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Many-body heat radiation and heat transfer in the presence of a nonabsorbing background medium

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Heat radiation and near-field radiative heat transfer can be strongly manipulated by adjusting geometrical shapes, optical properties, or the relative positions of the objects involved. Typically, these objects are considered as embedded in vacuum. By applying the methods of fluctuational electrodynamics, we derive general closed-form expressions for heat radiation and heat transfer in a system of \( N \) arbitrary objects embedded in a passive nonabsorbing background medium. Taking into account the principle of reciprocity, we explicitly prove the symmetry and positivity of transfer in any such system. Regarding applications, we find that the heat radiation of a sphere as well as the heat transfer between two parallel plates is strongly enhanced by the presence of a background medium. Regarding near- and far-field transfer through a gas like air, we show that a microscopic model (based on gas particles) and a macroscopic model (using a dielectric contrast) yield identical results. We also compare the radiative transfer through a medium like air and the energy transfer found from kinetic gas theory.

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I. INTRODUCTION

The derivation of Planck's law of thermal radiation more than a century ago set off a thorough and still ongoing development in the field of heat radiation and radiative heat transfer between objects in thermal nonequilibrium [1]. Over the years, and following the seminal works by Rytov [2,3], considerable physical insight has been gained after conceptually relating these phenomena to the presence of charge and current fluctuations inside the objects, or to the fluctuating electromagnetic field in such systems. The Rytov theory, based on the fluctuation-dissipation theorem, relates the electromagnetic field radiated by an object at a given temperature to its sources, i.e., to the fluctuating electric currents inside it. It allows us to have a clear physical intuition of several phenomena, such as the Casimir-Lifshitz dispersion forces [4–6] occurring between any polarizable objects. These forces, in the Rytov’s spirit, have been recently extended to nonequilibrium systems [7–19], for which lately experimental tests have been done using trapped atomic Bose-Einstein condensates [20,21], and have been proposed in other systems [22]. Another important outcome of the Rytov theory is the development of a more general framework for heat radiation and for near-field heat transfer [11,23].

While Planck’s theory of black-body radiation offers a precise theoretical description for the heat transfer on macroscopic length scales, it was found by Polder and Van Hove in the early 1970s that the situation on the submicron or even nanoscale can be fundamentally different [24]. The two theoreticians were prompted to reinvestigate the theory of radiative heat transfer between closely spaced bodies after being confronted with new experimental measurement results for the heat transfer between two chromium layers at the time [25]. In their famous work, they presented a general formalism based on the fluctuation-dissipation theorem for the heat transfer between macroscopic planar bodies of arbitrary dielectric properties. Their ground-breaking results for the heat transfer across a vacuum gap revealed a strong increase of many orders of magnitude for diminishing the gap width due to evanescent wave contribution in good agreement to experimental data.

In recent years, the field of heat radiation and heat transfer has regained considerable interest due to significant progress on the theoretical side, including improved general formalisms [10,11,13,14,16,17,19,23,26,27] and new powerful numerical methods [28–30] also for systems with temperature gradients [31]. The common concept underlying all these works is the fluctuation-dissipation theorem [32], which is used to describe the correlations of the fluctuating electromagnetic field of the radiating bodies in thermal nonequilibrium. Despite the generality of the available theories, a closed-form expression for the heat transfer between three objects in vacuum has only recently been given [17], and has already provided several interesting applications to heat transfer amplification [33] and guiding [34]. Aside from the advances in general formalism, various applications have been investigated in recent nonequilibrium studies [35–43]. On the experimental side, the development of high-precision measurement devices has verified many theoretical predictions concerning radiative heat transfer [44–49]. The mentioned theories share the similarity that they are restricted to the case of objects being embedded in a vacuum environment.

In this work, we revisit and extend the scattering formalism for radiative heat transfer between objects in vacuum that was presented in Ref. [14] to be applicable to an arrangement of \( N \) objects embedded in a passive nonabsorbing background medium (e.g., a fluid or gas). Specifically, we find compact trace formulas for the radiation in \( N \)-body systems, which are valid in either vacuum, or in a passive background medium.
We show that the symmetry and positivity of heat transfer, known for two bodies [14,30], is also valid for three or more bodies, and in the presence of the passive medium. We give several examples, including the radiation of a sphere in a background medium, as well as the near-field heat transfer between two parallel plates separated by a gap filled with a passive medium. In both cases, the background medium can drastically change the energy transfer. We also analyze the transfer between parallel plates in the presence of a dilute gas, both directly (by considering scattering from individual gas particles) as well as effectively (by assigning a dielectric function to the gas). The two approaches share a common limit of dilution.

II. DISCUSSION OF THE SETUP AND EXPERIMENTAL RELEVANCE

The setup under study consists of \( N \) objects labeled by \( \alpha = 1 \ldots N \) at time-independent, homogeneous temperatures \( \{ T_0 \} \) in a passive nonabsorbing background medium as schematically illustrated in Fig. 1. The requirement of vanishing absorption of the background medium results in appreciable technical simplifications regarding the following theoretical derivations. In practice, we expect the derived predictions to be valid as long as absorption in the background medium is negligible. We thus neglect absorption and emission of the medium [50]. Technically, we treat the background medium as an enclosing passive body occupying the infinite space complementary to the arrangement of objects. In this respect, the medium can, in principle, be any kind of weakly absorbing liquid or gas. Our general considerations are also valid if the background medium is inhomogeneous, as, e.g., in density gradients or through adsorption near object surfaces. For specific examples, we take it to be homogeneous, isotropic, and local. Despite being nonabsorbing, the background medium contains the famous environment dust [32], so that eventually, all radiation is absorbed by it at far distances. Due to this, as far as radiation is concerned, only the temperature of environment far away from all objects is relevant (denoted \( T_{\text{env}} \) and assumed homogeneous). This is important, as the liquid or gas may acquire spatially dependent temperatures near the objects, which, again, are irrelevant for the electromagnetic field radiation.

In an experimental realization, the medium as well as the ensemble of \( N \) objects may be considered as inside surrounding material walls, which are kept at fixed temperature \( T_{\text{env}} \) [8,11,51]. The surrounding walls should be far away from the \( N \)-body system so that only far-field waves emitted by the walls impinge on the \( N \)-body system. The walls should also be nonregular (black) to produce an isotropic radiation at the \( N \)-body location, and to allow us to neglect the backscattered radiation generated by the \( N \)-body system impinging on the walls. To avoid such multiple reflections, a tiny absorption in the background medium is advantageous. Such requirements allow us to treat the radiation produced by the surrounding walls as an isotropic black-body radiation, independent of the dielectric permittivity of the walls, with temperature \( T_{\text{env}} \). The mathematical description of such a setup is identical to the one using the environment dust [32]. As the temperature of the background medium may spatially vary in the considered setup, it is worth stressing that \( T_{\text{env}} \) is not equal to its temperature in the vicinity of the objects. Since it is nonabsorbing, its temperature in the vicinity of the objects is irrelevant.

It is worth commenting here that the energy transport via electromagnetic fields is in competition with the energy flux evoked by thermal conduction of the medium. We suspect metamaterials to be suitable candidates to reduce the latter contribution to the total net energy flux, as these kinds of structures can be successfully thermally insulated [52,53]. In particular, we think of structural defects in the composition of the medium, e.g., tiny vacuum gaps milled into a slab made of metamaterial, to inhibit the propagation of phonons through the medium and thus promoting the idea of thermal isolation.

III. FLUCTUATIONAL ELECTRODYNAMICS IN A PASSIVE BACKGROUND MEDIUM

In this section, we generalize the formalism introduced in Refs. [14,23] to be applicable to an arrangement of objects that is surrounded by a passive nonabsorbing background medium. In the given nonequilibrium situation, each object is assumed to be at local equilibrium, such that the spontaneous charge and current fluctuations within the individual object satisfy the fluctuation-dissipation theorem at the appropriate temperature (see Appendix A for details). The system under study is characterized in terms of its macroscopic material properties, i.e., through its electric and magnetic response \( \varepsilon(\omega; \mathbf{r}, \mathbf{r}’) \) and \( \mu(\omega; \mathbf{r}, \mathbf{r}’) \), which can in general be nonlocal complex tensors, \( \varepsilon(\omega; \mathbf{r}, \mathbf{r}’) \equiv \varepsilon_1(\omega; \mathbf{r}, \mathbf{r}’) \). Operator products generally include a matrix multiplication in \( 3 \times 3 \) space, as well as an integral in \( \mathbb{R}^3 \) of the common spatial argument. The response functions constitute the potential operator of the system \( \nabla \equiv \frac{\mu}{\varepsilon}(\varepsilon - \mathbb{I}) + \nabla \times (\mathbb{I} - \frac{1}{\mu})\nabla \times \) as defined in the
In this equation, we separated the zero-point term $\alpha$ objects. For object assumption, together with symmetries of microreversibility [32,55,59,60], the potential operator is nonzero everywhere in space and we need to make a distinction of cases

$$V(\mathbf{r}, \mathbf{r}') = \begin{cases} V_\alpha(\mathbf{r}, \mathbf{r}'), & \mathbf{r}, \mathbf{r}' \in V_\alpha, \\ V_b(\mathbf{r}, \mathbf{r}'), & \mathbf{r}, \mathbf{r}' \notin \{V_\alpha\}. \end{cases}$$  

Here, we introduced the potential $V_\alpha(\mathbf{r}, \mathbf{r}')$ of individual object $\alpha$, which is confined to its volume $V_\alpha$ and associated with the response functions $\varepsilon_b(\omega; \mathbf{r}, \mathbf{r}')$ and $\mu_b(\omega; \mathbf{r}, \mathbf{r}')$. For simplicity, we restrict to the case where the potential $V$ does not connect points in the background medium with those in the objects, to allow for sharp boundaries between medium and objects. The background medium, described by the potential $V_b$ and electric and magnetic responses $\varepsilon_b(\omega; \mathbf{r}, \mathbf{r}')$ and $\mu_b(\omega; \mathbf{r}, \mathbf{r}')$, respectively, is assumed nonabsorbing. This assumption, together with symmetries of microreversibility [32] make its electric and magnetic response Hermitian [57]

$$\varepsilon_b = \varepsilon_b^\dagger, \quad \mu_b = \mu_b^\dagger.$$  

Furthermore, we introduce $G_b$, which is the Green’s function in Eq. (1), for $\nabla = V_b$. It takes the role which the free Green’s function has for objects placed in vacuum. For mathematical reasons, we define the potential $V_b$ also inside $\{V_\alpha\}$, where it can take any (nonabsorbing) form in order for the following theoretical steps to be valid. In practice, one can choose a convenient form of $V_b$ inside $\{V_\alpha\}$. As done in the examples provided below, for the case where the background is homogeneous and local, we will naturally choose that the potential takes the same value inside the medium as outside.

