Pion distribution amplitude from lattice QCD

The MIT Faculty has made this article openly available. Please share how this access benefits you. Your story matters.

<table>
<thead>
<tr>
<th>Citation</th>
<th>Zhang, Jian-Hui, Jiunn-Wei Chen, Xiangdong Ji, Luchang Jin, and Huey-Wen Lin. “Pion Distribution Amplitude from Lattice QCD.” Physical Review D 95, no. 9 (May 31, 2017).</th>
</tr>
</thead>
<tbody>
<tr>
<td>As Published</td>
<td><a href="http://dx.doi.org/10.1103/PhysRevD.95.094514">http://dx.doi.org/10.1103/PhysRevD.95.094514</a></td>
</tr>
<tr>
<td>Publisher</td>
<td>American Physical Society</td>
</tr>
<tr>
<td>Version</td>
<td>Final published version</td>
</tr>
<tr>
<td>Accessed</td>
<td>Mon Feb 04 19:15:11 EST 2019</td>
</tr>
<tr>
<td>Citable Link</td>
<td><a href="http://hdl.handle.net/1721.1/110250">http://hdl.handle.net/1721.1/110250</a></td>
</tr>
<tr>
<td>Terms of Use</td>
<td>Article is made available in accordance with the publisher's policy and may be subject to US copyright law. Please refer to the publisher's site for terms of use.</td>
</tr>
<tr>
<td>Detailed Terms</td>
<td></td>
</tr>
</tbody>
</table>
Pion distribution amplitude from lattice QCD

Jian-Hui Zhang,1,* Jiunn-Wei Chen,2,3,‡ Xiangdong Ji,4,5,‡ Luchang Jin,6,§ and Huey-Wen Lin7,8,∥

1Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany
2Department of Physics, Center for Theoretical Sciences, and Leung Center for Cosmology and Particle
Astrophysics, National Taiwan University, Taipei, Taiwan 106
3Center for Theoretical Physics, Massachusetts Institute of Technology,
Cambridge, Massachusetts 02139, USA
4Tsung-Dao Lee Institute, and College of Physics and Astronomy, Shanghai Jiao Tong University,
Shanghai 200240, People’s Republic of China
5Department of Physics, Center for Fundamental Physics, Department of Physics, University of Maryland,
College Park, Maryland 20742, USA
6Physics Department, Brookhaven National Laboratory, Upton, New York 11973, USA
7Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824, USA
8Department of Computational Mathematics, Science and Engineering, Michigan State University,
East Lansing, Michigan 48824, USA

(Received 17 February 2017; published 31 May 2017)

We present the first lattice-QCD calculation of the pion distribution amplitude using the large-momentum effective field theory (LaMET) approach, which allows us to extract light cone parton observables from a Euclidean lattice. The mass corrections needed to extract the pion distribution amplitude from this approach are calculated to all orders in $m^2_P/P^2$. We also implement the Wilson-line renormalization which is crucial to remove the power divergences in this approach, and find that it reduces the oscillation at the end points of the distribution amplitude. Our exploratory result at 310-MeV pion mass favors a single-hump form broader than the asymptotic form of the pion distribution amplitude.

DOI: 10.1103/PhysRevD.95.094514

I. INTRODUCTION

Hadronic light cone distribution amplitudes (DAs) play an essential role in the description of hard exclusive processes involving large momentum transfer. They are crucial inputs for processes relevant to measuring fundamental parameters of the Standard Model and probing new physics [1]. The QCD factorization theorem and asymptotic freedom allow us to separate the short-distance physics incorporated in the hard quark and gluon subprocesses from the long-distance physics incorporated in the process-independent hadronic DAs. While the short-distance hard quark and gluon subprocesses are calculable perturbatively, the hadronic DAs are intrinsically non-perturbative. To determine them, we must resort to experimental measurements, lattice calculations or QCD models.

The simplest and most extensively studied hadronic DA is the twist-2 DA of the pion. It represents the probability amplitude of finding the valence $q\bar{q}$ Fock state in the pion with the quark (antiquark) carrying a fraction $x$ $(1-x)$ of the total pion momentum. The pion light cone distribution amplitude (LCDA) is defined as

$$\phi_x(x) = \frac{i}{f_\pi} \int_0^1 \frac{d\xi}{2\pi} e^{i(x-1)\xi P} \langle \pi(P) | \bar{\psi}(0) \lambda \cdot \gamma_5 \Gamma(0, \xi \lambda) \psi(\xi \lambda) | 0 \rangle$$

