**Upper bound on list-decoding radius of binary codes**

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Upper bound on list-decoding radius of binary codes

Yury Polyanskiy

Abstract—Consider the problem of packing Hamming balls of a given relative radius subject to the constraint that they cover any point of the ambient Hamming space with multiplicity at most $L$. For odd $L \geq 3$ an asymptotic upper bound on the rate of any such packing is proven. The resulting bound improves the best known bound (due to Blinovsky’1986) for rates below a certain threshold. The method is a superposition of the linear-programming idea of Ashikhmin, Barg and Litsyn (that was used previously to improve the estimates of Blinovsky for $L = 2$) and a Ramsey-theoretic technique of Blinovsky. As an application it is shown that for all odd $L$ the slope of the rate-radius tradeoff is zero at zero rate.

Index Terms—Combinatorial coding theory, list-decoding, converse bounds

I. MAIN RESULT AND DISCUSSION

One of the well-studied problems in information theory asks to find the maximal rate at which codewords can be packed in binary space with a given minimum distance between codewords. Operationally, this (still unknown) rate gives the capacity of the binary input-output channel subject to adversarial noise of a given level. A natural generalization was considered by Elias and Wozencraft [1], [2], who allowed the decoder to output a list of size $L$. In this paper we provide improved upper bounds on the latter question.

Our interest in bounding the asymptotic tradeoff for the list-decoding problem is motivated by our study of fundamental limits of joint source-channel communication [3]. Namely, in [4, Theorem 6] we proposed an extension of the previous result in [3, Theorem 7] that required bounding rate for the list-decoding problem.

We proceed to formal definitions and brief overview of known results. For a binary code $C \subset \mathbb{F}_2^n$ we define its list-size $L$ decoding radius as

$$\tau(L) = \frac{1}{L} \max \{ \tau \geq 0 : \forall x \in \mathbb{F}_2^n, |C \cap \{ x + B^n_r \}| \leq L \},$$

where Hamming ball $B^n_r$ and Hamming sphere $S^n_r$ are defined as

$$B^n_r = \{ x \in \mathbb{F}_2^n : |x| \leq r \},$$

$$S^n_r = \{ x \in \mathbb{F}_2^n : |x| = r \}$$

with $|x| = \sum_{i=1}^n x_i$ denoting the Hamming weight of $x$. Alternatively, we may define $\tau(L)$ as follows:

$$\tau(L) = \frac{1}{L} \min \left\{ \text{rad}(S) : S \in \binom{C}{L+1} \right\} - 1,$$

where $\text{rad}(S)$ denotes radius of the smallest ball containing $S$ (known as Chebyshev radius):

$$\text{rad}(S) = \min_{y \in S} \max_{x \in S} |y - x|.$$ 

The asymptotic tradeoff between rate and list-decoding radius $\tau_L$ is defined as usual:

$$\tau^*_L(R) = \limsup_{n \to \infty} \max_{C : |C| \geq 2^n} \frac{\tau_L(C)}{n},$$

$$R_L^*(\tau) = \limsup_{n \to \infty} \frac{1}{n} \log |C|.$$ 

The best known upper (converse) bounds on this tradeoff are as follows:

- List size $L = 1$: The best bound to date was found by McEliece, Rodemich, Rumsey and Welch [5]:

$$R_1^*(R) \leq R_{LP2}(2R),$$

$$R_{LP2}(\delta) = \min \log 2 - h(\alpha) + h(\beta),$$

where $h(x) = -x \log x - (1-x) \log(1-x)$ and minimum is taken over all $0 \leq \beta \leq \alpha \leq 1/2$ satisfying

$$2^{\alpha(1-\alpha) - \beta(1-\beta)} \leq \delta.$$ 

For rates $R < 0.305$ this bound coincides with the simpler bound:

$$\tau_1^*(R) \leq \frac{1}{2} \delta L P1(R),$$

$$\delta L P1(R) = \frac{1}{2} - \sqrt{\beta(1-\beta)}, \quad R = \log 2 - h(\beta),$$

where $\beta \in [0, \frac{1}{2}]$.

- List size $L = 2$: The bound found by Ashikhmin, Barg and Litsyn [6] is given as:

$$R_2^*(R) \leq \log 2 - h(2\tau) + R_{up}(2\tau, 2\tau),$$

$$R_{up}(\delta, \alpha)$$ is the best known upper bound on rate of codes with minimal distance $\delta n$ constrained to live on Hamming spheres $S^n_{\alpha n}$. The expression for $R_{up}(\delta, \alpha)$ can be obtained by using the linear programming bound from [5] and applying Levenshtein’s monotonicity, cf. [7, Lemma 4.2(6)]. The resulting expression is

$$R_2^*(\tau) \leq \left\{ \begin{array}{ll}
R_{LP2}(2\tau), & \tau \leq \tau_0 \\
\log 2 - h(2\tau) + h(u(\tau)), & \tau > \tau_0,
\end{array} \right.$$ 

where $\tau_0 \approx 0.1093$ and

$$u(\tau) = \frac{1}{2} - \sqrt{\frac{1}{4} - \left( \sqrt{\tau - 3\tau^2} - \tau \right)^2}.$$ 

2This result follows from optimizing [6, Theorem 4]. It is slightly stronger than what is given in [6, Corollary 5].

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\[ j \] denotes the set of all subsets of $C$ of size $j$. 


For list sizes $L \geq 3$: The original bound of Blinovsky [8] appears to be the best (before this work):

$$
\tau^*_L(R) \leq \left\lfloor \frac{L}{2} \right\rfloor \frac{(2^{1/2} - 2)}{(L - 2)} \left( (1 - 2^{-1/2}) \right)^{L/2}, \quad R = 1 - h(\lambda),
$$

(10)

where $\lambda \in [0, 1]$. Note that [8] also gives a non-constructive lower bound on $\tau^*_L(R)$. Results on list-decoding over non-binary alphabets are also known, see [9], [10].

In this paper we improve the bound of Blinovsky for lists of odd size and rates below a certain threshold. To that end we will mix the ideas of Ashikhmin, Barg and Litsyn (namely, extraction of a large spectrum component from the code) and those of Blinovsky (namely, a Ramsey-theoretic reduction to study of symmetric subcodes).

To present our main result, we need to define exponent of Krawtchouk polynomial $K_{\lambda n}(\xi n) = \exp\{nE(\xi) + o(n)\}$. For $\xi \in [0, 1/2 - \sqrt{2}/(1 - \beta)]$ the value of $E(\xi)$ was found in [11]. Here we give it in the following parametric form, cf. [12] or [13, Lemma 4]:

$$
E(\xi) = \xi \log(1 - \omega) + (1 - \xi) \log(1 + \omega) - \beta \log \omega
$$

(11)

$$
\xi = \frac{1}{2} (1 - (1 - \beta) \omega - \beta \omega^{-1}),
$$

(12)

where

$$
\omega \in \left[ \frac{\beta}{1 - \beta} \sqrt{1 - \beta} \right].
$$

Our main result is the following:

**Theorem 1.** Fix list size $L \geq 2$, rate $R$ and an arbitrary $\beta \in [0, 1/2]$ with $h(\beta) \leq R$. Then any sequence of codes $C_n \subset \{0, 1\}^n$ of rate $R$ satisfies

$$
\lim_{n \to \infty} \tau_L(C_n) \leq \max_{j, \xi_0} \xi_0 g_j \left( 1 - \frac{\xi_1}{2 \xi_0} \right) + (1 - \xi_0) g_j \left( \frac{\xi_1}{2 (1 - \xi_0)} \right),
$$

