Helicoid-like minimal disks and uniqueness

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Helicoid-like minimal disks and uniqueness

By Jacob Bernstein at Stanford and Christine Breiner at Cambridge

Abstract. We show that for an embedded minimal disk in $\mathbb{R}^3$, near points of large curvature the surface is bi-Lipschitz with a piece of a helicoid. Additionally, a simplified proof of the uniqueness of the helicoid is provided.

1. Introduction

This paper gives a condition for an embedded minimal disk to look like a piece of a helicoid. Namely, if such a disk has boundary in the boundary of a ball and has large curvature, then, in a smaller ball, it is bi-Lipschitz to a piece of a helicoid. Moreover, the Lipschitz constant can be chosen as close to 1 as desired (compare with [21], Proposition 2).

Theorem 1.1. Given $\varepsilon, R > 0$ there exists $R' \geq R$ so: Suppose $0 \in \Sigma' \in \mathcal{E}(1, 0, R's)$, and $(0, s)$ is a blow-up pair (see Section 2.2). Then there exists $\Omega$, a subset of a helicoid, so that $\Sigma$, the component of $\Sigma' \cap B_{R_s}$ containing 0, is bi-Lipschitz with $\Omega$, and the Lipschitz constant is in $(1 - \varepsilon, 1 + \varepsilon)$.

Here $\mathcal{E}(1, g, R)$ denotes the space of embedded minimal surfaces $\Sigma \subset \mathbb{R}^3$ of genus $g$ and with smooth, connected boundary, $\partial \Sigma \subset \partial B_R(0)$. We say a surface has genus $g$ if it is diffeomorphic to a punctured, compact, oriented genus $g$ surface, though in this paper we restrict attention to disks, i.e. $g = 0$.

Colding and Minicozzi, in their work on the shapes of embedded minimal disks [7]–[10] (see also [12] for a non-technical overview), show that a $\Sigma$ as in the above theorem looks, on a scale relative to $R$, roughly like the helicoid—an essentially qualitative description. Theorem 1.1 sharpens this description, though on a much smaller scale, giving a quantitative description of such a disk near a point of large curvature. Examples constructed by Colding and Minicozzi [5], Khan [19], Kleene [20], and Hoffman and White [17] demonstrate that this sharper description cannot hold on the outer scale $R$—we refer the interested reader to [2].

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To prove Theorem 1.1 we argue by contradiction, coupling the lamination theory of Colding and Minicozzi of [10] with the so-called uniqueness of the helicoid:

**Theorem 1.2.** Any \( \Sigma \in \mathcal{E}(1, 0) \) is either a plane or a helicoid.

Here \( \mathcal{E}(1, g, \infty) = \mathcal{E}(1, g) \) is the space of complete, embedded minimal surfaces in \( \mathbb{R}^3 \) with genus \( g \) and one end. Notice there is no a priori assumption that elements are properly embedded. This is because Colding and Minicozzi prove in [13], Corollary 0.13, that a complete, embedded minimal surface of finite topology is automatically properly embedded, a fact we use throughout.

Theorem 1.2 was first proved by Meeks and Rosenberg in [22]. Their argument uses, in an essential manner, the theory of Colding and Minicozzi, in particular the lamination theory and one-sided curvature estimate of [10]. In this paper, we will provide a new and more geometric proof of this fundamental theorem. Our argument makes direct use of the results of Colding and Minicozzi on the geometric structure of embedded minimal disks from [7]–[10]. Importantly, the geometric decomposition described in Theorem 1.4 below, and hence the entire proof, can be extended to arbitrary \( \Sigma \in \mathcal{E}(1, g) \) for \( g > 0 \). Indeed, in [1] we prove the following generalization of Theorem 1.2:

**Theorem 1.3.** Let \( \Sigma \in \mathcal{E}(1, g) \). Then \( \Sigma \) is conformally a once punctured, compact Riemann surface. Moreover, if \( \Sigma \) is non-flat, it is asymptotic to a helicoid.

In their paper, Meeks and Rosenberg first use the lamination theory to deduce that (after a rotation) a homothetic blow-down of a non-flat \( \Sigma \in \mathcal{E}(1, 0) \) is, away from some Lipschitz curve, a foliation of flat parallel planes transverse to the \( x_3 \)-axis. This gives, in a very weak sense, that the surface is asymptotic to a helicoid, which they use to conclude that the Gauss map of \( \Sigma \) omits the north and south poles. The asymptotic structure, a result of Collin, Kusner, Meeks and Rosenberg regarding the parabolicity of minimal surfaces [14] and some very delicate complex analytic arguments, are combined to show that \( \Sigma \) is conformally equivalent to \( \mathbb{C} \). Finally, by looking at the level sets of the log of the stereographic projection of the Gauss map and using a Picard type argument, they show that this holomorphic map does not have an essential singularity at \( \infty \) and is, in fact, linear. The Weierstrass representation then implies \( \Sigma \) is the helicoid.

The geometric approach of our paper allows for a more direct argument. Recall, the helicoid contains a central “axis” of large curvature away from which it consists of two multivalued graphs spiraling together, one strictly upward, the other downward. Note that the known embedded genus-one helicoids—constructed by Weber, Hoffman and Wolf [25] and by Hoffman and White [18]—behave similarly. We first show that this is the structure of any non-flat \( \Sigma \in \mathcal{E}(1, 0) \):

**Theorem 1.4.** There exist \( \varepsilon_0 > 0 \) and disjoint subsets of \( \Sigma \), \( \mathcal{A} \) and \( \mathcal{S} \), with \( \Sigma = \mathcal{A} \cup \mathcal{S} \) such that:

1. After possibly rotating \( \mathbb{R}^3 \), \( \mathcal{S} \) can be written as the union of two (oppositely oriented) strictly spiraling multivalued graphs \( \Sigma^1 \) and \( \Sigma^2 \).
2. In \( \mathcal{A} \), \( |\nabla \Sigma x_3| \geq \varepsilon_0 \).

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Remark 1.5. We say $\Sigma^i (i = 1, 2)$ is a multivalued graph if it can be decomposed into $N$-valued $e$-sheets (see Definition 2.2) with varying center. That is, $\Sigma^i = \bigcup_j \Sigma^i_j$ where each $\Sigma^i_j = y^i_j + \Gamma_{u^i_j}$ is an $N$-valued $e$-sheet. Strict spiraling then means that $(u^i_j)_0 \neq 0$ for all $j$. A priori, the axes of the multivalued graphs vary, a fact that introduces additional book-keeping. For the sake of clarity, we assume that each $\Sigma^i$ is an $\infty$-valued $e$-sheet—i.e. $\Sigma^i$ is the graph, $\Gamma_{u^i}$, of a single $u^i$ with $u^i_0 \neq 0$.

In order to establish this decomposition, we first use Colding and Minicozzi’s results on the structure of embedded minimal disks to obtain the existence of two infinite-sheeted multivalued graphs, $\Sigma^1$, $\Sigma^2$, that spiral together. Then, using a result of [6], we show that such graphs can be approximated asymptotically in a manner that allows one to show that far enough out along each sheet of the multivalued graphs, the sheet strictly spirals—giving the region $\mathcal{R}_S$. An application of the proof of Rado’s theorem [23] then implies $|\nabla_{\Sigma} x_3| \neq 0$ on $\mathcal{R}_A$, the subset of $\Sigma$ away from the two multivalued graphs. Finally, a Harnack inequality gives the uniform lower bound. In showing this, one obtains:

**Proposition 1.6.** On $\Sigma$, after a rotation of $\mathbb{R}^3$, $\nabla_{\Sigma} x_3 \neq 0$ and, for all $c \in \mathbb{R}$, $\Sigma \cap \{x_3 = c\}$ consists of exactly one properly embedded smooth curve.

Thus, $z = x_3 + ix_3^3$ is a holomorphic coordinate on $\Sigma$. Using the stereographic projection of the Gauss map, $g$, we show that $z$ maps onto $\mathbb{C}$ and so $\Sigma$ is conformally the plane. This follows from control on the behavior of $g$ in $\mathcal{R}_S$ due to the strict spiraling. Indeed, away from a small neighborhood of $\mathcal{R}_A$, $\Sigma$ is conformally the union of two closed half-spaces with $\log g = h$ providing the identification. It then follows that $h$ is also a conformal diffeomorphism which gives Theorem 1.2.

### 2. Global geometric structure of $\Sigma$

To study elements of $\mathcal{E}(1,0)$ we rely heavily on Colding and Minicozzi’s structural results for embedded minimal disks. Much of this can be found in the series of papers [7]–[10], with more technical analysis in [4]. For a detailed overview of the theory, the interested reader should consult the survey [11]. We have gathered the major results we use in Appendix A. Additionally, for the convenience of the reader we refer, when possible, to Appendix A rather than directly to the papers of Colding and Minicozzi.

