Semisimple and G-Equivariant Simple Algebras Over Operads

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Semisimple and $G$-Equivariant Simple Algebras Over Operads

Pavel Etingof

Abstract Let $G$ be a finite group. There is a standard theorem on the classification of $G$-equivariant finite dimensional simple commutative, associative, and Lie algebras (i.e., simple algebras of these types in the category of representations of $G$). Namely, such an algebra is of the form $A = \text{Fun}_H(G, B)$, where $H$ is a subgroup of $G$, and $B$ is a simple algebra of the corresponding type with an $H$-action. We explain that such a result holds in the generality of algebras over a linear operad. This allows one to extend Theorem 5.5 of Sciarappa (arXiv:1506.07565) on the classification of simple commutative algebras in the Deligne category $\text{Rep}(S_t)$ to algebras over any finitely generated linear operad.

Keywords Simple algebra · Semisimple algebra · Operad · Equivariant

1 Semisimple Algebras Over Operads

1.1 Algebras

Let $C$ be a linear operad over a field $F$ [1]. E.g., $C$ can be the operad of commutative associative unital algebras, associative unital algebras, or Lie algebras (the latter if $1/2 \in F$).

Recall [1] that a $C$-algebra is a vector space $A$ over $F$ with a collection of linear maps $\alpha_n : C(n) \to \text{Hom}_F(A^{\otimes n}, A)$ compatible with the operadic structure. Clearly, a direct product of finitely many $C$-algebras is a $C$-algebra.

Given a $C$-algebra $A$, we can define the space $E_A \subset \text{End}_F(A)$ spanned over $F$ by operators of the form $\alpha_n(c)(a_1, \ldots, a_{n-1}, ?, a_j, \ldots, a_{n-1})$ for various $n \geq 2$, $c \in C(n)$, and $a_i \in A$. By the definition of an operad, $E_A$ is a (possibly non-unital) subalgebra of $\text{End}_F(A)$.

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We also denote by $L_A$ the image of $C(1)$ in $\text{End}_F(A)$. Clearly, $L_A$ is a unital subalgebra and $L_A E_A = E_A L_A = E_A$. Thus $R_A := L_A + E_A$ is a unital subalgebra of $\text{End}_F(A)$, and $E_A$ is an ideal in $R_A$.

**Lemma 1.1** One has $1 E_A \oplus \ldots \oplus A_m = E_A \oplus \ldots \oplus E_A$.

**Proof** It is clear that $E_A \oplus \ldots \oplus A_m \subset E_A \oplus \ldots \oplus E_A$. Let $a_i \in A_r$, $c \in C(n)$, and $b = \alpha_n(c)(a_1, \ldots, a_{j-1}, ?, a_j, \ldots, a_{n-1}) \in E_A$. Let $b' := (0, \ldots, b, \ldots, 0)$ (where $b$ is at the $r$-th place). Then we have $b' = \alpha_n(c)(a'_1, \ldots, a'_{j-1}, ?, a'_j, \ldots, a'_{n-1})$, where $a'_i = (0, \ldots, a_i, \ldots, 0)$. Hence $b' \in E_A \oplus \ldots \oplus A_m$. Thus $E_A \oplus \ldots \oplus A_m \supset E_A \oplus \ldots \oplus E_A$. \hfill $\square$

### 1.2 Ideals

By an *ideal* in a $C$-algebra $A$ we mean a subspace $I \subset A$ such that for any $n \geq 1$, $c \in C(n)$, $j \in [1, n]$, and $T \in A^{\otimes j-1} \otimes I \otimes A^{\otimes n-j}$ one has $\alpha_n(..., x, ..., y, ...) = 0$ once $x \in A_i$ and $y \in A_j$ with $j \neq i$, which implies the statement.

**Lemma 1.2** (i) $I \subset A$ is an ideal if and only if it is an $R_A$-submodule of $A$.

(ii) $A = A_1 \oplus \ldots \oplus A_m$ as an $R_A$-module if and only if it is so as a $C$-algebra.

**Proof** (i) This follows directly from the definition.

(ii) The “if” direction is clear. To prove the “only if” direction, note that by (i) $A_i$ are ideals in $A$, hence $\alpha_n(..., x, ..., y, ...) = 0$ once $x \in A_i$ and $y \in A_j$ with $j \neq i$, which implies the statement. \hfill $\square$

It is clear that if $I \subset A$ is an ideal then $A/I$ is a $C$-algebra, and $E_{A/I}, L_{A/I}, R_{A/I}$ are homomorphic images of $E_A, L_A, R_A$ in $\text{End}_F(A/I)$.

### 1.3 Simple and Semisimple Algebras

From now on we assume that $A$ is a finite dimensional $C$-algebra. We say that $A$ is *simple* if any ideal in $A$ is either 0 or $A$ (i.e., $A$ is a simple $R_A$-module), and $E_A \neq 0$.

**Lemma 1.3** If $A$ is a simple $C$-algebra then $E_A = R_A$, and it is a central simple algebra (over some finite field extension of $F$).