(13)

where maximization is over $\xi_0$ satisfying

$$
0 \leq \xi_0 \leq \frac{1}{2} - \sqrt{\beta (1 - \beta)}
$$

(14)

and $j$ ranging over $\{0, 1, 3, \ldots, 2k + 1, \ldots, L\}$ if $L$ is odd and over $\{0, 2, \ldots, 2k, \ldots, L\}$ if $L$ is even. Quantity $\xi_1 = \xi_1(\xi_0, \delta, R)$ is a unique solution of

$$
R + h(\beta) - 2E(\xi_0) = h(\xi_0) - \xi_0 h \left( \frac{\xi_1}{2 \xi_0} \right) - (1 - \xi_0) h \left( \frac{\xi_1}{2 (1 - \xi_0)} \right),
$$

(15)

on the interval $[0, 2\xi_0(1 - \xi_0)]$ and functions $g_j(\nu)$ are defined as

$$
g_j(\nu) = \frac{1}{L + j} \left( L \nu - E [\lfloor 2W - L - j \rfloor] \right), W \sim \text{Bino}(L, \nu)
$$

(16)

As usual with bounds of this type, cf. [14], it appears that taking $h(\beta) = R$ can be done without loss. Under such choice, our bound outperforms Blinovsky’s for all odd $L$ and all rates small enough (see Corollary 3 below). The bound for $L = 3$ is compared in Fig. 1 with the result of Blinovsky numerically. For larger odd $L$ the comparison is similar, but the range of rates where our bound outperforms Blinovsky’s becomes smaller, see Table I.

**Evaluation of Theorem 1** is computationally possible, but is somewhat tedious. Fortunately, for small $L$ the maximum over $\xi_0$ and $j$ is attained at $\xi_0 = \frac{1}{2} - \sqrt{\beta (1 - \beta)}$ and $j = 1$. We rigorously prove this for $L = 3$:

**Corollary 2.** For list-size $L = 3$ we have

$$
\tau^*_L(R) \leq \frac{3}{4} \delta - \frac{1}{16} \left( \frac{2(\delta - \xi_1)^3}{\delta^2 + \xi_1^3} \right),
$$

(17)

where $\delta \in (0, 1/2]$ and $\xi_1 \in [0, 2\delta (1 - \delta)]$ are functions of $R$ determined from

$$
R = h \left( \frac{1}{2} - \sqrt{\delta (1 - \delta)} \right),
$$

(18)

$$
R = \log 2 - \delta h \left( \frac{\xi_1}{2\delta} \right) - (1 - \delta) h \left( \frac{\xi_1}{2 (1 - \delta)} \right).
$$

(19)

Another interesting implication of Theorem 1 is that it allows us to settle the question of slope of the curve $R^*_L(\tau)$ at zero rate. Notice that Blinovsky’s converse bound (10) has a negative slope, while his achievability bound has a zero slope. Our bound always has a zero slope for odd $L$ (but not for even $L$, see Remark 2 in Section II-C).

<Fig. 1. Comparison of bounds on $R^*_L(\tau)$ for list size $L = 3$>

<table>
<thead>
<tr>
<th>List size $L$</th>
<th>Range of rates</th>
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<tr>
<td>$L = 3$</td>
<td>$0 &lt; R \leq 0.361$</td>
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<tr>
<td>$L = 5$</td>
<td>$0 &lt; R \leq 0.248$</td>
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<tr>
<td>$L = 7$</td>
<td>$0 &lt; R \leq 0.184$</td>
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<tr>
<td>$L = 9$</td>
<td>$0 &lt; R \leq 0.136$</td>
</tr>
<tr>
<td>$L = 11$</td>
<td>$0 &lt; R \leq 0.100$</td>
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* This is computation of (13) with $h(\beta) = R$. 

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Notice that proofs of each of the two Corollaries below contain different relaxations of the bound (13), e.g. (22), which are easier to evaluate. Notice also that in Table I for the last two entries ($L = 9, 11$) at the high endpoint of rate the maximum over $\xi_0$ is attained not at $\frac{1}{2} - \sqrt{\beta (1 - \beta)}$. 

---

[Notice that proofs of each of the two Corollaries below contain different relaxations of the bound (13), e.g. (22), which are easier to evaluate. Notice also that in Table I for the last two entries ($L = 9, 11$) at the high endpoint of rate the maximum over $\xi_0$ is attained not at $\frac{1}{2} - \sqrt{\beta (1 - \beta)}$.]
Corollary 3. Fix arbitrary odd \( L \geq 3 \). There exists \( R_0 = R_0(L) > 0 \) such that for all rates \( R < R_0 \) we have
\[
\tau^*_L(R) \leq g_1(\delta_{LP1}(R)) ,
\]
where \( g_1(\cdot) \) is a degree-\( L \) polynomial defined in (16). In particular,
\[
\frac{d}{dr} \overline{R}^*_L(r) = 0 ,
\]
where the zero-rate radius is \( \tau^*_L(0) = \frac{1}{3} - 2^{-L-1}(\frac{L}{2}) \).

Before closing our discussion we make some additional remarks:

1) The bound in Theorem 1 can be slightly improved by replacing \( \delta_{LP1}(R) \), that appears in the right-hand side of (14), with a better bound, a so-called second linear-programming bound \( \delta_{LP2}(R) \) from [5]. This would enforce the usage of the more advanced estimate of Litsyn [15, Theorem 5] and complicate analysis significantly. Notice that \( \delta_{LP2}(R) \neq \delta_{LP1}(R) \) only for rates \( R \geq 0.305 \). If we focus attention only on rates where new bound is better than Blinovsky’s, such a strengthening only affects the case of \( L = 3 \) and results in a rather minuscule improvement (for example, for rate \( R = 0.33 \) the improvement is \( \approx 3 \cdot 10^{-5} \)).

2) For even \( L \) it appears that \( b(\beta) = R \) is no longer optimal. However, the resulting bound does not appear to improve upon Blinovsky’s.

3) When \( L \) is large (e.g. 35) the maximum in (13) is not always attained by either \( j = 1 \) or \( \xi_0 = \delta_{LP1}(R) \). It is not clear whether such anomalies only happen in the region of rates where our bound is inferior to Blinovsky’s.

4) The result of Corollary 3 follows by weakening (13) (via concavity of \( g_j \), Lemma 8) to
\[
\limsup_{n \to \infty} \tau_L(C_n) \leq \max_{j, \xi_0} g_j(\delta_{LP1}(R)) .
\]
The \( R < R_0(L) \) condition is only used to show that the maximum is attained at \( j = 1 \). Note also that weakening (22) corresponds to omitting the extra Elias-Bassalygo type reduction, which is responsible for the extra optimization over \( \xi_1 \) in (13).

Finally, at the invitation of anonymous reviewer we give our intuition about why our bound outperforms Blinovsky’s for odd \( L \). It is easiest to compare with the weakening (22) of our bound. Now compare the two proofs:

1) Blinovsky [8] first uses Elias-Bassalygo reduction to restrict attention to a subcode \( C' \) situated on a Hamming sphere of radius \( \approx \delta_{GV}(R) = h^{-1}(1 - R) \). Then he proves an upper bound for \( \tau_L(C') \) valid as long as \( |C'| \gg 1 \) via a Plotkin-type argument together with a great symmetrization idea.