**2.1. Preliminaries.** Throughout, let $\Sigma \in \mathcal{E}(1,0)$ be non-flat. Recall $\Sigma \in \mathcal{E}(1,g,R)$ is an embedded minimal surface with genus $g$ and so that $\partial \Sigma \subset \partial B_R(0)$ is connected and $\Sigma \in \mathcal{E}(1,g)$ is a complete, embedded minimal surface with genus $g$ and one end. Here $B_r(y)$ is the Euclidean ball of radius $r$ centered at $y$; for a point $p \in \Sigma$ we denote an intrinsic ball in $\Sigma$ of radius $R$ centered at $p$ by $\mathcal{R}_R(p)$. We let $|A|^2$ represent the norm squared of the second fundamental form on $\Sigma$. Let

$$(2.1) \quad C_\delta(y) = \{x | (x_3 - y_3)^2 \leq \delta^2((x_1 - y_1)^2 + (x_2 - y_2)^2)\} \subset \mathbb{R}^3$$

be the complement of a cone and set $C_\delta = C_\delta(0)$. We denote a polar rectangle by

$$(2.2) \quad S_{\rho_1,\rho_2}^{\theta_1,\theta_2} = \{(\rho, \theta) | r_1 \leq \rho \leq r_2, \theta_1 \leq \theta \leq \theta_2\}.$$
For a real-valued function, \( u \), defined on a polar domain \( \Omega \subset \mathbb{R}^+ \times \mathbb{R} \), define the map \( \Phi_u : \Omega \to \mathbb{R}^3 \) by \( \Phi_u(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta, u(\rho, \theta)) \). In particular, if \( u \) is defined on \( S_{\theta_1, \theta_2}^{\rho_1, \rho_2} \), then \( \Phi_u(S_{\theta_1, \theta_2}^{\rho_1, \rho_2}) \) is a multivalued graph over the annulus \( D_{\rho_2} \setminus D_{\rho_1} \). We define the separation of the graph \( u \) by \( w(\rho, \theta) = u(\rho, \theta + 2\pi) - u(\rho, \theta) \). Thus, \( \Gamma_u := \Phi_u(\Omega) \) is the graph of \( u \), and \( \Gamma_u \) is embedded if and only if \( w \equiv 0 \). Finally, we say a graph \( \Gamma_u \) strictly spirals if \( u \not\equiv 0 \).

Recall that \( u \) satisfies the minimal surface equation if

\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0. 
\]

The graphs of interest to us will also satisfy the following flatness condition:

\[
|\nabla u| + \rho |\text{Hess} u| + 4\rho \frac{|\nabla w|}{|w|} + \rho^2 \frac{|\text{Hess} w|}{|w|} \leq \varepsilon < \frac{1}{2\pi}.
\]

Note that if \( w \) is the separation of a \( u \) satisfying (2.3) and (2.4), then \( u \) and \( w \) satisfy uniformly elliptic second order equations. Thus, if \( \Gamma_u \) is embedded then \( w \) has point-wise gradient bounds and satisfies a Harnack inequality.

In Colding and Minicozzi’s work, multivalued minimal graphs are a basic building block used to study the structure of minimal surfaces. We also make heavy use of them and so introduce some notation.

**Definition 2.1.** A multivalued minimal graph \( \Sigma_0 \) is a weak \( N \)-valued \((\varepsilon)\)-sheet (centered at \( y \in \Sigma \) on the scale \( s > 0 \)) if \( \Sigma_0 = \Gamma_u + y \) and \( u \), defined on \( S_{\rho, \infty}^{-N, N} \), satisfies (2.3), has \( |\nabla u| \leq \varepsilon \), and \( \Sigma_0 \subset C_\varepsilon(y) \).

We will often need more control on the sheets as well as a normalization at \( \infty \):

**Definition 2.2.** A multivalued minimal graph \( \Sigma_0 \) is a (strong) \( N \)-valued \((\varepsilon)\)-sheet (centered at \( y \) on the scale \( s \)), if \( \Sigma_0 = \Gamma_u + y \) is a weak \( N \)-valued \( \varepsilon \)-sheet centered at \( y \) and on scale \( s \) and, in addition, \( u \) satisfies (2.4) and \( \lim_{\rho \to \infty} \nabla u(\rho, 0) = 0 \).

Using Simons’ inequality, [4], Corollary 2.3, shows that on the one-valued middle sheet of a 2-valued graph satisfying (2.4), the hessian of \( u \) has faster than linear decay and hence \( u \) has an asymptotic tangent plane. This Bers-like result implies that the normalization at \( \infty \) in the definition of an \( \varepsilon \)-sheet is well defined. As an additional consequence, for an \( \varepsilon \)-sheet,

\[
|\nabla u| \leq C\varepsilon^{-5/12}.
\]

Notice that if \( \Sigma_0 = \Gamma_u \) is a weak \( \varepsilon \)-sheet then because \( u \) has bounded gradient and satisfies (2.3), it is the solution to a uniformly elliptic second order equation. As such, one can apply standard elliptic estimates to gain further (interior) regularity. In particular, looking on a sub-graph (and possibly slightly rotating to normalize behavior at \( \infty \)) one has that \( \Sigma_0 \) contains an \( \varepsilon \)-sheet:
Proposition 2.3. Given $N \in \mathbb{Z}^+$ and $\varepsilon > 0$, sufficiently small (depending on $N$), there exist $N_0$ and $C$ depending only on $\varepsilon$ and $N$ so: Suppose that $\Sigma = \Gamma_u$ is a weak $N_0$-valued $\varepsilon/C$-sheet centered at 0 and on scale $s/C$. Then (possibly after a small rotation) $\Sigma$ contains an $N$-valued $\varepsilon$-sheet $\Sigma_0 = \Gamma_{u_0}$ on the scale $s$.

Proof. [7], Proposition II.2.12, and standard elliptic estimates give an $N \in \mathbb{Z}^+$ and $\delta_\varepsilon > 0$ depending only on $\varepsilon$ so that if $u$ satisfies (2.3) and $|\nabla u| \leq \varepsilon/4$ on $S_{e^{-\pi N}, \pi N}$ and $\Gamma_u \subset C_{\delta_\varepsilon}$, then on $S_{1, \infty}$ we have the sum of all the terms of (2.4) bounded by $\varepsilon/2$. Hence by the above (and a rescaling) we see that for $N_0 = N + N$ and $C = \max\{2e^{N_0}, \delta_\varepsilon^{-1}\}$, $u$ satisfies (2.4) on $S_{s/2, \infty}$. At this point we do not a priori know that $\lim_{\rho \to \infty} \nabla u(\rho, 0) = 0$. However, there is an asymptotic tangent plane. Thus, we may need a small rotation (the size of which is controlled by $\varepsilon$) to make this parallel to the $x_1$-$x_2$ plane. Notice this rotation affects the inner scale and the bound on (2.4). Nevertheless, for small enough rotations we may replace $s/2$ by $s$ and obtain $\Sigma_0$. □

2.2. Initial sheets. We now use Colding and Minicozzi’s work to establish the existence of a suitable number of $\varepsilon$-sheets within $\Sigma$. First, we give a condition for the existence of $\varepsilon$-sheets. Roughly, all that is required is a point with large curvature relative to nearby points. This is made precise by:

Definition 2.4. The pair $(y, s)$, $y \in \Sigma$, $s > 0$, is a $(C)$ blow-up pair if

$$
\sup_{\Sigma \cap B_s(y)} |A|^2 \leq 4|A|^2(y) = 4C^2s^{-2}.
$$

The existence of a blow-up pair in an embedded minimal disk forces the surface to spiral nearby and this extends outward (see Theorem A.1). As a consequence, after a suitable rotation, we obtain a weak sheet near the pair; by Proposition 2.3 this contains an $\varepsilon$-sheet. Hence, near any blow-up pair there is an $\varepsilon$-sheet:

Theorem 2.5. Given $N \in \mathbb{Z}^+$ and $\varepsilon > 0$ sufficiently small, there exist $C_1, C_2 > 0$ so: Suppose that $(0, s)$ is a $C_1$ blow-up pair of $\Sigma$. Then there exists (after a rotation of $\mathbb{R}^3$) an $N$-valued $\varepsilon$-sheet $\Sigma_0 = \Gamma_{u_0}$ on the scale $s$. Moreover, the separation over $\partial D_3$ of $\Sigma_0$ is bounded below by $C_2 s$.

Proof. Using $\varepsilon$ and $N$, let $N_0$ and $C$ be given by Proposition 2.3. Then to find the desired $\varepsilon$-sheet we must ensure the existence of a weak $N_0$-valued $\varepsilon/C$-sheet near 0.