with the normalization $\int_0^1 dx \phi_x(x) = 1$, where the two quark fields are separated along the light cone with $x^\mu = (1, 0, 0, -1)/\sqrt{2}$, and $x$ $(1-x)$ denotes the momentum fraction of the quark (antiquark). The twist-2 pion DA can be constrained from experimental measurements of e.g. the pion form factor [2], and then as an input can be used to test QCD in, for example, Ref. [7] suggested a “double-humped” shape for the pion DA, which is very different from the asymptotic form, while other QCD models (for example, large-$N_c$ Regge model [8], QCD sum rule calculations [9], Nambu-Jona-Lasinio model [10], Dyson-Schwinger equations [11], truncated Gegenbauer expansion [12], just to name a few) do not suggest such a feature. Unfortunately, lattice calculations have traditionally only been able to extract the lowest few moments of the pion DA after using the operator product expansion (OPE). The highest moment ever calculated on the lattice is the second
moment [13–17], and most calculations struggled with the 
noise-to-signal ratio. Reference [18] took the moment 
results from lattice-QCD calculations and reconstructed 
the pion DA using a specific parametrization; however, the 
errors propagating from the lattice calculations are rela-
tively large, preventing them from discriminating between 
the QCD models. Calculating moments beyond the lowest 
two on the lattice is much more difficult due to the breaking 
of rotational symmetry by discretization, which induces 
divergent mixing coefficients to lower moments such that 
the noise-to-signal becomes a big problem. It was proposed 
to use a smeared source to reduce the discretization error 
[19], or to use another scale to replace the lattice cutoff in 
the mixing. For example, by using a heavy-light current in 
the OPE for the current-current correlator, the scale in the 
mixing parameters is replaced by the heavy-quark mass 
[20] or the gradient-flow scale in the proposal of Ref. [21]. 
Having an alternative approach to calculate the pion DA 
with better precision and quantifiable systematics is highly 
desirable so that it can be used to make predictions in other 
harder-to-calculate processes, such as $B \rightarrow \pi \pi$.

Recently, a new approach has been proposed to calculate 
the full $x$ dependence of parton quantities, such as parton 
distributions, distribution amplitudes, etc. [22]. The method 
is based on the observation that, while in the rest frame of the 
nucleon, parton physics corresponds to light cone correla-
tions, the same physics can be obtained through time-
independent spatial correlations in the infinite-momentum 
frame (IMF) of the hadron after a matching procedure. This 
has been incorporated into a large-momentum effective field 
theory (LaMET) [23]. According to the LaMET, for a given 
light cone observable such as the PDF or the DA, one can 
construct a time-independent quasibariable which 
depends on the hadron momentum, but approaches the light 
cone observable if the infinite momentum limit is taken prior 
to a UV regularization. In practical lattice computations, 
what one calculates is the quasiobservable at large but finite 
nucleon momentum with UV regularization imposed first. 
The difference between the quasi- and light cone observable 
is just the order of limits. This is similar to an effective field 
theory setup. One can then convert the quasi observable 
to the light cone one through a factorization or matching 
formula [23] (there exist also other approaches to extract 
light cone quantities from Euclidean ones, see e.g. [24–28]). 
There have been many studies on factorization [29] and 
determinations of the one-loop corrections needed to 
connect finite-momentum quasidistributions to light cone 
distributions for nonsinglet leading-twist PDFs [30], 
generalized parton distributions (GPDs) [31], transversity GPDs 
[32] and pion DA [31] in the continuum. Reference [33] also 
explores the renormalization of quasidistributions, and 
establishes that the quasidistribution is multiplicatively 
renormalizable at two-loop order. There are also proposals 
to improve the quark correlators to remove linear divergences 
in the one-loop matching [34], to improve the nucleon 
source to get higher nucleon momenta on the lattice [35], and 
to use the nonperturbative evolution of quasidistributions as a 
guide for the extrapolation of lattice results at moderate 
momentum to infinite momentum [36,37]. In Refs. [38,39], 
it was shown that the power divergence present in the long-
link matrix elements can be removed by a mass renormal-
ization in the auxiliary $z$-field formalism, in the same way as 
the renormalization of power divergence for an open Wilson 
line. After the Wilson-line renormalization, the long-link 
matrix elements are improved such that they contain at most 
logarithmic divergences. A nonperturbative determination 
of the mass counterterm can, for example, be done following 
the procedure based on the static-quark potential for the 
renormalization of Wilson loop in Ref. [40].

