Some power series involving involutions in Coxeter groups

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SOME POWER SERIES INVOLVING INVOLUTIONS IN COXETER GROUPS

G. LUSZTIG

ABSTRACT. Let $W$ be a Coxeter group. We show that a certain power series involving a sum over all involutions in $W$ can be expressed in terms of the Poincaré series of $W$. (The case where $W$ is finite has been known earlier.)

INTRODUCTION

0.1. Let $V'$ be an $n$-dimensional vector space over the finite field $F_q$ and let $V = F_q^2 \otimes F_q V'$, an $n$-dimensional vector space over $F_q^2$. Let $F : V \to V$ be the $F_q$-linear isomorphism $\lambda \otimes v \mapsto \lambda^l \otimes v$ where $\lambda \in F_q^2, v \in V'$. Let $F$ be the set of all flags $V = (V_1 \subset V_2 \subset V_n)$ in $V$ where for $i = 1, \ldots, n$, $V_i$ there exists an $i$-dimensional $F_q^2$-subspace of $V$; let $F'$ be the analogous set defined in terms of $V', F_q$ instead of $V, F_q^2$. Note that $F$ induces a bijection $F \to F'$,

$$V_i = (V_1 \subset V_2 \subset V_n) \mapsto F(V_i) = (F(V_1)F(V_2)F(V_n)).$$

The standard way to count the number $|F|$ elements in $F$ is to fix $V_0 \in F$ and to partition $F$ into pieces indexed by the symmetric group $S_n$; the piece corresponding to $w \in S_n$ is the set of all $V_0 \in F$ such that $V_0, V_1$ are in relative position (this piece contains exactly $q^{2l(w)}$ elements where $l : S_n \to N$ is the standard length function). This yields the expression $|F| = \sum_{w \in S_n} q^{2l(w)}$. Similarly we have $|F'| = \sum_{w \in S_n} q^{l(w)}$.

Another way to compute $|F|$ without a choice of $V_0 \in F$ is to partition $F$ into pieces indexed by the set $I$ of involutions in $S_n$; the piece corresponding to $z \in I$ is the set of all $V_0 \in F$ such that $V_0, F(V_0)$ are in relative position $z$ (this piece contains exactly $|F'| q^{l(z)}(q^2 - 1)^{\phi(z)}$ where $\phi(z)$ is the number of $(-1)$-eigenvalues of $z$ in the reflection representation of $S_n$). (If we had taken a $z$ in $S_n - I$ we would have got an empty piece.) This yields the expression $|F| = |F'| \sum_{z \in I} q^{l(z)}(q^2 - 1)^{\phi(z)}$. We deduce that the sum

$$\sum_{z \in I} q^{l(z)}(q^2 - 1)^{\phi(z)}$$

(a kind of weighted Poincaré polynomial based on involutions) is equal to a quotient of two ordinary Poincaré polynomials:

$$\sum_{w \in S_n} q^{2l(w)} / \sum_{w \in S_n} q^{l(w)}.$$

The purpose of this paper is to generalize this equality to the case of Coxeter groups.
0.2. Let \((W, S)\) be a Coxeter group such that the canonical set of generators \(S\) of \(W\) is finite and let \(w \mapsto w^*\) be an involutive automorphism of \(W\) preserving \(S\). Let \(l : W \to \mathbf{N}\) be the usual length function and let \(\leq\) be the standard partial order on \(W\). The Poincaré series of \(W\) is the formal power series \(P(u) = \sum_{w \in W} u^{l(w)} \in \mathbf{Z}[u]\) where \(u\) is an indeterminate. We set \(P_*(u) = \sum_{w \in W, w^* = w^{-1}} u^{l(w)} \in \mathbf{Z}[u]\). Let \(I_* = \{w \in W; w^* = w^{-1}\}\). It is easy to show (see for example [LV]) that there is a unique function \(\phi : I_* \to \mathbf{N}\) such that for any \(I\) \(\phi(1) = 0\) and such that for any \(w \in I_*\) and any \(s \in S\) with \(sw < w\) we have \(\phi(w) = \phi(sw) + 1\) (if \(sw = ws^*\)) and \(\phi(w) = \phi(sw^*)\) (if \(sw \neq ws^*\)). We set

\[
\mathcal{R}(u) = \sum_{z \in I_*} u^{l(z)}(\frac{u - 1}{u + 1})^{\phi(z)} \in \mathbf{Z}[u].
\]

(The definition of \(\mathcal{R}(u)\) appeared in [LV] in the case where \(W\) is finite, but the definition clearly makes sense in the general case.) We say that \(W, *\) has property \(X\) if

\[
\mathcal{R}(u) = P(u^2)/P_*(u).
\]

In [LV] it was proved (case by case) that \(W, *\) has property \(X\) if \(W\) is finite. In [MW] it is proved that \(W, 1\) has property \(X\) if \(W\) is an affine Weyl group of type \(A\) and the question of the validity of \(X\) for any \(W, *\) is raised. We state the following result.