Making use of the identities $\text{Im}[\mathbb{G}] = -\mathbb{G}\text{Im}[\mathbb{G}^{-1}]\mathbb{G}^\dagger$ and $\text{Im}[\nabla - V_b] = -\text{Im}[\nabla - V_b]$, which can be directly found from Eq. (1), we can rewrite the equilibrium correlator given in Appendix A, Eq. (A5), according to its originating thermal sources [14,23]

$$C_{\text{eq}} = (\mathbf{E} \otimes \mathbf{E}^\dagger)^0 + \sum_a C_\alpha(T) + C_{\text{env}}(T).$$  

In this equation, we separated the zero-point term $(\mathbf{E} \otimes \mathbf{E}^\dagger)^0 = a_0 \text{Im}[\mathbb{G}]$ and the contributions of object $\alpha$ and the environment are identified, respectively, with

$$C_\alpha(T) = a(T)\mathbb{G}\text{Im}[\Delta V_\alpha]\mathbb{G}^\dagger,$$  

$$C_{\text{env}}(T) = -a(T)\mathbb{G}\text{Im}[\mathbb{G}^{-1}]\mathbb{G}^\dagger.$$  

The amplitude factors $a(T)$ and $a_0$ are given in Appendix A, Eqs. (A6) and (A7). We introduced the potential difference $\Delta V = V_b - V_b = \sum_a \Delta V_\alpha$, which is only nonzero inside the objects. For object $\alpha$, it reads as specifically

$$\Delta V_\alpha(\mathbf{r}, \mathbf{r}') = \begin{cases} V_\alpha(\mathbf{r}, \mathbf{r}') - V_b(\mathbf{r}, \mathbf{r}'), & \mathbf{r}, \mathbf{r}' \in V_\alpha, \\ 0, & \text{else}. \end{cases}$$  

Having identified the different sources of radiation in Eq. (4), we use the key assumption of local equilibrium in fluctuational electrodynamics, such that we can change the temperatures of the different sources independently to arrive at the field correlator in the nonequilibrium situation [14,23]

$$C_{\text{eq}}((T_a), T_{\text{env}}) = C_{\text{eq}}(T_{\text{env}}) + \sum_a [C_\alpha(T_a) - C_\alpha(T_{\text{env}})].$$

In this step, we eliminated the environment contribution in Eq. (4) by introducing the equilibrium correlator $C_{\text{eq}}$. It is again important to note that the environment temperature $T_{\text{env}}$ is measured far away from all objects. In other words, the value of the medium temperature near the objects is irrelevant. This is seen most explicitly in Eq. (6), which denotes the radiation from sources in the environment: due to the infinitesimal character of $\text{Im}[\mathbb{G}^{-1}]$ (the “dust”), only the (large) regions far away contribute.

While the fluctuating electromagnetic field is subject to the laws of quantum mechanics [compare the appearance of the reduced Planck constant $\hbar$ in Eq. (A5)], its scattering at the objects obeys classical scattering theory. The scattering by arbitrarily shaped particles is described by the $T$-matrix approach introduced in detail in Ref. [55]. For the objects in the background medium, the $T$ operator is defined slightly differently compared to the vacuum case. We derive it by starting from the Lippmann-Schwinger equation [58]

$$\mathbf{E}_b = \mathbf{E}_b + G_b \Delta V \mathbf{E}_b^\text{sc}.$$  

Equation (9) expresses the general solution $\mathbf{E}_b$ of the Helmholtz equation with the objects present in the background medium [55]

$$\left[\mathbb{H}_0 - V - \frac{\omega^2}{c^2}\right] \mathbf{E}_b = 0.$$  

Note that $\mathbf{E}_b$ is the solution to the Helmholtz equation in the absence of any object, i.e., where $\nabla = V_b$. Starting from Eq. (9), we can iteratively substitute for $\mathbf{E}_b^\text{sc}$ to obtain the formal expression

$$\mathbf{E}_b = \mathbf{E}_b + G_b \Delta V \mathbf{E}_b + G_b \Delta V G_b \Delta V \mathbf{E}_b^\text{sc} + \cdots$$

$$= \mathbf{E}_b + G_b \Delta V \mathbf{E}_b.$$  

Solving Eqs. (9) and (11) for $T$, we find for the $T$ operator of objects in the background medium

$$T = \Delta V \frac{1}{1 - G_b \Delta V}.$$  

It is related to the Green’s function $\mathbb{G}$ via

$$\mathbb{G} = G_b + G_b T G_b.$$  

Since the Green’s function, based on the principle of microscopic reversibility, is generally symmetric [32,55,59,60], i.e., $G_{ik}(\mathbf{r}, \mathbf{r}') = G_{ki}(\mathbf{r}', \mathbf{r})$, the $T$ operator enjoys the same symmetry. Furthermore, we point out that $T$ is the scattering operator of the entire collection of objects in the background medium, whereas we shall use $T_a$ for object $\alpha$ in isolation. In Table I, we provide the equivalent quantities of a system in vacuum to those in a nonabsorbing background medium to easily convert any formula between these two. In particular, Eqs. (12) and (13) approach the known vacuum expressions for $V_b = 0$. 

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TABLE I. Overview of equivalent quantities for a system in vacuum as presented in Ref. [14] and in a nonabsorbing background medium as discussed here.

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<th>Quantities and equivalences</th>
<th>Vacuum</th>
<th>Medium</th>
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<td>$G_0$</td>
<td>$\mathcal{V}$</td>
<td>$T = \frac{1}{\mathcal{V} - G_0\mathcal{V}}$</td>
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<tr>
<td>$G_b$</td>
<td>$\Delta \mathcal{V}$</td>
<td>$T = \Delta \mathcal{V} \frac{1}{\mathcal{V} - G_b \Delta \mathcal{V}}$</td>
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IV. THREE OR MORE OBJECTS: TOTAL HEAT ABSORPTION IN A MEDIUM

Application of the formalism presented in the previous section allows us to specify the energy fluxes in a many-body system of $N$ arbitrary objects in a passive nonabsorbing background medium. The many-body heat radiation and heat transfer in a medium will constitute the main results of this paper. One measurable quantity in the system is the total heat $H^{(β)}$ absorbed by object $β$ in the presence of all other objects. Using the representation of the nonequilibrium field correlator in Eq. (8), we can write [14]

$$H^{(β)}([T_α], T_{\text{env}}) = \sum_α (H^{(α)}_α(T_α) - H^{(β)}_α(T_{\text{env}})).$$ (14)

Note that $H^{(β)}$ (the heat transfer rate) is the component of radiation emitted by object $α$ and absorbed by another object $β$ in a many-body system. On the other hand, $H^{(β)}$ (the "self"-emission) describes the heat emitted by object $β$ in the presence of all other objects. The subtractive term in Eq. (14) implicitly reflects the principle of detailed balance: at global thermal equilibrium, i.e., $[T_α] = T_{\text{env}}$, all radiative fluxes cancel each other. In order to evaluate the total heat absorbed by object $β$, we require general expressions for these quantities which will be given in the following subsections.

A. Heat transfer rate $H^{(β)}_α$

We first derive the generalized heat transfer rate $H^{(β)}_α$ for $α \neq β$ in a many-body system of $N$ objects. The situation is schematically illustrated in Fig. 2. In the following, we attach the single-scattering operator $T_α$ to object $α$ and $T_β$ to object $β$. Moreover, $T_{\text{env}}$ will denote the composite $T$ operator of the residual $N - 2$ objects. (In a three-body system, it is the operator of the third object.) Using multiple scatterings (see Appendix B), we place the $N - 2$ objects described by $T_{\text{env}}$ into the field $\tilde{E}^{\text{iso}}_{b,α}$ radiated by the isolated object $α$:

$$\tilde{E}^{\text{sc}}_{b,α,α} = (1 + G_b T_{\text{env}}) \frac{1}{1 - G_b T_α G_b T_{\text{env}}} \tilde{E}^{\text{iso}}_{b,α}.$$ (15)

Finally, we add object $β$ into this field and obtain the full scattering solution

$$E^{\text{sc}}_{b,α} = (1 + G_b T_β) \frac{1}{1 - G_b T_α G_b T_β} \tilde{E}^{\text{sc}}_{b,α,α}. $$ (16)