The first attempts to apply the LaMET approach to 
compute parton observables were the direct lattice comput-
tations of the unpolarized, helicity and transversity iso-
vector quark distributions [41–46]. Although the current 
lattice systematics are not yet fully accounted for, a sea-
flavor asymmetry has been qualitatively seen in both the 
unpolarized and linearly polarized cases, part of which has 
been confirmed in the updated measurements by the STAR 
[47] and PHENIX [48] Collaborations. The Drell-Yan 
experiments at FNAL (E1027 + E1039) and future EIC 
data will be able to give more insight into the sea 
systematics in the transversely polarized nucleon.

In this paper, we present the first direct lattice-QCD 
results for the Bjorken-$x$ dependence of the pion DA using 
lattice gauge ensembles with $N_f = 2 + 1 + 1$ highly 
Improved staggered quarks (HISQ) [49] (generated by 
the MILC Collaboration [50]) and clover valence fermions 
with pion mass 310 MeV. In the framework of LaMET, 
the pion LCDA $\phi(x)$ can be studied from the IMF limit of 
the following quasi-distribution amplitude (quasi-DA)

$$
\tilde{\phi}(x, P_z) = \frac{i}{f_\pi} \int \frac{dz}{2\pi} e^{-ixP_zz} \langle \pi(P)|\tilde{\psi}(0)\gamma_\parallel \gamma_3 \Gamma(0,z)\psi(z)|0\rangle
$$

with the two quark fields separated along the spatial $z$ 
direction. As shown in Ref. [31], the pion LCDA can be 
related to the quasi-DA by the following matching formula:

$$
\tilde{\phi}(x, \Lambda, P_z) = \int_0^1 dyZ_\phi(x, y, \Lambda, \mu, P_z)\phi(y, \mu) + O(\frac{\Lambda^2_{QCD}}{P_z^2}, \frac{m_\pi^2}{P_z^2}),
$$

where $\Lambda = \pi/\alpha$ is the UV cutoff for the quasi-DA with $\alpha$ 
the lattice spacing. $\mu$ denotes the MS renormalization scale 
of the pion LCDA. Using Eq. (3), we will be able to recover 
the pion LCDA.

The paper is organized as follows: We will start by 
discussing the finite-momentum corrections for the quasi-
DA computed on the lattice in Sec. II and then present the 
lattice results in Sec. III. We first show the results without
Wilson-line renormalization to remove the power divergence and then explore the impact of Wilson-line renormalization where the mass counterterm is determined by using the static-quark potential for the renormalization of Wilson loop discussed in Ref. [40]. Finally we summarize in Sec. IV. The details of the finite-momentum corrections are given in the Appendixes.

II. FINITE-\(P_z\) CORRECTIONS FOR PION QUASIDISTRIBUTION AMPLITUDE

In this section, we present the finite-momentum corrections needed for the calculation of pion DA. In the limit \(P_z \rightarrow \infty\), the matching becomes the most important \(P_z\) correction. The factor \(Z_{\phi}\) has been computed up to one loop in Ref. [31] using a momentum-cutoff regulator instead of a lattice regulator. Therefore, this \(Z\) factor is accurate up to the leading logarithm but not for the numerical constant. Determining this constant requires a calculation using lattice perturbation theory with the same lattice action. Determining this constant requires a calculation using lattice perturbation theory with the same lattice action. Determining this constant requires a calculation using lattice perturbation theory with the same lattice action. Determining this constant requires a calculation using lattice perturbation theory with the same lattice action.

At tree level, the \(Z_{\phi}\) factor is just a delta function. Up to one-loop level, we can write

\[
Z_{\phi}(x, y) = \delta(x - y) + \frac{\alpha_s}{2\pi} Z_{\phi}(x, y) + \mathcal{O}(\alpha_s^2),
\]

such that

\[
\tilde{\phi}(x) \equiv \phi(x) + \frac{\alpha_s}{2\pi} \int dy \tilde{Z}_{\phi}(x, y) \phi(y).
\]

Since the difference between \(\tilde{\phi}(x)\) and \(\phi(x)\) starts at the loop level, we can rewrite the above equation as

\[
\phi(x) = \tilde{\phi}(x) - \frac{\alpha_s}{2\pi} \int dy \tilde{Z}_{\phi}(x, y) \tilde{\phi}(y)
\]

with an error of \(\mathcal{O}(\alpha_s^2)\) [29]. As in the parton distribution, \(\tilde{Z}_{\phi}(x, y)\) can be written as

\[
\tilde{Z}_{\phi}(x, y) = (Z_{\phi}^{(1)}(x, y) - C\delta(x - y)),
\]