**Proof** Since $A$ is a faithful simple $R_A$-module, $R_A$ is central simple. Since $E_A \neq 0$ and $E_A$ is an ideal in $R_A$, we have $E_A = R_A$. \hfill $\square$

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1Categorically, it is more natural to regard the direct sum $A_1 \oplus \ldots \oplus A_m$ as a direct product, but there is no difference since it is finite. So, we will call it a direct product, but use the sign $\oplus$ instead of $\times$ to emphasize that our constructions are linear over a field.

2Note that this recovers the standard definition for commutative, associative, and Lie algebras. Moreover, while in the commutative and associative case, the condition $E_A \neq 0$ is automatic for $A \neq 0$ because of the unit axiom, in the Lie case it is needed (as an abelian Lie algebra is not simple). Note also that if $C(n) = 0$ for $n \neq 1$ (i.e., when $C$ is an ordinary algebra), then $E_A = 0$ automatically, so there are no simple $C$-algebras, even though there may exist simple $C$-modules.
We say that $A$ is semisimple if $A$ is a direct product of a finite (possibly empty) collection of simple $C$-algebras: $A = A_1 \oplus ... \oplus A_m$.

**Lemma 1.4** Let $A = A_1 \oplus ... \oplus A_m$ be a semisimple $C$-algebra with simple constituents $A_i$. Then the only ideals in $A$ are $\oplus_{i \in S} A_i \subset A$, where $S \subset [1, m]$.

**Proof** Clearly, the subspaces in the lemma are ideals. Conversely, let $I \subset A$ be an ideal. Let $a = (a_1, ..., a_m) \in I$. By Lemmas 1.1 and 1.3, the projection operator $P_i : A \to A$ to $A_i$ along $\oplus_{j \neq i} A_j$ is contained in $E_A$. Thus, $P_ia = (0, ..., a_i, ..., 0) \in I$. This implies the statement. \qed

### 1.4 The Radical

Let $A'$ be the maximal semisimple quotient of $A$ as an $R_A$-module (it exists by the standard theory of finite dimensional algebras). Let $\overline{A}$ be the quotient of $A'$ by the kernel of the action of $E_A$ (which is an $R_A$-submodule of $A$). Define the radical $\text{Rad}(A)$ of $A$ to be the kernel of the projection of $A$ onto $\overline{A}$. So the radical of $A/\text{Rad}(A) = \overline{A}$ is zero. In particular, if $A$ is a semisimple $C$-algebra, then $\text{Rad}(A) = 0$.

**Theorem 1.5** (i) $\overline{A}$ is a semisimple $C$-algebra. In particular, $\text{Rad}(A) = 0$ if and only if $A$ is semisimple.

(ii) If $I \subset A$ is an ideal, then $A/I$ is a semisimple $C$-algebra if and only if $I$ contains $\text{Rad}(A)$.

**Proof** (i) By the definition, $\overline{A}$ is a semisimple $R_A$-module, such that $E_A$ acts by nonzero on all its simple summands. Hence by Lemma 1.2(ii), $\overline{A}$ is a semisimple $C$-algebra.

(ii) The “if” direction holds by (i) and Lemma 1.4. To prove the “only if” direction, let $I \subset A$ be an ideal such that $A/I$ is a semisimple $C$-algebra: $A/I = A_1 \oplus ... \oplus A_m$. Then by Lemma 1.2(ii) $A/I$ is a semisimple $R_{A/I}$-module and hence $R_A$-module, with simple constituents $A_i$, and the action of $E_A$ on $A_i$ is nonzero. Thus $I \supset \text{Rad}(A)$. \qed

### 2 $G$-Equivariant Simple Algebras Over Operads

Now let $G$ be a finite group, and $A$ be a $C$-algebra with an action of $G$. Let us say that $A$ is a simple $G$-equivariant $C$-algebra if the only $G$-invariant ideals in $A$ are 0 and $A$, and $E_A \neq 0$.

**Lemma 2.1** (i) If $B$ is a simple $C$-algebra then we have $\text{Aut}(B^\otimes n) = S_n \ltimes \text{Aut}(B)^n$.

(ii) If $A$ is a simple $G$-equivariant $C$-algebra then $A$ is semisimple as a usual $C$-algebra. Moreover, $G$ acts transitively on the simple constituents of $A$, and in particular they are all isomorphic.

**Proof** (i) Clearly, $S_n \ltimes \text{Aut}(B)^n$ acts on $B^\otimes n$, so we need to show that any automorphism $g$ of $B^\otimes n$ belongs to this group. By Lemma 1.4, the minimal (nonzero) ideals of $B^\otimes n$ are the $n$ copies of $B$. So they must be permuted by $g$, inducing an element $s \in S_n$. \qed
Thus $g s^{-1}$ is an automorphism preserving all the copies of $B$. So $g s^{-1} \in \text{Aut}(B)^n$, as desired.