In this equation, $T_{\text{env}}$ is the composite $T$ operator of all objects except object $β$. Insertion of Eq. (15) into (16) yields an explicit form of the multiple-scattering operator $O_α$,

$$E^{\text{sc}}_{b,α} = O_α E^{\text{iso}}_{b,α}. $$ (17)

$$O_α = (1 + G_b T_β) \frac{1}{1 - G_b T_α G_b T_β} \times \left((1 + G_b T_{\text{env}}) \frac{1}{1 - G_b T_α G_b T_{\text{env}}} \right). $$ (18)

This is identical (but more explicit) to the representation of $O_α$ in Appendix B, Eq. (B4). Note that the operator does not depend on the specific choice of object $β$ as can be directly seen from Eq. (B4). The representation of the multiple-scattering operator in Eq. (18) can be used to derive a general trace formula for the heat transfer rate $H^{(β)}_α$ in a many-body system. Following the derivation for two objects in vacuum in Ref. [14], we can express $H^{(β)}_α$ in terms of three well-known quantities: the radiation operator $R_α$, the multiple-scattering operator $O_α$, and the free Green’s function $G_0$. The radiation operator describes the emitted field by object $α$ embedded in the background medium and is defined according to

$$R_α = G_0 \operatorname{Im}[T_α] - T_α \operatorname{Im}[G_b] T_α G_b^*. $$ (19)

Explicitly, the heat transfer rate is given by an integral over the volume of object $β$ as

$$H^{(β)}_α = \frac{2\hbar}{\pi} \int_0^{∞} d\omega \frac{\omega}{e^{\hbar \omega/k_B T_α}} - 1 \times \operatorname{Im} \sum_i \int d^3r \left( R_α R_{α,α} R_{α} G_0^{-1} \right)_i (r, r). $$ (20)

This expression directly finds the absorbed energy by integrating over the object’s volume, as the integrand gives the dissipated energy ($G_0^{-1}$ enters from the conversion of electric field to current density [14]). It can also be reverted to a surface integral of the surface normal of the Poynting vector, evaluated

FIG. 2. Illustration of the quantity $H^{(β)}_α$ in a many-body system embedded in a passive nonabsorbing background medium. The heat radiation emitted by object $α$ (red) at temperature $T_α$ is partially absorbed by object $β$ (blue) at temperature $T_β$. The residual objects are marked by a gray color.
in the background medium. Because of the nonabsorptive properties of the background medium, this Poynting vector is well defined.

With regard to Eq. (18), \( O_\beta R_\alpha O_\alpha^{\dagger} G_0^{\dagger} G_0^{\dagger} (r, r) \) will carry the operator product \( G_b^\dagger G_0^{\dagger} G_0^{\dagger} \) on its rightmost position. Using the identity \( G_0^{\dagger} = G_0^{\dagger} + V_b \), which can be directly found from Eq. (1), we can rewrite this product according to

\[
G_b^\dagger G_0^{\dagger} G_0^{\dagger} = 1 + G_b^\dagger V_b. \tag{21}
\]

As a result, we can split up the operator to be integrated over the volume of object \( \beta \) into a sum of three terms

\[
O_\alpha R_\alpha O_\alpha^{\dagger} G_0^{\dagger} G_0^{\dagger} = O_\alpha R_\alpha O_\alpha^{\dagger} + O_\alpha R_\alpha O_\alpha^{\dagger} + O_\alpha R_\alpha O_\alpha^{\dagger} V_b,
\]

where we defined

\[
O_\alpha R_\alpha O_\alpha^{\dagger} = \frac{1}{1 - T_\alpha^\beta G_b^\dagger G_\alpha G_b}; \tag{22}
\]

\[
O_\alpha R_\alpha O_\alpha^{\dagger} = \frac{1}{1 - T_\alpha^\beta G_b^\dagger G_\alpha G_b}; \tag{23}
\]

\[
O_\alpha R_\alpha O_\alpha^{\dagger} = \frac{1}{1 - T_\alpha^\beta G_b^\dagger G_\alpha G_b}. \tag{24}
\]

The two parts differ precisely by the operator in the rightmost position:

\[
O_\alpha R_\alpha O_\alpha^{\dagger} = \ldots T_\beta' \tag{25}
\]

\[
O_\alpha R_\alpha O_\alpha^{\dagger} = \ldots T_\gamma' \quad (\gamma \neq \beta). \tag{26}
\]

Note that \( T_\gamma \) can be the scattering operator of any object except for object \( \beta \). As a consequence, \( O_\beta R_\alpha O_\alpha^{\dagger} G_0^{\dagger} G_0^{\dagger} (r, r) \) is identically zero if \( r \) is located inside object \( \beta \) and does not contribute to the volume integral in Eq. (20). This is because \( T_\gamma (r, r) \) can only be nonzero if both arguments are located within the volume \( V_\gamma \). Furthermore, in Appendix C, we show that the third term in Eq. (22) does not contribute either for the case of a nonabsorbing background medium. Thus, the heat transfer rate \( H_\alpha^{(\beta)} \) in Eq. (20) is given in terms of \( O_\beta R_\alpha O_\alpha^{\dagger} G_b^\dagger G_b \) only. Because this term is only nonzero if \( r \) is located inside \( V_\beta \), we can extend the integration range over all space to obtain a trace

\[
H_\alpha^{(\beta)} (T) = \frac{2\hbar}{\pi} \int_0^\infty d\omega \frac{\omega}{e^{\hbar \omega/T} - 1} \operatorname{Im} \operatorname{Tr} [O_\beta R_\alpha O_\alpha^{\dagger} G_b^\dagger G_b]. \tag{27}
\]

This trace is now understood over spatial coordinate \( r \) as well as over matrix indices \( i \). Using the cyclic properties of the trace, we give the final representation of the heat transfer rate \( H_\alpha^{(\beta)} \) in explicit form

\[
H_\alpha^{(\beta)} = \frac{2\hbar}{\pi} \int_0^\infty d\omega \frac{\omega}{e^{\hbar \omega/T} - 1} \operatorname{Im} \operatorname{Tr} \left\{ [T_\beta G_b^\dagger G_b T_\beta] - [T_\alpha G_b^\dagger G_b T_\alpha] \right\} \frac{1}{1 - G_b^\dagger G_b T_\beta T_\alpha G_b^\dagger G_b T_\alpha} \frac{1}{1 - G_b^\dagger G_b T_\beta G_b T_\alpha} \frac{1}{1 - G_b^\dagger G_b T_\beta T_\alpha G_b^\dagger G_b T_\alpha} \right\}. \tag{28}
\]

B. Positivity and symmetry of transfer in a many-body system

The positivity and symmetry of transfer already known for two objects [14,30] can be generalized to a many-body system by means of Eq. (28). In this section, we provide these properties, while the technical details are given in Appendix D. As naturally expected, the heat transfer rate \( H_\alpha^{(\beta)} \) is a non-negative number

\[
H_\alpha^{(\beta)} \geq 0. \tag{29}
\]

Based on the principle of reciprocity, the function also obeys the symmetry relation

\[
H_\alpha^{(\beta)} (T) = H_{\beta}^{(\alpha)} (T), \tag{30}
\]

stating that the roles of emitter and absorber can be interchanged in the presence of an arbitrary number of passive scatterers. The symmetry and positivity of the heat transfer can be used to show that the net heat flux between a warm object \( \alpha \) and a cold object \( \beta \) is always positive, i.e., that heat is always transferred from the warmer object to the colder one:

\[
H_\alpha^{(\beta)} (T) - H_{\beta}^{(\alpha)} (T) \geq 0 \quad \text{if} \quad T_\alpha \geq T_\beta. \tag{31}
\]

The statement is a direct confirmation of the second law of thermodynamics. Our derivation relies on the assumption that the objects’ potentials \( \{ V_\alpha \} \) do not depend on temperature.

C. “Self”-emission \( H_\beta^{(\beta)} \)

In order to give the total heat absorption of object \( \beta \) in Eq. (14), we also need to specify its heat emission \( H_\beta^{(\beta)} \) in the proximity of the \( N - 1 \) other objects. The situation under consideration is schematically illustrated in Fig. 3. The “self”-emission \( H_\beta^{(\beta)} \) is given by Eq. (20), where we need to integrate the operator \( O_\beta R_\beta O_\beta^{\dagger} G_\beta^{\dagger} G_\beta^{\dagger} (r, r) \) over the volume \( V_\beta \). This time, we split the multiplescattering operator \( \bigotimes_\beta \) appearing on the right-hand side differently, and after repeating the steps before Eq. (27), we
find

\[ H_\beta^{(β)}(T_β) = -\frac{2h}{π} \int_0^∞ dω \frac{ω}{e^{\frac{ω}{kT_β}} - 1} \text{Im} \text{Tr}[ G_β R_β G_β^{-1} ] . \]

(32)

The trace is now taken over the self-part “s” of the operator \( G_β R_β G_β^{-1} \) defined correspondingly to the vacuum

\[ H_\beta^{(β)} = -\frac{2h}{π} \int_0^∞ dω \frac{ω}{e^{\frac{ω}{kT_β}} - 1} \text{Im} \text{Tr} \left\{ (1 + G_β T_β) \frac{1}{1 - G_β T_β G_β T_β} G_β [\text{Im}[T_β] - \text{Im}[G_β T_β] - \text{Im}[G_β] T_β] \right\} . \]

(34)

Note that \(-H_\beta^{(β)}\) is positive if the object emits energy. By introducing the operator

\[ \mathbb{W}_β ≡ G_β^{-1} \frac{1}{1 - G_β T_β G_β T_β} , \]

(35)

we can rewrite the function \( H_\beta^{(β)} \) in a more compact way, where now the Green’s function \( G_β \) appears [61]

\[ H_\beta^{(β)} = -\frac{2h}{π} \int_0^∞ dω \frac{ω}{e^{\frac{ω}{kT_β}} - 1} \text{Tr}[\text{Im}[G_β] \mathbb{W}_β R_β \mathbb{W}_β^†] . \]