with the first term coming from gluon emission and the second term from the quark self-energy diagram, \(C = \int_{-\infty}^{\infty} dx' Z_{\phi}^{(1)}(x', y)\). [This implies \(\int dx \phi(x) = \int \tilde{dy} \tilde{\phi}(x)\) at one loop, which follows from the conservation of the nonsinglet axial current when quark masses are neglected.] Using this, Eq. (6) becomes

\[
\phi(x) = \tilde{\phi}(x) - \frac{\alpha_s}{2\pi} \int_{-\infty}^{\infty} dy [Z_{\phi}^{(1)}(x, y) \tilde{\phi}(y) - Z_{\phi}^{(1)}(y, x) \tilde{\phi}(x)],
\]

where for simplicity we have extended the integration range of \(y\) to infinity, which introduces an error at higher order. The expression for the matching factor \(Z_{\phi}^{(1)}(x, y)\) is given in Appendix A.

For a finite \(P_z\), we need to take into account the \(\mathcal{O}(m_N^2/P_z^2)\) meson-mass and \(\mathcal{O}(\Lambda_{QCD}^2/P_z^2)\) higher-twist corrections. Following a procedure similar to Ref. [45], we can derive the mass corrections to all orders in \(m_N^2/P_z^2\), which leads to the following relation between the pion DAs (for details see Appendix B):

\[
\phi(x) = \sqrt{1 + 4c \sum_{n=0}^{\infty} \frac{(4c)^n}{\int_{2n+1}^{\infty}}} \times \left[ (1 + (-1)^n) \tilde{\phi} \left( \frac{1}{2} - \frac{f_{2n+1}^n(1 - 2x)}{4(4c)^n} \right) + (1 - (-1)^n) \tilde{\phi} \left( \frac{1}{2} + \frac{f_{2n+1}^n(1 - 2x)}{4(4c)^n} \right) \right], \quad (9)
\]

where \(c = m_N^2/4P_z^2\) and \(f_+ = \sqrt{1 + 4c + 1} \).

The \(\mathcal{O}(\Lambda_{QCD}^2/P_z^2)\) correction can be derived in the same way as in Ref. [45], since the twist-4 operator involved is the same. The twist-4 effect can be implemented by adding a \(\tilde{\phi}_{\text{twist-4}}\) contribution to \(\tilde{\phi}\), such that

\[
\tilde{\phi}(x, \Lambda, P_z) \rightarrow \tilde{\phi}(x, \Lambda, P_z) + \tilde{\phi}_{\text{twist-4}}(x, \Lambda, P_z),
\]

where

\[
\tilde{\phi}_{\text{twist-4}}(x, \Lambda, P_z) = \frac{1}{8\pi} \int_{-\infty}^{\infty} dz \Gamma_0(-ixzP_z)\langle \pi(P)| O_u(z)|0 \rangle,
\]

\(\Gamma_0\) is the incomplete Gamma function and

\[
O_u(z) = \int_0^z dz_1 \bar{\psi}(0)[\gamma^\mu \gamma_5 \Gamma(0, z_1)D_\mu \Gamma(z, z_1)] + \int_0^z dz_2 \bar{\psi}(0)[\gamma^\mu D^\nu \Gamma(z_2, z_1)D_\nu \Gamma(z_1, z)\psi(\lambda)
\]

with \(\lambda^\mu = (0, 0, 0, -1)\). Equations (8)–(10) take into account the one-loop, mass and higher-twist corrections, respectively. We need to implement them step by step to achieve the final pion DA. For the higher-twist corrections, instead of computing them directly on the lattice, we only parametrize and fit them as a \(1/P_z^2\) correction after we have removed other leading-\(P_z\) corrections, as was done in Ref. [45].

III. NUMERICAL RESULTS AND DISCUSSION

In this section, we report the first results of a lattice-QCD calculation of the \(x\)-dependence of the pion DA. We use clover valence fermions on gauge ensembles with \(2 + 1 + 1\) flavors (degenerate up/down, strange and charm) with \(L \approx 3\) fm, corresponding to \(m_NL \approx 4.5\). The HISQ
ensembles are hypercubic (HYP)-smere [51] and the
clover parameters are tuned to recover the lowest pion mass
of the staggered quarks in the sea.\textsuperscript{1} HYP smearing has been
shown to significantly improve the discretization effects on
operators and shift their corresponding renormalizations
toward their tree-level values (near 1 for quark bilinear
operators). The results shown in this work are done using
correlators calculated from three source locations on 986
configurations. For each positive $z$-momentum $P_z$, the
matrix elements are averaged with their corresponding
$-P_z$ to improve the signal.