2) Our bound (following Ashikhmin, Barg and Litsyn [6]) instead uses a Kalai-Linial [11] reduction to select a subcode \( C'' \) situated on a Hamming sphere of radius \( \approx \delta_{LP1}(R) \). We then proceeded to prove a (Plotkin-type) upper bound on a strange quantity:
\[
\tau^*_L(C'') = \frac{1}{n} \left( \max \left\{ \text{rad}(\{0\} \cup S) : S \in \binom{C}{L} \right\} - 1 \right) ,
\]
which corresponds to a requirement that the code contain not more than \( L - 1 \) codewords in any ball of radius \( \tau^*_L \), but only for those balls that happen to also contain the origin.

Notice that the sphere returned by Kalai-Linial is bigger than that of Elias-Bassalygo (which is the reason our bound deteriorates at large rates), but the good thing is that the subcode \( C'' \) has another codeword \( c_0 \) at the center of the Hamming sphere. Now, intuitively \( \tau^*_L \) is roughly equivalent to \( \tau_{L-1} \). The zero-rate (Plotkin) radius for a list-\( L \) decoding of binary codes on Hamming sphere \( S^*_\xi_0 \) is given by
\[
p_L(\xi) = \frac{E \left[ \min(W_{\xi, L} + 1 - W_{\xi}) \right]}{L + 1} , \quad W_{\xi} \sim \text{Bino}(L + 1, \xi) .
\]
So intuitively, we expect that Blinovsky’s bound should give
\[
\tau^*_L(R) \lesssim p_L(\delta_{GV}(R))
\]
while our bound should give
\[
\tau^*_L(R) \lesssim p_{L-1}(\delta_{LP1}(R)) .
\]
Finally, it is easy to check that for even \( L \) we have \( p_L = p_{L-1} \), while for odd \( L, p_L > p_{L-1} \). This is the main intuitive reason why our bound succeeds in improving Blinovsky’s, but only for odd \( L \).

II. PROOFS

A. Proof of Theorem 1

Consider an arbitrary sequence of codes \( C_n \) of rate \( R \). As in [6] we start by using Delsarte’s linear programming to select a large component of the distance distribution of the code. Namely, we apply result of Kalai and Linial [11, Proposition 3.2]: For every \( \beta \) with \( h(\beta) \leq R \) there exists a sequence \( \epsilon_n \to 0 \) such that for every code \( C \) of rate \( R \) there is a \( \xi_0 \) satisfying (14) such that
\[
A_{\xi_0}(C) \triangleq \frac{1}{|C|} \sum_{x,x' \in C} 1\{|x - x'| = \xi_0 n\} \geq \exp\{n(R + h(\beta) - 2E_\beta(\xi_0) + \epsilon_n)\} .
\]
Without loss of generality (by compactness of the interval \([0, 1/2 - \sqrt{\beta(1 - \beta)}] \) and passing to a proper subsequence of codes \( C_{n_k} \)) we may assume that \( \xi_0 \) selected in (23) is the same for all blocklengths \( n \). Then there is a sequence of subcodes \( C'_n \) of asymptotic rate
\[
R' \geq R + h(\beta) - 2E_\beta(\xi_0)
\]
such that each \( C'_n \) is situated on a sphere \( c_0 + S_{\xi_0} \) surrounding another codeword \( c_0 \in C \). Our key geometric result is: If there are too many codewords on a sphere \( c_0 + S_{\xi_0} \) then it is possible to find \( L \) of them that are includable in a small ball that also contains \( c_0 \). Precisely, we have:
Lemma 4. Fix $\xi_0 \in (0,1)$ and positive integer $L$. There exist a sequence $\epsilon_n \to 0$ such that for any code $C_n \subset S_{\xi_0n}$ of rate $R' > 0$ there exist $L$ codewords $c_1, \ldots, c_L \in C_n'$ such that
\[
\frac{1}{n} \text{rad}(0, c_1, \ldots, c_L) \leq \theta(\xi_0, R', L) + \epsilon_n, \tag{24}
\]
where
\[
\theta(\xi_0, R', L) \triangleq \max_j \theta_j(\xi_0, R', L), \tag{25}
\]
\[
\theta_j(\xi_0, R', L) \triangleq \xi_0 g_j \left(1 - \frac{\xi_1}{2\xi_0} \right) + (1 - \xi_0) g_j \left(\frac{\xi_1}{2(1 - \xi_0)} \right), \tag{26}
\]
with $\xi_1 = \xi_1(\xi_0)$ found as unique solution on interval $[0, 2\xi_0(1 - \xi_0)]$ of
\[
R' = h(\xi_0) - \xi_0 h \left(\frac{\xi_1}{2\xi_0} \right) - (1 - \xi_0) h \left(\frac{\xi_1}{2(1 - \xi_0)} \right), \tag{27}
\]
functions $g_j$ are defined in (16) and $j$ in maximization (25) ranging over the same set as in Theorem 1.

Equipped with Lemma 4 we immediatly conclude that
\[
\limsup_{n \to \infty} \tau_L(C_n) \leq \max_{\xi_0 \in [0,\delta]} \theta(\xi_0, R + h(\beta) - 2E(\xi_0), L). \tag{28}
\]
Clearly, (28) coincides with (13). So it suffices to prove Lemma 4.

B. Proof of Lemma 4

Let $T_L$ be the $(2^L - 1)$-dimensional space of probability distributions on $\mathbb{F}_2^L$. If $T \in T_L$ then we have
\[
T = (t_v, v \in \mathbb{F}_2^L) \quad t_v \geq 0, \sum_v t_v = 1.
\]
We define distance on $T_L$ to be the $L_\infty$ one:
\[
\|T - T'\| \triangleq \max_{v \in \mathbb{F}_2^L} |t_v - t'_v|.
\]
Permutation group $S_L$ acts naturally on $\mathbb{F}_2^L$ and this action descends to probability distributions $T_L$. We will say that $T$ is symmetric if
\[
T = \sigma(T) \iff t_v = t_{\sigma(v)} \quad \forall v \in \mathbb{F}_2^L
\]
for any permutation $\sigma : [L] \to [L]$. Note that symmetric $T$ is completely specified by $L + 1$ numbers (weights of Hamming spheres in $\mathbb{F}_2^L$):
\[
\sum

Next, fix some total ordering of $\mathbb{F}_2^n$ (for example, lexicographic). Given a subset $S \subset \mathbb{F}_2^n$ we will say that $S$ is given in ordered form if $S = \{x_1, \ldots, x_{|S|}\}$ and $x_1 < x_2 < \cdots < x_{|S|}$ under the fixed ordering on $\mathbb{F}_2^n$. For any subset of codewords $S = \{x_1, \ldots, x_L\}$ given in ordered form we define its joint type $T(S)$ as an element of $T_L$ with
\[
t_v \triangleq \frac{1}{n} \left| \{j : x_1(j) = v_1, \ldots, x_L(j) = v_j\} \right|,
\]
where here and below $y(j)$ denotes the $j$-th coordinate of binary vector $y \in \mathbb{F}_2^L$. In this way every subset $S$ is associated to an element of $T_L$. Note that $T(S)$ is symmetric if and only if the $L \times n$ binary matrix representing $S$ (by combining row-vectors $x_j$) has the property that the number of columns equal to $[1, 0, \ldots, 0]^T$ is the same as the number of columns $[0, 1, \ldots, 0]^T$ etc. For any code $C \subset \mathbb{F}_2^n$ we define its average joint type:
\[
\bar{T}_L(C) = \frac{1}{L!} \left(\binom{2^n}{L}\right) \sum_{S \subseteq \binom{L}{L}} \sigma(T(S)).
\]
Evidently, $\bar{T}_L(C)$ is symmetric.