**Theorem 0.3.** \(W, *\) has property \(X\).

The proof is given in 1.11. It is based on results in [LV], [LI]. The most complicated part of the proof is a property stated in Proposition 1.8. In Section 2 we give an alternative proof of this property assuming that \(W\) is a Weyl group. (The same proof applies in the case where \(W\) is the “Weyl group” associated to a Kac-Moody Lie algebra and \(*\) is induced by an involutive automorphism of that Lie algebra.)

1. The polynomials \(X^z_y\)

1.1. Let \(W, S, \*, l, I_*\) be as in 0.2. Let \(u\) be an indeterminate.

Let \(A = \mathbf{Z}[u, u^{-1}]\). Let \(\mathfrak{H}\) be the free \(A\)-module with basis \((T_w)_{w \in W}\) with the unique \(A\)-algebra structure with unit \(T_1\) such that

\[
T_wT_{w'} = T_{ww'} \text{ if } l(ww') = l(w) + l(w') \text{ and } (T_s + 1)(T_s - u^2) = 0 \text{ for all } s \in S.
\]

(This is an Iwahori-Hecke algebra.) Let \(M\) be the free \(A\)-module with basis \(\{a_w; w \in I_*\}\). According to [LV] (when \(W\) is a Weyl group) and [LI] (in the general case) there is a unique \(\mathfrak{H}\)-module structure on \(M\) (extending the \(A\)-module structure) such that for any \(s \in S\) and any \(z \in I_*\), we have

(i) \(T_s a_z = ua_z + (u + 1)a_{sz}\) if \(sz = zs^* > z\);
(ii) \(T_s a_z = (u^2 - u - 1)a_z + (u^2 - u)a_{sz}\) if \(sz = zs^* < z\);
(iii) \(T_s a_z = a_{szs^*}\) if \(sz \neq zs^* > z\);
(iv) \(T_s a_z = (u^2 - 1)a_z + u^2a_{szs^*}\) if \(sz \neq zs^* < z\).

For any \(m \in M\) we write \(m = \sum_{z \in I_*} (a_z : m)a_z\) where \((a_z : m) \in A\) are zero for all but finitely many \(z\).

**Proposition 1.2.** There is a unique collection of polynomials \(X^z_y \in \mathbf{Z}[u]\) indexed by \((z, y) \in I_* \times W\) such that

...
(★) $X^z_y = \delta_{z,1}$ for all $z \in I_s$ and such that for any $s \in S, y \in W$ with $y > ys$ and any $z \in I_s$ we have

(i) $X^z_y = uX^{ys}_{ys} + (u + 1)X^{zs}_{ys}$ if $sz = zs^* > z$;

(ii) $X^z_y = (u^2 - u - 1)X^{ys}_{ys} + (u^2 - u)X^{zs}_{ys}$ if $sz = zs^* < z$;

(iii) $X^z_y = X^{ys}_{ys^*}$ if $sz \neq zs^* > z$;

(iv) $X^z_y = (u^2 - 1)X^{zs}_{ys} + u^2X^{s^*s}_{ys}$ if $sz \neqzs^* < z$.

Moreover, we have $X^z_y \in u^{l(z)}\mathbb{Z}[u]$ for any $(z, y) \in I_s \times W$.

For $(z, y) \in I_s \times W$ we set $X^z_y = (a_1 : T_ya_z) \in A$. Then clearly (★) holds. If $s \in S, y \in W$ satisfy $y > ys$, then $T_y = T_ys$; hence applying $T_ys$ to the formulas 1.1(i)--(iv) we see that $X^z_y$ satisfy (i)--(iv) (but they are in $A$ instead of $\mathbb{Z}[u]$). Now from (★) and (i)--(iv) we see by induction on $l(y)$ that $X^z_y$ are uniquely determined and that $X^z_y \in u^{l(z)}\mathbb{Z}[u]$ for any $(z, y) \in I_s \times W$. The proposition is proved.

1.3. The $A$-module $M$ in 1.1 can be viewed as an $A$-submodule of $\hat{M}$ which consists of all formal $A$-linear combinations of the elements $\{a_z ; z \in I_s\}$. The $A$-module structure on $M$ extends in an obvious way to an $A$-module structure on $\hat{M}$. We show that

(a) $T_y\left(\sum_z a_z\right) = u^{2l(y)} \sum_z a_z$ for any $y \in W$.

It is enough to prove this in the case where $y = s \in S$. For any $z \in I_s$ we set $s \cdot z = sz$ if $sz = zs^*$ and $s \cdot z = szs^*$ if $sz \neq zs^*$. From the definition for any $z \in I_s$ such that $sz < z$ we have

$T_s(a_z + a_{sz}) = u^2(a_z + a_{sz})$;

hence

$T_s\left(\sum_{z \in I_s} a_z\right) = \sum_{z \in I_s; sz < z} T_s(a_z + a_{sz}) = \sum_{z \in I_s; sz < z} u^2(a_z + a_{sz}) = u^2 \sum_{z \in I_s} a_z$.