\section*{A. Results from pion quasidistribution amplitude}

We begin with the pion quasi-DA without the Wilson-line
renormalization, and then we follow similar steps to those
listed in our previous work on nucleon parton distribution
functions: First, we implement the one-loop and mass
corrections whose formulas are detailed in the previous
sections, and extrapolate to the infinite-momentum limit via
\begin{equation}
\alpha(x) + \beta(x)/P_z^2 [\text{and thereby remove the higher-twist terms}
\text{that come in at } O(A^2_{\text{QCD}}/P_z^2)].
\end{equation}
The true light cone pion DA
should be recovered. Figure 1 shows the results for the pion
DA at $\mu = 2$ GeV after implementing one-loop and mass
corrections at different momenta $P_z = 2, 3$ (in units of $2\pi/L$)
to the quasi-DA.\textsuperscript{2} We then extrapolate using these two
momenta to the infinite-momentum limit using the form
\begin{equation}
\alpha(x) + \beta(x)/P_z^2, \text{ shown in red, where a linear divergence is}
\text{present in the one-loop matching kernel (later, we will show}
\text{improved results for the pion DA where the power divergence}
\text{is removed by taking into account the Wilson-line normalization).}
\end{equation}
The dashed line is the asymptotic form $6x(1-x)$.

Our resulting curves are symmetric around $x = 1/2$, as
expected from the symmetry of the pion DA under the
interchange $x \leftrightarrow 1-x$. The pion DA has often been
expanded in terms of Gegenbauer polynomials in past
studies, and the dashed curve here contains only the zeroth
Gegenbauer polynomial. The other three curves are broader
than the asymptotic form, indicating contribution from
higher Gegenbauer polynomials.

We note several interesting features of this result.
First, the pion DA is expected to vanish outside the region
$x \in [0, 1]$ after taking the IMF limit. We see the $P_z = 2$
ulate quasi-DA is nonzero for $x \in [1, 1.7]$, and this range
shrinks to $x \in [1, 1.4]$ for $P_z = 3$. A similar pattern is
observed for the region $x < 0$. The distributions are moving
in the right direction as the pion DA will vanish outside $[0, 1]$ with $P_z \rightarrow \infty$. However, after taking the IMF limit via
extrapolation formula $\alpha(x) + \beta(x)/P_z^2$, we find there is still
residual distribution outside $x \in [0, 1]$. This is likely due to
using the approximation Eq. (8), where the cancellation
among $\hat{g}(x)$ outside the $x \in [0, 1]$ region is between an
order result and a perturbative expression, and is therefore
incomplete.\textsuperscript{3} This can be improved by including the higher-order
matching and going to larger momentum, which we
will explore more extensively in future work.

Second, the results near $x = 0$ and $x = 1$ are not reliable.
There are unphysical peaks and dips due to the linear
divergence in the one-loop matching in these regions,
which become smaller as $P_z$ becomes larger. The smallest-$x$
region is dominated by the smallest nonzero momentum
fraction, which is proportional to $1/L$ (where $L$ is
the lattice length along boosted-momentum direction), due to the
finite box size. To improve results near these regions
would require large momentum and large box size.

Third, the unphysical oscillatory behavior near $x = 0$
and $x = 1$ is largely due to the presence of a linear
divergence in the one-loop matching for the bare
long-link matrix element. In Refs. \cite{38,39}, it has been
shown that the power divergence (in the $a \rightarrow 0$ limit) in the
long-link operator can be removed to all orders by a mass
counterterm $\delta m$ (in the auxiliary $z$-field description of
the Wilson line), which is the same as in the renormalization
of an open Wilson line. After the Wilson-line renormalization,
the pion quasi-DA is improved such that it contains at most
logarithmic divergences. We will investigate this improved
quasi-DA numerically in the rest of the paper.

\textsuperscript{1}Other studies using the same setup are done in Refs. [52–55],
and no exceptional-configuration behavior was observed.

\textsuperscript{2}For this work, we initially calculate the pion quasi-DA for
three momenta, $P_z = 1, 2, 3$ (in units of $2\pi/L$), but the
corrections term for the smallest-momentum distribution is less
well-behaved, as observed in the nucleon PDF case [45]; thus, we
drop it in the rest of this work.