Our proof crucially depends on a (slight extension of the) brilliant idea of Blinovsky [8]:

Lemma 5. For every $L \geq 1$, $K \geq L$ and $\delta > 0$ there exist a constant $K_1 = K_1(L, K, \delta)$ such that for all $n \geq 1$ and all codes $C \subset \mathbb{F}_2^n$ of size $|C| \geq K_1$ there exists a subcode $C' \subset C$ of size at least $K$ such that for any $S \in \binom{C}{L}$ we have
\[
\|T(S) - \bar{T}_L(C')\| \leq \delta. \tag{29}
\]

Remark 1. Note that if $S' \subset S$ then every element of $T(S')$ is a sum of $\leq 2^L$ elements of $T(S)$. Hence, joint types $T(S')$ are approximately symmetric also for smaller subsets $|S'| < L$.

Proof. We first will show that for any $\delta_1 > 0$ and sufficiently large $|C|$ we may select a subcode $C'$ so that the following holds: For any pair of subsets $S, S' \subset C'$ s.t. $|S| = |S'| \leq L$ we have:
\[
\|T(S) - T(S')\| \leq \delta_1. \tag{30}
\]
Consider any code $C_1 \subset \mathbb{F}_2^n$ and define a hypergraph with vertices indexed by elements of $C$ and hyper-edges corresponding to each of the subsets of size $L$. Now define a $\delta_1/2$-net on the space $T_L$ and label each edge according to the closest element of the $\delta_1/2$-net. By a theorem of Ramsey there exists $K_L$ such that if $|C| \geq K_L$ then there is a subset $C' \subset C$ such that $|C'| \geq K$ and each of the internal edges, indexed by $\binom{C'}{L}$, is assigned the same label. Thus, by triangle inequality (30) follows for all $S, S' \in \binom{C_1}{L}$.

Next, apply the previous argument to show that there is a constant $K_{L-1}$ such that for any $C_2 \subset \mathbb{F}_2^n$ of size $|C_2| \geq K_{L-1}$ there exists a subcode $C'_2$ of size $|C'_2| \geq K_L$ satisfying (30) for all $S, S' \in \binom{C_2}{L-1}$. Since $C'_2$ satisfies the size assumption on $C_1$ made in previous paragraph, we can select a further subcode $C''_2 \subset C'_2$ of size $\geq K_L$ so that for $C''_2$ property (30) holds for all $S, S'$ of size $L$ or $L - 1$.

Continuing similarly, we may select a subcode $C'$ of arbitrary $C$ such that (30) holds for all $|S| = |S'| \leq L$ provided that $|C| \geq K_1$.

Next, we show that (30) implies
\[
\|T(S_0) - \bar{T}(S_0)\| \leq C\delta_1,
\]
where $S_0 \in \binom{C}{L}$ is arbitrary and $C = C(L)$ is a constant depending on $L$ only.

Now to prove (31) let $T(S_0) = \{t_v, v \in \mathbb{F}_2^L\}$ and consider an arbitrary transposition $\sigma : [L] \to [L]$. It will be clear that our proof does not depend on what transposition is chosen, so
for simplicity we take \( \sigma = \{(L - 1) \leftrightarrow L \} \). We want to show that (30) implies
\[
|t_v - t_{\sigma(v)}| \leq \delta_1. \quad \forall v \in \mathbb{F}_2^n
\]
(32)
Since transpositions generate permutation group \( S_L \), (31) then follows. Notice that (32) is only informative for \( v \) whose last two digits are not equal, say \( v = [v_0, 0, 1] \). Suppose that \( S_0 = \{c_1, \ldots, c_L\} \) given in the ordered form. Let
\[
S = \{c_1, \ldots, c_{L-1}\}, \quad S' = \{c_1, \ldots, c_{L-2}, c_L\}
\]
(33)
(34)
Joint types \( T(S) \) and \( T(S') \) are expressible as functions of \( T(S_0) \) in particular, the number of occurrences of element \([v_0, 0] \) in \( S \) is \( t_{[v_0,0,1]} + t_{[v_0,0,0]} \) and in \( S' \) is \( t_{[v_0,0,0]} + t_{[v_0,1,0]} \). Thus, from (30) we obtain:
\[
|(t_{[v_0,0,1]} + t_{[v_0,0,0]}) - (t_{[v_0,0,0]} + t_{[v_0,1,0]})| \leq \delta
\]
(35)
implying (32) and thus (31).

Finally, we show that (31) implies (29). Indeed, consider the chain
\[
\|T(S) - \bar{T}_L(C')\| = \left\| T(S) - \frac{1}{L!} \cdot \left( \frac{C'}{L} \right) \sum_{S' \in C'} \sigma(T(S')) \right\|
\]
(36)
\[
\leq \frac{1}{L!} \cdot \left( \frac{C'}{L} \right) \sum_{S' \in C'} \|T(S) - \sigma(T(S'))\|
\]
(37)
\[
\leq \frac{1}{L!} \cdot \left( \frac{C'}{L} \right) \sum_{S' \in C'} \|T(S') - \sigma(T(S'))\|
\]
(38)
where (36) is by convexity of the norm, (37) is by triangle inequality and (38) is by (30) and (31). Consequently, setting \( \delta_1 = \frac{\delta}{1+\delta} \) we have shown (29).

Before proceeding further we need to define the concept of an average radius (or a moment of inertia):
\[
\bar{\mathrm{rad}}(x_1, \ldots, x_m) \triangleq \min_y \frac{1}{m} \sum_{i=1}^m |x_i - y|.
\]
Note that the minimizing \( y \) can be computed via a per-coordinate majority vote (with arbitrary tie-breaking for even \( m \)). Consider now an arbitrary subset \( S = \{c_1, \ldots, c_L\} \) and define for each \( j \geq 0 \) the following functions
\[
h_j(S) \triangleq \frac{1}{n} \mathrm{rad}(0, \ldots \hat{0}, c_1, \ldots, c_L) \text{ \( j \) times}.
\]
It is easy to find an expression for \( h_j(S) \) in terms of the joint-type of \( S \):
\[
h_j(S) = \frac{1}{L + j} \left( E[W] - E[|2W - L - j|] \right)
\]
(39)
\[
P[W = w] = \sum_{v:|v|=w} t_v,
\]
(40)
where \( t_v \) are components of the joint-type \( T(S) = \{t_v, v \in \mathbb{F}_2^L\} \). To check (39) simply observe that if one arranges \( L \) codewords of \( S \) in an \( L \times n \) matrix and also adds \( j \) rows of zeros, then computation of \( h_j(S) \) can be done per-column: each column of weight \( w \) contributes
\[
\min(w, L + j - w) = w - |2w - L - j| +
\]
to the sum. In view of expression (39) we will abuse notation and write
\[
h_j(T(S)) \triangleq h_j(S).
\]

We now observe that for symmetric codes satisfying (29) average-radii \( h_j(S) \) in fact determine the regular radius:

**Lemma 6.** Consider an arbitrary code \( C \) satisfying conclusion (29) of Lemma 5. Then for any subset \( S = \{c_1, \ldots, c_L\} \subset C \) we have
\[
\left| \bar{\mathrm{rad}}(0, c_1, \ldots, c_L) - n \cdot \max_j h_j(\bar{T}_L(C)) \right| \leq 2^L(1 + \delta n),
\]
(41)
where \( j \) in maximization (41) ranges over \( \{0, 1, 3, \ldots, 2k + 1, \ldots, L\} \) if \( L \) is odd and over \( \{0, 2, \ldots, 2k, \ldots, L\} \) if \( L \) is even.

