Rotating frame

$$\underline{u}_{fixed} = \underline{u}_{rot} + \underline{\Omega}\times r$$

$$\underline{a}_{fixed} = \underline{a}_{rot} + 2\underline{\Omega}\times\underline{u}_{rot} + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})$$

$$= \underline{a}_{rot} + 2\underline{\Omega}\times\underline{u}_{rot} - \Omega^2\underline{R}$$

Momentum equation (rotating)

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + (\underline{g} + \Omega^2\underline{R}) - 2\underline{\Omega}\times\underline{u} + \nu\nabla^2\underline{u} + \frac{\nu}{3}\nabla(\nabla\cdot\underline{u})$$

$$= -\frac{1}{\rho}\nabla p + \underline{g} - 2\underline{\Omega}\times\underline{u} + \nu\nabla^2\underline{u} + \frac{\nu}{3}\nabla(\nabla\cdot\underline{u})$$

centrifugal force sucked into gravity term

Coriolis force

- Turns stuff to the right (in N.H.)
- Does no work

Vorticity equation (rotating, earth-centric coord)

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega} + 2\underline{\Omega})\cdot\nabla\underline{u} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nu\nabla^2\underline{\omega}$$

Kelvin's circulation theorem (rotating)

$$\frac{D\Gamma_a}{Dt} = 0, \quad \Gamma_a = \int_A (\underline{\omega} + 2\underline{\Omega})\cdot d\underline{A}$$

Simplified momentum equations

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \nu\frac{\partial^2 u}{\partial z^2}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \nu\frac{\partial^2 v}{\partial z^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Know scaling arguments that got us here

Inviscid, simplified momentum equations

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

f-plane approximation

$$f = f_0 = 2\Omega\sin\phi_0$$

β-plane approximation

$$f = f_0 + \beta y = 2\Omega\sin\phi_0 + \frac{2\Omega\cos\phi_0}{r}\, y$$

Balances & flows from simplified, inviscid mo. eq.

- inertial oscillations

$$-\frac{u_\theta^2}{r} = fu_\theta$$

- geostrophy

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = fv$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -fu$$

- gradient wind (can be less than or greater than geostrophic wind)

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + fu_\theta$$

$$u_\theta = f\left(Ro, \frac{1}{\rho f^2 r}, \frac{\partial p}{\partial r}\right)$$

- describes cyclonic/anti-cyclonic highs/lows
- high pressure $\frac{\partial p}{\partial r}$ less than low pressure $\frac{\partial p}{\partial r}$
- high pressure systems have gentle winds in center

- cyclostrophic wind

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

- isallobaric wind

$$-u\frac{\partial u}{\partial x} = fv$$

$$\frac{\partial u}{\partial t} = fv$$

Geostrophy plus friction

$$\frac{1}{\rho}\frac{\partial p}{\partial x} - fv - \nu_H \nabla^2 u = 0$$

- Three way balance
- cross-isobaric flow

Balances & flows from simplified vorticity equation

- Taylor-Proudman

$$2\underline{\Omega} \cdot \nabla \underline{u} = 0 \qquad \text{(barotropic)}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

$\Rightarrow$ vertical rigidity

- Thermal wind

$$2\underline{\Omega} \cdot \nabla \underline{u} + \frac{1}{\rho^2}\nabla\rho \times \nabla p = 0 \qquad \text{(baroclinic)}$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{g}{f\rho}\underline{k} \times \nabla\rho$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{g\rho_0\alpha}{f\rho}\underline{k} \times \nabla T$$

based on relationship between ρ & T

Ekman Layer

- non-rotating boundary layers grow, Ekman layer does not

- mass transport

$$\underline{M}_{EK} = \frac{\underline{\tau} \times \underline{k}}{f}$$

- Ekman spiral

- Ekman pumping & suction

$$w_{EK} = \frac{1}{\rho f} \nabla \times \underline{\tau} \cdot k$$

Sverdrup transport

- explained using Kelvin's circulation theorem

$$V = \frac{1}{h \rho \beta} (\nabla \times \underline{\tau}) \cdot \underline{k}$$

- wind-driven circulation sensitive to curl of wind stress
- return flow in the western boundary current