\textsuperscript{3}Although the difference here is formally of higher order, it
might have a sizable numerical effect.
B. Results from the improved pion quasidistribution amplitude

The improved pion quasi-DA without power divergence can be defined as [39]

\[
\bar{\phi}_{\text{imp}}(x, P_z) = \frac{i}{f_\pi} \int \frac{d^2z}{2\pi} e^{-i(x-1)P_zz - \delta m|z|} \times \langle \pi(P)|\bar{\psi}(0)\gamma_\tau\Gamma(0, z)\psi(z)|0\rangle, \tag{13}
\]

where \(\delta m\) should be determined nonperturbatively through studying the Wilson-line renormalization. It is worthwhile to comment that since the mass counterterm \(\delta m\) cancels all power divergence in the improved pion quasi-DA,\(^4\) when we do the perturbative matching between Eqs. (13) and (1), we need to remove the linear divergence present in the one-loop matching kernel for consistency. Moreover, as shown in Ref. [39] and below, \(\delta m\) is negative, the exponential factor \(e^{-\delta m|z|}\) then increases the weight of matrix elements with relatively large \(z\), and thereby increases the contribution at relatively small momentum when Fourier transforming to momentum space. It is therefore important to properly account for the higher-twist corrections.

We first explore the nonperturbative determination of \(\delta m\) discussed in Ref. [40] using the static-quark potential for the renormalization of Wilson loop. The Wilson loop \(W(t, r)\) of width \(r\) and length \(t\) is long in the \(t\)-direction such that higher excitations are sufficiently suppressed. The quark potential is then obtained as

\[
V(r) = -\frac{1}{a} \lim_{t \to \infty} \frac{\langle \text{Tr}[W(t, r)] \rangle}{\langle \text{Tr}[W(t - a, r)] \rangle},
\]

where \(a\) is the lattice spacing and the cusp anomalous dimensions from the four sharp corners of the Wilson loop are canceled between numerator and denominator. When \(r\) is larger than the confinement scale but shorter than the string breaking scale,\(^5\) the lattice data should be described by the energy of the static quark pairs

\[
V(r) = \frac{c_1}{r} + c_2 + c_3 r,
\]

where the \(c_1\) term is the Coulomb potential which dominates at short distance, \(c_3\) term is the confinement linear potential. The \(c_2\) term is twice the rest mass of the heavy quark, and we expect \(c_2 = \tilde{c}/a + \mathcal{O}(\Lambda_{\text{QCD}})\). Thus, the \(\delta m\) counterterm that cancels the linear divergence in the Wilson line is

\[
\delta m = -\frac{\tilde{c}}{2a} = -\frac{c_2}{2} + \mathcal{O}(\Lambda_{\text{QCD}}). \tag{16}
\]

\(^4\)At perturbative one-loop, it appears as a linear divergence, but more-divergent power divergences can appear at higher loops.

\(^5\)The onset of string breaking can be estimated by \(V(r) > 2m_B - m_Y = 1.1\ \text{GeV}\).

This leads to

\[
\delta m = -260 \pm 200 \text{ MeV}, \tag{17}
\]

where we have used the fitted value \(\delta m = -0.16/a\) from Fig. 2, which is 0.38 times of the one-loop value computed in Ref. [39], and we estimate the error by the size of \(\Lambda_{\text{QCD}}\). The error can be reduced by performing the computation at different \(a\) to extract the \(1/a\)-dependent term in \(c_2\).

As mentioned before, once the improved pion DA of Eq. (13) is used with \(\delta m\) determined nonperturbatively, the linear divergence in the one-loop matching kernel will be canceled by the \(\delta m\) counterterm as shown in Eq. (A6). In Ref. [39], it was demonstrated that in the limit \(\Lambda/P_z \to \infty\), only the Wilson-line self-energy diagram is divergent among the “real diagrams” [i.e. \(Z_\phi^{(1)}(x, y)\) of Eq. (7)] in one loop and in the Feynman gauge. Therefore, in a lattice perturbation theory calculation, one only needs to calculate this diagram, which is linearly divergent \((\propto \Lambda/P_z)\). Using the simplest version of gauge-field discretization, one finds the matching between the momentum and lattice cutoffs is \(\Lambda = \pi/a + \mathcal{O}(a^2)\). This result holds not only for the nonsinglet quasi-PDF operator used in Ref. [39], but also for the pion quasi-DA in this work. The “virtual diagrams” (i.e. \(C\) of Eq. (7)) will contain logarithmic divergence from the quark self-energy diagram, which can be removed by adding counterterms in the lattice action or treating the integration limits of \(C\) carefully. In Eq. (A6), the \(\Lambda/P_z \to \infty\) limit is not taken, so \(C\) is finite. We find that the difference between taking this limit and not is small, certainly within the error induced by the uncertainty of \(\delta m\).