(ii) Let $I$ be kernel of the projection from $A$ to its maximal semisimple quotient $A'$ as an $R_A$-module. Then by Lemma 1.2(i), $I$ is a $G$-invariant ideal in $A$, and $I \neq A$. Hence $I = 0$, and $A$ is a semisimple $R_A$-module. So by Lemma 1.2(ii), $A = A_1 \oplus \ldots \oplus A_m$ is a semisimple $C$-algebra. Thus by Lemma 1.4, the minimal ideals of $A$ are the $A_i$. So they are permuted by $G$. Moreover, the action of $G$ on these ideals must be transitive, as every orbit gives a nonzero $G$-invariant ideal.

Now let $B$ be a simple $C$-algebra, $H$ a subgroup of $G$, and $\phi : H \to \text{Aut}(B)$ a homomorphism. Let $A = \text{Fun}_H(G, B)$ be the space of $H$-invariant functions on $G$ with values in $B$. Then it is clear that $A$ has a natural structure of a simple $G$-equivariant $C$-algebra, isomorphic to $B^{|G/H|}$ as a usual $C$-algebra. Note that the stabilizer of any minimal ideal of $A$ is a subgroup of $G$ conjugate to $H$.

**Theorem 2.2** Any simple $G$-equivariant $C$-algebra $A$ is of the form $A = \text{Fun}_H(G, B)$. Moreover, the subgroup $H$ is defined by $A$ uniquely up to conjugation in $G$, and $\phi$ is defined uniquely up to conjugation in $\text{Aut}(B)$.

**Proof** By Lemma 2.1(ii), $G$ acts transitively on the set of minimal ideals in $A$, and they are all isomorphic to some simple $C$-algebra $B$. Thus, the result follows from Lemma 2.1(i) and the standard classification of transitive homomorphisms $G \to S_n \rtimes \text{Aut}(B)^n$. Namely, let $H$ be the stabilizer of one of the copies of $B$. Then $H$ acts on $B$ through some homomorphism $\phi : H \to \text{Aut}(B)$. Moreover, we have a canonical $G$-equivariant linear map $\psi : A \to \text{Fun}_H(G, B)$ corresponding via Frobenius reciprocity to the $H$-stable projection $A \to B$ to the chosen copy of $B$ along the direct product of all the other copies. It is easy to check using Lemma 2.1 that $\psi$ is an isomorphism of $G$-equivariant $C$-algebras. The rest is easy. \qed

**Remark 2.3**

1. Note that in the examples of commutative, associative, and Lie algebras we obtain the classical theorems about classification of simple $G$-equivariant algebras of these types.

2. Lemma 2.1 and Theorem 2.2 don’t hold without the assumption $E_A \neq 0$. E.g., one may take $A$ to be any irreducible representation of $G$ equipped with the zero Lie bracket.

3. The results of this section extend verbatim to the case when $G$ is any group (not necessarily finite), or is an affine algebraic group over $F$. Namely, as in the finite group case, the classification of simple $G$-equivariant algebras reduces to classification of transitive homomorphisms $G \to S_n \rtimes \text{Aut}(B)^n$, which are paramertized by finite index subgroups $H$ of $G$ and homomorphisms $\phi : H \to \text{Aut}(B)$ up to conjugation.

**Remark 2.4** While the question of classification of $G$-equivariant simple algebras over operads is natural in its own right, the motivation for writing this note was to provide a more general context for the results of [2]. Namely, Lemma 2.1 and Theorem 2.2 allow one to extend the main result of [2] (Theorem 5.5 on the classification of simple commutative algebras in the Deligne category $\text{Rep}(S_t)$) to algebras over a finitely generated linear operad $C$ over $\mathbb{C}$. Informally speaking, this generalization says that for transcendental $t$ any such algebra is obtained by induction from $\text{Rep}(G) \boxtimes \text{Rep}(S_{t-k})$ of an interpolation $B$ of a family of
$G \times S_{n-k}$-equivariant simple algebras $B_n$, defined for some strictly increasing sequence of positive integers $n$ and depending algebraically on $n$.

This gives a classification of simple $C$-algebras in $\text{Rep}(S_r)$ whenever a classification of ordinary simple $C$-algebras (and their automorphisms) is available. For instance, in the case of associative unital algebras, $B = \text{End}(V)$, where $V$ is an object of $\text{Rep}(S_r)$, and in the case of Lie algebras $B = \mathfrak{sl}(V)$, $\mathfrak{o}(V)$, or $\mathfrak{sp}(V)$, where in the second case $V$ is equipped with a nondegenerate symmetric form and in the third case with a nondegenerate skew-symmetric form.

The proof of this generalization is similar to the proof of Theorem 5.5 of [2], which covers the case of commutative unital algebras (in which case $B = \mathbb{C}$), but is somewhat more complicated since in general $\text{Aut}(B) \neq 1$. The finite generation assumption for $C$ is needed to validate the constructibility arguments of [2], Section 4. This will be discussed in more detail elsewhere.

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**References**