This completes the proof of (a).

For $\hat{m} \in \hat{M}$ we write $\hat{m} = \sum_{z \in I_s} (a_z : \hat{m})a_z$ where $\{a_z : \hat{m}\} \in A$. Applying (a) to (a) we obtain for any $y \in W$:

(b) $\sum_{z \in I_s} X^z_y = u^{2l(y)}$.

Note that in the left-hand side we have $X^z_y = 0$ for all but finitely many $z$. Indeed, if $(z, y) \in I_s \times W$ and $l(z) > 2l(y)$, then $X^z_y = 0$. (This can be seen from Proposition 1.7 or directly from definitions by induction on $l(y)$.)

Proposition 1.4. Let $s \in S, z \in I_s$ be such that $sz < z$ and let $y \in W$. We have:

(i) $(u + 1)X^z_y = -uX^{ys}_{ys} + X^{zs}_{ys}$ if $sz = zs^*, ys > y$;

(ii) $(u + 1)X^z_y = u^2X^{zs}_{ys} + (u^2 - u - 1)X^{ys}_{ys}$ if $sz = zs^*, ys < y$;

(iii) $X^z_y = X^{zs}_{ys^*}$ if $sz \neq zs^*, ys > y$;

(iv) $X^z_y = u^2X^{s^*s}_{ys} + (u^2 - 1)X^{s^*s}_{ys}$ if $sz \neq zs^*, ys < y$.

From 1.2(i) we obtain $X^{2s}_{ys} = uX^{ys}_{ys} + (u + 1)X^z_y$ if $sz = zs^*, ys > y$. Clearly this yields (i). In 1.2(ii) we substitute $X^{zs}_{ys} = -\frac{u}{u + 1}X^{zs}_{ys} + \frac{1}{u + 1}X^z_y \in \mathbb{Q}(u)$ (with $ys < y$)
which follows from (i); we obtain
\[ X^z_y = (u^2 - u - 1)\left(-\frac{u}{u + 1}X^{sz}_{sy} + \frac{1}{u + 1}X^{sz}_{sy}\right) + (u^2 - u)X^{sz}_{sy} \]
\[ = \frac{u^2}{u + 1}X^{sz}_{sy} + \frac{u^2 - u - 1}{u + 1}X^{sz}_{sy} \]
so that (ii) holds. Now (iii) clearly follows from 1.2(iii). In 1.2(iv) we substitute
\[ X^z_{y^s} = X^{szs^*}_y \]
which follows from (iii); we obtain (iv). The proposition is proved.

From the proposition we deduce the following result.

**Corollary 1.5.** Let \( z \in I_s, y \in W, s \in S \). We have:

(i) \( X^z_y + X^z_{y^s} = u^2(X^{szs^*}_y + X^{szs^*}_y) \) if \( sz \neq zs^* < z \);

(ii) \( (u + 1)(X^z_y + X^z_{y^s}) = (u^2 - u)(X^{sz}_y + X^{sz}_y) \) if \( sz = zs^* < z \).

1.6. For any \( \sigma \in \mathfrak{I} \), \( s \in A \) we write \( h = \sum_{w \in W} [T_w : h]T_w \) where \( [T_w : h] \in A \) are zero for all but finitely many \( w \).

For \( z' \in I_s, s \in A \) we show:

(i) \( T_sT_{z'}T_s = cT_{z'} + c'T_{z''} \) where \( c, c' \in \mathbb{Z}[u] - \{0\} \), if \( z' \neq z's^* \);

(ii) \( T_sT_{z'}T_s = T_{z's^*} \) if \( z' \neq z's^* > z \);

(iii) \( T_sT_{z'}T_{s^*} = u^4T_{sz's^*} + (u^2 - 1)^2T_{z'} + u^2(u^2 - 1)T_{sz'} + u^2(u^2 - 1)T_{z's^*} \) if \( s' \neq z's^* < z' \).

In (i) assume first that \( z' > z' \). Then \( T_sT_{z'}T_{s^*} = T_sT_{z's^*} = u^2T_{z'} + (u^2 - 1)T_{sz'} \).

Now assume that in (i) we have \( sz < z \). Then
\[ T_sT_{z'}T_{s^*} = T_sT_{T_{z's^*}T_{s'}T_{s^*}} = T_sT_{z'}T_{s^*} = u^2T_{z'} + (u^2 - 1)T_{sz'} + \]
\[ = u^2T_{z'} + u^2(u^2 - 1)T_{sz'} + (u^2 - 1)^2T_{z's^*} \]

Now (ii) is obvious. In (iii) we have
\[ T_sT_{z'}T_{s^*} = T_sT_{T_{z's^*}T_{s'}T_{s^*}} \]
\[ = u^4T_{sz's^*} + (u^2 - 1)^2T_{z'} + u^2(u^2 - 1)T_{sz'} + u^2(u^2 - 1)T_{z's^*} \]
as desired.