The resulting pion DA using the improved pion quasi-DA of Eq. (13) with the central value of \(\delta m\) is shown in Fig. 3. The unphysical oscillations near \(x = 0\) and \(x = 1\) are largely removed. There are still small kinks in the unphysical region, but they are expected to vanish when

FIG. 2. The energy of the static-quark pairs fit to the functional form of Eq. (15). The point at \(r = 1\) is excluded from the fit to reduce discretization error. If we further exclude the \(r = 2\) point, then \(c_2\) is increased by 15%, still in the range of Eq. (17).
higher-order matching is taken into account and the $P_z \to \infty$ limit is approached.

The final result that includes the lattice statistical uncertainties, finite-$P_z$ corrections and the uncertainty of $\delta m$ estimated in Eq. (17) is presented in Fig. 4. Also shown in the same figure are the model calculation from the Dyson-Schwinger equation (DSE) [11], from the truncated Gegenbauer expansion fit to the Belle data for the $Dyson$-Schwinger equation (DSE) [11], from the truncated Gegenbauer polynomial expansion up to the eighth form factor [12] and from parametrizations of the pion DA with the parameters fit to lowest-moment calculations from lattice QCD in [13]. For the fit to the Belle data, we use the Gegenbauer polynomial expansion up to the eighth moment given in Ref. [12] and run to 2 GeV. For the fit to the lattice moment calculations, we have chosen two different parametrizations. One is simply a truncation of the Gegenbauer polynomial expansion of the pion DA to the second order $\phi(x) = 6x(1-x)[1 + a_2C_2^{\gamma}(2x-1)]$ (labeled “Param 1”) with the value of $a_2$ taken from [13]. The other is $\phi(x) = A(x(1-x))^B$ with $A$ and $B$ determined from the normalization condition and the second moment of the pion DA (labeled “Param 2”). The second parametrization is close to the DSE result, but differs from the first parametrization. The difference between them can be viewed as a rough estimate of errors from the truncation and reflects uncertainties in the parametrization, which are currently underestimated even though both bands have smaller errors than ours. A direct calculation of the $x$-dependence will help to resolve such uncertainties. Of course, this can be achieved only when the direct calculation reaches a sufficiently high accuracy, which is difficult at the current stage but might be improved in the foreseeable future. Nonetheless, the results of our direct calculation at 310-MeV pion mass is in agreement within errors with DSE, Belle data fit result and the parametrized reconstruction of pion DAs in the region near $x = 1/2$, although the two parametrized forms differ from each other. The uncertainty of our distribution is dominated by the $\delta m$ uncertainty, which can be largely removed by performing calculations at different lattice spacing. As before, we still have residual distribution outside the $[0, 1]$ region, which should vanish when larger momenta are reached and higher-order matching is taken into account in the future. Also, as is typical in an exploratory study, the pion mass in this work is still heavier than its physical value. However, the study of Ref. [56] shows that the leading chiral correction for $\phi_\pi(x)$ is proportional to $m_\pi^2$ with the chiral logarithm $m_\pi^2 \ln m_\pi^2$ completely absorbed by $f_{\pi}$. This property will simplify the chiral extrapolation in future computations. It is encouraging that our current result is qualitatively similar to other determinations using lattice-moment parametrization, models and fits to experimental data, and also favors a single-hump distribution in $\phi_\pi(x)$.

**IV. SUMMARY AND OUTLOOK**

In this work, we presented the first lattice-QCD calculation of the pion distribution amplitude using the large-momentum effective field theory (LaMET) approach. We derived the mass-correction formulation needed for the pion quasidistribution amplitude. We also implemented the Wilson-line renormalization in this work, which is important to remove the power divergences in LaMET approach, and found that it reduces the oscillation at the end points of the distribution amplitude. Finally, our result at 310-MeV pion mass shows similar behavior as previous studies done using DSE, a fit to the Belle data and as parametrizations with latest lattice moment result, and favors a single-hump structure.

However, in the current study, we have not accounted for all possible systematic uncertainties, and there are multiple
improvements that can be done in future studies. For example, in our work, it is clear that larger boosted momentum is needed for the pion distribution amplitude to make the result outside the physical region consistent with 0 than for the unpolarized nucleon parton distribution function. Finer lattice spacing would help reduce the uncertainty in the counterterm determined by the Wilson-loop study. Larger lattice box and also higher-order matching would reduce the unphysical kinks near $x = 1$ and 0. Last but not least, we hope this work will encourage following works to extensively study the distribution amplitude of the pion and other hadrons.