To that end, use $\varepsilon/C$ and $N_0$ with Theorem A.3 to obtain $C_1', C_2' > 0$. That is, if $(0, t/2)$ is a $C_1'$ blow-up pair in $\Sigma$, then (up to rotating $\mathbb{R}^3$) there is a weak $N_0$-valued $\varepsilon/C$-sheet centered at 0 and on scale $t$. Thus, (up to a further small rotation) one has an $N$-valued $\varepsilon$-sheet, $\Sigma_0 = \Gamma_{u_0}$, centered at 0 and on scale $Ct$. The proof of [8], Proposition 4.15, provides a constant $C_2 > 0$ (depending only on $C$) so that $w_0(Ct, 0) \geq C_2 Ct$. Finally, if we set $C_1 = 2C_1'C$ then $(0, s)$ being a $C_1$ blow-up pair implies that $\left(0, \frac{s}{2C}\right)$ is a $C_1'$ blow-up pair. This gives the result. □

Once there is one $\varepsilon$-sheet, $\Sigma_1$, in $\Sigma$, a barrier argument shows that between the sheets of $\Sigma_1$, $\Sigma$ consists of exactly one other $\varepsilon$-sheet. Namely, by Theorem A.4, the parts of $\Sigma$ that
lie in between an \( \epsilon \)-sheet make up a second multivalued graph. Furthermore, the one-sided curvature estimate of [10] (see Appendix A.5) gives gradient estimates which, coupled with Proposition 2.3, reveal that this graph contains an \( \epsilon \)-sheet. Thus, near a blow-up point, \( \Sigma \) contains two \( \epsilon \)-sheets spiraling together.

We now make the last statement precise. Suppose \( u \) is defined on \( S_{1/2, \infty}^{\pi N - 3\pi, \pi N + 3\pi} \) and \( \Gamma_u \) is embedded. We define \( E \) to be the region over \( D_{1/2} \setminus D_1 \) between the top and bottom sheets of the concentric sub-graph of \( u \). That is:

\[
E = \{ (\rho \cos \theta, \rho \sin \theta, t) | \quad 1 \leq \rho \leq \infty, -2\pi \leq \theta < 0, u(\rho, \theta - \pi N) < t < u(\rho, \theta + (N + 2)\pi) \}.
\]

Using Theorem A.4, Theorem 2.5, and the one-sided curvature estimate:

**Theorem 2.6.** Given \( \epsilon > 0 \) sufficiently small, there exist \( C_1, C_2 > 0 \) so: Suppose \((0, s)\) is a \( C_1 \) blow-up pair. Then there exist two \( 4 \)-valued \( \epsilon \)-sheets \( \Sigma_i = \Gamma_{u_i} \) \((i = 1, 2)\) on the scale \( s \) which spiral together \( \text{(i.e. } u_1(s, 0) < u_2(s, 0) < u_1(s, 2\pi)\text{)} \). Moreover, the separation over \( \partial D_s \) of \( \Sigma_i \) is bounded below by \( C_2\epsilon \).

**Remark 2.7.** We refer to \( \Sigma_1, \Sigma_2 \) as \( (\epsilon)\)-blow-up sheets associated with \((y, s)\).

**Proof.** Fix \( \epsilon_0 > 0 \) as in Theorem A.4. For \( \epsilon < \epsilon_0 \) and \( N = 4 \), choose \( N_0 \) and \( C \) as given by Proposition 2.3. With \( \tilde{N} = 10 + N_0 \) denote by \( C'_1, C'_2 \) the constants given by Theorem 2.5 \((\text{with } \tilde{N} \text{ replacing the } N \text{ in the theorem})\). Thus, if \((0, r)\) is a \( C'_1 \) blow-up pair then there exists an \( \tilde{N} \)-valued \( \epsilon \)-sheet \( \Sigma'_1 = \Gamma_{u'_1} \) on scale \( r \) inside of \( \Sigma \) with \( w(r, \theta) \geq C'_1 r \). Applying Theorem A.4 to \( u'_1 \), we see that \( \Sigma \cap E \setminus \Sigma'_1 \) is given by the graph of a function \( u'_2 \) defined on \( S_{2r, \infty}^{\pi N + 2\pi, \pi N - 2\pi} \). As long as we can control the gradient of \( u'_2 \) this will give us a weak sheet.

To that end, let \( z = \epsilon/C \); use the one-sided curvature estimate \((\text{Corollary A.8)}\) to choose \( C > \delta_0 > 0 \). By (2.5) and the fact that the initial \( \tilde{N} \) sheets of \( u'_1 \) exist outside a cone, there exists \( \tilde{C} > 1 \), depending on \( \epsilon, \delta_0 \), and the constant in (2.5), such that \( |\nabla u'_1| \leq \delta_0 \) on \( S_{2C, \infty}^{\pi N + \pi, \pi N - \pi} \) and \( u'_1 \) restricted to this domain is contained in \( C_{\delta_0} \setminus B_{\tilde{C}r} \). Thus, the gradient of \( u'_2 \) restricted to \( S_{2C, \infty}^{\pi N + 3\pi, \pi N - 3\pi} \) is bounded by \( \epsilon/C \). Moreover, since \( \tilde{N} - 1 \) sheets of \( u'_1 \) are inside of \( C_{\delta_0} \), the \( \tilde{N} - 3 \) concentric sheets of \( u'_2 \) are also in \( C_{\delta_0} \), Thus, the graph of \( u'_2 \) on \( S_{2C, \infty}^{\pi N_0, \pi N_0} \) is a weak \( N_0 \)-valued \( \epsilon/C \) sheet centered at 0 and on scale \( 2\tilde{C}r \). Proposition 2.3 then gives that the graph of \( u'_2 \) over \( S_{2C\tilde{C}, \infty}^{4\pi, 4\pi} \) is an \( \epsilon \)-sheet. Notice no rotation is needed here as \( u'_1 \) is already an \( \epsilon \)-sheet.

Let \( u_1 \) and \( u_2 \) be given by restricting \( u'_1 \) and \( u'_2 \) to \( S_{2C\tilde{C}, \infty}^{4\pi, 4\pi} \) and define \( \Sigma_i = \Gamma_{u_i} \). Set \( C_1 = 2C\tilde{C}C'_1 \), so if \((0, s)\) is a \( C_1 \) blow-up pair then \( \Sigma \) will exist on scale \( s \). Integrating (2.4), the lower bound \( C'_2 \) gives a lower bound on initial separation of \( \Sigma_i \). We find \( C_2 \) by noting that if the initial separation of \( \Sigma_2 \) was too small there would be two sheets between one sheet of \( \Sigma_1 \). \( \square \)

**2.3. Blow-up pairs.** Since \( \Sigma \) is not a plane, we can always find at least one blow-up pair \((y, s)\) — an immediate consequence of Lemma A.6. We then use this initial pair to find a sequence of blow-up pairs forming an “axis” of large curvature. The key results we need
are Lemma A.6, which says that as long as curvature is large enough in a ball, measured relative to the scale of the ball, we can find a blow-up pair in the ball, and Corollary A.5, which guarantees points of large curvature above and below blow-up points. Colding and Minicozzi, in [13], Lemma 2.5, provide a good overview of this process of decomposing $\Sigma$ into blow-up sheets. The main result is the following:

**Theorem 3.2.** For $1/2 > \gamma > 0$ and $\epsilon > 0$ both sufficiently small, let $C_1$ be given by Theorem 2.6. Then there exists $C_{\text{in}} > 4$ and $\delta > 0$ so: If $(0, s)$ is a $C_1$ blow-up pair then there exist $(y_+, s_+)$ and $(y_-, s_-)$, $C_1$ blow-up pairs, with $y_+ \in \Sigma \cap B_{C_{\text{in}} s}(B_{2s} \cup C_0)$, $x_3(y_+) > 0 > x_3(y_-)$, and $s_+ \leq \gamma |y_+|$. Hence, given a blow-up pair, we can iteratively find a sequence of blow-up pairs ordered by height and lying in a cone, with distance between subsequent pairs bounded by a fixed multiple of the scale.

### 3. Asymptotic helicoids

[6], Lemma 14.1, and the gradient decay (2.5) show that $\epsilon$-sheets can be approximated by a combination of planar, helicoidal, and catenoidal pieces. Precisely, there is a “Laurent expansion” for the almost holomorphic function $u_\gamma - iu_\gamma$. This result allows us to bound the oscillation on broken circles $C(\rho) := S^{\gamma, \pi}_\rho$ of $u_\rho$, which yields asymptotic lower bounds for $u_\rho$.

**Lemma 3.1.** Given $\Gamma_\rho$, a 3-valued $\epsilon$-sheet on scale 1, set $f = u_\rho - iu_\gamma$. Then for $r_1 \geq 1$ and $\xi = \rho e^{i\xi}$ with $(\rho, \theta) \in S^{\gamma, \pi}_{2r_1, \infty}$,

$$f(\rho, \theta) = c\zeta^{-1} + g(\zeta)$$

where $c = c(r_1, u) \in \mathbb{C}$ and $|g(\zeta)| \leq C_0 r_1^{-1/4}|\zeta|^{-1} + C_0 \epsilon r_1^{-1}|w(r_1, -\pi)|$.

For the proof of this lemma, see [6], Lemma 14.1, noting that $\epsilon$-sheets satisfy the necessary hypotheses. Using this result we bound the oscillation.