**Proof.** For joint-types of size \( L \) and all \( j \geq 0 \) we clearly have (cf. expression (39))
\[
|h_j(T_1) - h_j(T_2)| \leq 2^{L-1}||T_1 - T_2||, \quad \forall T_1, T_2 \in T_L.
\]
(42)
We also trivially have
\[
\frac{1}{n} \bar{\mathrm{rad}}(0, c_1, \ldots, c_L) \geq h_j(S) \quad \forall j \geq 0.
\]
(43)
Thus from (29) and (42) we already get
\[
\frac{1}{n} \bar{\mathrm{rad}}(0, c_1, \ldots, c_L) \geq \max_j h_j(\bar{T}_L(C)) - 2^{L-1} \delta.
\]
(44)
It remains to show
\[
\frac{1}{n} \bar{\mathrm{rad}}(0, c_1, \ldots, c_L) \leq \max_j h_j(\bar{T}_L(C)) + \delta + \frac{2^L}{n}.
\]
(45)
This evidently requires constructing a good center \( y \) for the set \( \{0, 1, \ldots, c_L\} \). To that end fix arbitrary numbers \( q = (q_0, \ldots, q_L) \in [0, 1]^L \). Next, for each \( v \in \mathbb{F}_2^L \) let \( E_v \subset [n] \) be all coordinates on which restriction of \( \{c_1, \ldots, c_L\} \) equals \( v \). On \( E_v \) put \( y \) to have a fraction \( q_v \) of ones and remaining set to zeros (rounding to integers arbitrarily). Proceed for all \( v \in \mathbb{F}_2^L \) and call resulting vector \( y(q) \in \mathbb{F}_2^n \).

Denote for convenience \( e_0 = 0 \). We clearly have
\[
\bar{\mathrm{rad}}(c_0, c_1, \ldots, c_L) \leq \min_{q} \frac{E_v}{p} \sum_{i=0}^L p_i |c_i - y(q)|,
\]
(46)
where \( p = (p_0, \ldots, p_L) \) is a probability distribution.

Denote
\[
T(S) = \{t_v, v \in \mathbb{F}_2^L\}
\]
(47)
\[
\bar{T}_L(C) = \{\bar{t}_v, v \in \mathbb{F}_2^L\}
\]
We proceed to computing $|c_i - y(q)|$.

$$|c_i - y(q)| \leq n \sum_{v \in \mathbb{F}_2^n} t_v(q_{|v|}) 1\{v(i) = 0\} + (1 - q_{|v|}) 1\{v(i) = 1\} + 2^L,$$  

(48)

where $2^L$ comes upper-bounding the integer rounding issues and we abuse notation slightly by setting $v(0) = 0$ for all $v$ (recall that $v(i)$ is the $i$-th coordinate of $v \in \mathbb{F}_2^n$).

By (29) we may replace $t_v$ with $\tilde{t}_v$ at the expense of introducing $2^L n$ error, so we have:

$$|c_i - y(q)| \leq n \sum_{v \in \mathbb{F}_2^n} \tilde{t}_v(q_{|v|}) 1\{v(i) = 0\} + (1 - q_{|v|}) 1\{v(i) = 1\} + 2^L (1 + \delta n).$$  

(49)

Next notice that the sum over $v$ only depends on whether $i = 0$ or $i \neq 0$ (by symmetry of $\tilde{t}_v$). Furthermore, for any given weight $w$ and $i \neq 0$ we have

$$\sum_{v:|v| = w} 1\{v(i) = 1\} = \binom{L}{w} w \bigg( \frac{L}{n} \bigg)^w .$$

Thus, introducing the random variable $W$, cf. (39),

$$P[W = w] \triangleq \sum_{v:|v| = w} \tilde{t}_v,$$

we can rewrite:

$$\sum_{v \in \mathbb{F}_2^n} \tilde{t}_v(q_{|v|}) 1\{v(i) = 0\} + (1 - q_{|v|}) 1\{v(i) = 1\} = \frac{1}{L} \mathbb{E}[\tilde{W} + (L - 2W)q_{\tilde{W}}].$$  

(50)

For $i = 0$ the expression is even simpler:

$$\sum_{v \in \mathbb{F}_2^n} \tilde{t}_v(q_{|v|}) 1\{v(0) = 0\} + (1 - q_{|v|}) 1\{v(0) = 1\} = \mathbb{E}[q_{\tilde{W}}].$$

Substituting derived upper bound on $|c_i - y(q)|$ into (45) we can see that without loss of generality we may assume $p_1 = \cdots = p_L$, so our upper bound (modulo $O(\delta)$ terms) becomes:

$$\min_{q} \max_{p_1 \in [0, L^{-1}]} (1 - Lp_1) \mathbb{E}[q_{\tilde{W}}] + p_1 \mathbb{E}[\tilde{W} + (L - 2\tilde{W})q_{\tilde{W}}] = \min_{q} \max_{p_1 \in [0, L^{-1}]} p_1 \mathbb{E}[\tilde{W}] + \mathbb{E}[q_{\tilde{W}}(1 - 2\tilde{W}p_1)].$$

By von Neumann’s minimax theorem we may interchange min and max, thus continuing as follows:

$$\max_{p_1 \in [0, L^{-1}]} \min_{q} \mathbb{E}[\tilde{W}] + \mathbb{E}[q_{\tilde{W}}(1 - 2\tilde{W}p_1)]$$  

(51)

and

$$\max_{p_1 \in [0, L^{-1}]} p_1 \mathbb{E}[\tilde{W}] - \mathbb{E}[2\tilde{W}p_1 - 1]^+] .$$  

(52)

The optimized function of $p_1$ is piecewise-linear, so optimization can be reduced to comparing values at slope-discontinuities and boundaries. The point $p_1 = 0$ is easily excluded, while the rest of the points are given by $p_1 = \frac{\lambda - 1}{2\lambda}$ with $j$ ranging over the set specified in the statement of Lemma\(^4\). So we continue (52) getting

$$= \max_{j} \frac{1}{L + j} \left( \mathbb{E}[W] - \mathbb{E}[2W - L - j]^+ \right).$$  

(53)

We can see that expression under maximization is exactly $h_j(\tilde{T}_L(C))$ and hence (44) is proved. \(\square\)

**Lemma 7.** There exist constants $C_1, C_2$ depending only on $L$ such that for any $C \subset \mathbb{F}_2^n$ the joint-type $\tilde{T}_L(C)$ is approximately a mixture of product Bernoulli distributions\(^5\), namely:

$$\left\| \tilde{T}_L(C) - \frac{1}{n} \sum_{i=1}^n \text{Bern}^{\otimes L}(\lambda_i) \right\|_1 \leq \frac{C_1}{|C|},$$  

(54)

where $\lambda_i = \frac{1}{n} \sum_{c \in C} 1\{c(i) = 1\}$ be the density of ones in the $i$-th column of a $|C| \times n$ matrix representing the code. In particular,

$$\left| h_j(\tilde{T}_L(C)) - \frac{1}{n} \sum_j g_j(\lambda_j) \right| \leq \frac{C_2}{|C|},$$  

(55)

where functions $g_j$ were defined in (16).

**Proof.** Second statement (55) follows from the first via (42) and linearity of $h_j(T)$ in the type $T$, cf. (39). To show the first statement, let $M = |C|$, $M_i = \lambda_i M$ and $p_w$ total probability assigned to vectors $v$ of weight $w$ by $\tilde{T}_L(C)$. Then by computing $p_w$ over columns of $M \times n$ matrix we obtain

$$p_w = \frac{1}{n} \sum_{i=1}^n \binom{M_i}{w} \binom{M-M_i}{L-w} \binom{M}{L} .$$

By a standard estimate we have for all $w = \{0, \ldots, L\}$:

$$\binom{M_i}{w} \binom{M-M_i}{L-w} \binom{M}{L} = \binom{L}{w} \lambda_w^w (1 - \lambda_i)^{L-w} + O(\frac{1}{M}),$$

with $O(\cdot)$ term uniform in $w$ and $\lambda_i$. By symmetry of the type $\tilde{T}_L(C)$ the result (54) follows. \(\square\)

**Lemma 8.** Functions $g_j$ defined in (16) are concave on $[0, 1]$.

**Proof.** Let $W_\lambda \sim \text{Bino}(L, \lambda)$ and $V_\lambda \sim \text{Bino}(L - 1, \lambda)$. Denote for convenience $\lambda = 1 - \lambda$ and take $j_0$ to be an integer

\(^4\)The difference between odd and even $L$ occurs due to the boundary point $p_1 = \frac{1}{L}$ not being a slope-discontinuity when $L$ is odd, so we needed to add it separately.

\(^5\)Distribution $\text{Bern}^{\otimes L}(\lambda)$ assigns probability $\lambda^{v(1 - \lambda)^{L-|v|}}$ to element $v \in \mathbb{F}_2^n$. 
between 0 and $L$. We have then
\[
\frac{\partial}{\partial \lambda} \mathbb{E}[W_{\lambda} - j_0]\]
\[= \sum_{w=j_0+1}^L \left( \frac{L}{w} \right) (w-j_0) \lambda^w \bar{\lambda}^{L-w} \{w\lambda^{w-1} - (L-w)\bar{\lambda}^{w-1}\} \tag{56}\]
\[= L(j_0+1)\lambda^{j_0}\bar{\lambda}^{L-j_0-1} + \sum_{w=j_0+1}^{L-1} \left[ \left( \frac{L}{w+1} \right) (w+1-j_0)(w+1) \right.\]
\[\left. - \left( \frac{L}{w} \right)(w-j_0)(L-w) \right] \lambda^w \bar{\lambda}^{L-w-1} \tag{57}\]
\[= L\left( \frac{L-1}{j_0} \right) \lambda^{j_0} \bar{\lambda}^{L-1-j_0} + \sum_{w=j_0+1}^{L-1} \left( \frac{L-1}{w} \right) \lambda^w \bar{\lambda}^{L-1-w}\tag{58}\]
\[\mathbb{P}[\lambda \geq j_0], \tag{59}\]
where in (57) we shifted the summation by one for the first term under the sum in (56), and in (58) applied identities \(\binom{L}{w+1} = \binom{L}{w} \frac{L-w}{w+1} = \binom{L}{w} \frac{L-w}{w+1} \). Similarly, if $\theta \in [0, 1]$ we have
\[\frac{\partial}{\partial \lambda} \mathbb{E}[w\lambda - j_0 - \theta] = \mathbb{P}[\lambda \geq j_0 + 1] + L(1-\theta)\mathbb{P}[\lambda = j_0]. \tag{60}\]
Similarly, one shows we will need it later in Lemma 9:
\[\frac{\partial}{\partial \lambda} \mathbb{P}[\lambda \geq j_0] = L\mathbb{P}[\lambda = j_0 - 1]. \tag{61}\]
Since clearly the function in (60) is strictly increasing in $\lambda$ for any $j_0$ and $\theta$ we conclude that
\[\lambda \mapsto \mathbb{E}[w\lambda - j_0 - \theta]\]
is convex. This concludes the proof of concavity of $g_j$. \[\square\]

**Proof of Lemma 4.** Our plan is the following:
1. Apply Elias-Bassalygo reduction to pass from $C''_n$ to a subcode $C''_{n}$ on an intersection of two spheres $S_{\xi_0n}$ and $y + S_{\xi_1n}$.
2. Use Lemma 5 to pass to a symmetric subcode $C''_{n} \subset C''$
3. Use Lemmas 7-8 to estimate maxima of average radii $h_j$ over $C''_{n}$.
4. Use Lemma 6 to transport statement about $h_j$ to a statement on $\tau_L(C''_{n})$.

We proceed to details. It is sufficient to show that for some constant $C = C(L)$ and arbitrary $\xi > 0$ estimate (24) holds with $\epsilon_n = C\delta$ whenever $n \geq n_0(\delta)$. So we fix $\delta > 0$ and consider a code $C' \subset S_{\xi_0n} \subset F_{\mathbb{F}_2}$ with $|C'| \geq \exp(nR' + o(n))$.

Note that for any $r$, even $m$ with $m/2 \leq \min(r, n-r)$ and arbitrary $y \in S^m_r \cap S^m_r$ is isometric to the product of two lower-dimensional spheres:
\[\{y + S^m_m \cap C' \} = |C'|\left( \frac{\xi_0 n}{\xi_0 n - m/2} \right) \left( \frac{n(1-\xi_0)}{m/2} \right). \tag{62}\]

Therefore, we have for $r = \xi_0 n$ and valid $m$:
\[
\sum_{y \in S^m_m} |\{y + S^m_m \cap C'\}| = |C'|\left( \frac{\xi_0 n}{\xi_0 n - m/2} \right) \left( \frac{n(1-\xi_0)}{m/2} \right).
\]
Consequently, we can select $m = \xi_1 n - o(n)$, where $\xi_1$ defined in (27), so that for some $y \in S^m_m$:
\[|\{y + S^m_m \cap C'\}| > n \cdot \tag{63}\]
Note that we focus on solution of (27) satisfying $\xi_1 < 2\xi_0(1-\xi_0)$. For some choices of $R, \delta$ and $\xi_0$ choosing $\xi_1 > 2\xi_0(1-\xi_0)$ is also possible, but such a choice appears to result in a weaker bound.

Next, we let $C'' = \{y + S^m_m \cap C'\}$. For sufficiently large $n$ the code $C''$ will satisfy assumptions of Lemma 5 with $K \geq \frac{1}{2}$. Denote the resulting large symmetric subcode $C''$.

Note that because of (59) column-densities $\lambda_i$’s of $C''$, defined in Lemma 7, satisfy (after possibly reordering coordinates):
\[\sum_{i=1}^{\xi_0 n} \lambda_i = \xi_1 n/2 + o(n), \quad \sum_{i > \xi_0 n} \lambda_i = \xi_1 n/2 + o(n). \tag{64}\]

Therefore, from Lemmas 7-8 we have
\[h_j(\mathbb{T}(C'')) \leq \xi_0 g_j \left( 1 - \frac{\xi_1}{2\xi_0} \right) \tag{65}\]
\[+ (1-\xi_0) g_j \left( 1 - \frac{\xi_1}{2(1-\xi_0)} \right) + \epsilon_n + \frac{C_1}{|C''|}, \tag{66}\]
where $\epsilon_n \to 0$. Note that by construction the last term in (63) is $O(\delta)$. Also note that the first two terms in (63) equal $\theta_j$ defined in (25).

