be regarded as part of the unknown acceleration, i.e.,

$$\frac{d^2 \mathbf{X}'}{dt^2} +\ 2\boldsymbol{\Omega}\times\frac{d\mathbf{X}'}{dt}\ +\ \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{X}' = \mathbf{F}'/M. \tag{28}$$

If we compare the left side here with Eqs. (22) - (27) it is evident that this equation states that the rotated acceleration is equal to the rotated force/$M$, i.e.,

$$\left(\frac{d^2 \mathbf{X}}{dt^2}\right)' = \mathbf{F}'/M \tag{29}$$

which is well and true (and the same as Eq. 25). However, the left side of Eq. (28) taken all at once is *not* the acceleration that we observe or seek to analyze when we use a rotating reference frame — the acceleration we observe in a rotating frame is $d^2\mathbf{X}'/dt^2$, the first term only. Once we solve for $d^2\mathbf{X}'/dt^2$, it follows that the Coriolis and centrifugal terms are, figuratively or literally, sent to the right side of the equation of motion where they are interpreted as if they were forces. Therein lies the fate of the Coriolis and centrifugal terms and there too is the seed of our possible confusion regarding the nature of these terms.

When the Coriolis and centrifugal terms are regarded as forces — it seems apt to call them 'apparent forces' if we don't have to be concise — they have some peculiar and even slightly troubling properties. Since the centrifugal and Coriolis forces are exactly proportional to mass, they are inertial forces, just like gravity (which is fine). They differ from the usual, central forces $\mathbf{F}$, including gravity, in the crucial respect that there is no physical interaction that causes the Coriolis or centrifugal force and hence there is no reaction force. Since there is no physical origin for these two forces, neither is there a physical explanation.[16] The origin and the explanation of these terms is the transformation law for acceleration that we have just worked through, combined with the fact that we seek to observe or analyze the acceleration seen in the rotating frame. As we will see in the following Section 3, the Coriolis and centrifugal forces disrupt global momentum and energy conservation that are an implicit property of the inertial frame equation of motion and central forces. Similarly, we note here only that the rotating frame equation of motion does not preserve Galilean invariance since the Coriolis force involves the velocity and not just the velocity derivatives. Thus the velocity $\mathbf{V}'$ has to be regarded as an absolute velocity with respect to the rotating Earth.[17] It is important to be aware of these properties of the rotating frame equation of motion, and also to be assured that in most practical analysis of geophysical flows they are of no consequence. What is of real importance is that the rotating frame equation of motion offers a very significant gain in simplicity compared to the inertial frame equation of motion, as we will see in Section 4.3.

Once these conceptual questions are put aside, we can see that the Coriolis and centrifugal (apparent) forces have a simple, direct effect within the momentum equation for a single particle. From the vector relation Eq. (27) it is evident that the Coriolis force is normal to the rotating frame velocity, $d\mathbf{X}'/dt$, and to the rotation vector, $\mathbf{\Omega}$; the Coriolis force causes the particle velocity to change direction but not magnitude, and is thus a deflecting force. The centrifugal force is in a direction perpendicular to and directed away from the axis of rotation; the centrifugal force can have any direction with respect to the velocity. How these forces effect dynamics in simplified conditions will be considered further in Sections 3 and 5.

# 3 A comparison of rotating and inertial reference frames.

The logical path to the rotating frame momentum equation is very direct; if Eq. (23) holds in a given reference frame, then Eq. (26) holds exactly in a reference frame that rotates at the rate $\Omega$ with respect to the first frame. Thus whenever we use a rotating reference frame we should in principle take account of the Coriolis and centrifugal terms. However, as we will see in this section, there are common circumstances in which the error incurred by ignoring these terms will be entirely negligible. For other problems, and especially those involving the large scale motion of the atmosphere and ocean noted in the opening paragraph and in Section 5, rotation effects are of qualitative importance.

To appreciate the content of the transformation rules we will analyze two truly elementary motions; prescribed motion in a rotating frame, and a projectile problem. These two problems are configured so that the inertial frame dynamics is very simple and familiar and the only issue is how these motions look when they are observed from a rotating reference frame. (Readers who find these examples just too elementary may skip the bulk of this section, but should take a look at the section summary.) The main point to be made here is that the rotating frame description of a given motion may be remarkably different from the inertial frame description of the same motion, and yet both descriptions are exactly self-consistent with the appropriate equation of motion.

## 3.1 Circular motion and polar coordinates

Many geophysical flows and rotational phenomenon are analyzed most efficiently with cylindrical polar coordinates reviewed here very briefly. The vertical coordinate is exactly the $z$ or $x_3$ of Cartesian coordinates, and so we consider only the horizontal position, which can be specified by a distance from the origin, r, and the angle, $\theta$ between the radius vector and (arbitrarily) the $x_1$ axis. The corresponding unit vectors are given in terms of the time-independent Cartesian unit vectors that define the master frame by

$$\mathbf{e_r} = cos(\theta)\mathbf{e_x} + sin(\theta)\mathbf{e_y} \;\; \text{and,} \;\; \mathbf{e}_\theta = -sin(\theta)\mathbf{e_x} + cos(\theta)\mathbf{e_y}.$$

Notice that these unit vectors are time-dependent since $\theta$ is time-dependent. The position vector is

$$\mathbf{X} = r\mathbf{e_r}$$

and the velocity is

$$\frac{d\mathbf{X}}{dt} = \frac{dr}{dt}\mathbf{e_r} + r\frac{d\mathbf{e_r}}{dt} = \frac{dr}{dt}\mathbf{e_r} + r\omega\mathbf{e}_\theta,$$

where $\omega = d\theta/dt$. Continuing in a similar way the equation of motion is

$$\frac{d^2\mathbf{X}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\omega^2\right]\mathbf{e_r} + \left[2\omega\frac{dr}{dt} + r\frac{d\omega}{dt}\right]\mathbf{e}_\theta = \frac{F_r}{M}\mathbf{e_r} + \frac{F_\theta}{M}\mathbf{e}_\theta, \tag{30}$$

which, notice, has terms that look just like the centrifugal and Coriolis accelerations (though this equation holds in the master frame where centrifugal and Coriolis accelerations do not arise). To find the rotating frame acceleration is particularly simple; the unit vectors are identical in the master and rotating frames, as is the radius. The only thing different is that the rotation rate is decomposed as

$$\omega = \omega' + \Omega,$$

where $\Omega$ is the (constant) rotation rate of the rotating reference frame and $\omega'$ is the relative rotation rate that we see from the rotating reference frame. Substituting this in Eq. (30) and moving the terms containing $\Omega$ to the right side yields a rather formidable-looking expression that will be useful in the analysis that follows:

$$\frac{d^2\mathbf{X}'}{dt^2} = \left[\frac{d^2r'}{dt^2} - r'\omega'^2\right]\mathbf{e}'_\mathbf{r} + \left[2\omega'\frac{dr'}{dt} + r'\frac{d\omega'}{dt}\right]\mathbf{e}'_\theta =$$

$$\left[r'\Omega^2 + 2\Omega\omega' r' + \frac{F'_r}{M}\right]\mathbf{e}'_\mathbf{r} + \left[-2\Omega\frac{dr'}{dt} + \frac{F'_\theta}{M}\right]\mathbf{e}'_\theta. \tag{31}$$

You should verify the algebra leading to this last result as it shows most clearly how the factor 2 arises on the Coriolis term during the transformation from inertial to rotating frames.

## 3.2 To get a feel for the Coriolis force

The Coriolis force associated with Earth's rotation is not something that we experience through our own senses under everyday conditions.[1] For example, the Coriolis force on a runner, with $V = 5$ m sec$^{-1}$, is very, very small compared to gravity, $2\Omega VM \approx 10^{-4}gM$. On the other hand, the same runner making a moderately sharp turn, radius R = 15 m, will undoubtedly feel the centrifugal force, $(V^2/R)M \approx 0.15gM$, and will compensate instinctively for the tipping moment by leaning toward the center of the turn.

It can be very enlightening to experience the Coriolis force in the same immediate way, i.e., to feel it in your bones, at least once. To accomplish this will require a platform having a rotation rate that exceeds Earth's by a factor of O($10^4$). A typical merry-go-round has a rotation rate of $\Omega = 2\pi/12$ sec$^{-1}$ = 0.5 rad sec$^{-1}$ that is just right. We are going to calculate the forces that you would feel while sitting or walking about on this merry-go-round, and to do that will also need to guess your mass, say $M = 75$ kg. This analysis of forces is trivial in that all we need to do is evaluate Eqs. (30) and (31). In another respect it is slightly subtle in so far as very similar looking terms have very different interpretations in one frame compared with the other. To understand that there is more to the analysis than relabeling and reinterpreting terms in an arbitrary way, it will be helpful for you to make a sketch of each case and pay close attention to the acceleration, especially.

### 3.2.1 Zero relative velocity

To start let's presume that you are sitting quietly near the outside radius $r = 6$ m of a merry-go-round that it is rotating at a steady rate, $\Omega$. How does the momentum balance of your motion depend upon the reference frame, i.e, whether inertial or rotating, used to describe this motion?

Viewed from an **inertial frame** outside of the merry-go-round (fixed stars are not required given the rapid rotation rate), the polar coordinate momentum balance Eq. (30) with $\omega = \Omega$ and $dr/dt = d\omega/dt = F_\theta = 0$ reduces to a two term radial balance,

$$-r\Omega^2 M = F_r, \tag{32}$$

in which a centripetal acceleration ($\times M$) is balanced by an inward-directed radial force, $F_r$. We can readily evaluate the former and find $-r\Omega^2 M = F_r = -112$ N, which is quite noticeable (this corresponds to the weight on a mass of $F_r/g = 11.5$ kg).

Viewed from the **rotating reference frame** (your seat on the merry-go-round), you are stationary and of course not accelerating. To evaluate the rotating frame momentum equation, Eq. 31, we thus set $\omega' = 0, r' =$ constant, and are left with a two term radial force balance,

$$0 = r'\Omega^2 M + F'_r. \tag{33}$$

The physical conditions are unchanged and thus $F'_r = F_r = -112$ N just as before (recall that $r' = r$ and $F'_r = F_r$ in these coordinates).

What has changed is that the term $r'\Omega^2 M$ is now on the right side of the momentum equation where we call it the centrifugal force. Within the rotating frame, the evidence for and indeed the reality of a centrifugal force is quite vivid; it appears that you are being pushed outwards by a gravity-like force that is opposed by $F_r'$. The centrifugal force acts on every stationary object in your rotating frame with a magnitude that is proportional to the mass of that object and hence produces a radial acceleration on every stationary object that depends only upon the radius, $r'$. For example, a plumb line (a weight hanging on a string and at rest in this frame) makes an angle to the vertical of $\text{asin}(r'\Omega^2/g)$, where the vertical direction and $g$ are in the absence of rotation. The centrifugal force thus contributes to the direction and magnitude of the gravitational acceleration an important point that we will return to in Section 4.1.

### 3.2.2 With relative velocity

Most merry-go-rounds have signs posted which insist that riders remain in their seats once the ride begins. This is a sensible rule, of course, but if your goal is to get a feel for the Coriolis force then you will have to go for a (very cautious) walk on the merry-go-round. We will presume that the relative velocity, i.e., your walking velocity, is specified, and then calculate the force that must be exerted by the merry-go-round upon you as a consequence.