**Lemma 3.2.** Suppose $\Gamma_\rho$ is a 3-valued $\epsilon$-sheet on scale 1. Then for $\rho \geq 2$, there exists a universal $C$ so:

$$\osc_{C(\rho)} u_\rho \leq C\rho^{-1/4} + C\epsilon|w(\rho, -\pi)|.$$  

**Proof.** Using Lemma 3.1 and $u_\rho(\rho, \theta) = -\text{Im} \zeta f(\rho, \theta)$ for $\zeta = \rho e^{i\theta}$, we compute:

$$\osc_{C(\rho)} u_\rho = \sup_{|\zeta| = \rho} \text{Im}(-c - \zeta g(\zeta)) - \inf_{|\zeta| = \rho} \text{Im}(-c - \zeta g(\zeta))$$

$$\leq 2 \sup_{|\zeta| = \rho} |\zeta| |g(\zeta)| \leq 4C_0 \rho^{-1/4} + 2C_0 \epsilon|w(\rho/2, -\pi)|.$$  

The last inequality comes from Lemma 3.1, setting $2r_1 = \rho$. Finally, integrate (2.4) to get the bound $|w(\rho/2, -\pi) \leq 4|w(\rho, -\pi)|$ and choose $C$ sufficiently large. □
Integrating $u_0$ around $C(\rho)$ gives $w(\rho, -\pi)$, which yields a lower bound on $\sup_n u_0$ in terms of the separation. The oscillation bound of (3.2) then gives a lower bound for $u_0$. Indeed, for $\varepsilon$ sufficiently small and large $\rho$, $u_0$ is positive.

**Proposition 3.3.** There exists an $\varepsilon_0$ so: Suppose $\Gamma_u$ is a 3-valued $\varepsilon$-sheet on scale 1 with $\varepsilon < \varepsilon_0$ and $w(1, \theta) \geq C_2 > 0$. Then there exists $C_3 = C_3(C_2) \geq 2$, so that on $S_{C_3, \infty}^{-\varepsilon, \pi}$:

$$u_0(\rho, \theta) \geq \frac{C_2}{8\pi} \rho^{-\varepsilon}. \tag{3.3}$$

**Proof.** Since \( \int_{-\pi}^{\pi} u_0(\rho, \theta) \, d\theta = w(\rho, -\pi) \) we see $w(\rho, -\pi) \leq 2\pi \sup_{C(\rho)} u_0$. Using the oscillation bound (3.2) then gives the lower bound:

$$w(\rho, -\pi) - 2\pi C\rho^{-1/4} \leq 2\pi \inf_{C(\rho)} u_0. \tag{3.4}$$

Pick $\varepsilon_0$ so that $2\pi C\varepsilon_0 \leq 1/2$. Integrating (2.4) yields $w(\rho, \theta) \geq w(1, \theta)\rho^{-\varepsilon} \geq C_2 \rho^{-\varepsilon}$. Thus,

$$\inf_{C(\rho)} u_0 \geq \frac{C_2}{4\pi} \rho^{-\varepsilon} - C \rho^{-1/4}. \tag{3.5}$$

Since $\varepsilon < 1/4$, just choose $C_3$ large. \( \Box \)

### 4. Decomposition of $\Sigma$

In order to decompose $\Sigma$, we use the explicit asymptotic properties found above to show that, away from the “axis,” $\Sigma$ consists of two strictly spiraling graphs. In particular, this implies that all intersections of $\Sigma$ with planes orthogonal to the $x_3$-axis have exactly two ends. The proof of Rado’s theorem [23] then gives that $V_{\Sigma \chi_3}$ is non-vanishing and so each level set consists of one unbounded smooth curve. A curvature estimate and a Harnack inequality then give the lower bound on $|V_{\Sigma \chi_3}|$.

**4.1. Two technical lemmas.** Presently, we know that near a blow-up pair there exist two associated 4-valued $\varepsilon$-sheets on some multiple of the blow-up scale. Moreover, the one-sided curvature estimate guarantees that for any two blow-up pairs, the part of $\Sigma$ between the associated $\varepsilon$-sheets and far from the blow-up pairs consists of two multivalued graphs with good gradient bounds. However, in order to determine the region $\mathcal{R}_S$, we must ensure that each of these “in between” sheets is an $\varepsilon$-sheet on a suitable scale. To do so, we will need two technical lemmas. The first gives a bound on the number of sheets between the blow-up sheets associated to nearby blow-up pairs. The second will imply that far enough out all of these sheets are $\varepsilon$-sheets.

**Lemma 4.1.** Given $K$, there is an $N$ so that: If $(y_1, s_1)$ and $(y_2, s_2)$ are $C$ blow-up pairs of $\Sigma$ with $y_2 \in B_{Ks_1}(y_1)$, then the number of sheets between the associated blow-up sheets is at most $N$.

**Proof.** Note that for a large, universal constant $C'$ the area of $B_{C'Ks_1}(y_1) \cap \Sigma$ gives a bound on $N$, so it is enough to uniformly bound this area. The chord-arc bounds of [13]
give a uniform constant \( \gamma \) depending only on \( C' \) so that \( B_{C'K_0}(y_1) \cap \Sigma \) is contained in \( \mathcal{B}_{\gamma K_0}(y_1) \) the intrinsic ball in \( \Sigma \) of radius \( \gamma K_0 \). Furthermore, [13], Lemma 2.26, gives a uniform bound on the curvature of \( \Sigma \) in \( \mathcal{B}_{\gamma K_0}(y_1) \) and hence, by area comparison, a uniform bound on the area of \( \mathcal{B}_{\gamma K_0}(y_1) \). Since \( B_{C'K_0}(y_1) \cap \Sigma \subset \mathcal{B}_{\gamma K_0}(y_1) \) it also has uniformly bounded area. \( \square \)

We now prove the extension property. Essentially, the existence of a single \( \varepsilon \)-sheet and the one-sided curvature estimate imply that outside of a wide cone \( \Sigma \) consists of the union of weak \( \varepsilon \)-sheets. Thus, an initial 4-valued \( \varepsilon \)-sheet extends to an \( N \)-valued \( \varepsilon \)-sheet on a fixed multiple of the initial scale. Results along these lines can be found in [4], Section 5, and [9], Section II.3.

**Lemma 4.2.** There exists \( \varepsilon_0 > 0 \) so: Given \( N > 4 \) and \( \varepsilon_0 > \varepsilon > 0 \) there exists an \( \mathcal{R} = \mathcal{R}(\varepsilon, N) > 1 \) so that if \( \Sigma \) contains a 4-valued \( \varepsilon \)-sheet, \( \Sigma_0 \), centered at 0 and on scale \( s \) then there exists an \( N \)-valued \( \varepsilon \)-sheet on scale \( \mathcal{R}s \), \( \Sigma_1 \subset \Sigma \). Moreover, \( \Sigma_1 \) may be chosen so its 4-valued middle sheet contains \( \Sigma_0 \\{x_1^2 + x_2^2 \leq \mathcal{R}^2 s^2 \} \).

**Proof.** By rescaling we may assume that \( s = 1 \). For \( \varepsilon_0 \) sufficiently small, Corollary A.8 guarantees that the component of \( \Sigma \cap C \setminus B_2 \) meeting \( \Sigma_0 \) is the graph of some function \( u \) defined on a polar domain \( \Omega_0 \subset \mathbb{R}^+ \times \mathbb{R} \) with \( |\nabla u| \leq 1 \). Moreover, this gradient bound, together with the curvature bound, implies that \( \| \text{Hess}_u \| \leq C_0 / \rho \) (here \( C_0 \) is determined by the one-sided curvature estimate). Notice that by assumption we may normalize the angular coordinate so \( S_{2,0,4\pi} \subset \Omega_0 \). Set \( \Omega_1 = \Phi^{-1}_u(C_{\varepsilon/2} \setminus B_3) \). The distance (as subsets of \( \mathbb{R}^3 \)) between \( C_{\varepsilon/2} \setminus B_3 \) and \( \partial C_{\varepsilon} \setminus B_2 \) is bounded below by \( \varepsilon/10 \). Because \( \Phi_u(\Omega_0) \) is a graph with gradient bounded by 3, the intrinsic distance (in \( \Sigma \)) between \( \partial \Phi_u(\Omega_0) \) and \( \partial \Phi_u(\Omega_1) \) is bounded below by \( 2d_1 = \varepsilon/50 \). Thus, as \( \Phi_u \) is a distance non-decreasing map, \( \text{dist}(\partial \Omega_0, \Omega_1) \geq 2d_1 \).

The separation, \( w \), of \( u \) is defined on

\[ \Omega_{\varepsilon} = \Omega_1 \cap (\Omega_1 - (0, 2\pi)) \text{ and } \text{dist}(\partial \Omega_0, \Omega_1) \geq 2d_1. \]

Because the gradient and hessian of \( u \) are bounded, standard computations give that \( w \) solves a uniformly elliptic and uniformly bounded second order equation in \( \Omega_0 \) (see [4], Section 3). As \( w > 0 \), we have a Harnack inequality, e.g. [15], Theorems 9.20 and 9.22. Thus, for all \( x \in \Omega_1 \),

\[ \sup_{B_{d_1}(x)} \inf_{B_{d_1}(x)} w \leq C_H \]

where \( C_H > 2 \) depends only on \( d_1 \) and \( C_0 \).