(36)

Taking into account that the product of the two positive semidefinite operators \( \text{Im}[G_β] \) and \( \mathbb{W}_β R_β \mathbb{W}_β^† \) is also positive semidefinite, the positivity of the heat radiation in a many-body system including the background medium is found:

\[ -H_\beta^{(β)} ≥ 0 . \]

(37)

As a final remark, we note again that all formulas derived in this section are also applicable to the case where the objects are embedded in vacuum. For the vacuum case, the definition of \( T \) in case [14]. It is the part of \( G_β R_β G_β^{-1} \) (r, r) which is finite inside \( V_β \) (again, the term with \( V_β \) on the rightmost position does not contribute as the background medium is nonabsorbing):

\[ G_β R_β G_β^{-1} = \frac{1}{1 - T_β G_β T_β} \]

(33)

The final representation of the self-emission in a many-body system reads as

\[ \text{the first line of Table I should be applied, and the Green’s function } G_β \text{ should be replaced by the free Green’s function } G_0. \]

V. PARTIAL-WAVE REPRESENTATION

In the previous section, we derived trace formulas for heat radiation and heat transfer in a many-body system in operator notation. These formulas are basis independent and hold for any geometry. They are also valid if the potential of the passive background medium depends on space, as could for example be the case if the objects’ surfaces are covered by a wetting film. In this section, we apply the techniques presented in Refs. [14,56] to expand the formulas in partial-wave bases. Therefore, we assume that the background medium is homogeneous, isotropic, and local to allow for an expansion of the Green’s function \( G_β \) in partial waves. In this way, the traces of operators will turn into sums over matrix elements with respect to partial-wave indices.

A. Partial-wave expansion of the Green’s function and the \( T \) operator in the background medium

For an isotropic, homogeneous, and local background medium, the dielectric permittivity and magnetic permeability tensors reduce to constant scalars, e.g., \( ε_β = ε_b \). In these circumstances, a simple conversion relates the Green’s function in the background medium to the free Green’s function:

\[ G_β \left( \frac{ω}{c}, r - r' \right) = μ_b G_0 \left( \frac{ω}{c}, \sqrt{ε_b μ_b}, r - r' \right) . \]

(38)

For a nonmagnetic background medium, i.e., \( μ_b = 1 \), only the speed of light outside of the objects is redefined, and the expansion of the free Green’s function in eigenfunctions of the Helmholtz equation given in Ref. [14] can be used. For the sake of completeness, we omit this expansion (suppressing the \( ω \) dependence)

\[ G_0(r - r') = \sum_μ \begin{cases} \mathbb{E}^{\text{out}}_μ(r) \otimes \mathbb{E}^{\text{reg}}_{μ}(r') & \text{if } ξ_μ(r) > ξ_μ^*(r'), \\ \mathbb{E}^{\text{reg}}_μ(r) \otimes \mathbb{E}^{\text{out}}_μ(r') & \text{if } ξ_μ(r) < ξ_μ^*(r'). \end{cases} \]

(39)

The free Green’s function is expanded in terms of “regular” and “outgoing” waves propagating through vacuum, which are solutions of the wave equation, differing by their behavior at the coordinate origin. While “regular” waves are nonsingular,
“outgoing” waves are typically singular at the origin and obey outgoing boundary conditions for $\xi_1 \to \infty$. Recall that $\xi_1$ is the “radial” coordinate which gives rise to a distinction of cases, depending on which of the two arguments of the Green’s function has a greater value of $\xi_1$ [56]. In this way, the outgoing waves are always evaluated for the larger argument and do not encounter the divergence at $\xi_1 = 0$. The wave index $\mu$ runs over polarization (electric and magnetic) and indices of vector functions depending on the specific basis. The function $\sigma(\mu)$ is a permutation among these indices, fulfilling $\sigma[\sigma(\mu)] = \mu$ [14].

Furthermore, we define the matrix elements of the $T$ operator by

$$T_{\mu\nu} = i \int d^3r \int d^3r' E_{b,\nu}(r') \mathcal{T}(r,r') E_{b,\mu}^*(r') ,$$

(40)

where we apply a regular wave propagating through the background medium (denoted by the index “b”) to both sides of the $T$ operator. The waves in the homogeneous background medium are related to those in vacuum by

$$E_{b,\mu}(\omega) \mathcal{r} = \sqrt{\mu_b} E_{\mu}(\omega) \mathcal{r} .$$

(41)

As in the vacuum case, the matrix elements $T_{\mu\nu}$ in Eq. (40) obey

$$T_{\mu\nu} = T_{\nu(\mu)\sigma(\mu)}$$

(42)

due to the symmetry of the $T$ operator. Besides, both expansions of $\mathcal{G}_b$ and $\mathcal{T}$, respectively, do not contain complex conjugations of waves, which ensures manifest analyticity of these quantities in the upper complex frequency plane.

### B. Heat transfer and “self”-emission

The general formulas for the heat transfer rate $H^{(b)}_a$ and the self-emission $H^{(b)}_\nu$ in an arrangement of $N$ objects in operator notation are given in Eqs. (28) and (34). Applying the techniques of partial-wave expansion presented in this section to these two equations, we obtain

$$H^{(b)}_a = \frac{2\hbar}{\pi} \int_0^\infty d\omega \frac{\omega}{e^{\omega/T_a} - 1} \text{Re} \sum_{\mu\nu} \int \left[ (\tau^\dagger_{\mu\nu} + \tau^\dagger_{\nu\mu}) = \frac{1}{2} \right. \left. \frac{1}{1 - \nu^\dagger_{\mu\nu} U_{\nu\mu}} \right] \left( 1 + u^\dagger_{\mu\nu} U_{\nu\mu} \right) \left( 1 - u^\dagger_{\mu\nu} U_{\nu\mu} \right) ,$$

(43)

$$H^{(b)}_\nu = \frac{2\hbar}{\pi} \int_0^\infty d\omega \frac{\omega}{e^{\omega/T_b} - 1} \text{Re} \sum_{\mu\nu} \int \left[ (\tau^\dagger_{\mu\nu} + \tau^\dagger_{\nu\mu}) = \frac{1}{2} \right. \left. \frac{1}{1 - \nu^\dagger_{\mu\nu} U_{\nu\mu}} \right] \left( 1 + u^\dagger_{\mu\nu} U_{\nu\mu} \right) \left( 1 - u^\dagger_{\mu\nu} U_{\nu\mu} \right) ,$$

(44)

where we introduced the redefined matrices

$$\tau^\dagger_{\mu\nu} \equiv e^{-i\phi_{\nu}} \tau_{\mu\nu} , \quad \nu^\dagger_{\mu\nu} \equiv U_{\nu\mu} e^{i\phi_{\nu}} .$$

(45)

The phase factors $e^{i\phi_{\nu}} = e^{i\phi_{\nu}(\omega)}$ for evanescent wave contribution vary for each basis individually. The matrix

$$\Pi^\nu_{\mu\nu} = \delta_{\nu\mu} \delta_{\mu\nu}$$

(46)

acts as a projector on propagating waves. In the spherical basis, the phase factors are unity, and, since no evanescent waves exist, the projector $\Pi^\nu_{\mu\nu}$ is the identity so that the above formulas simplify in that case.

$U$ is the translation matrix defined by

$$E_{b,\mu}(r) = \sum_{\nu} U_{\mu\nu} (X_{\nu}) E_{b,\nu}^*(r).$$

(47)

A system is a microscopical gas model which we compare to the macroscopic approach, where a medium with dielectric constant $\varepsilon_b$ is placed between the plates. Finally, we compare the radiative transfer between two plates to the energy transfer found from kinetic gas theory. Throughout this section, the medium is supposed to be nonmagnetic, i.e., $\mu_b = 1$ in all applications.

### A. Heat radiation of a sphere in the presence of a nonabsorbing background medium

#### 1. General formula

The heat radiation of a homogeneous sphere in vacuum was computed in Ref. [62], and analyzed in detail in Ref. [14]. We revisit the heat radiation of a sphere and investigate how it changes due to the presence of a homogeneous background medium of dielectric constant $\varepsilon_b$. The energy emitted by an arbitrary isolated object is obtained by setting $\mathcal{T}_b = 0$ in the general formula for the self-emission in Eq. (44) (where we include a minus sign to get the emitted energy):

$$H = \frac{-2\hbar}{\pi} \int d\omega \frac{\omega}{e^{\omega/T} - 1} \text{Re} \left[ \text{Tr} [\mathcal{T} + \mathcal{T}^\dagger] \right].$$

(48)

Note that the index “pr” upon the trace indicates that all partial-wave indices involved run over propagating modes and therefore all phase factors $e^{i\phi_b}$ turn unity. It is naturally appropriate to write the heat radiation of a sphere in the spherical wave basis, were only propagating modes exist, and
the index “pr” is dropped, 

\[ H_s = -\frac{2\hbar}{\pi} \int_0^\infty d\omega \frac{\omega}{e^{\frac{\omega}{\kappa T}} - 1} \sum_{P,l,m} \left( \text{Re} T_i^P + |T_i^P|^2 \right). \]

(49)

In this equation, the summation is over the polarization \( P \) and the quantum numbers \( l \) and \( m \). The matrix elements of \( T \) for a homogeneous sphere in a nonabsorbing background medium are well known and sometimes referred to as Mie coefficients [63] (see Appendix E for details). They do not depend on \( m \).