Finally, by Lemma 6 we get that for any codewords $c_1, \ldots, c_L \in C''$, some constant $C$ and some sequence $\epsilon_n \to 0$ the following holds:
\[\frac{1}{n} \text{rad}(0, c_1, \ldots, c_L) \leq \theta(\xi_0, R', L) + \epsilon_n + C \delta. \tag{67}\]

By the initial remark, this concludes the proof of Lemma 4. \[\square\]

**C. Proof of Corollary 3**

**Lemma 9.** For any odd $L = 2a + 1$ there exists a neighborhood of $x = \frac{1}{2}$ such that
\[\max_j g_j(x) = g_1(x), \tag{68}\]
maximum taken over $j$ equal all the odd numbers not exceeding $L$ and $j = 0$. We also have for some $c > 0$
\[g_1(x) = \frac{1}{2} - 2^{L-1} \left( \frac{L}{L-1} \right)^2 + O(2x - 1)^2, \quad x \to \frac{1}{2} \tag{69}\]

**Proof.** First, the value $g_1(1/2)$ is computed trivially. Then from (60) we have
\[\frac{d}{dx} g_j(x) = \frac{L}{L+j} \left( 1 - 2p \left[ V_x \geq \frac{L+j}{2} \right] \right), \tag{70}\]
where $j \geq 1$ and $V_x \sim \text{Bino}(x, L-1)$. This implies (65). For future reference we note that (69) (below) and (61) imply
\[\frac{d}{dx} g_0(x) = 1 - 2p[V_x \geq \frac{L+1}{2}] - p[V_x = \frac{L-1}{2}], \quad V_x \sim \text{Bino}(x, L-1). \tag{71}\]
By continuity, (64) follows from showing
\[ g_1(1/2) > \max_{j \in \{0, \ldots, L \}} g_j(1/2). \]  
(68)

Next, consider \( W_x \sim \text{Bino}(x, L) \) and notice the upper-bound
\[ g_j(x) \leq \frac{1}{L + j} \mathbb{E} [W_x 1\{W_x \leq a\} + (L + j - W_x) 1\{W_x > a + 1\}]. \]

Then, substituting expression for \( g_1(x) \) we get
\[ g_1(x) - g_0(x) = \frac{1}{L} (\mathbb{P}[W_x \geq a + 1] - g_1(x)) \]  
(69)
\[ g_1(x) - g_j(x) \geq \left( \frac{j - 1}{L + j} \right) (g_1(x) - \mathbb{P}[W_x > a + 1]). \]  
(70)

Thus, to show (68) it is sufficient to prove that for \( x = 1/2 \)
we have
\[ \mathbb{P}[W_{1/2} > a + 1] < g_1(1/2) < \mathbb{P}[W_{1/2} \geq a + 1]. \]  
(71)

The right-hand inequality is trivial since \( \mathbb{P}[W_{1/2} \geq a + 1] = 1/2 \) while from (65) we know \( g_1(1/2) < 1/2 \). The left-hand inequality, after simple algebra, reduces to showing
\[ \sum_{u=0}^{a-1} \binom{2a + 1 - 2u}{2a + 1} < \binom{2a + 1}{a}. \]  
(72)

Notice, that
\[ (n - 2u) \binom{n}{u} = n \left[ \binom{n-1}{u} - \binom{n-1}{u-1} \right] \]  
and therefore
\[ \sum_{u \leq \ell} (n - 2u) \binom{n}{u} = n \binom{n-1}{\ell}. \]

Plugging this identity into the right-hand side of (72) we get
\[ \sum_{u=0}^{a-1} \binom{2a + 1 - 2u}{2a + 1} = (2a + 1) \binom{2a}{a - 1} < (2a + 1) \binom{2a + 1}{a}, \]  
(73)
completing the proof of (72).

Proof of Corollary 3. We first show that (20) implies (21). To that end, fix a small \( \epsilon > 0 \) so that \( 1/2 - \epsilon \) belongs to the neighborhood existence of which is claimed in Lemma 9. Choose rate so that \( \delta_{L, P_1}(R) = 1/2 - \epsilon \) and notice that this implies
\[ R = h(\epsilon^2 + o(\epsilon^2)). \]  
(74)

By Lemma 9, the right-hand side of (20) is
\[ \tau_L^*(0) - \text{const} \cdot \epsilon + o(\epsilon), \]
which together with (74) implies (21).

To prove (20) we use Theorem 1 with \( \delta = \delta_{L, P_1}(R) \). Next, use concavity of \( g_j \)'s (Lemma 8) to relax (13) to
\[ \limsup_{n \to \infty} \tau_L(C_n) \leq \max_{j \geq \delta} g_j(\xi_0). \]

From (66) and (67) it is clear that \( \xi_0 \mapsto g_j(\xi_0) \) is monotonically increasing for all \( j \geq 0 \) on the interval \([0, 1/2]\). Thus, we further have
\[ \limsup_{n \to \infty} \tau_L(C_n) \leq \max_{j \geq \delta} g_j(\delta_{L, P_1}(R)). \]  
(75)

Bound (75) is valid for all \( R \in [0, 1] \) and arbitrary (odd/even \( L \)). However, when \( R \) is small (say, \( R < R_0 \)) and \( L \) is odd, \( \delta_{L, P_1}(R) \) belongs to the neighborhood of \( 1/2 \) in Lemma 9 and thus (20) follows from (75) and (64).

Remark 2. It is, perhaps, instructive to explain why Corollary 3 cannot be shown for even \( L \) (via Theorem 1). For even \( L \) the maximum over \( j \) of \( g_j(1/2 - \epsilon) \) is attained at \( j = 0 \) and
\[ g_0(\frac{1}{2} - \epsilon) = \tau_L^*(0) + \epsilon c^2 + o(\epsilon^3), \epsilon \to 0 \]  
(76)

Therefore, for \( \delta_{L, P_1}(R) = \frac{1}{2} - \epsilon \) we get from (76) that the right-hand side of (75) evaluates to
\[ \tau_L^*(0) - \text{const} \cdot \epsilon^2 \log \frac{1}{\epsilon}. \]  
(77)

Thus, comparing (77) with (74) we conclude that for even \( L \) our bound on \( R_{L, P_1}(\tau) \) has negative slope at zero rate. Note that Blinovsky’s bound (10) has negative slope at zero rate for both odd and even \( L \).

D. Proof of Corollary 2

Proof. Instead of working with parameter \( \delta \) we introduce \( \beta \in [0, 1/2] \) such that
\[ \delta = \frac{1}{2} - \sqrt{\beta(1 - \beta)}. \]

We then apply Theorem 1 with \( h(\beta) = R \). Notice that the bound on \( \xi_0 \) in (14) becomes
\[ 0 \leq \xi_0 \leq \delta. \]

By a simple substitution \( \omega = \sqrt{\frac{\beta}{1 - \beta}} \) we get from (11)
\[ E_\beta(\delta) = \frac{1}{2} (\log 2 - h(\delta) + h(\beta)). \]

Therefore, when \( \xi_0 = \delta \) we notice that
\[ R + h(\beta) - 2E_\beta(\xi_0) = R - \log 2 + h(\delta) \]
implying that defining equation for \( \xi_1 \), i.e. (15), coincides with (19).

Next for \( L = 3 \) we compute
\[ g_0(\nu) = \nu (1 - \nu), \]  
(78)
\[ g_1(\nu) = \frac{3}{4} \nu - \frac{1}{2} \nu^3, \]  
(79)
\[ g_3(\nu) = \frac{1}{2} \nu. \]  
(80)

Note that the right-hand side of (17) is precisely equal to
\[ \delta g_1 \left( \frac{1 - \xi_1}{2\delta} \right) + (1 - \delta) g_1 \left( \frac{\xi_1}{2(1 - \delta)} \right). \]
So this corollary simply states that for $L = 3$ the maximum in (13) is achieved at $j = 1, \xi_0 = \delta$. Let us restate this last statement rigorously: The maximum

$$
\max_{j \in \{0, 1, 2\}} \max_{\xi_0 \in \delta} \xi_0 g_j \left( 1 - \frac{x}{2\xi_0} \right) + (1 - \xi_0) g_j \left( \frac{x}{2(1 - \xi_0)} \right)
$$

is achieved at $j = 1, \xi_0 = \delta$. Here $x = x(\xi_0, \beta)$ is a solution of

$$
2(h(\beta) - E\beta(\xi_0)) = h(\xi_0) - \xi_0 h \left( \frac{x}{2\xi_0} \right) - (1 - \xi_0) h \left( \frac{x}{2(1 - \xi_0)} \right).
$$

(81)

For notational convenience we will denote the function under maximization in (81) by $g_j(\xi_0, x)$.