Now given \( \varepsilon \), pick \( N_0 \) and \( C \) from Proposition 2.3. With \( \varepsilon = \varepsilon / C \), pick \( \delta_1 \) as in the remark following Corollary A.8. We are free to shrink \( \delta_1 \), and so assume \( \delta_1 \leq \varepsilon / C \). By (2.5), there exists \( R_0 = R_0(\varepsilon) \) such that \( S_{R_0,2\pi} \subset \Omega_1 \) and the graph of \( u \) on this polar rectangle is a 2-valued weak \( \delta_1 \)-sheet on scale \( R_0 \). Thus, on \( \Omega_2 = \Phi_u^{-1}(C_{\delta_1/2} \setminus B_2 R_0) \), \( |\nabla u| \leq \varepsilon / C \). Thus, by Proposition 2.3 we only need to find an \( \mathcal{R} > R_0 \) so that \( S_{\mathcal{R}/C, \varepsilon} \subset \Omega_2 \).

By hypothesis, \( |w|(1,0) \leq 2\varepsilon \). Thus, integrating (2.4), gives \( |w|(\rho, 0) \leq 2\varepsilon \rho^\varepsilon \). By increasing \( R_0 \), if necessary, we can assume \( \Phi_u(S_{R_0,2\pi}) \subset C_{\delta_1/2} \setminus B_{R_0} \) — recall the middle
4-valued sheet of $u$ is actually an $\varepsilon$-sheet and thus satisfies (2.5). Define $N^\pm(\rho)$ to be the number of sheets, at radius $\rho$, in $\Omega_2$ between the $\theta = 0$ sheet and the top (respectively bottom) of $\Omega_2$. Note that by assumption, the $\theta = 0$ sheet lies in $C_{0/2}$. We claim there exists $R_1 > R_0$ so $N^\pm(\rho) \geq N_0 + N$ for all $\rho \geq R_1$. To that end, iterated application of (4.1), gives

$$
(4.2) \quad |w|((\rho, \theta) \leq 2\rho^\varepsilon C_H^{2N(\rho)/d_1} = 2\rho^\varepsilon \tilde{C}_H^{N(\rho)}
$$

for $(\rho, \theta) \in \Omega_2$. Without loss of generality, we treat only $N(\rho) = N^+(\rho)$. Then

$$
\delta_1 \rho / 2 \leq \sum_{k=1}^{N(\rho)} w(\rho, 2\pi N(\rho) - 2\pi k) \leq 2\rho^\varepsilon N(\rho) C_H^{N(\rho)} \leq 4\rho^\varepsilon \tilde{C}_H^{2N(\rho)}
$$

where the last inequality comes from the fact that $xb^x \leq 2b^{2x}$ for $x \geq 0$ when $b > 2$. Since $\varepsilon < 1/2$, $\rho^{1-\varepsilon} \leq 4\tilde{C}_H^{2N(\rho)}$. As $\tilde{C}_H$ depends only on $\varepsilon$, for $R_1 = R_1(\varepsilon, N)$ sufficiently large, one has for $\rho \geq R_1$ that $N(\rho) \geq N_0 + N$. Thus, set $\tilde{R} = CR_1$. \hfill \square

4.2. Decomposition. To prove Theorem 1.4 we first construct $\mathcal{H}_S$.

**Lemma 4.3.** There exist constants $C_1$, $R_1$ and a sequence $(y_i, s_i)$ of $C_1$ blow-up pairs of $\Sigma$ so that: $x_3(y_i) < x_3(y_{i+1})$ and for $i \geq 0$, $y_{i+1} \in B_{R_1}(y_i)$, while for $i < 0$, $y_{i-1} \in B_{R_1}(y_i)$. Moreover, if $\mathcal{H}_A$ is the connected component of $\bigcup_i B_{R_1}(y_i) \cap \Sigma$ containing $y_0$ and $\mathcal{H}_S = \Sigma \setminus \mathcal{H}_A$, then $\mathcal{H}_S$ has exactly two unbounded components, which are (oppositely oriented) strictly spiraling, multivalued graphs $\Sigma^1$ and $\Sigma^2$. In particular, $V_*x_3 \neq 0$ on the two graphs.

**Proof.** Fix $\varepsilon < \varepsilon_0$ where $\varepsilon_0$ is smaller than the constant given by Theorem 2.6, Proposition 3.3, and Lemma 4.2. Using this $\varepsilon$, from Theorem 2.6 we obtain the blow-up constant $C_1$ and denote by $C_2$ the lower bound on initial separation. Suppose $0 \in \Sigma$ and that $(0, 1)$ is a $C_1$ blow-up pair. From Theorem 2.8 there exists a constant $C_{in}$ so that there are $C_1$ blow-up pairs $(y_+, s_+)$ and $(y_-, s_-)$ with $x_3(y_-) < x_3(y_+) + y_\pm \in B_{C_{in}}$. Note by Lemma 4.1 that there is a fixed upper bound $N$ on the number of sheets between the blow-up sheets associated to $(y_\pm, s_\pm)$ and the sheets $\Sigma^i_0$ ($i = 1, 2$) associated to $(0, 1)$.

By Theorem 4.2, there exists an $\tilde{R}$ so that all the $N$ sheets above and the $N$ sheets below $\Sigma^i_0$ are $\varepsilon$-sheets centered on the $x_3$-axis on scale $\tilde{R}$. Call these pairs of 1-valued sheets $\Sigma^i_j$, and their associated graphs $u^i_j$, with $-N \leq j \leq N$. Integrating (2.4), we obtain from $C_2$ and $N$ a value, $C_2'$, so that for all $\Sigma^i_j$, the separation over $\partial D_{\tilde{R}}$ is bounded below by $C_2'$. The non-vanishing of the right-hand side of (3.3) is scaling invariant, so there exists a $C_3$ such that: on each $\Sigma^i_j$, outside of a cylinder centered on the $x_3$-axis of radius $\tilde{R}C_3$, $(u^i_j)_0 \neq 0$. The chord-arc bounds of [13] (i.e. Theorem 0.5) then allow us to pick $R_1$ large enough so the component of $B_{R_1} \cap \Sigma$ containing 0 contains this cylinder, the points $y_+$, $y_-$ and meets each $\Sigma^i_j$. Crucially, all the statements in the theorem are invariant under rescaling. Thus, to finish the proof, we note that once we find an initial blow-up pair, we can use Theorem 2.8 to iteratively construct a sequence of $C_1$ blow-up pairs $(y_k, s_k)$ and choose $R_1$ as described above. As $\Sigma$ is not flat, there is an $\varepsilon_0$ so that $\sup_{B_{\varepsilon_0}} |A|^2 \geq 4C_1^{-2}$. By Lemma A.6, this gives the existence of an initial blow-up pair. \hfill \square
With the given decomposition, we now show that each level set of $x_3$, outside of a large ball, consists of exactly two proper curves.

**Lemma 4.4.** For all $h$, there exist $\alpha, \rho_0 > 0$ so that for all $\rho > \rho_0$ the set $\Sigma \cap \{x_3 = c\} \cap \{x_1^2 + x_2^2 = \rho^2\}$ consists of exactly two points for $|c - h| \leq \alpha$.

**Proof.** First note, for $\rho_0$ large, the intersection is never empty by the strong half-space theorem [16], the maximum principle and because $\Sigma$ is properly embedded. Without loss of generality we may assume $h = 0$ with $0 \in \mathbb{Z}^0 = \Sigma \cap \{x_3 = 0\}$ and $|A|^2(0) \neq 0$. Let $R_1$ and the set of blow-up pairs be given by Lemma 4.3 and $\Sigma^i$ the unbounded components of $\check{\Sigma}_S$. There then exists $\rho_0$ so for $2\rho > \rho_0$, $\{x_1^2 + x_2^2 = \rho^2\} \cap \mathbb{Z}^0$ lies in the set $\Sigma^1 \cup \Sigma^2$. If no such $\rho_0$ existed then, since the blow-up pairs lie within a cone, there would exist $\delta > 0$ and a subset of the blow-up pairs $(y_i, s_i)$ so $0 \in B_{\delta R_1}(y_i)$. However, [13], Lemma 2.26, with $K_1 = \delta R_1$, would then imply $|A|^2(0) \leq K_2 s_i^{-2}$ for all $i$, i.e. $|A|^2(0) = 0$, a contradiction. Now, for some small $\alpha$ and $\rho > \rho_0$, $Z^\epsilon \cap \{x_1^2 + x_2^2 = \rho^2\}$ lies in $\Sigma^1 \cup \Sigma^2$ for all $|c| < \alpha$, and so $\{x_1^2 + x_2^2 = \rho^2\} \cap \{-\alpha < x_3 < \alpha\} \cap \Sigma$ consists of the union of the graphs of $u^1$ and $u^2$ over the circle $\partial D_\rho$, both of which are monotone increasing in height. \[ \square \]

As $x_3$ is harmonic on $\Sigma$, Proposition 1.6 is an immediate consequence of the previous result and the proof of Rado’s theorem. Recall Rado’s theorem [23] implies that any minimal surface whose boundary is a graph over the boundary of a convex domain is a graph over that domain. The proof of this reduces to showing that a non-constant harmonic function on a closed disk has an interior critical point if and only if the level curve of the function through that point meets the boundary in at least 4 points, which is exactly what we use. We now show Theorem 1.4:

**Proof.** First of all, set $\check{\Sigma}_S = \Sigma^1 \cup \Sigma^2$, the infinite-valued graphs defined by Lemma 4.3. It will then suffice to adjoin the bounded components of $\check{\Sigma}_S$ to $\check{\Sigma}_A$, and to show that $|V_{\Sigma x^3}|$ is bounded below on the new $\Sigma$. Suppose that $(0, 1)$ is a blow-up pair. By the chord-arc bounds of [13], there exists $\gamma$ large enough so that the intrinsic ball of radius $\gamma R_1$ contains $\Sigma \cap B_{R_1}$. [13], Lemma 2.26, implies that curvature is bounded in $B_{2\gamma R_1} \cap \Sigma$ by some $K = K(\gamma R_1)$. The function $v = -2 \log |V_{\Sigma x^3}| \geq 0$ is well defined and smooth by Proposition 1.6 and standard computations give $\Delta v = |A|^2$. Then, since $|V_{\Sigma x^3}| = 1$ somewhere in the component of $B_1(0) \cap \Sigma$ containing 0, we can apply a Harnack inequality (e.g. [15], Theorems 9.20 and 9.22) to obtain an upper bound for $v$ on the intrinsic ball of radius $\gamma R_1$ that depends only on $K$ and $\gamma R_1$. Consequently, there is a lower bound $\varepsilon_0$ on $|V_{\Sigma x^3}|$ in $\Sigma \cap B_{R_1}$. Since this bound is scaling invariant, the same bound holds around any blow-up pair. Finally, any bounded component, $\Omega$, of $\check{\Sigma}_S$ has boundary in $\check{\Sigma}_A$ and so, since $v$ is subharmonic, $|V_{\Sigma x^3}| \geq \varepsilon_0$ on $\Omega$. Thus, by adjoining all such bounded $\Omega$ to $\check{\Sigma}_A$ we obtain Theorem 1.4. \[ \square \]

5. Concluding uniqueness

Since $V_{\Sigma x^3}$ is non-vanishing and the level sets of $x_3$ in $\Sigma$ consist of a single curve, the map $z = x_3 + x_3^3 : \Sigma \rightarrow \mathbb{C}$ is a global holomorphic coordinate (here $x_3^3$ is the harmonic conjugate of $x_3$). Additionally, $V_{\Sigma x^3} \neq 0$ implies that the normal of $\Sigma$ avoids $(0, 0, \pm 1)$. Thus, the stereographic projection of the Gauss map, denoted by $g$, is a holomorphic map $g : \Sigma \rightarrow \mathbb{C} \setminus \{0\}$. By monodromy, there exists a holomorphic map $h = h_1 + ih_2 : \Sigma \rightarrow \mathbb{C}$ so
that \( g = e^h \). We will use \( h \) to show that \( z \) is actually a conformal diffeomorphism between \( \Sigma \) and \( \mathbb{C} \). As the same is then true for \( h \), embeddedness and the Weierstrass representation imply \( \Sigma \) is the helicoid.

5.1. Structure of \( h \). As \( \nabla_{\Sigma} e_3 \) is the projection of \( e_3 \) onto \( T\Sigma \), one can compute \( |\nabla_{\Sigma} e_3| \) by comparing it to the projection of \( e_3 \) onto the unit normal of \( \Sigma \). This gives the following relation between \( \nabla_{\Sigma} e_3 \), \( g \) and \( h \):

\[
|\nabla_{\Sigma} e_3| = 2 \frac{|g|}{1 + |g|^2} \leq 2e^{-|h|}. \tag{5.1}
\]

An immediate consequence of (5.1) and the decomposition of Theorem 1.4 is that there exists \( \gamma_0 > 0 \) so on \( \mathcal{R}_A \), \( |h_1(z)| \leq \gamma_0 \). This imposes strong rigidity on \( h \):

**Proposition 5.1.** Let \( \Omega_{\pm} = \{ x \in \Sigma \mid \pm h_1(x) \geq 2 \gamma_0 \} \) then \( h \) is a proper conformal diffeomorphism from \( \Omega_{\pm} \) onto the closed half-spaces \( \{ z \mid \pm \Re z \geq 2 \gamma_0 \} \).

**Proof.** Let \( \gamma > \gamma_0 \) be a regular value of \( h_1 \). Such \( \gamma \) exist by Sard’s theorem and indeed form a dense subset of \( (\gamma_0, \infty) \). We claim that the set of smooth curves \( Z = h^{-1}(\gamma) \) has exactly one component. Note that \( Z \) is non-empty by (2.5) and (5.1). By construction, \( Z \) is a subset of \( \mathcal{R}_S \) and, up to choosing an orientation, \( Z \) lies in the graph of \( u^1 \), which we will henceforth denote as \( u \). Let us parametrize one of the components of \( Z \) by \( \phi(t) \), non-compact by the maximum principle, and write \( \phi(t) = \Phi_u(\rho(t), \theta(t)) \). Note, \( h_2(\phi(t)) \) is monotone in \( t \) by the Cauchy-Riemann equations and because \( \gamma_0 \) is a regular value of \( h_1 \) and so \( \nabla_{\Sigma} h_1(\phi(t)) \equiv 0 \).

At the point \( \Phi_u(\rho, \theta) \) we compute

\[
g(\rho, \theta) = \frac{1}{\sqrt{1 + |\nabla u|^2 - 1}} \left( u_\rho(\rho, \theta) + i \frac{u_\theta(\rho, \theta)}{\rho} \right) e^{i \theta}. \tag{5.2}
\]

Since \( u_\theta(\rho(t), \theta(t)) > 0 \), there exists a function \( \tilde{\theta}(t) \) with \( \pi < \tilde{\theta}(t) < 2\pi \) such that

\[
|\nabla u|(\rho(t), \theta(t)) e^{i \tilde{\theta}(t)} = -u_\rho(\rho(t), \theta(t)) - i \frac{u_\theta(\rho(t), \theta(t))}{\rho(t)}. \tag{5.3}
\]

Thus, \( h_2(\phi(t)) = \theta(t) + \tilde{\theta}(t) \).

We now claim that, up to replacing \( \phi(t) \) by \( \phi(-t) \), \( \lim_{t \to \infty} h_2(\phi(t)) = R < \infty \). Without loss of generality, we need only rule out \( \lim_{t \to \infty} h_2(\phi(t)) = R < \infty \). Suppose this occurred, then by the monotonicity of \( h_2, h_2(\phi(t)) < R \). The formula for \( h_2(\phi(t)) \) implies that, for \( t \) large, \( \phi(t) \) lies in one sheet. The decay estimates (2.5) together with (5.1) imply \( \rho(t) \) cannot became arbitrarily large and so the positive end of \( \phi \) lies in a compact set. Thus, there is a sequence of points \( p_j = \phi(t_j) \), with \( t_j \) monotonically increasing to \( \infty \), so \( p_j \to p_\infty \in \Sigma \). By the continuity of \( h_1, p_\infty \in Z \), and since \( h_2(p_j) \) is monotone increasing with supremum \( R \), \( h_2(p_\infty) = R \), and so \( p_\infty \) is not in \( \phi \). However, \( p_\infty \in Z \) implies \( h'(p_\infty) \equiv 0 \) and so \( h \) re-
stricted to a small neighborhood of $p_\infty$ is a diffeomorphism onto its image, contradicting $\phi$ coming arbitrarily close to $p_\infty$.

Thus, the formula for $h_2(\phi(t))$, its monotonicity, and the bound on $\tilde{\theta}$ show that $\theta(t)$ must extend from $-\infty$ to $\infty$. We now conclude that there are at most a finite number of components of $Z$. Namely, since $\theta(t)$ runs from $-\infty$ to $\infty$ we see that every component of $Z$ must meet the ray $\eta(\rho) = \Phi_u(\rho, 0)$, $\rho \in (\rho_0, \infty)$, where $\rho_0$ is the smallest value so $\eta \in \mathcal{R}_S$. Again, the gradient decay of (2.5) says that the set of intersections of $Z$ with $\eta$ lies in a compact set, and so consists of a finite number of points. Now, suppose there was more than one component of $Z$. Looking at the intersection of $Z$ with $\eta$, we order these components innermost to outermost; parametrize the innermost curve by $\phi_1(t)$ and the outermost by $\phi_2(t)$. Pick $\tau$ a regular value for $h_2$, and parametrize the component of $h_2^{-1}(\tau)$ that meets $\phi_1$ by $\sigma(t)$, writing $\sigma(t) = \Phi_u(\rho(t), \theta(t))$ in $\mathcal{R}_S$. From the formula for $h_2$, $|\theta(t) - \tau| \leq 2\pi$. Again, $\sigma(t)$ cannot have an end in a compact set, so $\rho(t) \to \infty$. Hence, $\sigma$ must also intersect $\phi_2$ contradicting the monotonicity of $h_1$ on $\sigma$.