2. Gold sphere

We first consider a sphere which is made up of gold using the Drude model [54]

\[ \varepsilon_{Au}(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\omega_r)}, \]

(50)

with \( \omega_p = 9.03 \text{ eV} \) and \( \omega_r = 2.67 \times 10^{-2} \text{ eV} \). In Fig. 4, we show the change of the heat radiation due to the presence of a background medium by comparing it to the vacuum case. The presence of the background medium yields a strong amplification for the heat radiation of the gold sphere. Physically, this observation can be ascribed to a higher density of propagating waves that can traverse the interface between the two media (compare the angle of total internal reflection for planar surfaces with the modified condition \( k_\perp < \frac{\sqrt{2}}{c} \sqrt{\varepsilon_b} \)).

In the limit, where the radius \( R \) is the largest scale and much larger than the thermal wavelength \( \lambda_T = \hbar c / (\sqrt{\varepsilon_b} k_B T) \) (which is roughly 8 \( \mu \text{m} \) at room temperature in vacuum) and the skin depth \( \delta = c / (1 \sqrt{\varepsilon_b} c T) \), the heat radiation of the sphere approaches the classical Stefan-Boltzmann law

\[ \lim_{R \to \infty} H_s = 4\pi R^2 \sigma T^4 \varepsilon(T, \varepsilon_b) = 4\pi R^2 \frac{H_P}{A}, \]

(51)

with \( \sigma = \pi^2 k_B^4 / (60 c^4 \varepsilon^2) \) and emissivity factor \( \varepsilon(T, \varepsilon_b) \). The linear dependence of \( \varepsilon \) in the dielectric constant \( \varepsilon_b \) in the given range in Fig. 4 reflects the increase of propagating modes due to the presence of the background medium for large \( R \). In the last step, we introduced the heat radiation \( H_P \) per surface area \( A \) of a semi-infinite half-space at temperature \( T_p \) in the medium given by

\[ H_P = \frac{\hbar}{2\pi} \int_0^\infty d\omega \frac{\omega}{e^{\frac{\omega}{\kappa T}} - 1} \int_{k_\perp < \frac{\sqrt{2}}{c} \sqrt{\varepsilon_b}} d^2 k_\perp (2\pi)^2 \sum_p (1 - |r_P|^2). \]

(52)

The Fresnel coefficients \( r_P \) for the infinitely thick plate in a medium are given in Appendix F. As a test of the approach presented in this paper, we compared Eq. (52) numerically to the heat transfer between two half-spaces in vacuum given in Eq. (58) below. If, in that formula, the plate separation is set to zero, \( d = 0 \), and the absorbing half-space is assigned the nonabsorbing medium’s dielectric constant \( \varepsilon_b \), the results are numerically identical.

Interestingly, the radiation, normalized to the corresponding vacuum case, depends pronounced on \( R \), and is largest for small \( R \). It means that the radiation of nanoparticles is most strongly affected by the presence of the background medium. For radii even smaller than shown in Fig. 4, the curves saturate and reach plateau values. The local maximum around \( R = 1 \mu \text{m} \) in Fig. 4 shows that, when lowering the contrast \( \varepsilon_{Au}/\varepsilon_b \) between the gold sphere and the background medium, the global maximum of the heat radiation, seen in the graph, is shifted to smaller values of \( R \).

We show the ratio \( H_s(\varepsilon_b)/H_s^{\text{vac}} \) as a function of \( \varepsilon_b \) in Fig. 5. The radiation increases monotonically for increasing \( \varepsilon_b \) in the given interval, and for the given values of \( R \) (we omit the discussion of practical relevance regarding large and real values of \( \varepsilon_b \)). Remarkably, for \( \varepsilon_b \to \infty \), the curves seem to approach finite values, which may be considered counterintuitive; For objects in vacuum, the radiation or transfer vanishes for \( \varepsilon \to \infty \). It is insightful to start with a small vacuum gap between object and medium, which is easily
The overall curve is very similar to the one of gold, however, the radiation of a SiO₂ sphere in vacuum outnumbers the one of gold: the radiation of a SiO₂ sphere by at least one order of magnitude. Indeed, the curves in Fig. 6 follow a dependence as \( R \rightarrow \infty \) for small \( \varepsilon_s \), which physically accounts for the reduced speed of light in the medium. Since \( \varepsilon_b \rightarrow \infty \), we expect that (experimental) results for very different \( \varepsilon_s \) and \( \varepsilon_b \), with its respective magnetic responses \( \mu_s \) and \( \mu_b \). The term in Eq. (53) is commonly written in terms of the dipole polarizability of the sphere in a homogeneous background medium [55]

\[
\alpha_b \equiv \frac{(\varepsilon_s - \varepsilon_b)}{(\varepsilon_s + 2\varepsilon_b)} R^3. \tag{54}
\]

For a nonmagnetic sphere, i.e., \( \mu_s = 1 \), the heat radiation is then given by

\[
\lim_{R \rightarrow \infty} H_s = \frac{4h}{c^3 \pi} \int_0^\infty \frac{\omega d\omega}{\varepsilon_s/\varepsilon_b T - 1} \text{Im} \alpha_b. \tag{55}
\]

For small dielectric constants \( \varepsilon_b - 1 \ll 1 \), we can expand the dipole polarizability in Eq. (54) to obtain its vacuum correspondence

\[
\lim_{\varepsilon_b \rightarrow 1} \alpha_b = \frac{(\varepsilon_s - 1)}{(\varepsilon_s + 2)} R^3 + O(\varepsilon_b - 1). \tag{56}
\]

Since \( \alpha_b \) approaches a finite value for \( \varepsilon_b \rightarrow 1 \), the heat radiation in the medium is then directly proportional to \( \varepsilon_b^{3/2} \), which physically accounts for the reduced speed of light in the medium. Indeed, the curves in Fig. 6 follow a dependence \( \varepsilon_b^{3/2} \) for small \( R \) well.

**B. Heat transfer between two semi-infinite bodies**

The heat transfer between two semi-infinite bodies separated by a vacuum gap has been thoroughly studied by many authors [24,26,64,65]. Recently, setups consisting of three slabs in vacuum were investigated [17,33]. In this section, we use two different approaches to study the transfer between
two surfaces in the presence of a nonabsorbing medium. We start by ascribing a homogeneous value of $\varepsilon_b$ to the medium between the surfaces. In a second approach, we investigate the situation with a single polarizable atom located inside the gap (which is then homogeneously distributed to mimic a gas). Among other things, we show that the two approaches yield identical results for the dilute limit, where $\varepsilon_b$ is close to unity.

1. Two semi-infinite plates in the presence of a homogeneous background medium

The setup of two semi-infinite plates with a medium in-between is schematically illustrated in Fig. 8. We are interested in the heat transfer $H^{1\rightarrow2}$ between the two plates, and start from the general formula for the heat transfer rate $H^{(\beta)}_\alpha$ in Eq. (43), where for two objects $\tau_{\alpha\beta} = 0$. The total heat transferred from object 1 to object 2 can be written as the difference of the heat transfer rates between the objects, where we already use the symmetry to write

$$H^{1\rightarrow2} = H^{(2)}_1 - H^{(2)}_2.$$

As found in Sec. IV, we saw that the heat transfer in the nonabsorbing background medium is given by a redefinition of the involved (scattering) operators compared to the vacuum case. The heat transfer between two plates in vacuum is well known [24,26,64,65], and we can adopt its form to describe the radiative transfer through the nonabsorbing medium by redefining the Fresnel coefficients and the ranges of propagating and evanescent waves

$$\frac{H^{1\rightarrow2}}{A} = \frac{\hbar}{8\pi^3} \int_0^\infty d\omega \omega [n_1(\omega) - n_2(\omega)] \sum_{\nu} \int d^2 k_\perp \left\{ \frac{(1 - |r^P_{1,\nu}|^2)(1 - |r^P_{2,\nu}|^2)}{|1 - e^{2ik_\nu d}r^P_{1,\nu}r^P_{2,\nu}|^2} \theta_{\nu} + 4 \text{Im} [r^P_{1,\nu}] \text{Im} [r^P_{2,\nu}] e^{-2k_\nu d} \theta_{\nu} \right\}.$$

We introduced the step functions $\theta_{\nu} = \theta(\frac{\omega}{c} \sqrt{\varepsilon_b} - k_\perp)$ and $\theta_{ev} = \theta(k_\perp - \frac{\omega}{c} \sqrt{\varepsilon_b})$ to distinguish between propagating and evanescent wave contributions in a medium, as well as $n_\alpha(\omega) = (e^{\frac{\omega}{c} h \varepsilon_b} - 1)^{-1}$. Note that the wave vector $k_\perp$ perpendicular to the symmetry axes of the plates is measured within the background medium. Its absolute value is related to the vacuum wave vector by $k_\perp = k_\nu \sqrt{\varepsilon_b}$. The Fresnel reflection coefficients $r^P$, given in Appendix F, describe waves moving from the medium to the respective plate. For completeness of our presentation, we show in Fig. 9 the well-known heat transfer for two SiC plates and two gold plates in vacuum, i.e., for $\varepsilon_b = 1$. Optical properties of SiC are taken to be [66]

$$\varepsilon_{\text{SiC}} = \varepsilon_\infty \frac{\omega^2 - \omega_{\text{LO}}^2 + i\gamma \omega}{\omega^2 - \omega_{\text{TO}}^2 + i\gamma \omega}$$

with $\varepsilon_\infty = 6.7$, $\omega_{\text{LO}} = 0.12$ eV, $\omega_{\text{TO}} = 0.098$ eV, and $\gamma = 5.88 \times 10^{-4}$ eV. For any plate separation $d$, the heat transfer between the SiC plates exceeds the transfer between the gold plates. In the near field $d \ll \lambda_T$, the evanescent wave contribution dominates due to photon tunneling processes [27], and the transfer scales as $1/d^2$ (not seen yet for gold). The known near-field enhancement is also valid if the vacuum gap is filled with a homogeneous and nonabsorbing medium. We find the following approximation that appears to be asymptotically exact in the limit where $d$ is the smallest length scale (cf. the dashed line for $\varepsilon_b = 1$)

$$H^{1\rightarrow2}/A \approx \frac{\hbar}{8\pi^3} \frac{1}{d^2} \text{Im} \left[ \varepsilon_{\text{SiC}} \right]$$

of the involved (scattering) operators compared to the vacuum case. The heat transfer between two plates in vacuum is well known [24,26,64,65], and we can adopt its form to describe the radiative transfer through the nonabsorbing medium by redefining the Fresnel coefficients and the ranges of propagating and evanescent waves

$$\frac{H^{1\rightarrow2}}{A} = \frac{\hbar}{8\pi^3} \int_0^\infty d\omega \omega [n_1(\omega) - n_2(\omega)] \sum_{\nu} \int d^2 k_\perp \left\{ \frac{(1 - |r^P_{1,\nu}|^2)(1 - |r^P_{2,\nu}|^2)}{|1 - e^{2ik_\nu d}r^P_{1,\nu}r^P_{2,\nu}|^2} \theta_{\nu} + 4 \text{Im} [r^P_{1,\nu}] \text{Im} [r^P_{2,\nu}] e^{-2k_\nu d} \theta_{\nu} \right\}.$$