For \( s \in S \) and \( z, z' \in I_s \) we show:

(a) If \( (z : T_sza_{z'}) \neq 0 \), then \( [T_z : T_sT_{z'T_{s'}}] \neq 0 \).

If \( z' = z's^* \), then by Proposition 1.2 we have \( z = z' \) or \( z = sz' \) and (a) follows from (i).

If \( z' = z's^* > z' \), then by Proposition 1.2 we have \( z = z' \) or \( z = sz's^* \) and (a) follows from (ii).

If \( z' = z's^* < z' \), then by Proposition 1.2 we have \( z = z' \) or \( z = sz's^* \) and (a) follows from (iii).

This proves (a).

We shall also need a variant of (a). Assume that \( s \neq s' \in S \) are such that \( s' = s^* \), \( s, s' \) generate a finite (dihedral) subgroup of \( W \) and \( \sigma \) is the longest element in that subgroup.

Then:

(b) If \( z, z' \in I_s \) and \( (z : T_za_{z'}) \neq 0 \), then \( [T_z : T_{z'T_{s'}}] \neq 0 \).

(c) \( (a_1 : T_{z'a_1}) = u^{l(\sigma)} \).

This can be proved using the results in [L1] which describe the action of \( T_\sigma \) on \( M \).

We have the following result.

**Proposition 1.7.** If \( z \in I_s, y \in W \) and \( (z : T_ya_1) \neq 0 \), then \( [T_z : T_yT_{y-1}] \neq 0 \).
We argue by induction on \( l(y) \). If \( y = 1 \) the result is obvious. Assume now that \( y \neq 1 \). We can find \( s \in S \) such that \( sy < y \). We have
\[
(a_z : T_y a_1) = \sum_{z' \in \Pi^*} (a_z : T_s a_{z'}) (a_{z'} : T_{sy} a_1).
\]

By assumption there exists \( z' \in \Pi^* \) such that \( (a_z : T_s a_{z'}) \neq 0 \) and \( (a_{z'} : T_{sy} a_1) \neq 0 \). By 1.6(a) we then have \([T_z : T_s T_z T_{s*}] \neq 0\) and by the induction hypothesis we have \([T_{z'} : T_s T_{y^*} T_{s*}] \neq 0\). We have
\[
[T_z : T_y T_{y^*} T_{s*} - 1] = \sum_{w \in W} [T_z : T_s T_w T_{s*}][T_w : T_s T_{y^*} T_{s*} - 1].
\]

This is a sum of terms in \( N[u-1] \) (the set of polynomials in \( u-1 \) with coefficients in \( N \)) and the term corresponding to \( w = z' \) is nonzero; hence the sum is nonzero. The proposition is proved.

**Proposition 1.8.** For any \( y \in W \) we have \( X^1_y = \delta_{y,y*} u^{l(y)} \).

From \( [2] \) 10.4(a) we have
\[
(a)
[T_1 : T_y T_{y^*} T_{y_1} - 1] = \delta_{y_1,y_1} u^{2l(y)} \text{ for any } y_1 \in W.
\]

We prove the proposition by induction on \( l(y) \). If \( y = 1 \) the result is obvious. Assume now that \( y \neq 1 \). We can write \( y = \sigma y' \) where \( \sigma, y' \in W \), \( l(y) = l(\sigma) + l(y') \) and the following holds: if \( y \neq y' \), then \( \sigma \in S \); if \( y = y' \), then there exists \( s, s' \in S \) such that \( s^r = s'^r \), \( s, s' \) generate a finite subgroup of \( W \) and \( \sigma \) is the longest element in that subgroup, so that \( \sigma = \sigma^r \). (We use \([2] \) A1(a).) By the induction hypothesis we have
\[
T_{y'} a_1 = \delta_{y',y*} u^{l(y')} a_1 + \sum_{z \in \Pi^*, z \neq 1} (a_z : T_{y'} a_1) a_z.
\]

Since \( T_y = T_{\sigma} T_{y'} \), it follows (also using \( a \)) that
\[
(b)
T_y a_1 = \delta_{y',y*} u^{l(y')} T_{\sigma} a_1 + \sum_{z \in \Pi^*, z \neq 1} (a_z : T_{y'} a_1) T_{\sigma} a_z,
\]
\[
T_y T_{y^*} T_{y_1} - 1 = \delta_{y',y*} u^{2l(y')} T_{\sigma} T_{\sigma^*} + \sum_{w \in W; w \neq 1} [T_w : T_{y'} T_{y^*} T_{y_1} - 1] T_{\sigma} T_w T_{\sigma^*}.
\]

Note that by \( a \) we have
\[
[T_1 : T_{\sigma} T_{\sigma^*}] = \delta_{\sigma, \sigma^*} u^{2l(\sigma)}.
\]

Hence \( \gamma := \delta_{y',y^*} u^{2l(y')} T_{\sigma} T_{\sigma^*} \) is \( \delta_{y',y} u^{2l(y')} \delta_{\sigma, \sigma^*} u^{2l(\sigma)} \), that is \( \delta_{y,y} u^{2l(\gamma)} \). If \( y \neq y^* \) we have \( \gamma = 0 \) since we have either \( \delta_{y',y^*} = 0 \) or \( \delta_{\sigma, \sigma^*} = 0 \). If \( y = y^* \), then \( \sigma = \sigma^* \) hence \( y' = y^r \) and \( \gamma = u^{2l(y^r)} \).