**Azimuthal relative velocity:** Let's assume that you walk azimuthally so that $r = 6$ m and constant. A reasonable walking pace under the circumstance is about $U_w = 1.5$ m s$^{-1}$, which corresponds to a relative rotation rate $\omega' = 0.25$ rad sec$^{-1}$, and recall that $\Omega = 0.5$ rad sec $^{-1}$. Let's assume that you walk in the direction of the merry-go-round rotation so that $\omega = \Omega + \omega' = 0.75$ rad sec$^{-1}$.

From the **inertial frame** momentum equation (30) we can readily calculate that the inward-directed radial force required to maintain this greater centripetal acceleration is then

$$-r\omega^2 M = F_r \approx -253 \text{ N},$$

or roughly twice the force required when you were seated. If you then reverse direction and walk at the same speed against the rotation of the merry-go-round, $F_r$ is reduced to about -28 N. This pronounced variation of $F_r$ is a straightforward consequence of the quadratic dependence of centripetal acceleration upon the rotation rate, $\omega$.

When this is viewed from the **rotating frame** we distinguish between the rotation rate of the merry-go-round, $\Omega$, and the relative rotation rate $\omega'$ due to your relative motion (walking speed). The radial component of the rotating frame momentum equation reduces to

$$-r'\omega'^2 M = (r'\Omega^2 + 2r'\Omega\omega')M + F_r'. \tag{34}$$

The term on the left is the comparatively small (relative) centripetal acceleration; the first term on the right is the usual centrifugal force, and the second term on the right, $2r'\Omega\omega'$, is the Coriolis force. In this circumstance, the Coriolis force could loosely be said to be a (relative)velocity-dependent component of the centrifugal force. The Coriolis force is quite noticeable, $2r'\Omega\omega' \pm 112$ N, with the sign determined by the direction of your motion relative to $\Omega$. For example, if $\Omega > 0$ and $\omega' > 0$ then the Coriolis force is positive (and radial) and hence the Coriolis force is to the right of and normal to the azimuthal relative velocity.

**Radial relative velocity:** If all is well to this point, then a cautious walk in the radial direction might be attempted. To isolate the effects of radial motion we will presume that your radial speed is constant at $dr'/dt = 1$ m s$^{-1}$ and that you walk along a radial line so that your rotation rate also remains constant at $\omega = \Omega$ (in practice this is very difficult to do for more than a few steps). The resulting forces are then in the azimuthal direction, and their magnitude and sense can most easily be interpreted in terms of the balance of angular momentum, $A = \omega r'^2 M$. In this circumstance the rate of change of $A$ has been fully specified,

$$\frac{dA}{dt} = 2\Omega r' \frac{dr'}{dt} M = r' F_\theta,$$

and must be accompanied by an azimuthal torque, $r'F_\theta$, that is exerted by the merry-go-round upon you.

Viewed from an **inertial frame**, the azimuthal component of the momentum balance reduces to

$$2\Omega \frac{dr}{dt} M = F_\theta, \tag{35}$$

where $F_\theta \approx -75$ N for the given data. The azimuthal force $F_\theta$ has the form of the Coriolis force, but remember that we are viewing the motion from an inertial frame. If the radial motion is inward so that $dr/dt < 0$, then $F_\theta$ must be negative, or opposite the direction of the merry-go-round rotation, since the angular momentum is necessarily becoming less positive. (Be sure that these signs are clear before going on to consider this motion from the rotating frame.)

In the **rotating frame** the momentum equation reduces to an azimuthal force balance,

$$0 = -2\Omega \frac{dr'}{dt} M + F'_\theta, \tag{36}$$

where $-2\Omega \frac{dr'}{dt} M$ is the Coriolis force, and $F'_\theta = F_\theta$ as before. In this circumstance the Coriolis force has exactly the same interpretation as an apparent force as does the centrifugal force that we considered in the example of steady, circular motion and Eq. (33). Namely, the force exerted by the merry-go-round, $F'_\theta$, appears to be balanced by a gravity-like (though relative-velocity-dependent) inertial force in the direction opposed to $F'_\theta$. Thus in the rotating frame (in which the only motion is your radial velocity) it seems as if you are being pushed up against a stationary merry-go-round. If the radial motion is inward, $\frac{dr'}{dt} \leq 0$, then the Coriolis force, $-2\Omega \frac{dr'}{dt} M \geq 0$, is again to the right of and normal to the relative velocity.

**Be careful!** I hope that you will have a chance to do this experiment some day, as you will learn from firsthand experience whether the Coriolis force is real or just a mathematical device. I also trust that you will exercise genuine caution; ask permission of the merry-go-round operator before you start walking around, and maintain a grip on something secure at all times. The Coriolis force is likely to be surprising, even when you understand all of this analysis.

### 3.3 An elementary projectile problem

A very simple projectile problem can reveal some other aspects of rotating frame dynamics. Assume that a projectile is launched at a speed $U_0$ and at an angle to the ground $\beta$ from a location $[x \;\; y] = [0 \;\; y_0]$. The only force presumed to act on the projectile after launch is the downward force of gravity, $-gM\mathbf{e_3}$, which is the same in either reference frame.

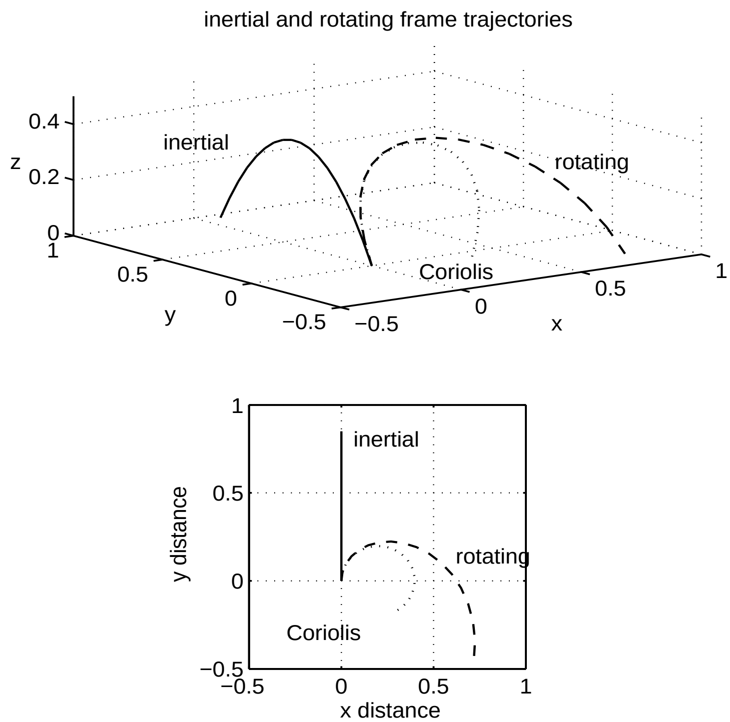


Figure 4: Trajectory of a particle launched with a horizontal velocity in the positive y-direction as seen from an inertial reference frame (solid line, displaced in the y-direction only), and as seen from a rotating frame (dashed, curves lines). The upper and lower panels are 3-dimensional and plan views. The dotted curve is with the Coriolis force only (the motivation for this is in Section 4). This trajectory has the form of a circle, and if the projectile had not returned to the surface, it would have made a complete loop back to the starting point.

### 3.3.1 From the inertial frame

The equations of motion and initial conditions in Cartesian components are linear and uncoupled;

$$\frac{d^2x}{dt^2} = 0; \quad x(0) = 0, \quad \frac{dx}{dt} = 0, \tag{37}$$

$$\frac{d^2y}{dt^2} = 0; \quad y(0) = y_0, \quad \frac{dy}{dt} = U_0 \cos\beta,$$

$$\frac{d^2z}{dt^2} = -g; \quad z(0) = 0, \quad \frac{dz}{dt} = U_0 \sin\beta,$$

where $M$ has been divided out. The solution

$$\begin{aligned} x(t) &= 0, \\ y(t) &= y_0 + tU_0 \cos\beta, \\ z(t) &= t(U_0 \sin\beta - \frac{1}{2}gt) \end{aligned} \tag{38}$$

defined on the interval $0 < t < \frac{2U_0 \sin\beta}{g}$ requires no comment (Fig. 4).

### 3.3.2 From the rotating frame

How would this same motion look when viewed from a rotating reference frame?, and, How could we compute the motion from within a rotating reference frame?

The first question can be answered very simply by rotating the trajectory Eq. (38) via the rotation matrix

$$x'_i = D(\Omega t)_{ij} x_j, \tag{39}$$

with the result:

$$\begin{aligned} x'(t) &= (y_0 + tU_0 \cos\beta) \sin(\Omega t), \\ y'(t) &= (y_0 + tU_0 \cos\beta) \cos(\Omega t), \\ z'(t) = z \ &= \ t(U_0 \sin\beta - \frac{1}{2} g t), \end{aligned} \tag{40}$$

and valid over the same time interval as before. Notice that the $z$ component is unchanged in going to the rotating reference frame and recall that we presumed the rotation axis was aligned with $z$. This is quite general; motion that is parallel to the rotation vector $\mathbf{\Omega}$ is entirely unaffected by rotation. On the other hand, motion in the $(x, y)$-plane that is perpendicular to the rotation vector can be altered quite substantially, depending upon the phase $\Omega t$. In the case shown in Figure (4), the change of phase is 2.0 at the end of the trajectory, so that rotation effects are prominent.[18] One important aspect of the trajectory is not changed, however, and that is the (horizontal) radius,

$$\sqrt{x'^2 + y'^2} = \sqrt{x^2 + y^2},$$

since the coordinate systems have coincident origins (Fig. 5, upper)

How could we compute the trajectory in the rotating frame? The Cartesian component equations in the rotating frame are a bit awkward (the x-component only):

$$\frac{d^2 x'}{dt^2} = 2\Omega \frac{dy'}{dt} + \Omega^2 \frac{x'}{\sqrt{x'^2 + y'^2}}.$$

An elementary problem in the inertial frame transforms into a pair of coupled, nonlinear equations in the rotating frame ($z' = z$). We can always solve these equations numerically, but we are better off in this and many problems involving rotation to use cylindrical polar coordinates where we can take advantage of what we have already learned about the rotating frame solution. We know that

$$r' = r = y_0 + tU_0 \cos\beta,$$

and that the angle in the inertial frame, $\theta$, is constant in time since the motion is purely radial and for the specific case considered, $\theta = \pi/2$. The rotation rates are related by $\omega' = -\Omega$, and thus

$$\theta' = \pi/2 - \Omega t.$$

Both the radius and the angle increase linearly in time, and the horizontal trace of the trajectory is Archimedes spiral (Fig. 4, lower).