We introduced the step functions $\theta_{\nu} = \theta(\frac{\omega}{c} \sqrt{\varepsilon_b} - k_\perp)$ and $\theta_{ev} = \theta(k_\perp - \frac{\omega}{c} \sqrt{\varepsilon_b})$ to distinguish between propagating and evanescent wave contributions in a medium, as well as $n_\alpha(\omega) = (e^{\frac{\omega}{c} h \varepsilon_b} - 1)^{-1}$. Note that the wave vector $k_\perp$ perpendicular to the symmetry axes of the plates is measured within the background medium. Its absolute value is related to the vacuum wave vector by $k_\perp = k_\nu \sqrt{\varepsilon_b}$. The Fresnel reflection coefficients $r^P$, given in Appendix F, describe waves moving from the medium to the respective plate. For completeness of our presentation, we show in Fig. 9 the well-known heat transfer for two SiC plates and two gold plates in vacuum, i.e., for $\varepsilon_b = 1$. Optical properties of SiC are taken to be [66]

$$\varepsilon_{\text{SiC}} = \varepsilon_\infty \frac{\omega^2 - \omega_{\text{LO}}^2 + i\gamma \omega}{\omega^2 - \omega_{\text{TO}}^2 + i\gamma \omega}$$

with $\varepsilon_\infty = 6.7$, $\omega_{\text{LO}} = 0.12$ eV, $\omega_{\text{TO}} = 0.098$ eV, and $\gamma = 5.88 \times 10^{-4}$ eV. For any plate separation $d$, the heat transfer between the SiC plates exceeds the transfer between the gold plates. In the near field $d \ll \lambda_T$, the evanescent wave contribution dominates due to photon tunneling processes [27], and the transfer scales as $1/d^2$ (not seen yet for gold). The known near-field enhancement is also valid if the vacuum gap is filled with a homogeneous and nonabsorbing medium. We find the following approximation that appears to be asymptotically exact in the limit where $d$ is the smallest length scale (cf. the dashed line for $\varepsilon_b = 1$)

$$H^{1\rightarrow2}/A \approx \frac{\hbar}{8\pi^3} \frac{1}{d^2} \text{Im} \left[ \varepsilon_{\text{SiC}} \right]$$
three-body configuration can
be also calculated using the theory developed in Ref. [17], where an example of the nonequilibrium Casimir-Polder

\[
\Delta H_{1,2}^{\text{ev}} = \frac{\hbar}{4\pi^3} \int_0^\infty d\omega \frac{d^3k}{|\omega|} \sum_p \left[ |r_{1p}|^2 |r_{2p}|^2 + e^{2ikz(d-d_1)} r_{1p} r_{2p} + e^{2ikz(d_2-d_1)} r_{3p} r_{2p} \right] \theta_{1,2}\]

Again, these terms are linearized with respect to \( \alpha \). We provide the expressions for the reflection and transmission coefficients \( r_3 \) and \( t_3 \) of the atom in Appendix F. In Fig. 14, we illustrate the correction term \( \Delta H^{1-2} \) for two SiC plates at fixed separation \( d \) in the near field \( d = 10 \text{ nm} \ll \lambda_\gamma \) plotted over the atom position \( d_1 \). The two plates are held at the constant temperatures \( T_1 = 301 \text{ K} \) and \( T_2 = 300 \text{ K} \). We choose the two plate temperatures very closely (still with a huge temperature gradient for small gap widths \( d \)), to allow comparison of \( \Delta H^{1-2} \) to kinetic gas theory below. In the graph, the correction term is normalized by the polarizability \( \alpha \) of the atom. The symmetry of the heat transfer (see Sec. IV) is reflected by the fact that \( \Delta H^{1-2} \) is an even function of the atom position \( d_1 \), where the symmetry axis is located at the center of the gap (only given for identical plates). The influence of the atom on the heat transfer between the two plates is maximal when the atom is in close proximity to one of the plates. This can be explained by examining the evanescent contribution in Eq. (64). In the near field, the correction term is dominated by the electric polarization \( P = N \) of the reflection coefficient \( r_3^p \) of the atom [see Eq. (F2)]. Physically, the impact of the atom on the heat transfer is maximized by minimizing the light path of the electromagnetic waves emitted by one of the plates and reflected by the atom. In Eq. (64), this is achieved by minimizing \( d + d_1 \) or \( 2d - d_1 \) (see the distance dependence in the exponential functions of the two terms carrying the reflection coefficient \( r_3^p \)) so that the maxima are at \( d_1 = 0 \) and \( d_1 = d \).

In Fig. 15, we show the corresponding graph for a fixed plate separation \( d = 10 \mu \text{m} > \lambda_\gamma \). In this regime, the correction
term is dominated by the propagating wave modes in Eq. (63) and is more than six orders of magnitude smaller than in the near field. The correction term is now minimized when the atom is located at one of the plates. Moving the atom away from the plate leads to a global maximum at roughly $d_{1} \approx 1 \mu m$. In the center of the gap we observe another local maximum. The oscillatory behavior of the curve is due to interference effects of the reflected and transmitted waves.

3. Comparison of the approaches of Secs. VI B 1 and VI B 2 in the dilute limit

In Sec. VI B 1, we regarded the case of a homogeneous background medium filling the gap between the plates, while in Sec. VI B 2, we analyzed the situation of a single atom between the two plates. These cases are expected to share the limit of dilution, as, microscopically, a diluted medium consists of atoms. We integrate the results for the atom in Eqs. (63) and (64) over the position $d_{1}$ of the atom, assuming a homogeneous distribution. We then multiply the result by a finite-particle-number density, as dictated by the theory of Clausius-Mossotti (i.e., assuming the effect of the atoms to be additive):

$$\epsilon_{b} - 1 = 4\pi n_{b} \alpha_{b}. \quad (65)$$

With this relation, the two approaches can now be compared, as shown in Fig. 16 for the case of air (where we use $\epsilon_{air} = 1.00059$ and $n_{air} = 2.55 \times 10^{19} \text{cm}^{-3}$, which are typical numbers at sea level [67]). The two curves are (numerically) identical for any plate distance $d$. Coming from two completely different starting points, we have thus demonstrated that the two approaches are consistent and yield the same result in the dilute limit as expected.

4. Comparison to heat transfer via kinetic gas theory

With a medium present, the energy transferred via electromagnetic radiation is in competition to other mechanisms of thermal conduction, i.e., by phonons or collisions of gas atoms. For the case of a gas, such transfer can be expressed by simple formulas: For the regime where the distance between the plates is small compared to the mean-free path $l$ of the gas molecules, $d \ll l$, we can assume the particles of the medium to traverse the gap between the two plates unhindered and therefore use a simple model from kinetic gas theory to describe the energy flux evoked by collisions of the atoms with the two surfaces:

$$\lim_{d \to 0} \frac{H_{kin}}{A} \approx \frac{3}{2} k_{B} T n_{b} \bar{v}_{x}. \quad (66)$$

In this equation, $\Delta T = T_{1} - T_{2}$ is the temperature difference between the two plates, $n_{b}$ is the particle-number density of the dilute gas, and $\bar{v}_{x} = \sqrt{\frac{2 k_{B} T}{m}}$ is the mean velocity of the particles pointing in the direction normal to the plates, with molecule mass $m$. One may use $T = T_{1}$ or $T = T_{2}$ for an estimate, as $T_{1}$ and $T_{2}$ are almost equal. On the other hand, for distances $d$ much larger than the mean-free path, i.e., $d \gg l$, the kinetic part is described by Fourier’s law for thermal conduction, being proportional to $d^{-2}$ for the case of two plates:

$$\lim_{d \to \infty} \frac{H_{kin}}{A} = \frac{\kappa}{d}. \quad (67)$$

Here, $\kappa$ is the thermal conductivity. Since for air $l \approx 68 \text{nm}$ [68], the optical near field should be compared to Eq. (66), while the optical far field should be compared to Eq. (67). Recently, it was shown that the radiative heat exchange between two closely spaced bodies in vacuum can in principle surpass conductive heat transfer through air at room temperature [69]. Using the derived formula for the heat transfer in a background medium, we can explicitly analyze the case of two SiO$_2$ plates surrounded by air (again $\epsilon_{air} = 1.00059$ [67]). In Fig. 17, we show both the kinetic part $H_{kin}$ and the radiative part of energy transfer $H_{air}^{1/2}$. In general, the radiative part exceeds the kinetic part in both limits for small and large $d$. In the near field, the radiative energy flux grows as $d^{-2}$, and exceeds the kinetic part, which in the regime of Eq. (66) is independent of $d$. For the parameters chosen, the crossover is at roughly 10 nm. For larger plate temperatures, it
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VII. SUMMARY