Assume that for some \( z \in \Pi^* - \{1\} \), we have \( (a_z : T_{y'} a_1) \neq 0 \) and \( (a_1 : T_{\sigma} a_z) \neq 0 \). Then, by Proposition 1.7 we have \([T_1 : T_{y'} T_{y^*}] \neq 0\). Moreover, we have \([T_1 : T_{\sigma} T_{y'} T_{y^*}] \neq 0\). (When \( \sigma \in S \) this follows from 1.6(a). When \( \sigma \notin S \) this follows from 1.6(b).) Thus,
\[
\pi := \sum_{w \in W; w \neq 1} [T_w : T_{y'} T_{y^*} T_{y_1} - 1] T_{\sigma} T_w T_{\sigma^*}
\]

is a sum of terms in \( N[u-1] \), at least one of which is \( \neq 0 \), so that \( \pi \in N[u-1] - \{0\} \). Thus \([T_1 : T_y T_{y^*} T_{y_1} - 1] = \delta_{y,y^*} u^{2l(y)} + \pi \). By \( a \), \([T_1 : T_y T_{y^*} T_{y_1} - 1] = \delta_{y,y^*} u^{2l(y)} \). Thus, \( \pi = 0 \), a contradiction.
This contradiction shows that for any \( z \in I_\ast - \{1\} \) we have either \((a_z : T_ya_1) = 0 \) or \((a_1 : T_\sigma a_z) = 0 \). Hence from (b) we deduce that \((a_1 : T_\sigma a_1) = (a_1 : \delta_{y',y'^*}u^{l(y')}T_\sigma a_1) \). We have

\[
(a_1 : T_\sigma a_1) = \delta_{\sigma,\sigma^*}u^{l(\sigma)}.
\]

(When \( \sigma \in S \) this follows from 1.1(i)–(iv). When \( \sigma \notin S \) this follows from 1.6(c).) We deduce

\[
(a_1 : T_ya_1) = \delta_{y',y'^*}u^{l(y')}\delta_{\sigma,\sigma^*}u^{l(\sigma)} = \delta_{y',y'^*}u^{l(y)}.
\]

It remains to show that

\[
(c) \quad \delta_{y',y'^*} = \delta_{y,y'^*}.
\]

If \( y \neq y'^* \), then we have either \( y' \neq y'^* \) or \( \sigma \neq \sigma^* \); hence both sides of (c) are zero. If \( y = y'^* \), then \( \sigma = \sigma^* \) hence \( y' = y'^* \) hence both sides of (c) are 1. Thus (c) holds.

The proposition is proved.

**Proposition 1.9.** We have \( X^z_y \in u^{l(y)}Z[[u]] \) for any \((z, y) \in I_\ast \times W \).

This follows by induction on \( l(z) \) using 1.4(i)–(iv); to start the induction, we assume that \( z = 1 \) in which case the result follows from Proposition 1.8.

1.10. For any \( z \in I_\ast \) we set

\[
(b) \quad X^z = \sum_{y \in W} X^z_y \in Z[[u]].
\]

Note that, by Proposition 1.9, the sum in the right-hand side converges in \( Z[[u]] \). For \( z \in I_\ast \) we show that

\[
(c) \quad X^z = P_\ast(u)u^{l(z)}(\frac{u-1}{u+1})^{\phi(z)}.
\]

We argue by induction on \( l(z) \). If \( z = 1 \) we have, using Proposition 1.8:

\[
X^1 = \sum_{y \in W : y = y'^*} u^{l(y)} = P_\ast(u).
\]