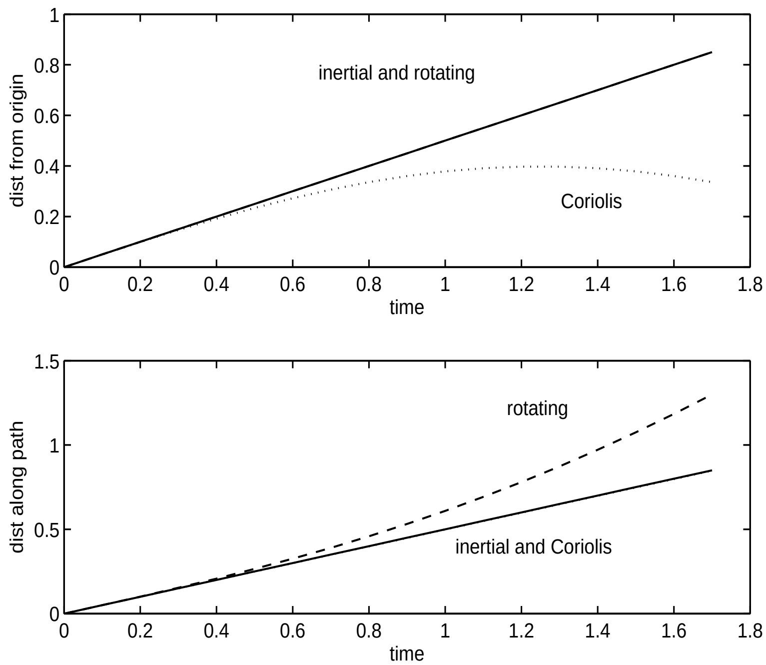


Figure 5: (upper) The distance from the origin in the horizontal plane for the trajectories of Fig. (4). The distance from the origin is identical for the inertial and rotating trajectories, and reduced for the Coriolis trajectory. (lower) The distance along the path in the horizontal plane for the same trajectories. The slope gives the speed of the particle. The inertial and Coriolis frame trajectories retain their initial speed and are identical; the rotating frame trajectory accelerates due to the centrifugal force.

It is interesting to see how the rotating frame momentum equation is consistent with the rotating frame trajectory. The projectile is obviously deflected to the right when viewed from the rotating frame, and from the azimuthal component of Eq. (31) we readily attribute this to the Coriolis force,

$$2\omega'\frac{dr'}{dt}M = -2\Omega\frac{dr'}{dt}M.$$

Notice that the horizontal speed and thus the kinetic energy increase with time (Fig. 5, lower). The rate of increase of rotating frame kinetic energy (per unit mass) is

$$\frac{d\mathbf{V'^2}/2}{dt} = \frac{d(U_0^2 + r'^2\Omega^2)/2}{dt} = \frac{dr'}{dt}r'\Omega^2$$

where the term on the right side is the work done by the centrifugal force. If the projectile hadn't returned to the ground, its speed would have increased without limit so long as the radius increased, a profoundly unphysical result of the rotating frame dynamics.[19]

## 3.4 Summary

This analysis of elementary motions can be summarized with three important points:

1) We can make an exact and self-consistent explanation of forced or free motion observed from a rotating frame in terms of Coriolis and centrifugal forces plus the usual inertial frame forces, **F**. The

complement of this is that there is nothing that occurs in the rotating frame dynamics that can not be understood in terms of inertial frame dynamics and forces **F**. Clearly then we can use either reference frame that best suits the circumstance. The interpretation of geophysical flow phenomenon is usually far simpler when viewed from an Earth-attached, rotating reference frame, though by design that was not the case for the two problems discussed in this section (the case for this is made clearer in Section 4.3).

2) Within the rotating frame, the Coriolis force can appear to be just as vivid as the familiar centrifugal force. In the example of azimuthal relative motion on a merry-go-round (Section 3.2.2) the magnitude of the Coriolis force can be understood as the relative-velocity dependent component of centrifugal force (loosely speaking); in the example of radial relative motion it is equal to the force associated with angular momentum balance. These cases have almost the feeling of a physical explanation of the Coriolis force, but it is probably more appropriate to regard them as a demonstration.

3) There is no physical agent or interaction that causes the Coriolis force and so there is no object that is accelerated in the opposite direction by a reaction force. In the same way, there is no source for the work done by the centrifugal force. Global accounting for momentum and energy thus breaks down when we interpret the Coriolis and centrifugal accelerations as if they were forces. This emphasizes that the origin of the Coriolis force is not to be found in a physical process or interaction, but rather in the kinematic, transformation laws of Section 2, i.e., in motion itself.

# 4 Application to rotating fluids and to Earth.

The equations of motion appropriate to the atmosphere and ocean differ from that considered up to now in two significant ways. First, it isn't just the reference frame that rotates, but the entire Earth, ocean and atmosphere, aside from the comparatively small (but very important!) relative motion of winds and ocean currents. One consequence of the solid body rotation of the Earth is that the horizontal component of the centrifugal force on a particle that is stationary in the rotating frame is canceled by a component of the gravitational mass attraction. Thus the centrifugal force does not appear in the rotating frame dynamical equations for the atmosphere and oceans, a welcome simplification (Section 4.1). Second, because the Earth is nearly spherical, the rotation vector is not perpendicular to the plane of horizontal motions except at the poles. This causes the horizontal component of the Coriolis force to vary with latitude (Section 4.2). Finally, in this section we also make the case that the rotating frame equations of motion are indeed simpler than the inertial frame equations when applied to geophysical flows. (Section 4.3).

## 4.1 Cancelation of the centrifugal force

To understand how the centrifugal force is canceled we consider briefly the balance of gravitational mass attraction and centrifugal forces on a rotating Earth. If Earth was a perfect, homogeneous sphere, the gravitational mass attraction at the surface, $\mathbf{g}*$, would be directed towards the center (Fig. 6). Because the Earth is rotating, every particle on the surface is subject also to a centrifugal force of magnitude $\Omega^2 R \sin\theta$, where $R$ is the nominal radius and $\theta$ is the colatitude ($\pi/2$ - latitude). This centrifugal force has a component parallel to the surface (a shear stress)

$$C_\theta = \Omega^2 R \sin\theta \cos\theta \tag{41}$$

that is directed towards the equator. $C_\theta$ is not large compared to $g*$, $C_\theta/g* \approx 0.002$ at most, but it is extremely persistent, having been present since the Earth's formation. A fluid can not sustain a shear stress without deforming, and over geological time this holds as well for the Earth's interior and crust. Thus it is highly plausible that the Earth long ago settled into an equilibrium configuration in which this $C_\theta$ is exactly balanced by a component of the gravitational (mass) attraction that is parallel to the displaced surface and poleward. If we continue to assume that the gravitational mass attraction is that of a sphere, then the required meridional slope of the displaced surface, $\eta$, is given by

$$\frac{g*}{R}\frac{d\eta}{d\theta} = C_\theta. \tag{42}$$

This can be integrated over latitude to yield the displacement

$$\eta(\theta) = \int_0^\theta \frac{\Omega^2 R^2}{g*} \sin\theta\cos\theta d\theta = \frac{\Omega^2 R^2}{2g*}\sin\theta^2 + constant, \tag{43}$$

which when added on to a sphere gives an oblate (flattened) spheroid, (Fig. 6), consistent with the observed shape of the Earth. If cut on a plane through the rotation vector, the result is an ellipse. Eq. (43) indicates a pole-to-equator rise of $\eta$ by about 11 km; precise measurements[20] show that Earth's equatorial radius $R_e = 6378.2$ is greater than the polar radius $R_p = 6356.7$ km by about 21.5 km or roughly twice as much as the estimate given by Eq. (42).[21] A convenient measure of flattening is $F = (R_e - R_p)/R_e$; for Earth, $F = 0.0033$. $F$ varies considerably among the planets due to variations in density and rotation rate.[22]

Closely related is the notion of 'vertical'. When we measure vertical we do so by means of a plumb bob that hangs in the direction of the gravitational acceleration, also called the plumb line, and that by definition is vertical. Following the discussion above we know that the gravitational acceleration is made up of two contributions, the first and by far the largest being mass attraction, $\mathbf{g}*$, with a much smaller contribution due to the centrifugal acceleration associated with the Earth's rotation, $\mathbf{C}$, Fig. (6). Just as on the merry-go-round, this centrifugal acceleration adds with the gravitational mass attraction to give the net acceleration, $\mathbf{g} = \mathbf{g}* + \mathbf{C}$, a vector (field) whose direction and magnitude we can measure with a plumb bob and by observing the period of a simple pendulum. A surface that is normal to the gravitational acceleration vector is said to be a level surface, and is given by the surface of a water body that is at rest (in the rotating frame), since a fluid at rest can sustain only normal stresses, i.e., pressure. Thus the measurements of vertical or level that we can readily make necessarily include the centrifugal force.[23] (The detailed shape of the Earth is thus a bit of a red-herring for the central purpose here, albeit an interesting one that is of great importance in other phenomenon.) The upshot of all of this is that the rotating frame equation of motion applied in an Earth-attached reference frame, Eq. (2), does not include a centrifugal force associated with Earth's rotation, and of course neither does it show the balancing, tangential component of the gravitational mass attraction.

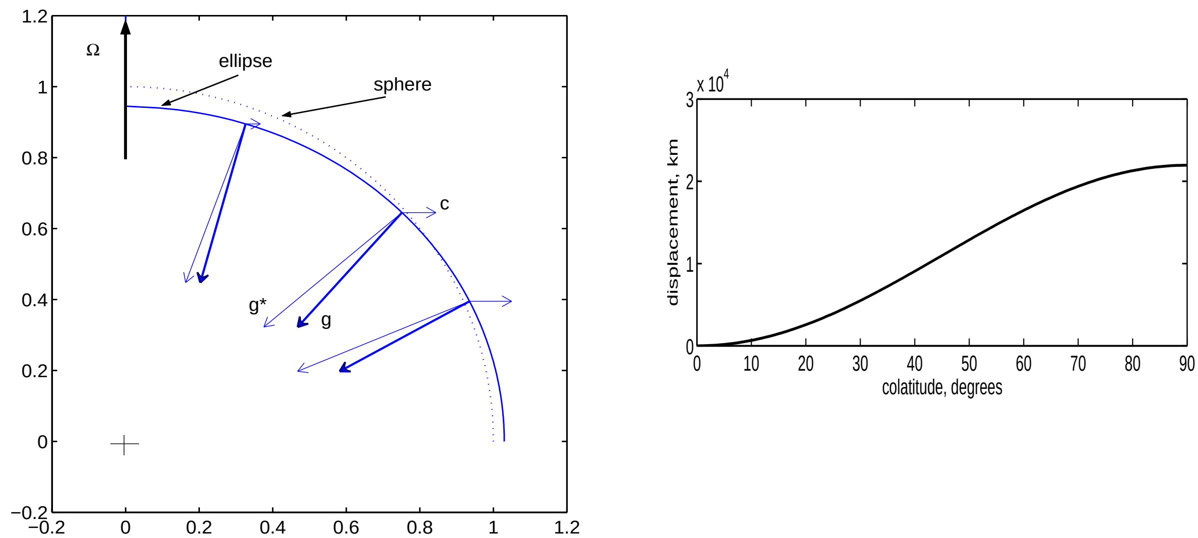


Figure 6: (left) The gravitational acceleration due to mass attraction is shown as the vector **g**∗ that points to the center of a spherical, homogeneous planet. The centrifugal acceleration, **C**, associated with the planet's rotation is directed normal to and away from the rotation axis. The combined gravitational and centrifugal acceleration is shown as the heavier vector, **g**. This vector is in the direction of a plumb line, and defines vertical. A surface that is normal to **g** similarly defines a level surface, and has the approximate shape of an oblate spheroid (the solid curve). The ellipse of this diagram has a flatness $F = 0.1$ that approximates Saturn. Earth's flatness is much less, only about 0.0033, but highly significant. (right) The displacement of Earth's surface that is required to give a horizontal component of gravitational mass attraction that just balances the horizontal component of centrifugal force due to Earth's rotation. This displacement added onto a sphere gives an excellent approximation to the actual shape of the Earth, nearly an oblate spheroid (this displacement is the distance between the sphere and the ellipse at left). The displacement at the equator (colatitude = 90 degrees) shown here, 21.5 km, is taken from observations and the integration constant of Eq. (43) was set to zero at the pole.