Closed formulas for heat radiation and radiative heat transfer in a many-body system embedded in a nonabsorbing background medium (described by real magnetic and electric permeabilities \( \mu_b \) and \( \epsilon_b \)) were derived. In contrast to the case of objects in vacuum (\( \mu_b = \epsilon_b = 1 \)), these formulas contain the classical scattering operators which describe scattering of objects in the corresponding background medium. The formulas show a number of general properties (such as positivity and symmetry). The presence of a nonabsorbing background medium can have very strong effects on the resulting radiation: Both the radiation of a sphere, as well as the heat transfer between two parallel plates, can be enhanced by orders of magnitude, this enhancement effect being especially strong for metals such as gold. For example, the radiation of a gold nanosphere in a medium with \( \epsilon_b \approx 6.5 \) is 100 times larger compared to vacuum, an effect which can be relevant for composite materials. The enhancement is generally smaller for near-field quantities, but can still be pronounced: the near-field transfer between two SiC plates is doubled at \( \epsilon_b \approx 8 \). For very large values of \( \epsilon_b \), the found results are expected to depend on the details of the interface between object and medium, an expectation which should be explored further in the future. Concerning near- and far-field transfer through a gas like air, both a microscopic model (based on gas particles) and a macroscopic model (using a dielectric constant) can be used with equal accuracy. The radiative transfer through air exceeds the energy transfer from kinetic gas theory for both very small and very large distances. The generality of the derived formalism provides incentive for future studies of heat radiation and radiative heat transfer in complex environments.

The theory derived in this paper can also be useful for possible implementation and improvement of the performances of the recently proposed nonequilibrium-based many-body quantum thermal machines \([70]\) and configurations to create and/or protect entanglement \([71]\).

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APPENDIX A: NONEQUILIBRIUM FIELD FLUCTUATIONS

1. Spectral density

Random processes of spontaneous fluctuations can in general be described by means of correlation functions. For any two field operators \( \hat{A} \) and \( \hat{B} \) in the Heisenberg picture, we employ the symmetrized expectation value \([72]\)

\[
\langle \hat{A}(t, \mathbf{r}) \hat{B}(t', \mathbf{r}') \rangle_s = \frac{1}{2} \langle \hat{A}(t, \mathbf{r}) \hat{B}(t', \mathbf{r}') + \hat{B}(t', \mathbf{r}') \hat{A}(t, \mathbf{r}) \rangle \quad (A1)
\]

In stationary conditions, the correlation function is invariant under time translation and depends only on the time difference \( t - t' \). Accordingly, we define the spectral density of fluctuations \( \langle A(r)B^*(r') \rangle_\omega \) by \([73]\)

\[
\langle \hat{A}(t, \mathbf{r}) \hat{B}(t', \mathbf{r}') \rangle_s = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t-t')} \langle A(r)B^*(r') \rangle_\omega \quad (A2)
\]

Note that the spectral density satisfies the reality condition

\[
\langle A(r)B^*(r') \rangle_\omega^* = \langle A(r)B^*(r') \rangle_{-\omega} \quad (A3)
\]

2. Electric field correlator

The spectral density of the electric field \( \mathbf{E} \) at points \( \mathbf{r} \) and \( \mathbf{r}' \) is given by

\[
C_{ij} = \langle E_i(r)E_j^*(r') \rangle_\omega \quad (A4)
\]

In global thermal equilibrium, the spectral density of the electric field is related to the imaginary part of the dyadic retarded Green’s function of the system via the fluctuation-dissipation theorem \([3,32]\):

\[
C_{ij}^{\text{eq}}(T; \mathbf{r}, \mathbf{r}') = [a(T) + a_0] \text{Im}[G_{ij}(\mathbf{r}, \mathbf{r}')]. \quad (A5)
\]

The two contributions are identified with the zero-point and thermal fluctuations with amplitude

\[
a(T) \equiv \text{sgn}(\omega) \frac{8\pi \hbar \omega^2}{c^2} \left[\exp(\hbar |\omega|/k_B T) - 1\right]^{-1}, \quad (A6)
\]

\[
a_0 \equiv \text{sgn}(\omega) \frac{4\pi \hbar \omega^2}{c^2}. \quad (A7)
\]
Note that \( c \) is the vacuum speed of light, \( \hbar \) is the reduced Planck constant, \( k_B \) is the Boltzmann constant, and \( T \) is the temperature of the equilibrium system.

**APPENDIX B: REDEFINED OPERATORS**

1. Radiation operator

The radiation operator \( R_{\alpha} \) specifies the radiation of an arbitrary object in isolation. Corresponding to the vacuum case in Ref. [14], we redefine this operator by

\[
C_{\alpha}(T_{\alpha}) = a(T_{\alpha}) R_{\alpha
\]}

\( R_{\alpha} \equiv G_b [\text{Im}[T_{\alpha}] - T_{\alpha} \text{Im}[G_b^1 T_{\alpha}^* G_b^*]]. \) \hspace{1cm} (B2)

Note that \( C_{\alpha} \) is the field correlator of the isolated object \( \alpha \) as defined in Eq. (5) (where the Green’s function \( G_b \) appears) and \( T_{\alpha} \) is the individual scattering operator of this object.

2. Multiple-scattering operator

The multiple-scattering operator \( O_{\alpha} \) describes the scattering of the field \( E_{\alpha}^{\text{iso}} \) radiated by the isolated object \( \alpha \) at all other objects in the system. For objects in a background medium, this operator is found to be

\[
E_{\alpha}^{\text{sc}} = O_{\alpha} E_{\alpha}^{\text{iso}}, \hspace{1cm} (B3)
\]

\[ O_{\alpha} = (1 + G_b T_{\alpha}) \frac{1}{1 - G_b T_{\alpha} G_b^1 T_{\alpha}}. \hspace{1cm} (B4) \]

For a detailed derivation of this important operator we refer the reader to Ref. [14]. Note that \( T_{\alpha} \) is the scattering operator of the single object \( \alpha \), while \( T_{\alpha}^\pi \) is the composite \( T \) operator of the residual objects.

**APPENDIX C: NONABSORBING BACKGROUND MEDIUM**

We want to prove that the third term \( O_{\alpha} R_{\alpha} O_{\beta} V_{\beta} \) in Eq. (22) does not contribute to the heat transfer rate \( H_{\beta}^{(\beta)} \) given in Eq. (20) for the case of a nonabsorbing background potential. The derivation of the “self”-emission \( H_{\beta}^{(\beta)} \) leads to a similar term that can be treated in the same way as presented in this appendix. Since the volume integral in Eq. (20) is restricted to the volume \( V_{\beta} \) of object \( \beta \), and we do not allow the potential \( V_{\beta} \) to connect points in the background medium and the objects, we can introduce a redefined background potential

\[
\tilde{V}_{\beta}(r, r') = \delta(r_{\beta} - r) \delta(r_{\beta} - r') V_{\beta}(r, r'). \hspace{1cm} (C1)
\]

Note that \( \delta(r_{\beta} - r) \) is the Heaviside step function, where \( r_{\beta} \) is the coordinate attached to the surface of object \( \beta \). Because \( \tilde{V}_{\beta}(r, r') \) is only nonzero if both arguments are located inside the volume of \( \beta \), the range of integration can be extended to all space to obtain the following operator trace:

\[
-\frac{2\hbar}{\pi} \int_0^\infty d\omega \frac{\omega}{e^\frac{\omega}{k_B T_{\beta}} - 1} \text{Im} \{O_{\alpha} R_{\alpha} O_{\alpha}^\dagger \tilde{V}_{\beta}^\dagger \}. \hspace{1cm} (C2)
\]

As discussed in the main text, the potential of the nonabsorbing background medium is Hermitian, \( \tilde{V}_b = \tilde{V}_b^\dagger \). Since \( R_{\alpha} \) is a Hermitian operator as well, which can be directly seen from its definition in Eq. (19), the operator to be traced in the preceding equation consists of a product of two Hermitian operators. By using the properties of the trace, we can prove that the trace of a product of two Hermitian operators is always real, although the product is not necessarily Hermitian in general. For two arbitrary Hermitian operators \( A \) and \( B \), we have [74]

\[
\text{Tr}(AB) = \text{Tr}((BA)^\dagger) = \text{Tr}(BA)^* = \text{Tr}(AB)^*, \hspace{1cm} (C3)
\]

where we used the invariance of the trace under transposition in the second step and the cyclicity of the trace in the last step. The last equality shows that \( \text{Tr}(AB) \) is a real number and we conclude \( \text{Im} \{\text{Tr}(AB)\} = 0 \). As a consequence, the imaginary part of the trace in Eq. (C2) is identically zero for any nonabsorbing background medium with Hermitian potential \( V_b \).