Assume now that \( z \neq 1 \). We can find \( s \in S \) such that \( sz < z \). From Corollary 1.5 we deduce that \( X^z = u^2X^{szs'^*} \) if \( sz \neq zsz^* \) and \((u+1)X^z = (u^2 - u)X^{sz} \) if \( sz = zsz^* \). Using the induction hypothesis we see that, if \( sz \neq zsz^* \) we have

\[
X^z = P_\ast(u)u^{l(szs'^*) + 2(\frac{u-1}{u+1})^{\phi(szs'^*)}},
\]

while if \( sz = zsz^* \) we have

\[
X^z = P_\ast(u)u^{l(sz) + 1(\frac{u-1}{u+1})^{\phi(sz)} + 1};
\]

the desired result follows.
1.11. We prove Theorem 0.3. The sum \( \sum_{(z,y) \in \mathbf{I}_* \times W} X^z_y \) is convergent in \( \mathbb{Z}[[u]] \) since \( X^z_y \in u^{\max(l(z),l(y))} \mathbb{Z}[u] \) (see Propositions 1.2 and 1.9). We can compute this sum in two different ways and we get the same result. Thus we have

\[
\sum_{z \in \mathbf{I}_*} \left( \sum_{y \in W} X^z_y \right) = \sum_{y \in W} \left( \sum_{z \in \mathbf{I}_*} X^z_y \right).
\]

By 1.3(b), the right-hand side is equal to \( \sum_{y \in W} u^{2l(y)} = \mathbf{P}(u^2) \). By 1.4(c), the left-hand side is equal to

\[
\sum_{z \in \mathbf{I}_*} \mathbf{P}_s(u)u^{l(z)} \left( \frac{u-1}{u+1} \right)^{\phi(z)} = \mathbf{P}_s(u)\mathcal{R}(u).
\]

Thus we have

\[
\mathbf{P}(u^2) = \mathbf{P}_s(u)\mathcal{R}(u).
\]

Theorem 0.3 is proved.

2. The case of Weyl groups

2.1. In this section we assume that \( W \) is the Weyl group of a connected adjoint simple algebraic group \( \mathbf{G} \) defined and split over the finite field \( \mathbf{F}_p \) with \( p \) elements (\( p \) is a prime number); we identify \( \mathbf{G} \) with \( \mathbf{G}(\mathbf{k}) \) where \( \mathbf{k} \) is an algebraic closure of \( \mathbf{F}_p \). Let \( F : \mathbf{G} \to \mathbf{G} \) be the “Frobenius map”; it is an abstract group isomorphism whose fixed point set is the group of \( \mathbf{F}_p \)-rational points of \( \mathbf{G} \). Let \( \mathcal{B} \) be the variety of Borel subgroups of \( \mathbf{G} \). The set of \( \mathbf{G} \)-orbits on \( \mathcal{B} \times \mathcal{B} \) (for the simultaneous conjugation action) are naturally indexed by \( W \); we denote by \( \mathcal{O}_w \) the \( \mathbf{G} \)-orbit indexed by \( w \in W \). The length function \( l : W \to \mathbf{N} \) is such that for any \( w \in W \) and any \( C \in \mathcal{B} \), the algebraic variety \( \{ B \in \mathcal{B}; (C,B) \in \mathcal{O}_w \} \) is an affine space over \( \mathbf{k} \) of dimension \( l(w) \). The subset \( S \) of \( W \) is then given by \( \{ s \in W; l(w) = 1 \} \). If \( B \in \mathcal{B} \), then \( F(B) \in \mathcal{B} \); moreover, \( B \mapsto F(B) \) is a bijection \( F : \mathcal{B} \to \mathcal{B} \).

For a finite set \( X \) we denote by \( |X| \) the cardinal of \( X \). If \( X \subseteq X' \) are sets and \( f : X' \to X' \) is a map such that \( f(X) \subseteq X \) we set \( X^f = \{ x \in X; f(x) = x \} \).

2.2. Let \( s \in S \). If \( B, B' \in \mathcal{B} \) we write \( B \sim_s B' \) if \( (B,B') \in \mathcal{O}_1 \cup \mathcal{O}_s \). This is an equivalence relation on \( \mathcal{B} \). A subgroup \( P \) of \( \mathbf{G} \) is said to be parabolic of type \( s \) if it is the union of all \( B \) in a fixed equivalence class for \( \sim_s \). Let \( \mathcal{P}_s \) be the set of parabolic subgroups of type \( s \) of \( \mathbf{G} \). For \( P \in \mathcal{P}_s \) we set \( \mathcal{B}_P = \{ B \in \mathcal{B}; B \subset P \} \); this is a projective line over \( \mathbf{k} \).

If \( P \in \mathcal{P}_s \) and \( B \in \mathcal{B} \), then there is a unique element \( y = \text{pos}(B,P) \in W \) such that \( y < y_s \) and \( (B,B') \in \mathcal{O}_y \cup \mathcal{O}_{ys} \) for any \( B' \in \mathcal{B}_P \); moreover, we have \( (B,B') \in \mathcal{O}_y \) for a unique \( B' \in \mathcal{B}_P \), denoted by \( P^B \).