## 4.2 Coriolis force for motions in a thin, spherical shell

The application of the Coriolis formalism to geophysical flows requires a further small elaboration because the rotation vector makes a considerable angle to the vertical except at the poles. An Earth-attached coordinate system is usually envisioned to have $\mathbf{e_x}$ in the east direction, $\mathbf{e_y}$ in the north direction, and $\mathbf{e_z}$ in the radial direction. The rotation vector $\mathbf{\Omega}$ thus makes an angle $\phi$ with respect to the local horizontal $x', y'$ plane, where $\phi$ is the latitude of the coordinate system and thus

$$\mathbf{\Omega} = 2\Omega\cos(\phi)\mathbf{e_y} + 2\Omega\sin(\phi)\mathbf{e_z}.$$

If $\mathbf{V}' = U'\mathbf{e_x} + V'\mathbf{e_y} + W'\mathbf{e_z}$, then the Coriolis force is

$$2\mathbf{\Omega}\times\mathbf{V}' = (2\Omega\cos(\phi)W' - 2\Omega\sin(\phi)V')\mathbf{e_x} + (2\Omega\sin(\phi)U' - 2\Omega\cos(\phi)W')\mathbf{e_y} + 2\Omega U'\sin(\phi)\mathbf{e_z}. \quad (44)$$

Large scale geophysical flows are very flat in the sense that the horizontal components of wind or current are very much larger than the vertical component, $U' \propto V' \gg W'$, simply because the oceans and the atmosphere are quite thin, having a depth to width ratio of about 0.001. The ocean and atmosphere are stably stratified in the vertical, which still further inhibits the vertical component of motion. For these large scale (in the horizontal) flows, the Coriolis terms multiplying $W'$ in the $x$ and $y$ component equations are thus very much smaller than the terms multiplied by $U'$ or $V'$ and as an excellent approximation may be ignored. The Coriolis terms that remain are those having the sine of the latitude, and the important combination

$$f = 2\Omega \sin\phi \tag{45}$$

is dubbed the Coriolis parameter. In the vertical component of the momentum equation the Coriolis term is usually much smaller than the gravitational acceleration, and so it too is usually dropped. The result is the thin fluid approximation of the Coriolis force,

$$2\mathbf{\Omega}\times\mathbf{V'} \approx \mathbf{f}\times\mathbf{V'} = fV'\mathbf{e_x} + fU'\mathbf{e_y}, \tag{46}$$

in which only the horizontal components due to horizontal motions have been retained ($\mathbf{f}$ is $f$ times the local vertical unit vector). Notice that the Coriolis parameter $f$ varies with the sine of the latitude, having a zero at the equator and maxima at the poles; $f < 0$ for southern latitudes.[24] For problems that involve particle displacements, $L$, that are very small compared to the radius of the Earth, $R$, a simplification of $f$ is often appropriate. The Coriolis parameter may be expanded in a Taylor series about a central latitude, $y_0$,

$$f(y) = f(y_0) + (y - y_0)\frac{df}{dy}\mid_{y_0} + HOT \tag{47}$$

and if the second term is demonstrably much smaller than the first term, which follows if $L \ll R$, then the second and higher terms may be dropped to leave $f = f(y_0)$, a constant. Under this so-called $f$-plane approximation[25] the period of inertial motions, $P = 2\pi/f$, is just a little bit less than 12 hrs at the poles, a little less than 24 hrs at 30 N or S, and infinite at the equator (recall the discussion of sidereal and solar days in Ref. 15). The period of inertial motions is sometimes said to be half of a 'pendulum day', the time required for a Foucault pendulum to precess through $2\pi$ radians.[26]

## 4.3 Why do we insist on the rotating frame equations?

We have emphasized that the rotating frame equation of motion has some inherent awkwardness, namely the Coriolis force and the loss of Galilean invariance. However, the gain in simplicity when analyzing the motions of the atmosphere and ocean more than compensates. The reasons are several, but primarily that the inertial frame velocity consists of the solid body rotation plus the relative velocity,[27] $\mathbf{V} = \mathbf{V_\Omega} + \mathbf{V'}$, with the former being very much larger than the latter; $V_\Omega = \Omega R_e \cos(\text{latitude})$, where $R_e$ is earth's radius, 6350 km, and thus $V_\Omega \approx O(500)$ m s$^{-1}$ near the equator. This very large velocity is accelerated centripetally, and is balanced by a centripetal force associated with the ellipsoidal shape of the Earth discussed in Section 4.1. This centripetal force is larger than the Coriolis force in the ratio $V_\Omega/V'$ that is O(10) or more. The inertial frame equations have to account for all of this explicitly and yet our interest is almost always the small relative motions of the atmosphere and ocean, since it is the relative motion that transports heat and mass over the Earth. In that important regard, the velocity associated with the solid body rotation of the Earth, atmosphere and ocean is invisible, no matter how large it is. As well, when we observe the winds and ocean currents we almost always do so from a reference frame that is fixed to the Earth. Given that our goal is to solve for or observe the relative velocity, then the rotating frame equations are generally much simpler and more appropriate than are the inertial frame equations. In this section we will analyze the free oscillations, also called inertial oscillations, to help make this point. The domain is presumed to be a small region centered on the pole so that latitude = 90 degrees, and the domain is, in effect, flat.

The inertial and rotating frame momentum equations are listed again for convenience using velocity in place of the previous time rate change of position,

$$\frac{d\mathbf{V}}{dt} = \mathbf{F}/M \tag{48}$$

$$\frac{d\mathbf{V}'}{dt} = -2\boldsymbol{\Omega}\times\mathbf{V}' \; - \; \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{X}' \; + \; \mathbf{F}'/M. \tag{49}$$

Now we are going to assume the result from Section 4.1 that there is a force $\mathbf{F}$ due to a tangential component of gravitational mass attraction that exactly balances the centrifugal force due to Earth's rotation, i.e.,

$$\mathbf{F}/M = \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{X}, \tag{50}$$

and similarly for the rotating frame. Gravity is otherwise normal to the vertical, and does not effect the horizontal motions of interest here. All other forces (Coriolis aside) are presumed to vanish. The momentum equations for the horizontal component of motion are then

$$\frac{d\mathbf{V}}{dt} = \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{X} \tag{51}$$

and

$$\frac{d\mathbf{V}'}{dt} \;\; = \;\; -\,2\boldsymbol{\Omega}\times\mathbf{V}'. \tag{52}$$

Thus Eq. (52) is the rotating frame equivalent of Eq. (51) and contains no new physics given that $\mathbf{V}_{\boldsymbol{\Omega}}$ is known. The difference in the two equations can be appreciated as we solve for free oscillations under the realistic condition that $\mathbf{V}' \ll \mathbf{V}_{\boldsymbol{\Omega}}$.

### 4.3.1 Inertial motions from an inertial frame

Since the motion is almost circular it is appropriate to use the cylindrical coordinate inertial frame momentum equation, Eq. (30) (horizontal only and dividing out the constant $M$):

$$\begin{aligned} \frac{d^2r}{dt^2} - r\omega^2 &= -\Omega^2 r, \\ 2\omega\frac{dr}{dt} + r\frac{d\omega}{dt} &= 0. \end{aligned} \tag{53}$$

The initial condition is presumed to be a small radial perturbation away from a balanced state given by the solid body velocity, $r = R_o$, and $\omega = \Omega$. Since there is no tangential force, the angular momentum,

$$A = \omega r^2 = \Omega R_o^2$$

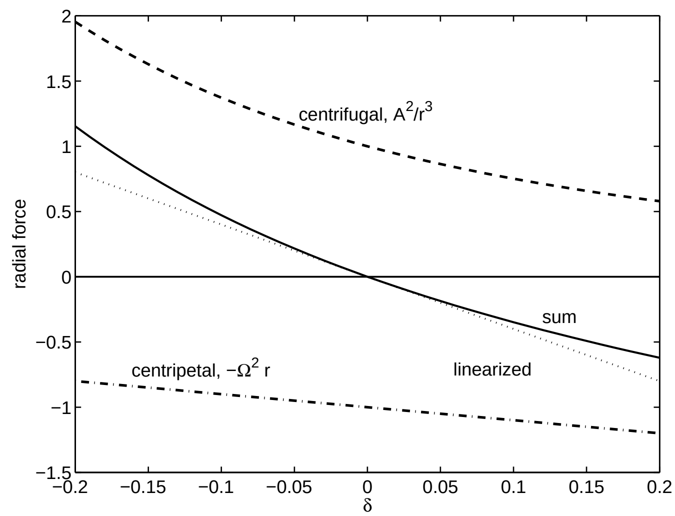


Figure 7: The terms of the right side of Eq. (55) shown as a function of $r$. These are proportional to the force on the particle as it is displaced away from its equilibrium position, $\delta = 0$. The force is restoring, and nearly linear in the displacement. Small displacements follow a linear oscillator equation having a frequency twice that given by the centripetal force, $-\Omega^2 r$, alone.

is a conserved quantity, and can be used to eliminate $\omega$ from the radial component equation,

$$\frac{d^2r}{dt^2} - \frac{A^2}{r^3} = -\Omega^2 r. \tag{54}$$

When we solve this equation for $r$ we implicitly move the centripetal acceleration term $A^2/r^3$ to the right side where it becomes a centrifugal force,

$$\frac{d^2r}{dt^2} = \frac{A^2}{r^3} - \Omega^2 r. \tag{55}$$

This is exactly the kind of manipulation that we used to derive the rotating frame equation of motion (Section 2.5) but here it seems that we are on the verge of a *faux pas* - speaking of centrifugal force in an inertial frame analysis. Indeed, we are, because it is convenient to solve for $d^2r/dt^2$ rather than the radial acceleration *per se* (the left side of Eq. 54).

As it stands, Eq. (55) is a nonlinear oscillator equation. Our interest is in the case of small displacements $\delta$ away from the balanced state, $r = R_o$, which leads to a significant simplification. Substitution of $r = R_o + \delta$ into Eq. (54) and expansion in the small parameter $\delta/R_o$ shows that small displacements are governed by the linear equation (Fig. 7)

$$\frac{d^2\delta}{dt^2} \approx -4\Omega^2\delta, \tag{56}$$

which indicates a simple harmonic oscillation at a frequency $2\Omega$. Of the factor four on the right hand side, three fourths came from the angular momentum constraint and the remainder from the linear r-dependence of the radial force.[28]

If the perturbation is a small radial impulse that gives a velocity $V_0$ then the solution for the radius is just

$$r(t) = R_o + \delta_0 \sin(2\Omega t), \tag{57}$$

where $\delta_0 = V_0/2\Omega$. The corresponding angular rotation rate can be found by using Eq. (57) together with the angular momentum conservation relation

$$\omega(t) = \Omega \frac{R_o^2}{(R + \delta_0 \sin(2\Omega t))^2} \approx \Omega(1 - 2\frac{\delta_0}{R_o}\sin(2\Omega t)). \quad (58)$$

When graphed, these show that the particle moves in an ellipsoidal orbit, Fig. (8, left), that crosses the (balanced) radius $r = R_o$ four times per complete orbit. The rotating frame turns through 180 degrees just as the particle returns to $r = R_o$ the second time, after completing a full cycle of the oscillation. Over the next cycle the rotating frame turns through another 180 degrees. Thus when viewed from the rotating frame (Fig. 8, right), the particle appears to be going around in a circle with a frequency of twice $\Omega$[29]. Had we taken the initial condition to be a small impulse in the azimuthal direction, the result would have been an altered phase of the oscillation, but all else the same.

### 4.3.2 Inertial motion from the rotating frame

How does this motion look when viewed from the rotating reference frame? It is convenient to expand the rotating frame momentum equation in Cartesian coordinates, and since we have resticted the analysis above to small displacements we can utilize the f-plane approximation that takes $f$ as a constant, and thus

$$\frac{du'}{dt} = fv' \quad (59)$$
$$\frac{dv'}{dt} = -fu'.$$

Given that the initial condition is an impulse $V_0$ in the y-direction then the solution is just

$$u' = V_0 \sin(ft), \quad v' = V_0 \cos(ft), \quad (60)$$
$$x' = \delta_0(1 - \cos(ft)), \quad y' = \delta_0 \sin(ft).$$

These correspond exactly to the result found in the inertial frame analysis (cf. Fig. 8, right), but were much simpler to obtain in large part because we did not have to deal with the solid body rotation but only the relative velocity that was of interest. The velocity of the particle seen from the rotating frame makes a clockwise rotation at a rate of $f$ in a direction that is opposite the rotation $\Omega$. From the rotating frame perspective, the rotation of the velocity vector is attributable to deflection by the Coriolis force. This kind of motion, termed an inertial oscillation,[30,31] is frequently observed in the upper ocean, especially, and will be seen again in Section 5.

## 5 Adjustment to gravity, rotation and friction.

The last problem we consider gives some insight into the establishment of a geostrophic momentum balance, which, as noted in the opening section, is the defining characteristic of large scale flows of the atmosphere and ocean. We will model the motion of a single particle[32] on a rotating Earth (so there is no centrifugal force) and that is subjected to a force that is suddenly turned on at $t = 0$. This force

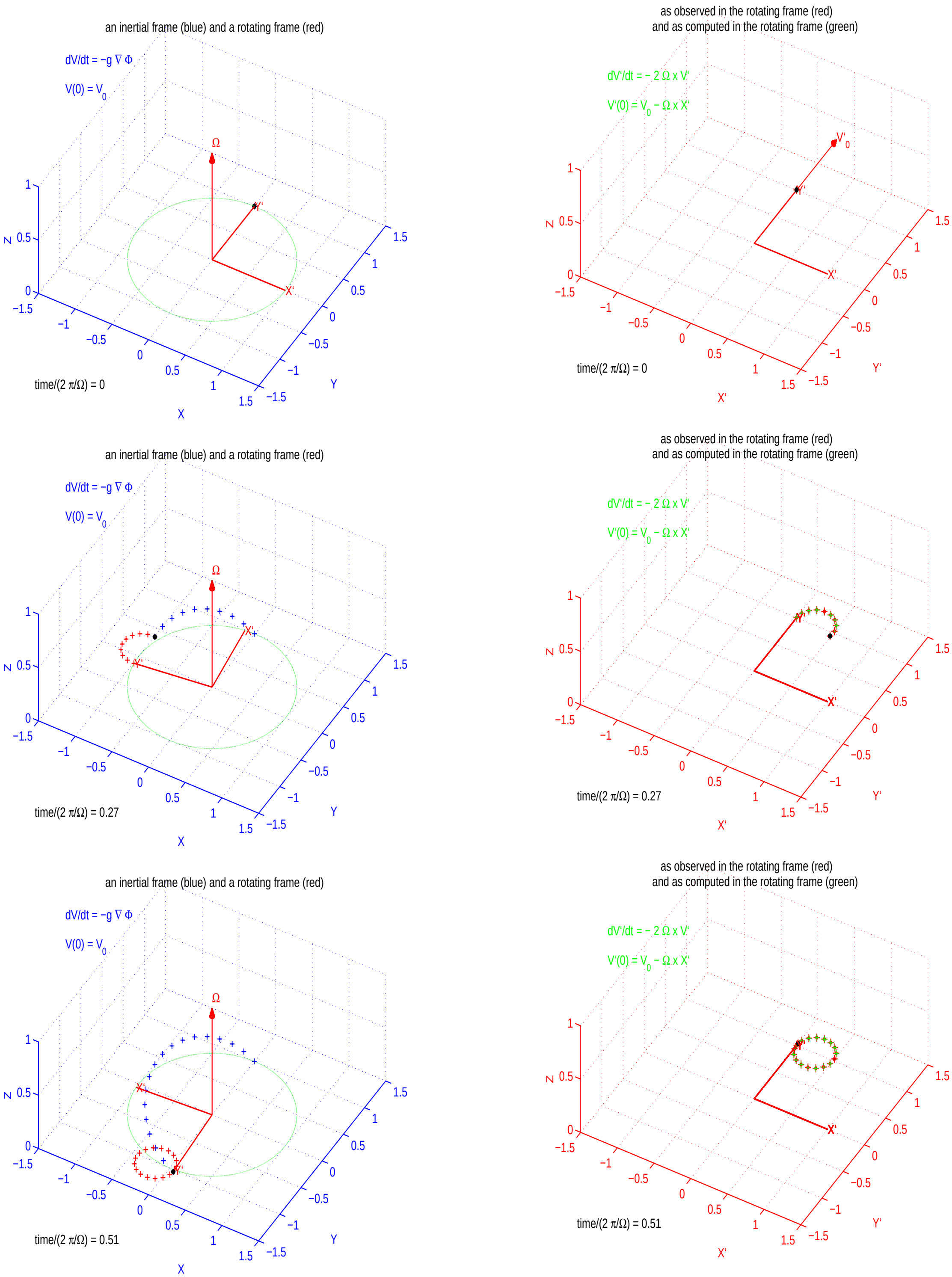


Figure 8: The two-dimensional trajectory of a particle subject to a centripetal force, $-r\Omega^2$ (as if it were on a frictionless parabolic surface). The initial velocity was a solid body rotation in balance with the centripetal force, and a small radial impulse was then superimposed. In this case the ratio $V'/V_\Omega \approx 0.2$, which is far larger than actually occurs. The left column shows the resulting ellipsoidal trajectory as seen from an inertial frame, along with the circular trajectory that is seen from a rotating frame (indicated by the rotating, solid unit vectors). The right column shows the trajectory as seen from the rotating frame only, along with the solution computed in the rotating frame shown as green dots. These latter lie exactly on top of the 'observed' trajectory and are very difficult to discern in this figure. See instead the script Coriolis.m[4] that includes this and a number of other cases.

could be a wind stress, a pressure gradient, or the buoyancy force on a relatively dense particle sitting on a sloping sea floor. This latter has the advantage of being similar to the gravitational force due to a tilted Earth's surface (Section 4.1), and so we will take the force to be buoyancy, $b = g\frac{\delta\rho}{\rho_o}$ times the bottom slope $\nabla h = \alpha \mathbf{e_y}$, with $\delta\rho$ the density anomaly of the particle with respect to its surroundings (assumed constant), $h$ the bottom depth and $\alpha$ the slope of the bottom. The depth $h$ is presumed to vary linearly in $y$ only and hence this buoyancy force will appear in the $y$-component equation only. If the particle is in contact with a sloping bottom, then it is plausible that the momentum balance should include a frictional term due to bottom drag. The task of estimating an accurate bottom drag for a specific case is beyond the scope here, and so we will represent bottom drag by the simplest linear[33] (or Rayleigh) drag law in which the drag is presumed to be proportional to and antiparallel to the velocity difference between the current and the bottom, i.e., bottom drag $= -k(\mathbf{V} - \mathbf{V_{bot}})$. Of course the ocean bottom is at rest in the rotating frame and so $\mathbf{V_{bot}} = 0$ and is omitted from here on. From observations of ocean currents we can infer that a reasonable value of $k$ for a density-driven current on a continental shelf is $k = 1/10$ days$^{-1}$. Thus $k$ is roughly an order of magnitude smaller than a typical mid-latitude value of $f$. Since $k$ appears in the momentum equations in the same way that $f$ does, we can anticipate that rotational effects will be dominant over frictional effects. The momentum equations from a rotating frame are then:

$$\begin{aligned} \frac{dU}{dt} &= fV - kU, \\ \frac{dV}{dt} &= -fU - kV + b\alpha, \end{aligned} \tag{61}$$

and we assume initial conditions $U(0) = 0, V(0) = 0$. The depth of the particle can be computed diagnostically from the $y$ position and the known slope. Notice that we have dropped the superscript prime that had previously been used to indicate the rotating frame variables and we have used the thin fluid approximation for the Coriolis terms. We also use the $f-$plane approximation that $f$ = constant since typical particle displacements are very small compared to the Earth's radius. The solutions of this linear model are not complex,

$$\begin{aligned} U(t) &= \frac{b\alpha}{k^2 + f^2}\left[f - \exp(-tk)(f\cos(ft) - k\sin(ft))\right], \\ V(t) &= \frac{b\alpha}{k^2 + f^2}\left[k - \exp(-tk)(f\sin(ft) + k\cos(ft))\right] \end{aligned} \tag{62}$$

though they do contain three parameters along with the time, and hence represent a fairly large parameter space. We are not interested in any one solution as much as we are in understanding the qualitative effects of rotation and friction upon the entire family of solutions. How can we display the solution to this end?

One approach of great generality is to rewrite the governing equations or the solution using nondimensional variables. This will serve to reduce the number of parameters to the fewest possible. To define these nondimensional variables we begin by noting that there are three external parameters in the problem (external in that they do not vary with a dependent variable): the buoyancy and bottom slope, $b\alpha$, which always occur in this combination, the Coriolis parameter, $f$, an inverse time scale, and the bottom friction coefficient, $k$, also an inverse time scale. To form a nondimensional velocity, $U^* = U/U_{geo}$, we have to make an estimate of the velocity scale as the product of the acceleration and the time scale $f^{-1}$ as $U_{geo} = (b\alpha)/f$ and thus $U^* = U/(b\alpha/f)$ and similarly for the $V$ component. To define a nondimensional time we need an external time scale and choose the inverse of the Coriolis parameter, $t^* = tf$, rather than $k^{-1}$, since we expect that rotational effects will dominate frictional effects in most cases of interest. Rewriting the governing equations in terms of these nondimensional variables gives

$$\frac{dU^*}{dt^*} = V^* - EU^*, \tag{63}$$

$$\frac{dV^*}{dt^*} = -U^* - EV^* + 1, \tag{64}$$

and initial conditions $U^*(0) = 0,\ V^*(0) = 0$. The solution

$$U^*(t^*) = \frac{1}{1+E^2}\left[1 - \exp(-Et^*)(\cos(t^*) - E\sin(t^*))\right], \tag{65}$$

$$V^*(t^*) = \frac{1}{1+E^2}\left[E - \exp(-Et^*)(\sin(t^*) + E\cos(t^*))\right],$$

$$U^* = \frac{U}{b\alpha/f}, \qquad t^* = tf, \qquad \text{and} \qquad E = k/f,$$

shows explicitly that the single nondimensional parameter $E = k/f$ serves to define the parameter space of this problem.[34] $E$, often termed the Ekman number, is the ratio of frictional to rotational forces on the particle. Thus we probably shouldn't speak of large friction or large rotation, but rather of large or small $E$.

## 5.1 Inertial and geostrophic motion

The solution is made up of two modes: a time-dependent, oscillatory part, $\propto \cos(t^*),\ \sin(t^*)$, which is the now familiar inertial oscillation.[35] These oscillations are the consequence of starting from rest (or not in a steady balance), and decay with an e-folding time of $E^{-1}$ (the dimensional period of the oscillation is $f/2\pi$ and the e-folding is in $1/E$ of these periods). If we had ignored bottom friction the inertial oscillations would persist. There is also a steady (or time mean) mode $\propto [1, E]$. If $E$ is small, then the long term displacement of the particle is in a direction almost perpendicular to the applied force, Fig. (9). If the applied force is due to a pressure gradient or buoyancy force, the time-mean velocity is said to be geostrophic, and is the first approximation to the momentum balance of most large scale currents and winds (tides being an exception). The effect of drag on the time-mean motion is to cause a down-slope component of order $E$ compared to the along slope component, and thus a particle will cross isobaths toward greater depth at an angle (Fig. 9 (right)) given by the Ekman number,

$$\frac{V*}{U*} = E.$$

In this linear model the amplitude of the response is directly proportional to the forcing, $U_{geo} = b\alpha/f$. For a dense water particle on a typical continental slope rough values are $b = g(\delta\rho)/\rho_0 \approx g1.0/1000 = 10^{-2}$ m sec$^{-2}$, $\alpha = 5\times10^{-3}$, and f at 45 degrees latitude = 1.0 $\times10^{-4}$ sec$^{-1}$, and thus a typical geostrophic density current has a speed $U_{geo} = 0.5$ m sec$^{-1}$.

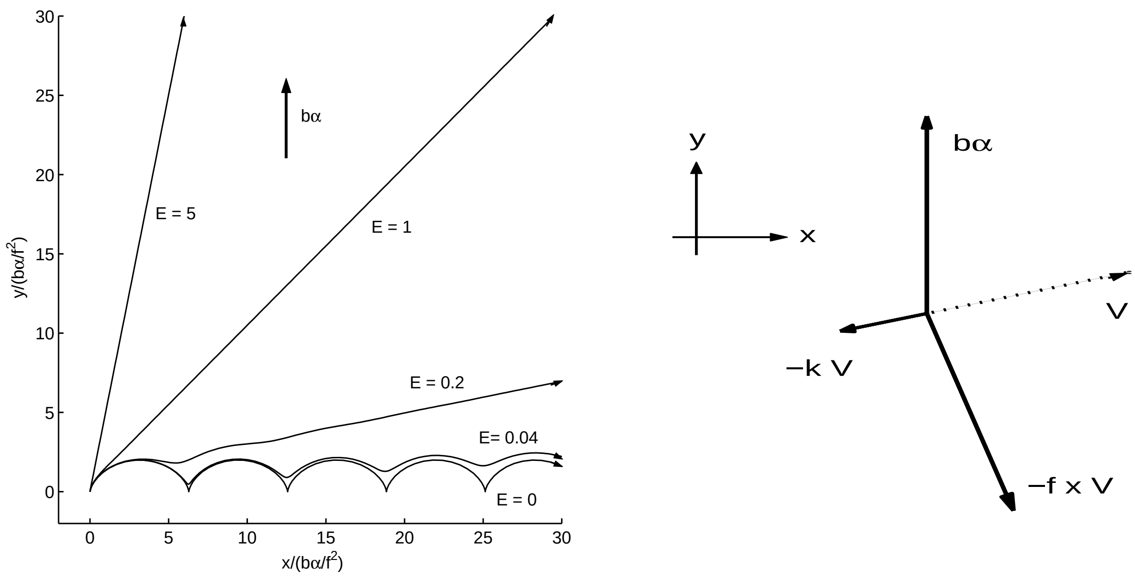


Figure 9: (left) Trajectories of dense particles released from rest onto a sloping bottom. The buoyancy force is toward positive $y$. The Ekman number, $E$, has the value shown. Notice that for values of $E$ small compared to 1 the long term displacement is nearly at right angles to the imposed force, indicative of geostrophic balance. (right) A plan view of the slope with vectors indicating the force balance (solid arrows) and the time-mean motion (the dashed vector) for the time-mean of the case $E = 0.2$. The angle of the velocity with respect to the isobaths is $E = k/f$, the Ekman number. The Coriolis force is labeled $-\mathbf{f} \times \mathbf{V}$ where $\mathbf{f}$ is $f$ times a vertical unit vector.

## 5.2 Energy budget

The energy budget for the particle makes an interesting diagnostic. To find the energy budget (per unit mass) we simply multiply the $x$-component momentum equation by $U$ and the $y$-component equation by $V$ and add:

$$\frac{d(U^2 + V^2)/2}{dt} = -\tilde{g}\alpha V - k(U^2 + V^2) \tag{66}$$

and then integrate with time to calculate the energy changes and net work;

$$\begin{aligned} (U^2 + V^2)/2 \quad &= \quad -\tilde{g}\Delta h \quad - \quad \int_o^t k(U^2 + V^2)dt \\ KE \quad &= \quad \Delta PE \quad - \quad FW \end{aligned} \tag{67}$$

with $FW$ the net frictional work (Fig. 10). The Coriolis force drops out of the energy budget since it is normal to the current and hence does no work. Nevertheless, the Coriolis force has a profound effect on the energy budget overall, as can be appreciated by noting that if $f = 0$, the particle would descend the gradient of the topography, releasing potential energy much faster than shown here, and of course also causing much greater frictional work. Thus, rotation has the effect of slowing the rate of potential energy release. If friction were zero, then the time-averaged motion would be perpendicular to the applied force and the particle would simply coast along with no energy exchanges. This is the single particle analog of geostrophic motion noted in the opening paragraph of Section 1.[37]

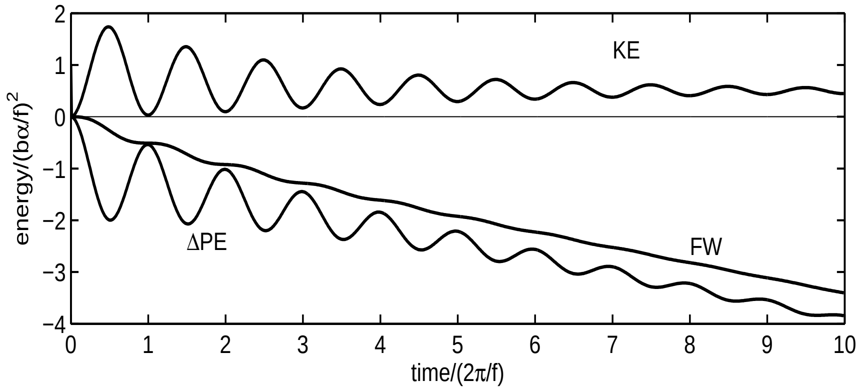


Figure 10: The energy budget for the trajectory of Fig. (9) having $E = 0.04$. These data are plotted in nondimensional form.

## 6 Summary and closing remarks.

This essay has been a rather slow and careful journey from Eq. (1) to Eq. (2)[38], with the goal being to understand better the dynamics of a rotating reference frame, and especially the Coriolis force that results. The first and essential step was to transform an acceleration defined in an inertial frame into a steadily rotating frame. The transformation law defined by Eqs. (26) or (27) can be derived without the need for approximation, and shows that there are two terms required in the transformation — the Coriolis term, $2\boldsymbol{\Omega} \times \mathbf{V}'$, and the centrifugal term, $\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{V}'$. In the special case of an Earth-attached reference frame, the centrifugal term is exactly canceled by a component of the gravitational mass attraction, and so drops out of the equation of motion, leaving only the Coriolis force. For small scale motions or motions strongly affected by friction, the Coriolis force may be entirely negligible. For the large scale motions of the atmosphere and ocean the effects of Earth's rotation are of qualitative importance. There are two important modes of motion that are directly attributable to rotation: inertial oscillations result when there is a balance between the time rate of change and the Coriolis force, and geostrophic flow results when a pressure gradient balances the Coriolis force. A geostrophic momentum balance is perhaps the distinguishing characteristic of the major wind and current systems of Earth's atmosphere and ocean outside of equatorial regions.

To close we will try to summarize a response to the last and most important of the questions raised in Section 1.2 — What is the Coriolis force, and what should we call it? Before we respond to this we should recognize that if we had the ability to compute trajectories in an inertial frame, we could then transform those trajectories into the rotating frame and would never have to consider the Coriolis force. An example of this procedure was shown in Section 3.2. However, inertial frame solutions are almost never attempted for oceanic and atmospheric flows, which in practice are much better analyzed from an Earth-attached rotating frame. Once we decide to use a rotating reference frame, then the centrifugal and Coriolis forces are exact consequences of the transformation of the equation of motion to a steadily rotating frame; there is nothing *ad hoc* or discretionary about the centrifugal and Coriolis force.[4]

What we call the Coriolis term, i.e., whether an acceleration or a force, *is* a matter of choice, though our usage should be consistent with our interpretation. The former is sensible in so far as the

Coriolis term arises from the transformation of acceleration, and too because it is independent of the mass of the particle. However, when we use an Earth-attached, rotating reference frame we seek to analyze (and necessarily observe) the acceleration seen in the rotating frame, $d^2\mathbf{X}'/dt^2$, and not the rotated acceleration (Section 2.5). If we reserve the phrase acceleration to mean the unknown that we solve in a momentum equation, then all the other terms, including the Coriolis term, are perforce, considered to be forces. Thus when we ascribe a direction to the Coriolis term it is almost always in the sense of a force, rather than an acceleration (e.g., as in the force balance of Fig. 9). These last considerations favor the usage 'Coriolis force', that we have followed here.

The Coriolis and centrifugal forces arise solely from the rotation of a reference frame, rather than as the result of a two-way interaction between physical objects, and this leads to some peculiar behavior. Recall the rotating frame, elementary trajectory of Section 3.2; the Coriolis force was deemed to deflect the particle, but we couldn't point to anything in the environment that revealed the corresponding reaction force. Even more alarming, the centrifugal force seemed to be an infinite source of work on the particle, at no cost. The rotating frame equation of motion thus does not support a global energy and momentum balance and neither does it preserve invariance to Galilean transformations. In that respect the Coriolis and centrifugal forces are clearly different from gravity or friction, say, and so it is also appropriate to call the Coriolis an 'apparent' force to acknowledge its origin in kinematics and in our use of an Earth-attached reference frame.

**Acknowledgments:** The author was supported primarily by the Seward Johnson Chair in Physical Oceanography, administered by the Massachusetts Institute of Technology — Woods Hole Oceanographic Institution Joint Program in Oceanography. Additional support during the period of manuscript preparation was provided by the U.S. Office of Naval Research. My sincere thanks to Prof. Jim Hansen of MIT for his insightful comments on an early draft of this manuscript, and to Shin Kida of the MIT/WHOI Joint Program for his careful reading of the present draft.

## References

[1] This geostrophic balance is evident in weather maps that depict both the pressure field (usually as the height of pressure surfaces) and the wind vector at the same level. Weather maps are available on the web from, among others, Fleet Numerical Meteorological and Oceanographic Center; www.fnmoc.navy.mil, and take a look at MyWxmap, say for North America, 500 hPa temperature, wind barbs and geopotential height.

You are probably asking yourself "What's it do right on the equator?" (S. Adams, *It's Obvious You Won't Survive by Your Wits Alone*, p. 107, Andrews and McNeil Press, Kansas City, Kansas, 1995). By symmetry we would expect that this deflection does not occur on the equator, and that large scale pressure and flow fields in the equatorial regions would be quite different from that at higher latitudes. What do the weather maps indicate?

[2] An informative history of the Coriolis force is by A. Persson, "How do we understand the Coriolis force?" Bull. Am. Met. Soc., **79**(7), 1373–1385 (1998), which includes an extensive bibliography of articles related to the atmospheric sciences.

[3] Central forces are those that act in a direction along the radius between particles and are instantaneous. This holds very accurately of the main forces that drive geophysical flows, but not

for the electromagnetic force between moving charged particles, and not for the Coriolis force either, as we will describe shortly.

[4] One authoritative view of the Coriolis and centrifugal forces is by Marion (Ref. 5), who describes the plight of a rotating observer as follows (the quotes are his):

> ... the observer must postulate an additional force - the centrifugal force. But the "requirement" is an artificial one; it arises solely from an attempt to extend the form of Newton's equations to a non inertial system and this may be done only by introducing a fictitious "correction force". The same comments apply for the Coriolis force; this "force" arises when attempt is made to describe motion relative to the rotating body.

[5] Each of the following classical mechanics texts have excellent discussions of noninertial reference frames. In order of increasing level: A. P. French, *Newtonian Mechanics* (W. W. Norton Co., 1971), J. D. Marion, *Classical Mechanics of Particles and Systems* (Academic Press, NY, 1965), A. L. Fetter and J. D. Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw-Hill, NY, 1990), C. Lanczos, *The Variational Principles of Mechanics* (Dover Pub., NY, 1949). Of these, French provides perhaps the best foundation in classical mechanics along with excellent discussions of highly relevant geophysical and astrophysical phenomenon.

[6] Textbooks on geophysical fluid dynamics emphasize mainly the consequences of Earth's rotation. A. E. Gill, *Atmosphere-Ocean Dynamics* (Academic Press, NY, 1982), J. R. Holton, *An Introduction to Dynamic Meteorology, 3rd Ed.* (Academic Press, San Diego, 1992). A particularly thorough account of the Coriolis force, including examples not emphasized here, is by B. Cushman-Roisin, *Introduction to Geophysical Fluid Dynamics* (Prentice Hall, Engelwood Cliffs, New Jersey, 1994).

[7] There are engineering and industrial applications of the Coriolis force as well. For example, the Coriolis force is exploited by transducers that measure angular velocity required for vehicle control systems, http://www.siliconsensing.com, and to measure mass transport in fluid flow, http://micromotion.com.

[8] There are several essays or articles that, like this one, aim to clarify the Coriolis force. A fine treatment in great depth is by H. M. Stommel and D. W. Moore, *An Introduction to the Coriolis Force* (Columbia Univ. Press, 1989). A detailed treatment of particle motion including the still unresolved matter of the southerly deflection of dropped particles is by M. S. Tiersten and H. Soodak, 'Dropped objects and other motions relative to a noninertial earth', Am. J. Phys., **68**(2), 129–142 (2000). Another is by D. H. McIntyre, "Using great circles to understand motion on a rotating sphere," Am. J. Phys., **68**(12), 1097–1105 (2000). This last article has an accompanying web page, http://www.physics.orst.edu/∼mcintyre/coriolis/

[9] The text is meant to be accompanied by four Matlab scripts — rotation.m, Coriolis.m, Coriolis-forced.m and partslope.m — that can be recovered from the Mathworks File Central archive, where the file name is Coriolis in the Earth Science section, or from the author's anonymous ftp site: http://www.whoi.edu/science/PO/people/jprice/misc/welcome.html

[10] This brief review of reference frame rotation serves mainly to introduce notation. If the idea of the rotation matrix is not already familiar, then see, e.g., Ch. 3 of M. L. Boas, *Mathematical methods in the physical sciences, 2nd edition*, John Wiley and Sons (1983), an excellent reference for undergraduate-level applied mathematics. Also, take the time to verify Eqs. (6) through (11) by direct experimentation; if these equations do not have a concrete meaning for you, then the remainder of this important section will be for naught.

[11] To see that this holds for all vectors, imaging taping vectors (paper arrows) onto a turntable — some in the middle, some on the outside, and with random orientations. Once the turntable is set into rotation, all of the vectors will necessarily rotate at the same rate, $\Omega$, regardless of their position or orientation (assuming that they are all parallel to the plane of the turntable). If instead we orient the vectors so that they are normal to the plane of the turntable, then rotation will have no effect upon their amplitude or direction.

[12] To warm up for this, compute the second time derivative of $\sin(\theta)a(t)$, where $\theta = const + \Omega t$ and $a(t)$ is some differentiable function of time and compare to Eq. (21). Now suppose that $\theta$ is an arbitrary function of time. The new term proportional to the acceleration of angular position is sometimes referred to as the Euler force when it appears in a momentum equation.

[13] Centrifugal and centripetal mean center-fleeing and center-seeking, respectively, and may be used indicate the sign of a radial force, for example. However, these phrases are very commonly used to mean the specific term $\omega^2 r$, i.e., centrifugal force when it is on the right side of an equation of motion and centripetal acceleration when it is on the left side.

[14] This phrase, 'fixed stars', sounds archaic but has a real significance in this context. The distant, fixed stars are presumed to be a proxy for the spatially-averaged mass of the universe, which, so far as is known, define an inertial reference frame. This grand idea was expressed by the German philosopher/physicist Ernst Mach, who insisted that only relative position or motion was meaningful, and that acceleration (linear or rotational) relative to the mass of the entire universe would give rise to inertia (see, e.g., M. Born, *Einstein's Theory of Relativity*, Dover Publications (1962)).

[15] The Earth's motion through space is much more complex than a spin around the polar axis. Among other things the Earth orbits the sun in a counterclockwise direction with a rotation rate of 1.9910 $\times 10^{-7}$ sec$^{-1}$. This is only about 0.3% of the rotation rate $\Omega$, but the question we pose here is does this orbital motion enter the Coriolis force, or otherwise effect the dynamics of the atmosphere and oceans? The short answer is no and yes. We have already fully accounted for the rotation of the Earth when we measured the rotation rate with respect to the fixed stars and found $\Omega = 7.2921 \times 10^{-5}$ sec$^{-1}$. Whether this rotation is due to a spin about an axis centered on the Earth or due to a solid body rotation about some displaced center is not relevant for the Coriolis force *per se*. However, since Earth's polar axis is tilted significantly from normal to the plane of the Earth's orbit, and since the polar axis remains nearly aligned on the North Star throughout an orbit, we can ascribe the rotation $\Omega$ to spin alone.

The orbital motion about the sun gives rise to tidal forces, which are small but important spatial variations of the centrifugal/gravitational balance that holds for the Earth and Sun as a whole (see, e.g., French[5]).

The inverse of Earth's rotation rate defines a period called the sidereal day, $2\pi/\Omega$ = 23 hrs, 56 min and 9 sec, the time interval between the meridian transit of a given, fixed star. Because the Earth's orbit is leisurely compared with its rotation rate, the duration of the sidereal day is within about 0.3% of the usual (solar) day, and the two are sometimes not distinguished. For rough numerical estimates that may be acceptable, but for our purpose the sidereal and solar days are qualitatively different, and it is the former that measures the Earth's rotation rate with respect to the mass of the universe (rather than with respect to the Sun only).

What is the rotation rate of the Moon? Hint - make a sketch of the Earth-Moon orbital system and consider what we observe of the Moon from Earth.

Earth's rotation rate varies slightly but detectably along with the Earth's moment of inertia, due in part to changes in the atmosphere and ocean circulation and mass distribution within the cryosphere, see B. F. Chao and C. M. Cox, "Detection of a large-scale mass redistribution in the terrestrial system since 1998," Science, **297**, 831–833 (2002), and R. M. Ponte and D. Stammer, "Role of ocean currents and bottom pressure variability on seasonal polar motion," J. Geophys. Res., **104**, 23393–23409 (1999). The inclination of Earth's axis of rotation varies by a few degrees on a time scale of several tens of thousands of years and the direction of the rotation axis precesses on a similar time scale due to gravitational interactions between the Earth, the Moon and the nearest planets. These small variations of the Earth's orbital parameters appear to be an important element of Earth's climate since they give rise to long-time scale variations in the amplitude of seasonality, see, e.g., J. A. Rial, "Pacemaking the ice ages by frequency modulation of Earth's orbital eccentricity," Science, **285**, 564–568 (1999).

[16] Other authors, and notably Stommel and Moore,[8] indicate that there is a possible "physical" understanding of the Coriolis force, and so perhaps this is a matter of what we call things, rather than what they are, since we all agree on Eq. (2). There is a positive point to be made here, however, and that is to emphasize the fundamental importance of the kinematics developed in Section 2.

[17] The steady linear translation of a reference frame has no effect upon the (Newtonian) dynamics observed from that reference frame. This is also called Galilean invariance. By contrast, a steady rotation rate renders a reference frame non-inertial, the departure depending upon the magnitude of the rotation rate as measured by the angular velocity of fixed (very distant) stars. Rotation of a reference frame is thus qualitatively different from the translation of a reference frame. This crucially important point is discussed further by J. Schwinger, *Einstein's Legacy*, Dover Publications (1986) and Born[14], both of which are highly recommended.

[18] A well-thrown baseball travels at about 45 m $s^{-1}$. How much will it be deflected as it travels over a distance of 30 m? Use the nominal Earth's rotation rate (as we will see in Section 4.2 this is appropriate for the north pole). A long-range artillery shell has an initial speed of about 700 m $s^{-1}$. Assuming the shell is launched at angle to the ground of 30 degrees, how much will it be deflected over its trajectory (ignoring air resistance)?

[19] Imagine riding along on the projectile. After launch (and before returning to the ground) the projectile is in free fall and is a zero-$g$ or weightless environment. This is the same whether the reference frame rotates or not. On the other hand, the view out the window would be quite surprising if the reference frame were rotating.

[20] A comprehensive source for physical data on the planets is by C. F. Yoder, "Astrometric and geodetic data on Earth and the solar system," Ch. 1, pp 1–32, of *A Handbook of Physical Constants: Global Earth Physics (Vol. 1).* American Geophysical Union (1995).

[21] This model underestimates the actual displacement by almost a factor of 2. The reason is mainly that the mass displaced from the pole towards the equator causes the gravitational mass attraction to differ from that of a perfect sphere (assumed in writing Eq. 42) by having a small equatorward component. The mathematics required to account for this change in the mass distribution take us far afield from our main topic; see Ch. 9 of Stommel and Moore.[8]

[22] The flatness of a rotating planet is $F = (R_e - R_p)/R_e \approx \Omega^2 R_e/g$. If the gravitational acceleration $g$ is written in terms of the planet's mean density, $\rho$ and the mean radius then

$$F = \frac{\Omega^2}{\frac{4}{3}\pi G \rho},$$

where $G = 6.67\times 10^{11}$ $m^3$ $kg^{-1}$ $s^{-2}$ is the universal gravitational constant. The rotation rate and the density vary a good deal among the planets, and consequently so does the flatness. The most extreme case is that of the gas giant, Saturn, which has a rotation rate a little more than twice that of Earth and a very low mean density, about one eighth of Earth's. Saturn's flatness is large enough, $F \approx 0.10$ that it can be discerned through a good backyard telescope (Fig. 6).

[23] This important point may be easier to appreciate in a simpler geometry. **Linear, Accelerated Motion.** Imagine a tank containing a fluid that is subjected to a constant, linear horizontal acceleration of magnitude $\beta g$, where $g$ is the nominal gravitational acceleration. To an observer in the frame of the accelerating tank, the entire effect of this constant acceleration is as if gravity has changed direction and magnitude. The new vertical direction can be observed by a plumb bob. Once the transients die away and the fluid settles into a state of rest in the frame of the accelerating tank, this new vertical direction will be perpendicular to the free surface of the fluid and surfaces of constant hydrostatic pressure will be parallel to the tilted free surface. The magnitude of the new $g$ will be greater than the nominal value by the factor $\sqrt{1+\beta^2}$. To an observer outside of the tank, a

fluid parcel will appear to be accelerating (horizontally) at a rate $\beta g$ due to the hydrostatic pressure gradient set up by the tilted free surface. **Circular Motion.** Now imagine that the tank is rotating steadily at a rate $\Omega$ about the center. When the fluid reaches a steady (solid body) rotation the free surface will assume a parabolic displacement, $\eta = \Omega^2 r^2/2g$, so that the resulting hydrostatic radial pressure gradient provides the centripetal force required to balance the centripetal acceleration. The remarks above regarding an observer who is within the rotating tank applies here as well.

[24] Consider motion that is either horizontal or vertical; can you relate the dependence of $f$ upon latitude to the orientation of vectors on a turntable noted in Ref. (11)?

[25] The next approximation to Eq. (47) is to retain the first order term, with the symbol $\beta$ often used to denote the first derivative, viz., $f = f_0 + (y - y_0)\beta$, the beta-plane approximation. A profound change in dynamics follows on this seemingly small change in the Coriolis parameter (see Refs. 6).

[26] The effect of Earth's rotation on the motion of a simple (one bob) pendulum, called a Foucault pendulum in this context, is treated in detail in many physics texts by e.g., Marion,[5] and will not be repeated here. Here are a few questions, however. Can you calculate the Foucault pendulum motion by rotating the inertial frame solution? How does the time required for precession through 360 degrees depend upon latitude? What happens when the pendulum's natural frequency (in the absence of Earth's rotation) equals the Earth's rotation rate? Given the rotated trajectory, can you show that the acceleration for very short times is consistent with the rotating frame equations of motion?

A Foucault pendulum can be easily made and used to make relevant observations. There are two properties of the pendulum that require some attention: First, the decay rate must be sufficiently slow that the pendulum will maintain most of its energy for a time of at least 20-30 min. This is most easily achieved by using a dense, smooth and symmetric bob having a weight of about half a kilogram or more, and suspended on a fine, smooth monofilament line. It is helpful if the length can be made several meters or more. Second, the pendulum should not interact appreciably with its mounting and certainly should not be turned by the mounting. This is hard to evaluate, but generally requires a very rigid support, and a bearing that can not exert torque, for example a needle bearing of some kind.

You should plan to bring a simple and rugged pocket pendulum with you on your merry-go-round ride (Section 3.2), where rotational effects are not the least bit subtle. How do these observations (even if qualitative) compare with your solution for a Foucault pendulum?

[27] This ignores other, even larger motions of the Earth, the orbital motion around the Sun (noted already in Ref. 15), the motion of our Solar System within the Milky Way Galaxy, and the motion of the local galaxy cluster (and see French[5]). These very large velocities are not associated with strong accelerations, and so are not relevant to our purpose here.

[28] Throughout this essay we consider exclusively the case of solid body rotation since that's what coordinate system rotation amounts to. It is possible to have other kinds of rotation that might describe, say, the radial dependence of the azimuthal velocity in a fluid vortex (a circular motion). There too the solid body rotation is an important case, but other kinds of vortices are also quite common. For example, the vortex that forms in a drain or that spills off of the edges of a paddle that has been pushed through water will have an azimuthal velocity that goes like

$$U_\theta \propto r^{-1},$$

(except very near the center and at great distances from the center). This kind of vortex is known as an irrotational or free vortex. It would be useful for you to carry through the kind of calculation we did for a solid body rotation to find the frequency of small perturbations to that kind of vortex flow. Now a more general problem - under what circumstances would a perturbation grow rather than oscillate, as we found above? A suggestion - consider simple power law velocity profiles, i.e. $U_\theta \propto r^n$, and find the circumstance (the $n$) that causes the frequency to become real so that a perturbation grows exponentially in time.

[29] Can you derive the inertial motion trajectory by rotating the inertial frame trajectory that corresponds to Eq. (51)? Assume the simplest initial conditions. Given the rotating frame trajectory, can you verify that the acceleration is indeed consistent with the rotating frame momentum equation?

[30] The name 'inertial oscillation' is very widely accepted but is not highly descriptive of the dynamics in either the inertial or rotating reference frame. For the rotating frame, 'Coriolis oscillation' might be more appropriate, and see D. R. Durran, "Is the Coriolis force really responsible for the inertial oscillation?" Bull. Am. Met. Soc., **74**(11), 2179–2184 (1993).

[31] We noted in Section 1.1 that the rotating frame equations of motion do not satisfy a global momentum conservation or Galilean invariance. The former can be seen by noting that if all forces except Coriolis were zero, and the initial condition included a velocity, then that velocity would be continually deflected and change direction (as an inertial oscillation) with nothing else showing a reaction force; i.e., global momentum would not be conserved. The failure is, however, a soft one in the sense that the Coriolis force is not a source of momentum or energy. And, when a 'real' force $\mathbf{F}$ produces a change of momentum, the corresponding reaction force $-\mathbf{F}$ generates the complementary change of momentum that would then undergo a compensating Coriolis deflection.

[32] The applicability of our single-particle model to the interpretation of geostrophic adjustment of a fluid continuum is, admittedly, unclear. I will make the following claim on behalf of the single-particle model: everything that happens in the single-particle model occurs as well in the fluid model. The converse is certainly not true, however, and once you understand this single-particle model you should continue on with a study of a fluid model. To experiment with a simple fluid model of geostrophic adjustment you might try the Matlab script geoadjPE.m, also available from the author's web page[4].

[33] A linear drag law of this sort is most appropriate as a model of viscous drag in a laminar boundary layer within which $\tau = \mu \frac{\partial U}{\partial z}$, where $\mu$ is the viscosity of the fluid. The boundary layer above a rough ocean bottom is almost always fully turbulent above a very thin, O($10^{-3}$ m), laminar sublayer. If the velocity used to estimate drag is measured or analyzed at a depth in the fully turbulent boundary layer then the appropriate drag law can be approximated as being independent of the viscosity and so is quadratic in the velocity, $\tau \propto \rho U^2$.

[34] This use of nondimensional variables, though highly economic of parameters, does add an additional layer of abstraction to the analysis, and an occasional reminder of the dimensional variables can be very helpful. We also have to remind the reader of our nondimensional scheme, as there are usually several plausible ways to accomplish the same thing. For example, in this case we could have plausibly used $1/k$ to nondimensionalize the time. How would this change the result?

[35] The Coriolis force is identical in form to the Lorentz force, $F = q\mathbf{V}\times\mathbf{B}$, on a moving, charged particle in a magnetic field $\mathbf{B}$. Aside from the bottom drag, the motion of a dense particle on a slope is analogous to that of a charged particle that is accelerated in a uniform electric field and a perpendicular magnetic field. For example, if the particle is moving through a uniform magnetic field only, it will move in a circular orbit with the cyclotron frequency, $qB/M$, analogous to an inertial oscillation at the frequency $2\Omega$ sin(latitude).

[36] How would geostrophic adjustment look if viewed from an inertial frame, as in Fig. 8? Consider that there is an initial, balanced solid body rotation, and then impose a small radial or azimuthal force. Compare your (qualitative) result with the solution computed by the script Coriolis-forced.m.[9]

[37] This solution of the single particle model might give the impression that a geostrophic balance is almost inevitable on a rotating Earth, but there are three restrictions that limit this result mainly to motions of the ocean and atmosphere (including on planets other than Earth) that are free, large scale, and extratropical. Our single particle model gives a clear and relevant indication of only the first of these. **Free Motions.** Frictional drag must be very small in an absolute sense, and more to the point, small compared to the rotation rate as measured by the Ekman number. This holds well

for the atmosphere and ocean outside of bottom boundary layers, often termed the free atmosphere (or ocean). **Large Scale Motions.** A second restriction is that the external force must persist for at least a significant fraction of an inertial period, and the domain must be large enough that boundaries do not block the motion over at least similar time scales. In the case of a dense water particle sliding down a continental slope this requires that the slope have a width and length of at least several tens of kilometers, which is quite common. On the other hand, the flow in a river is both strongly braked by friction and strongly constrained by river banks. Rotational effects may be very important (see A. Einstein, The cause of the formation of meanders in the courses of rivers and of the so-called Baer's law, In *Ideas and Opinions*, 249-253. Wings Books, NY, 1954) but a river flow is usually nowhere near being in geostrophic balance. **Extratropical.** The third restriction is to latitudes outside of the tropics, and here our little single particle model could be positively misleading in that it gives an answer that is qualitatively correct but happens to be irrelevant. For a given $k$, $E$ will become very large as the latitude goes to zero and hence we would conclude that friction will dominate the dynamics within some narrow equatorial zone. This is true enough for the single particle model, but something quite different holds in the atmosphere or ocean. The horizontal component of the Coriolis force vanishes on the equator, and so a mass anomaly generated in the upper ocean by changing winds can not adjust to a geostrophic state. Friction is not especially important, however. Instead, an equatorial mass anomaly will rather quickly disperse into large scale gravity waves that then propagate for thousands of kilometers along the equator. This rapid and far reaching response within a few degrees of the equator is a key element of the El Nino phenomenon (see Refs. 6) that could not have been anticipated from the single particle model considered here.

[38] T. S. Eliot could have had the Coriolis force in mind when he wrote (Little Gidding, 1942):

We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we began
And to know the place for the first time.