We proceed in two steps:

• First, we estimate the maximum over $\xi_0$ for $j = 0$ as follows:

$$
\max_{\xi_0} g_0(\xi_0, x) \leq \frac{\log 2 - R}{4\log 2} \left( 1 \left( 1 - \frac{\delta}{\max(1 - a_{\max})} \right) \right) + (1 - \delta)g_0(a_{\min}),
$$

(83)

where $a_{\max}, a_{\min} \leq \frac{1}{2}$ are given by

$$
a_{\max} = h^{-1}(\log 2 - R),
$$

(84)

$$
a_{\min} = h^{-1} \left( \log 2 - \frac{R}{1 - \delta} \right).
$$

(85)

• Second, we prove that for $j = 1$ function

$$
\xi_0 \mapsto g_j(\xi_0, x(\xi_0))
$$

is monotonically increasing.

Once these two steps are shown, it is easy to verify (for example, numerically) that $g_1(\delta, x(\delta))$ exceeds both $\frac{1}{2} \delta$ (term corresponding to $j = 3$ in (81)) and the right-hand side of (83) (term corresponding to $j = 0$). Notice that this relation holds for all rates. Therefore, maximum in (81) is indeed attained at $j = 1, \xi_0 = \delta$.

One trick that will be common to both steps is the following. From the proof of Lemma 4 it is clear that the estimate (24) is monotonic in $R'$. Therefore, in equation (82) we may replace $E\beta(\xi)$ with any upper-bound of it. We will use the well-known upper-bound, which leads to binomial estimates of spectrum components [15, (46)]:

$$
E\beta(\xi_0) \leq \frac{1}{2} \left( \log 2 + h(\beta) - h(\xi_0) \right).
$$

(86)

Furthermore, it can also be argued that maximum cannot be attained by $\xi_0$ so small that

$$
h(\beta) - \frac{1}{2} \left( \log 2 + h(\beta) - h(\xi_0) \right) < 0.
$$

So from now on, we assume that

$$
h^{-1}(\log 2 - h(\beta)) \leq \xi_0 \leq \delta,
$$

and that $x = x(\xi_0) \leq 2\xi_0(1 - \xi_0)$ is determined from the equation:

$$
\log 2 - R = \xi_0 h \left( \frac{x}{2\xi_0} \right) + (1 - \xi_0) h \left( \frac{x}{2(1 - \xi_0)} \right).
$$

(87)

(we remind $R = h(\beta)$).

We proceed to demonstrating (83). For convenience, we introduce

$$
a_1 \triangleq \frac{1 - x}{2\xi_0},
$$

(88)

$$
a_2 \triangleq \frac{x}{2 - 2\xi_0}.
$$

(89)

By constraints on $x$ it is easy to see that

$$
0 \leq a_2 \leq \min(a_1, 1 - a_1).
$$

Therefore, we have

$$
\log 2 - R = \xi_0 h(a_1) + (1 - \xi_0) h(a_2) \geq h(a_2)
$$

and thus $a_2 \leq a_{\max}$ defined in (84). Similarly, we have

$$
\log 2 - R = \xi_0 h(a_1) + (1 - \xi_0) h(a_2) \leq \xi_0 \log 2 + (1 - \xi_0) h(a_2),
$$

and since $\xi_0 \leq \delta$ we get that $a_2 \geq a_{\min}$ defined in (85).

Next, notice that $\frac{h(x)}{x(1-x)}$ is decreasing on $[0, 1/2]$. Thus, we have

$$
h(a_1) \geq g_0(a_1) 4 \log 2
$$

(90)

$$
h(a_2) \geq h(a_{\max}) \frac{g_0(a_{\max})}{g_0(a_{\min})} \geq \frac{\log 2 - R}{a_{\max}(1 - a_{\max})} g_0(a_{\min}) = c \cdot g_0(a_{\min}),
$$

(91)

where in the last step we introduced $c = 4 \log 2$ for convenience. Consequently, we get

$$
\log 2 - R
$$

= $\xi_0 h(a_1) + (1 - \xi_0) h(a_2) \geq
$$

(92)

$$
\geq 4 \log 2 \cdot \xi_0 g_0(a_1) + (1 - \xi_0) c \cdot g_0(a_2)
$$

(93)

$$
= 4 \log 2 \cdot \xi_0 g_0(a_1) + (1 - \xi_0) c \cdot 4 \log 2 \cdot g_0(a_2)
$$

(94)

$$
\geq 4 \log 2 \cdot \xi_0 g_0(a_1) + (1 - \xi_0) c \cdot 4 \log 2 \cdot g_0(a_{\min}).
$$

(95)

Rearranging terms yield (83).

We proceed to proving monotonicity of (82). The technique we will use is general (can be applied to $L > 3$ and $j > 1$), so we will avoid particulars of $L = 3, j = 1$ case until the final step.

Notice that regardless of the function $g(\nu)$ we have the equivalence:

$$
\frac{d}{d\xi_0} \xi_0 g(a_1) + (1 - \xi_0) g(a_2) \geq 0 \iff \frac{1}{2} \frac{dx}{d\xi_0} \left( g'(a_2) - g'(a_1) \right) \geq \int_{a_1}^{a_2} (1 - x)(-g''(x)) dx - g'(a_2),
$$

(96)

where we recall definition of $a_1, a_2$ in (88)-(89). Differentiating (87) in $\xi_0$ (and recalling that $R$ is fixed, while $x = x(\xi_0)$ is an implicit function of $\xi_0$) we find

$$
\frac{dx}{d\xi_0} = -2 \frac{\log -\frac{a_2}{a_1}}{\log \frac{1 - a_2}{a_1}} \frac{1}{1 - a_1} < 0.
$$

Next, one can notice that the map $(\xi_0, x, R) \mapsto (a_1, a_2)$ is a bijection onto the region

$$
\{(a_1, a_2) : 0 \leq a_1 \leq 1, 0 \leq a_2 \leq a_1(1 - a_1)\}.
$$

(97)
With the inverse map given by

\[ \xi_0 = \frac{a_2}{1 - a_1 + a_2}, \quad x = \frac{2a_2^2}{1 - a_1 + a_2}, \]

\[ R = \log 2 - \xi_0 h(a_1) - (1 - \xi_0) h(a_2). \]

Thus, verifying (96) can as well be done for all \(a_1, a_2\) inside the region (97). Substituting \(g = g_1\) into (96) we get that monotonicity in (82) is equivalent to a two-dimensional inequality:

\[ -2 \log \frac{1 - a_2}{a_1} \cdot (a_1^2 - a_2^2) \geq (2a_1^2 - \frac{4}{3}(a_1^3 - a_2^3) - 1) \log \frac{1 - a_2}{a - 2} \frac{a_1}{1 - a_1}. \quad (98)\]

It is possible to verify numerically that indeed (98) holds on the set (97). For example, one may first demonstrate that it is sufficient to restrict to \(a_2 = 0\) and then verify a corresponding inequality in \(a_1\) only. We omit mechanical details.

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REFERENCES


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