Thus, when $\gamma > \gamma_0$ is a regular value of $h_1$, $h_1^{-1}(\gamma)$ is a single smooth curve. We claim this implies that all $\gamma > \gamma_0$ are regular values. Suppose $\gamma' > \gamma_0$ were a critical value of $h_1$. However, as $h_1$ is harmonic, the proof of Rado’s theorem implies for $\gamma > \gamma_0$, a regular value of $h_1$ near $\gamma'$, $h_1^{-1}(\gamma')$ would have at least two components. Thus, $h : \Omega_+ \to \{z \mid \text{Re } z \geq 2\gamma_0 \}$ is a conformal diffeomorphism that maps boundaries onto boundaries, immediately implying that $h$ is proper on $\Omega_+$. An identical argument applies to $\Omega_-$. 

By looking at $z$, which already has well understood behavior away from $\infty$, we see that $\Sigma$ is conformal to $\mathbb{C}$ with $z$ providing an identification.

**Proposition 5.2.** The map $h \circ z^{-1} : \mathbb{C} \to \mathbb{C}$ is linear.

**Proof.** We first show that $z$ is a conformal diffeomorphism between $\Sigma$ and $\mathbb{C}$, i.e. $z$ is onto. This will follow if we show $x_3^* \gamma$ goes from $-\infty$ to $\infty$ on the level sets of $x_3$. The key fact is: each level set of $x_3$ has one end in $\Omega_+$ and the other in $\Omega_-$. This is an immediate consequence of the radial decay along level curves of $x_3$, which follows from the one-sided curvature estimate. Indeed, for any $\epsilon > 0$ there is a $\delta_\epsilon$ so that if $C_\delta_\epsilon$ contains a weak 2-valued $\delta_\epsilon$ sheet on scale 1 then all components of $\Sigma \cap (C_\delta \setminus B_2)$ can be expressed as graphs with gradient bounded by $\epsilon$. By the faster than linear gradient decay of (2.5) and a rescaling, such a weak sheet can always be found. Every level set of $x_3$ must lie in this set, and so far enough out, each point of the level set lies on a graph with gradient bounded by $\epsilon$. However, (5.2) implies that at such points $|h_1| \geq -1/4 \ln \epsilon$, forcing $x_3$ to run from $-\infty$ to $\infty$ along the curve $\partial \Omega_+$. Thus, $z(\partial \Omega_+)$ splits $\mathbb{C}$ into two components with only one, $V$, meeting $z(\Omega_+) = U$. After conformally straightening the boundary of $V$ (using the Riemann mapping theorem) and precomposing with $h_1^{-1}$, we obtain a map from a closed half-space into a closed half-space with the boundary mapped into the boundary. We claim that this map is necessarily onto, that is $U$ equals $\overline{V}$. Suppose it was not onto, then a Schwarz reflection would give a holomorphic map from $\mathbb{C}$ into a simply connected proper subset of $\mathbb{C}$. Because the latter is conformally a disk, Liouville’s theorem would imply this map was constant, a contradiction. As a consequence, if $p \to \infty$ in $\Omega_+$ then $z(p) \to \infty$, with the same true in $\Omega_-$. Thus, along each level set of $x_3$, $|x_3^*(p)| \to \infty$ and so $z$ is onto. Then, by the level set analysis in the proof of Proposition 5.1 and Picard’s theorem, $h \circ z^{-1}$ is a polynomial and is indeed linear. 

**References:**

Bernstein and Breiner, *Helicoid-like minimal disks and uniqueness*
5.2. Concluding uniqueness. After a translation in \( \mathbb{R}^3 \) and a rebasing of \( x_1^i \), \( h(p) = az(p) \) for some \( p \in \Sigma \). As \( dz \) is the height differential, the Weierstrass representation gives, on the curve parameterized by \( z = 0 + it \), that
\[
x_1(it) = |z|^{-2} \left( x_2 \sinh(x_2 t) \sin(x_1 t) - x_1 \cosh(x_2 t) \cos(x_1 t) \right)
\]
and
\[
x_2(it) = |z|^{-2} \left( x_2 \sinh(x_2 t) \cos(x_1 t) + x_1 \cosh(x_2 t) \sin(x_1 t) \right)
\]
where \( x = x_1 + ix_2 \). By inspection, this curve is only embedded when \( x_1 = 0 \), i.e. if \( x = ix_2 \). The factor \( x_2 \) corresponds to a homothetic rescaling and so \( \Sigma \) is the helicoid.

6. Local result

Consider two oriented surfaces \( \Sigma_1, \Sigma_2 \subset \mathbb{R}^3 \), so that \( \Sigma_2 \) is the graph of \( v \) over \( \Sigma_1 \). Then the map \( \phi : \Sigma_1 \rightarrow \Sigma_2 \) defined as \( \phi(x) = x + v(x)n(x) \) is smooth. Moreover, if \( v \) is small in a \( C^1 \) sense, \( \phi \) is an “almost isometry”.

**Lemma 6.1.** Let \( \Sigma_2 \) be the graph of \( v \) over \( \Sigma_1 \), with \( \Sigma_1 \subset B_R \), \( \partial \Sigma_1 \subset \partial B_R \) and \( |A_{\Sigma_1}| \leq 1 \). Then, for \( \varepsilon \) sufficiently small, \( |v| + |\nabla v| \leq \varepsilon \) implies \( \phi \) is a diffeomorphism with \( 1 - \varepsilon \leq \|d\phi\| \leq 1 + \varepsilon \).

**Proof.** For \( \varepsilon \) sufficiently small (depending on \( \Sigma_1 \)), \( \phi \) is injective. Working in \( \mathbb{R}^3 \), given orthonormal vectors \( e_1, e_2 \in T_x \Sigma_1 \) we compute

\[
(6.1) \quad d\phi_p(e_i) = e_i + \langle \nabla_x v(p), e_i \rangle n(p) + v(p) Dn_p(e_i).
\]

The last two terms are together controlled by \( \varepsilon \). Hence, \( 1 - \varepsilon < |d\phi_p(e_i)| < 1 + \varepsilon \). \( \square \)

**Proof of Theorem 1.1.** By rescaling we may assume that \( s = 1 \). We proceed by contradiction. Suppose no such \( R' \) existed for fixed \( \varepsilon, R \). That is, there exists a sequence of counter-examples; \( \Sigma_i' \in \mathcal{E}(1, 0, R_i), (0, 1) \) a \( C \) blow-up pair of each \( \Sigma_i' \) and \( R \leq R_i \rightarrow \infty \), but \( \Sigma_i \), the component of \( B_R \cap \Sigma_i' \) containing zero, is not close to a helicoid.

By definition, \( |A_{\Sigma_i'}(0)|^2 = C > 0 \) for all \( \Sigma_i' \) and so the lamination theory of Colding and Minicozzi implies that a subsequence of the \( \Sigma_i' \) converges smoothly with multiplicity one to \( \Sigma_{\infty}, \) a complete embedded minimal disk. Namely, in any ball centered at 0 the curvature of \( \Sigma_i \) is uniformly bounded by [13], Lemma 2.26. Furthermore, the chord-arc bounds of [13] give uniform area bounds and so by standard compactness arguments one has smooth convergence (possibly with multiplicity) to \( \Sigma_{\infty} \). If the multiplicity of the convergence is greater than 1, then one can construct a positive solution to the Jacobi equation (see [3], Appendix B). That implies \( \Sigma_{\infty} \) is stable, and thus a plane by Schoen’s extension of the Bernstein theorem [24], contradicting the curvature at 0. Thus, as \( \Sigma_{\infty} \in \mathcal{E}(1, 0) \) is non-flat, Theorem 1.2 implies it is a helicoid. We may, by rescaling, assume \( \Sigma_{\infty} \) has curvature 1 along its axis.

For any fixed \( R' \) a subsequence of \( \Sigma_i' \cap B_{R'} \) converges to \( \Sigma_{\infty} \cap B_{R'} \) in the smooth topology. And so, for any \( \varepsilon \), with \( i \) sufficiently large, we find a smooth \( v_i \) defined on a subset of \( \Sigma_{\infty} \) so that \( |v_i| + |\nabla_{\Sigma_{\infty}} v_i| < \varepsilon \) and the graph of \( v_i \) is \( \Sigma_i' \cap B_{R'} \). Choosing \( R' \) large enough to
ensure minimizing geodesics between points in Σ lie in Σ′ ∩ Br (using the chord-arc bounds of [13]), Lemma 6.1 gives the desired contradiction.

### Appendix A. Structural results of Colding and Minicozzi

For the convenience of the reader, we gather here some of the results of Colding and Minicozzi on the structure of embedded minimal surfaces. The foundation of their work is their description of embedded minimal disks in [7]–[10], which underpins their results for more general topologies in [3]. Roughly speaking, they show that embedded minimal disks (with boundary lying on the boundary of a ball) fall into precisely two classes. On the one hand, if the curvature of such a surface, Σ, is everywhere small, then the surface is nearly flat and hence modeled on a plane (i.e. is a single-valued graph). On the other hand, when Σ has (far from the boundary) a point with large curvature then it is modeled on a helicoid. That is, in a smaller ball Σ consists of two multivalued graphs that spiral together and that are glued along an “axis” of large curvature. In proving such a qualitative description, Colding and Minicozzi show many quantitative results about the behavior of the sheets and of the axis. It is these later results that we use and describe in more detail below.