Shallow water equations

$$\frac{Du}{Dt} = -g\frac{\partial h}{\partial x} + fv$$

$$\frac{Dv}{Dt} = -g\frac{\partial h}{\partial y} - fu$$

$$\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

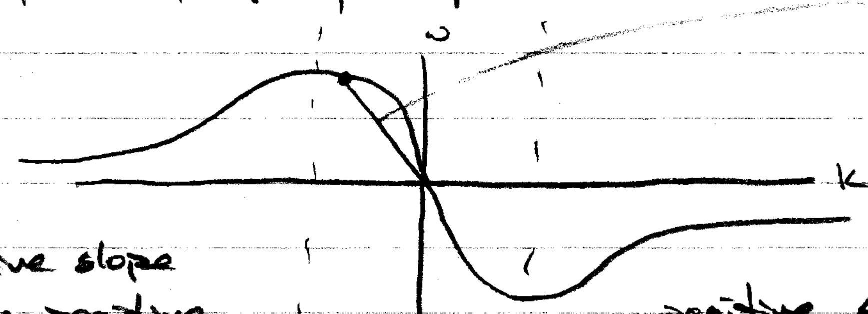
Wave kinematics

$$k = \frac{2\pi}{\lambda_x}, \quad l = \frac{2\pi}{\lambda_y}, \quad \omega = \frac{2\pi}{T}$$

phase speed $= \dfrac{\omega}{k}$

group speed $= \dfrac{\partial \omega}{\partial k}$ (carries information)

dispersion relation relates ω, k, l, & provide graphical information about phase & group speed



Slope of line connecting a point on the curve to the origin gives phase speed.

positive slope means positive group speed.

negative slope means negative group speed

positive slope means positive group speed

Shallow water potential vorticity equation

$$\frac{D}{Dt}\left(\frac{\omega_z+f}{h}\right)=0$$

obtained from cross-differentiating, subtracting, and simplifying the shallow water equations

Note: fixed-depth barotropic vorticity equation is

$$\frac{D}{Dt}(\omega_z+f)=0$$

Potential vorticity

$$\frac{\omega_z+f}{h} = \text{constant}$$

Potential vorticity is conserved following the flow

"Flow over a mountain" example

Shallow water gravity waves w/o rotation

- Obtained from non-rotating shallow water equations
- $\omega = \sqrt{g\bar{h}}\,K = cK$
- non-dispersive

Inertia-gravity waves

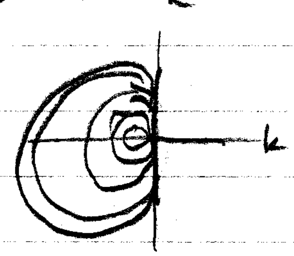
- obtained from rotating shallow water equations
- $\omega = 0, \pm\sqrt{f_0^2 + c^2K^2}$
- dispersive, except in limit of large K
- behave like non-rotating gravity waves at large K, $\omega = cK$
- behave like inertial oscillations at small K, $\omega = f_0$

Kelvin waves

- obtained from rotating shallow water equations w/ $v = 0$ (transverse velocity) and a lateral boundary
- $\omega = cK$
- non-dispersive

Constant depth, barotropic Rossby waves

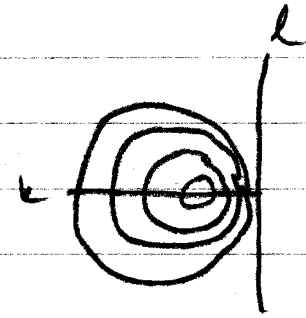
- obtained from barotropic vorticity equation ($\frac{D}{Dt}(\omega_z + f) = 0$)
- $\omega = -\dfrac{\beta k}{k^2 + l^2}$



- dispersive
- phase & group speeds can be in opposite directions

Shallow water Rossby waves

- obtained from conservation of potential vorticity
- $\omega = -\dfrac{\beta k}{k^2 + l^2 + \frac{1}{Rd^2}}$, $Rd = \dfrac{c}{f_0}$
- dispersive
- phase & group speeds can be in opposite directions
- Rd sets a length scale that forces a maximum phase speed.



Wave	dispersion relation	dispersive?	Comments
Gravity	$\omega = cK$	No	
Inertia-gravity	$\omega = 0$ $\omega = \pm\sqrt{f_0^2 + c^2K^2}$	Yes (for most k & l)	In limits of large & small K, behaves like gravity waves, and like inertial oscillations
Kelvin	$\omega = cK$	No	Leans against boundary, $v=0$ (transverse velocity)
Rossby (fixed depth)	$\omega = -\frac{\beta k}{k^2+l^2}$	Yes	Phase speed always negative, group speed either direction. No max phase speed & group speed
Rossby (shallow water)	$\omega = -\frac{\beta k}{k^2+l^2+\frac{1}{Rd^2}}$	Yes	As above, except limit on phase & group speed set by Rd