2.3. Let \( s, s' \in S \) and let \( P \in \mathcal{P}_s, P' \in \mathcal{P}_{s'} \). There is a unique element \( y = \text{pos}(P, P') \in W \) such that \( y < sy, y < ys' \) and \( (B,B') \in \mathcal{O}_y \cup \mathcal{O}_{sy} \cup \mathcal{O}_{ys'} \cup \mathcal{O}_{ys} \) for any \( B \in \mathcal{B}_P, B' \in \mathcal{B}_{P'} \). Assuming in addition that \( sy = ys' \), the map \( \mathcal{B}_P \to \mathcal{B}_{P'} \) given by \( B \mapsto P^B \) is an isomorphism of projective lines with inverse \( B' \mapsto P^{B'} \).
2.4. We shall fix $C \in \mathcal{B}^F$. We assume that $\ast$ has the following property: we can find an involutive automorphism of algebraic groups $\iota : G \rightarrow G$ such that:

- $\iota(C) = C$;
- if $(B, B') \in \mathcal{O}_w$, then $(\iota(B), \iota(B')) \in \mathcal{O}_{w^*}$.

Note that $\iota$ defines an involution $\mathcal{B} \rightarrow \mathcal{B}$ of algebraic varieties (denoted again by $\iota$). It commutes with $F : \mathcal{B} \rightarrow \mathcal{B}$.

2.5. For $(z, y) \in I_s \times W$ we set

$$\mathcal{X}_y^z = \{ B \in \mathcal{B}^F^2 ; (B, F(\iota(B))) \in \mathcal{O}_z, (C, B) \in \mathcal{O}_y \}.$$ 

This is a finite set.

**Proposition 2.6.** Let $(z, y) \in I_s \times W$, $s \in S$. Assume that $sz < z$. We have

(i) $(p + 1) |X_y^z| = -p |X_y^z| + |X_y^z|$, if $sz = zs^*$, $y < ys$;

(ii) $(p + 1) |X_y^z| = p^2 |X_y^z| + (p^2 - p - 1) |X_y^z|$, if $sz = zs^*$, $y > ys$;

(iii) $|X_y^z| = |X_{ys}^{zs}|$ if $sz \neq zs^*$, $y < ys$;

(iv) $|X_y^z| = p^2 |X_{ys}^{zs}| + (p^2 - 1) |X_y^{zs^*}|$, if $sz \neq zs^*$, $y > ys$.

In the setup of (iii), (iv) we have $l(z) = l(s) + l(szs^*) + l(s^*)$; hence for any $B \in \mathcal{X}_y^z$ we have $(B, \beta_B) \in \mathcal{O}_s$, $(\beta_B, B) \in \mathcal{O}_{szs^*}$, $(\beta_B', F(\iota(B))) \in \mathcal{O}_s$, for uniquely defined $\beta_B, \beta_B'$ in $\mathcal{B}$. It follows that $(B, F(\iota(\beta_B))) \in \mathcal{O}_s$, $(F(\iota(\beta_B')), F(\iota(B))) \in \mathcal{O}_{szs^*}$, $(F(\iota(B)), B) \in \mathcal{O}_s$, and from the uniqueness we see that $\beta_B' = F(\iota(\beta_B))$, $\beta_B = F(\iota(\beta_B'))$; hence $F^2(\beta_B) = \beta_B$.

Assume that we are in the setup of (iii). We have a bijection $\mathcal{X}_y^z \rightarrow \mathcal{X}_{ys}^{zs} \ast$ given by $B \mapsto \beta_B$. Hence (iii) holds.

Assume that we are in the setup of (iv). We have a map $\mathcal{X}_y^z \rightarrow \mathcal{X}_{ys}^{zs} \cup \mathcal{X}_{ys}^z$ given by $B \mapsto \beta_B$. Its fibre over a point in $\mathcal{X}_{ys}^{zs}$ has cardinal $p^2 - 1$ while its fibre over a point in $\mathcal{X}_{ys}^{zs}$ has cardinal $p^2$. We deduce that (iv) holds.

In the rest of the proof we assume that $sz = zs^*$. We set $y' = y$ if $y < ys$ and $y' = ys$ if $ys < y$. Let

$$Y = \{ P \in \mathcal{P}_s ; F^2(P) = P, \text{pos}(P, F(\iota(P))) = sz, \text{pos}(C, P) = y' \}.$$ 

For $P \in Y$ we have $P^C \in \mathcal{B}_P$, $(F(P))^C = F(P^C) \in \mathcal{B}_P^F(P)$.

We define a map $\mathcal{X}_y^z \cup \mathcal{X}_{ys}^{zs} \cup \mathcal{X}_{ys}^z \rightarrow Y$ by $B \mapsto P$ where $B \in P \in \mathcal{P}_s$.

The fibre of this map over $P \in Y$ is $\{ \beta \in \mathcal{B}^F ; B \subset P \}$, hence it has cardinal $p^2 + 1$.

We define a bijection $\mathcal{X}_y^z \cup \mathcal{X}_{ys}^{zs} \rightarrow Y$ by $B \mapsto P$ where $B \in P \in \mathcal{P}_s$.