**APPENDIX D: POSITIVITY AND SYMMETRY OF TRANSFER**

Equation (28) gives the general formula for the heat transfer rate \( H_{\beta}^{(\beta)} \) in the presence of \( N \) objects embedded in a passive nonabsorbing background medium. It is straightforward to prove the positivity and symmetry of this expression. For this purpose, we define the operator

\[
\mathcal{W}_{\alpha\beta} = G_b^{-1} \frac{1}{1 - G_b T_{\beta} G_b^1 T_{\alpha}} (1 + G_b^1 T_{\alpha}) \frac{1}{1 - G_b T_{\alpha} G_b^1 T_{\alpha}}. \hspace{1cm} (D1)
\]

Equation (28) can then be rewritten in compact form as

\[
H_{\alpha}^{(\beta)} = \frac{2\hbar}{\pi} \int_0^\infty d\omega \frac{\omega}{e^\frac{\omega}{k_B T_{\beta}} - 1} \text{Tr} \{R_{\beta}^* \mathcal{W}_{\alpha\beta} R_{\alpha} \mathcal{W}_{\alpha\beta}^\dagger \}. \hspace{1cm} (D2)
\]

For the positivity of transfer, we need to show that the radiation operator \( R_{\alpha} \) as defined in Eq. (19) is positive semidefinite. By virtue of the symmetry relation for the potential difference

\[
\Delta V_{\alpha}(-\omega) = \Delta V_{\alpha}(\omega), \hspace{1cm} (D3)
\]

we restrict ourselves to positive frequencies \( \omega \). For any body made of passive material, the imaginary part of the potential difference is positive semidefinite, i.e.,

\[
\text{Im} \{\Delta V_{\alpha}\} \geq 0. \hspace{1cm} (D4)
\]

Recalling that for any positive-semidefinite operator \( A \), the product \( BAB^\dagger \) is also positive semidefinite, we can write

\[
G_b^{-1} G_b \text{Im} \{\Delta V_{\alpha}\} G_b^* G_b^{-1} \geq 0. \hspace{1cm} (D5)
\]

Application of Eqs. (12) and (13) then yields the desired property for the radiation operator \( R_{\alpha} \):

\[
R_{\alpha} = G_b [\text{Im}[T_{\alpha}]] - T_{\alpha} \text{Im}[G_b^1 T_{\alpha}^*] G_b^* \geq 0. \hspace{1cm} (D6)
\]

Finally, as the product of the two positive-semidefinite operators \( R_{\beta}^* \) and \( \mathcal{W}_{\alpha\beta} R_{\alpha} \mathcal{W}_{\alpha\beta}^\dagger \) is also positive semidefinite, we conclude that the heat transfer rate \( H_{\alpha}^{(\beta)} \) in a many-body system including the background medium in Eq. (D2) is non-negative,

\[
H_{\alpha}^{(\beta)} \geq 0. \hspace{1cm} (D7)
\]

In order to prove the symmetry of the heat transfer, i.e., \( H_{\alpha}^{(\beta)}(T) = H_{\beta}^{(\alpha)}(T) \), we need to show the following equality of operators:

\[
\mathcal{W}_{\alpha\beta}^\dagger = \mathcal{W}_{\beta\alpha}^\dagger. \hspace{1cm} (D8)
\]
This relation can be straightforwardly proven by making use of the symmetry of \( G_b \) and \( T \) as well as the decomposition of the composite scattering operator \( T_{\beta} \) (see Appendix G). As a result, we can write
\[
H^\beta_\alpha(T_{\alpha}) = \frac{2\hbar}{\pi} \int_0^\infty d\omega \frac{\omega}{e^{\omega/k_T} - 1} \text{Tr}[\mathbb{R}_\beta \mathbb{W}_{\alpha\beta} \mathbb{W}_{\mu\alpha}]
= H^{\beta\alpha}_\alpha(T_{\alpha}). \tag{D9}
\]
In the second line we have used the cyclic property of the trace and, furthermore, we have taken the complex conjugate of the expression since it is real. The symmetry and positivity of the heat transfer in the presence of an arbitrary number of other objects can be used to show that the net heat flux between a warm object \( \alpha \) and a cold object \( \beta \) is always positive, i.e., that heat is always transferred from the warmer object to the colder one:
\[
H^\beta_\alpha(T_{\alpha}) - H^\beta_\alpha(T_{\beta}) \geq 0 \quad \text{if } T_{\alpha} \geq T_{\beta}. \tag{D10}
\]
This relation holds because of the monotonic increase of \((\exp[ih\omega/k_T] - 1)^{-1}\) with \( T \) for any \( \omega \). And it also relies on the assumption that the objects’ potentials \( \{\mathbb{W}_\alpha\} \) do not depend on temperature.

**APPENDIX E: SPHERICAL WAVE BASIS**

1. Partial waves and Green's function

   In the main text, we stated that the Green’s function \( G_b \), describing a homogeneous, isotropic, and local background medium, can be traced back to the partial-wave expansion of the free Green’s function \( G_0 \) given by the relation in Eq. (38). For explicit knowledge of the expansion of the free Green’s function in spherical waves, we refer the interested reader to Ref. [14].

2. **\( T \)** matrix of a sphere

   The scattering problem of a homogeneous sphere of radius \( R \) is an exactly solvable problem. The matrix elements of \( T \) are well known [63] and sometimes referred to as Mie coefficients. Considering spheres with isotropic and local background medium, can be traced back to the partial-wave expansion of the free Green’s function \( G_0 \) given by the relation in Eq. (38). For explicit knowledge of the expansion of the free Green’s function in spherical waves, we refer the interested reader to Ref. [14].

   Fresnel coefficients of the atom

   The Fresnel reflection and transmission coefficients of a polarizable atom in vacuum with dipole polarizability \( \alpha \) are straightforwardly derived from the definition of the \( T \) matrix in Eq. (40). For the reflection coefficients we obtain
\[
r^N = 2\pi i \frac{\omega^2}{c^2} \frac{1}{k_z} \left\{ \frac{2c^2k_z^2}{\omega^2} - 1 \right\}, \tag{F2}
\]
\[
r^M = 2\pi i \frac{\omega^2}{c^2} \frac{1}{k_z}, \tag{F3}
\]
and for the transmission coefficients we find
\[
t^N = 2\pi i \frac{\omega^2}{c^2} \frac{1}{k_z}, \tag{F4}
\]
\[
t^M = t^N. \tag{F5}
\]

**APPENDIX F: PLANE-WAVE BASIS**

1. Green’s function and \( T \) matrix of a plate

   The expansion of the Green’s function \( G_b \) in plane waves is found with the help of Eq. (38) and the representation of the vacuum waves given in Ref. [14]. For the \( T \) matrix of a plate in a homogeneous, isotropic and local background medium, we may resort to its representation in vacuum [14], where we only need to adjust the Fresnel coefficients as given below for the case of an infinitely thick plate.

   2. Fresnel coefficients of the plate

   The Fresnel reflection coefficients for an infinitely thick plate embedded in a background medium are given in Ref. [54]:
\[
r^N(k_{\perp},\omega) = \frac{\omega}{k_{\perp}} \sqrt{\varepsilon_b \mu_b - k_{\perp}^2} - \sqrt{\varepsilon_b \mu_b - k_{\perp}^2}, \tag{F1}
\]
r\(^M\) is obtained from \( r^N \) by interchanging \( \varepsilon_b \) and \( \mu_b \). In order to regain the Fresnel coefficients in the vacuum case, one has to set \( \varepsilon_b = 1 \).

   3. Fresnel coefficients of the atom

   The Fresnel reflection and transmission coefficients of a polarizable atom in vacuum with dipole polarizability \( \alpha \) are straightforwardly derived from the definition of the \( T \) matrix in Eq. (40). For the reflection coefficients we obtain
\[
r^N = 2\pi i \frac{\omega^2}{c^2} \frac{1}{k_z} \left\{ \frac{2c^2k_z^2}{\omega^2} - 1 \right\}, \tag{F2}
\]
\[
r^M = 2\pi i \frac{\omega^2}{c^2} \frac{1}{k_z}, \tag{F3}
\]
and for the transmission coefficients we find
\[
t^N = 2\pi i \frac{\omega^2}{c^2} \frac{1}{k_z}, \tag{F4}
\]
\[
t^M = t^N. \tag{F5}
\]
two objects is then obtained by iteratively substituting the composite scattering operator in Eq. (G3) for either of the two single-scattering operators $T_1$ and $T_2$, respectively.

2. Expansion for $N$ objects

An expansion for the $T_{\alpha \beta \gamma \ldots}$ operator for $N$ objects in terms of the single-scattering operators $[75,76]$ can be given by starting from the definition in Eq. (12). The expansion of the inverse operator in that equation into a power series yields

$$T_{\alpha \beta \gamma \ldots} = \Delta V \sum_{i=0}^{\infty} T_i G_b \Delta V^i$$

where the ellipsis stands for the remaining infinite terms. Notice that in the higher-order terms the same index of the potential introduced by object $\alpha$ may be repeated several times. By resummation of the infinite terms, we finally arrive at the expansion of $T$ in terms of the single-scattering operators, reading as

$$T_{\alpha \beta \gamma \ldots} = \sum_{i=1}^{N} T_i + \sum_{i=1, j \neq i}^{N,N} T_i G_b T_j + \sum_{i=1, j \neq k, j \neq k}^{N,N,N} T_i G_b T_j G_b T_k + \cdots$$

Again, the ellipsis represents the remaining higher-order terms. From this representation, it becomes apparent that the scattering operator for an arbitrary number of objects is always symmetric with respect to a permutation of indices, e.g., $T_{\alpha \beta \gamma \ldots} = T_{\gamma \beta \alpha \ldots}$.

Insertion of this relation into the power series leads to

$$T_{\alpha \beta \gamma \ldots} = \sum_{i=1}^{N} \Delta V_i + \sum_{i=1, j \neq i}^{N,N} \Delta V_i G_b \Delta V_j + \sum_{i=1, j \neq k, j \neq k}^{N,N,N} \Delta V_i G_b \Delta V_j G_b \Delta V_k + \cdots$$

where the ellipsis stands for the remaining infinite terms.

References:
