$$x: \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} \qquad (1)$$

$$y: \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} \qquad (2)$$

$$\frac{\partial}{\partial y}(1): \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - f\frac{\partial v}{\partial y} - v\frac{\partial f}{\partial y}$$

$$= -g\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) \qquad (3)$$

$$\frac{\partial}{\partial x}(2): \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + f\frac{\partial u}{\partial x} + u\frac{\partial f}{\partial x}$$

$$= -g\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) \qquad (4)$$

$$(4)-(3): \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)$$

$$-\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$+ u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0$$

Note: $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$+ \frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$+ f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 0 + \beta v = 0$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} 2\Omega \sin\phi$$

$$= 2\Omega \cos\phi \frac{\partial \phi}{\partial y}$$

$\Delta y = r\Delta\phi$

$$= 2\Omega \cos\phi \left(\frac{1}{r}\right)$$

$$\frac{\partial f}{\partial y} = \beta$$

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y}$$

$$+ \omega_z\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

$$\frac{D\omega_z}{Dt} + (\omega_z + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

From shallow water continuity equation

$$\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\Rightarrow \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{1}{h}\frac{Dh}{Dt}$$

Substitute:

$$\frac{D\omega_z}{Dt} - \left(\frac{\omega_z + f}{h}\right)\frac{Dh}{Dt} + \beta v = 0$$

Note that

$$\frac{Df}{Dt} = \cancelto{0}{\frac{\partial f}{\partial t}} + u\cancelto{0}{\frac{\partial f}{\partial x}} + v\frac{\partial f}{\partial y}$$

$$\frac{Df}{Dt} = v\frac{\partial f}{\partial y} = \beta v$$

Substitute:

$$\frac{D\omega_z}{Dt} - \left(\frac{\omega_z + f}{h}\right)\frac{Dh}{Dt} + \frac{Df}{Dt} = 0$$

$$\frac{D}{Dt}(\omega_z + f) - \left(\frac{\omega_z + f}{h}\right)\frac{Dh}{Dt} = 0$$

Multiply through by $\frac{-h}{h^2}$

$$\frac{(\omega_z + f)\frac{Dh}{Dt} - h\frac{D}{Dt}(\omega_z + f)}{h^2} = 0$$

$$\boxed{\frac{D}{Dt}\left(\frac{\omega_z + f}{h}\right) = 0}$$

Shallow water potential vorticity equation