We point out that their work is interior theory, that is it holds far from the boundary. In our application, the minimal disks are complete and without boundary which simplifies things somewhat and so the reader may wish to assume this and ignore the conditions regarding the boundary. We have also, where needed, changed notation to that used in the present paper.

#### A.1. Existence of multivalued graphs.

Definition 2.4 gives a condition that specifies the points and scales of an embedded minimal disk of large curvature that are of particular interest in the theory—recall we refer to these pairs as blow-up pairs. These are points of almost maximal curvature in a ball with scale which is inversely proportional to the curvature at the point. A good example of such blow-up pairs is provided by points on the axis of the helicoid, as there the scale is proportional to the separation between the sheets of the helicoid. This is the model behavior for any embedded minimal disk containing a blow-up pair—i.e. in any embedded minimal disk, near a blow-up pair one has a small multivalued graph forming on the scale of the pair.

**Theorem A.1** ([8], Theorem 0.4). Given \( N, \omega > 1 \) and \( \varepsilon > 0 \), there exists \( C = C(N, \omega, \varepsilon) > 0 \) so: Let \( 0 \leq \varepsilon(1, 0, R) \). If \( (0, 0_0) \) is a \( C \) blow-up pair for \( 0 < r_0 < R \), then there exist \( \tilde{R} < r_0/\omega \) and (after a rotation) an \( N \)-valued graph \( \Sigma_g \subset \Sigma \) over \( D_{\omega \tilde{R}} \setminus D_{\tilde{R}} \) with gradient \( \leq \varepsilon \), and \( \operatorname{dist}_\Sigma(0, \Sigma_g) \leq \tilde{R} \).

#### A.2. Extending the graphs.

Using the initial small multivalued graph, Colding and Minicozzi show that it can be extended, as a graph and within the surface \( \Sigma \), nearly all the way to the boundary of \( \Sigma \).

**Theorem A.2** ([7], Theorem 0.3). Given \( \tau > 0 \) there exist \( N, \Omega, \varepsilon > 0 \) so that the following hold: Let \( \Sigma \in \varepsilon(1, 0, R_0) \). If \( \Omega r_0 < 1 < R_0/\Omega \) and \( \Sigma \) contains an \( N \)-valued graph \( \Sigma_g \) over \( D_1 \setminus D_{r_0} \) with gradient \( \leq \varepsilon \) and \( \Sigma_g \subset C_\varepsilon \) then \( \Sigma \) contains a 2-valued graph \( \Sigma_d \) over \( D_{R_0/\Omega} \setminus D_{r_0} \) with gradient \( \leq \tau \) and \( (\Sigma_g)^\delta \subset \Sigma_d \).
Here \((\Sigma_g)^M\) indicates the “middle” 2-valued sheet of \(\Sigma_g\). Combining this with Theorem A.1, gives the existence of a multivalued graph near a blow-up pair that extends almost all the way to the boundary. Namely, [8], Theorem 0.2:

**Theorem A.3.** Given \(N \in \mathbb{Z}^+, \varepsilon > 0\), there exist \(C_1, C_2, C_3 > 0\) so: Let

\[
0 \in \Sigma \in \mathcal{E}(1, 0, R).
\]

If \((0, r_0)\) is a \(C_1\) blow-up pair then there exists (after a rotation) an \(N\)-valued graph \(\Sigma_g \subset \Sigma\) over \(D_{R/2} \setminus D_{2r_0}\) with gradient \(\leq \varepsilon\) and \(\Sigma \subset C_c\). Moreover, the separation of \(\Sigma_g\) over \(\partial D_{r_0}\) is bounded below by \(C_2r_0\).

Note that the lower bound on the initial separation is not explicitly stated in [8], Theorem 0.2, but is proved in [8], Proposition 4.15.

A.3. The second multivalued graph. Colding and Minicozzi show that, “between the sheets” of \(\Sigma_g\), \(\Sigma\) consists of exactly one other multivalued graph. That is we have at least that part of \(\Sigma\) looks like (a few sheets of) a helicoid. Precisely, one has [10], Theorem I.0.10:

**Theorem A.4.** Suppose \(0 \in \Sigma \in \mathcal{E}(1, 0, 4R)\) and \(\Sigma_1 \subset \Sigma \cap \Sigma_1\) is an \((N + 2)\)-valued graph of \(u_1\) over \(D_{2R} \setminus D_{r_1}\) with \(|\nabla u_1| \leq \varepsilon\) and \(N \geq 6\). There exist \(C_0 > 2\) and \(\varepsilon_0 > 0\) so that if \(R \geq C_0r_1\) and \(\varepsilon_0 \geq \varepsilon\), then \(E \cap \Sigma \setminus \Sigma_1\) is an (oppositely oriented) \(N\)-valued graph \(\Sigma_2\).

Here \(E\) is the region between the sheets of \(\Sigma_1\) and is the same as (2.7):

\[
(A.1) \quad \{(r \cos \theta, r \sin \theta, z) \mid 2r_1 < r < R, -2\pi < \theta < 0, u_1(r, \theta - N\pi) < z < u_1(r, \theta + N\pi)\}.
\]

By Theorem A.4, near a blow-up point there are two multivalued graphs that spiral together and extend within \(\Sigma\) almost all the way to the boundary of \(\Sigma\).

A.4. Finding blow-up pairs. The existence of two multivalued graphs spiraling together allows Colding and Minicozzi to use the following result from [9] in order to show there are regions of large curvature above and below the original blow-up pairs. This is [9], Corollary III.3.5:

**Corollary A.5.** Given \(C_1\) there exists \(C_2\) so: Let \(0 \in \Sigma \in \mathcal{E}(1, 0, 2C_2r_0)\). Suppose \(\Sigma_1, \Sigma_2 \subset \Sigma \cap C_1\) are graphs of \(u_i\) satisfying (2.4) on \(S_{r_0, 2r_0}^{\varepsilon, 2\pi}\), \(u_1(r_0, 0) < u_2(r_0, 0) < u_1(r_0, 0)\), and \(v \subset \partial \Sigma_0, 2r_0\) a curve from \(\Sigma_1\) to \(\Sigma_2\). (Here, \(\Sigma_0, 2r_0\) denotes the component of \(\Sigma \cap B_{2r_0}\) containing \(0\).) Let \(\Sigma_0\) be the component of \(\Sigma_0, C_2r_0 \setminus (\Sigma_1 \cup \Sigma_2 \cup v)\) which does not contain \(\Sigma_0, r_0\). Then

\[
(A.2) \quad \sup_{x \in \Sigma_0 \setminus B_{2r_0}} |x|^2 |A|^2(x) \geq 4C_1^2.
\]

By a standard blow-up argument if there is large curvature in a ball (measured with respect to the scale of the ball) then there exists a blow-up pair in the ball. This is [8], Lemma 5.1:
Lemma A.6. If $0 \in \Sigma \in \mathcal{E}(1,0,r_0)$ and $\sup_{B_{2r_0} \cap \Sigma} |A|^2 \geq 16C^2r_0^{-2}$ then there exists a pair $(y,r_1)$ with $y \in \Sigma$ and $r_1 < r_0 - |y|$ so $(y,r_1)$ is a $C$ blow-up pair.

Combining Theorem A.1, Corollary A.5, and Lemma A.6, gives the existence of blow-up pairs above and below an initial pair.

A.5. The one-sided curvature estimate. Using the above results, Colding and Minicozzi prove that an embedded minimal disk that is close to and on one side of a plane has uniformly bounded curvature. The precise statement of the one-sided curvature estimate is the following:

Theorem A.7 ([10], Theorem 0.2). There exists $\varepsilon > 0$ so that if

$$\Sigma \subset B_{2r_0} \cap \{x_3 > 0\} \subset \mathbb{R}^3$$

is an embedded minimal disk with $\partial \Sigma \subset \partial B_{2r_0}$, then for all components, $\Sigma'$ of $\Sigma \cap B_{r_0}$ which intersect $B_{2r_0}$, we have

(A.3) $\sup_{\Sigma'} |A_{\Sigma'}|^2 \leq r_0^{-2}$.

Rescaled catenoids show that the surface must be an embedded disk. As a consequence of this estimate, if an embedded minimal disk, $\Sigma$, contains a two-valued graph lying outside of a cone union a ball, then all components of $\Sigma$ in the complement of a larger cone (and larger ball) are multivalued graphs. The nearly flat two-valued graph takes the place of the plane in Theorem A.7. Precisely:

Corollary A.8 ([10], Corollary I.1.9). There exists $\delta_0$ so that the following holds: Let $\Sigma \in \mathcal{E}(1,0,2R)$. If $\Sigma$ contains a 2-valued graph $\Sigma_d \subset \mathcal{C}_{\delta_0}$ over $D_{2R} \setminus D_{r_0}$ with gradient $\leq \delta_0$, then each component of

$$(\mathcal{C}_{\delta_0} \cup B_{R/2}) \cap \Sigma \setminus B_{2r_0}$$

is a multivalued graph with gradient $\leq 1$.

Remark A.9. Schauder estimates imply that, by shrinking $\delta_0$, one has $|\nabla u| \leq \varepsilon$. 

References


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