We define a map $\mathcal{X}_{ys}^{zs} \cup \mathcal{X}_{ys}^z \rightarrow Y$ by $B \mapsto P$ where $B \subset P \in \mathcal{P}_s$. The fibre of this map over $P \in Y$ is $\{ \beta \in \mathcal{B}^F ; \beta \subset C, (\beta, F(\beta)) \in \mathcal{O}_{sz} \}$ which can be viewed as a set of $F_p$-rational points of the projective line $\mathcal{B}_P$ with Frobenius map $\beta \mapsto F(\beta)$, hence it has cardinal $p + 1$.

We see that:

$$|\mathcal{X}_y^z| + |\mathcal{X}_{ys}^{zs}| + |\mathcal{X}_{ys}^z| + |\mathcal{X}_{ys}^z| = (p^2 + 1)|Y|,$$

$$|\mathcal{X}_y^z| + |\mathcal{X}_{ys}^{zs}| = |Y|,$$

$$|\mathcal{X}_{ys}^{zs}| + |\mathcal{X}_{ys}^z| = (p + 1)|Y|.$$ 

Thus we have:

$$|\mathcal{X}_y^z| + |\mathcal{X}_{ys}^{zs}| + |\mathcal{X}_{ys}^z| + |\mathcal{X}_{ys}^z| = (p^2 + 1)(|\mathcal{X}_y^z| + |\mathcal{X}_{ys}^{zs}|),$$

$$|\mathcal{X}_{ys}^{zs}| + |\mathcal{X}_{ys}^z| = (p + 1)(|\mathcal{X}_y^z| + |\mathcal{X}_{ys}^{zs}|).$$ 

This implies that (i),(ii) hold.
Proposition 2.7. Let \((z, y) \in I_s \times W, s \in S\). Assume that \(ys < y\). We have
(i) \(X^z_y = pX^z_{ys} + (p + 1)X^{ys}_z\) if \(sz = zs^* > z\);
(ii) \(X^z_y = (p^2 - p - 1)X^z_{ys} + (p^2 - p)X^{ys}_z\) if \(sz = zs^* < z\);
(iii) \(X^z_y = X^{zs*}_y\) if \(sz \neq zs^* > z\);
(iv) \(X^z_y = (p^2 - 1)X^z_{ys} + p^2X^{zs*}_{ys}\) if \(sz \neq zs^* < z\).

The proposition is deduced from Proposition 2.6 in the same way as Proposition 1.4 was deduced from 1.2 (or rather in reverse).

Proposition 2.8. For any \(z \in I_s\) we have \(|X^z_1| = \delta_{z,1}|_i\).

We have
\[
X^z_i = \{B \in B^{F^2}; (B, F(\iota(B))) \in O_z, C = B\}.
\]
Since \(C = F(\iota(C))\) we have \((C, F(\iota(C))) \in O_1\); hence \(X^z_i\) is a single point \(C\) if \(z = 1\) and is empty if \(z \neq 1\). The proposition is proved.

Proposition 2.9. For any \(y \in W\) we have \(|X^1_y| = \delta_{y,y^*}l(y)|\).

We have
\[
X^1_y = \{B \in B^{F^2}; B = F(\iota(B)), (C, B) \in O_y\}
\]
\[
= \{B \in B; B = F(\iota(B)), (C, B) \in O_y\}.
\]
If \(B \in X^1_y\), then \((F(\iota(C)), F(\iota(B))) \in O_y^r\), hence \((C, B) \in O_y^r\) so that \(y = y^*\). Thus, if \(X^1_y \neq \emptyset\), then \(y = y^*\). Assume now that \(y = y^*\). Consider the \(l(y)\)-dimensional affine space \(B \in B; (C, B) \in O_y\) over \(k\). Since \(y = y^*\), this affine space is stable under \(B \mapsto F(\iota(B))\) which is the Frobenius map for an \(F^{r^n}\)-rational structure whose fixed point set is exactly \(X^1_y\). It follows that \(|X^1_y| = p^{l(y)}|\). The proposition is proved.

Proposition 2.10. For any \((z, y) \in I_s \times W\) we have \(|X^z_y| = X^z_y|_{u=p}\).

By Propositions 1.2 and 2.8, the two sides of the equality in the proposition satisfy the same inductive formulas; they also satisfy the same initial conditions \(|X^z_i| = X^z_i|_{u=p} = \delta_{z,1}|_i\) (see 2.8 and 1.2). Hence they are equal. The proposition is proved.

Proposition 2.11. For any \(y \in W\) we have \(X^1_y = \delta_{y,y^*}u^{l(y)}|_u\).

Combining Propositions 2.9 and 2.10 we see that \(X^1_y|_{u=p} = \delta_{y,y^*}u^{l(y)}|_{u=p}\). Since two polynomials in \(u\) which take equal values at any prime number are equal, we deduce that \(X^1_y = \delta_{y,y^*}u^{l(y)}|_u\). The proposition is proved.

Note that this proof gives an alternative approach to Proposition 1.8 in our case.

References