ACKNOWLEDGMENTS

JHZ thanks G. Bali, V. Braun, and M. Göckeler and A. Schäfer for valuable discussions and comments. The LQCD calculations were performed using the Chroma software suite [57]. We thank MILC Collaboration for sharing the lattices used to perform this study. Computations for this work were carried out in part on facilities of the USQCD Collaboration, which are funded by the Office of Science of the U.S. Department of Energy, and on the National Research Scientific Computing Center, a DOE Office of Science User Facility supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231. This work was partially supported by the U.S. Department of Energy via Grant No. DE-FG02-93ER-40762, a grant (Grant No. 11DZ2260700) from the Office of Science and Technology in Shanghai Municipal Government, grants from National Science Foundation of China (Grants No. 11175114, No. 11405104, No. 11655002), a DFG grant SCHW 458/20-1, the SFB/TRR-55 grant “Hadron Physics from Lattice QCD”, the MIT MISTI program, the Ministry of Science and Technology, Taiwan under Grants No. 105-2112-M-002-017-MY3 and No. 105-2918-I-002-003 and the CASTS of NTU. The work of JWC and XJ is supported in part by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics, within the framework of the TMD Topical Collaboration. L. C. J. is supported by the Department of Energy, Laboratory Directed Research and Development (LDRD) funding of BNL, under Contract No. DE-SC0012704.

APPENDIX A: ONE-LOOP MATCHING FOR QUASI-DA OF PION

In this appendix, we list the one-loop matching factors used throughout this paper. These factors have been obtained in Ref. [31]. However, as in Ref. [45], we keep a finite cutoff $\Lambda$ and do not take the limit $\Lambda \gg x P_z$.

For the pion distribution amplitude, expanding the matching factor $Z_\Phi(x, y, \Lambda, \mu, P_z)$ in Eq. (3) as

$$Z_\Phi(x, y, \Lambda, \mu, P_z) = \delta(x - y) + \frac{\alpha_s}{2\pi} Z_\Phi^{(1)}(x, y, \Lambda, \mu, P_z) + \cdots,$$

we have

$$Z_\Phi^{(1)}(x, y, \Lambda, \mu, P_z) / C_F = G_1(x, y, \Lambda, \mu, P_z) \theta(x < 0) + G_2(x, y, \Lambda, \mu, P_z) \theta(0 < x < y)$$
$$+ G_3(x, y, \Lambda, \mu, P_z) \theta(y < x < 1) + G_4(x, y, \Lambda, \mu, P_z) \theta(x > 1)$$

(A2)

with

$$G_1(x, y, \Lambda, \mu, P_z) = \frac{1}{x - y} + \frac{\Lambda(x, 1) - \Lambda(x, y)}{2 P_z(y - 1)(x - y)} + \frac{\Lambda(x, y) - \Lambda(x, 0)}{2 P_z(x - y) y} + \frac{\Lambda(x, y) + \Lambda(y, x)}{2 P_z(x - y)^2 y} + \left(\frac{1}{x - y} - \frac{x}{1 - y}\right) \ln(1 - x)$$
$$+ \left(\frac{1}{x - y} + \frac{1 - x}{y}\right) \ln(-x) + \left(\frac{x}{1 - y} - \frac{1 - x}{y} - \frac{2}{x - y}\right) \ln(y - x) - \left(\frac{1 - x}{2 y} + \frac{1}{2(x - y)}\right) \ln \frac{\Lambda(0, x)}{\Lambda(x, 0)}$$
$$+ \left(\frac{x}{2(1 - y) - 1 - x} - \frac{1}{2(1 - y)}\right) \ln \frac{\Lambda(1, x)}{\Lambda(x, 1)} + \left(\frac{x}{2(1 - y) - \frac{1 - x}{2 y} - \frac{1}{x - y}}\right) \ln \frac{\Lambda(x, y)}{\Lambda(y, x)},$$

$$G_2(x, y, \Lambda, \mu, P_z) = \frac{3}{2y} + \frac{1}{2(y - 1)} - \frac{2}{x - y} + \frac{2}{2 P_z(y - 1)(x - y)} + \frac{\Lambda(x, 0)}{2 P_z(1 - y)(x - y)} + \frac{(x + y - 2x)(\Lambda(x, y) + \Lambda(y, x))}{4 P_z(x - y)^2 y(1 - y)}$$
$$+ \left(\frac{x - 1}{y - x} + \frac{1}{y - x}\right) \ln \frac{P_z^2}{\mu^2} + \left(\frac{x - 1}{y - x} + \frac{1}{y - x}\right) \ln(4x) + \left(\frac{x}{x - 1} - \frac{1}{y - x}\right) \ln(1 - x)$$
$$+ \left(\frac{x - 1}{y} + \frac{2}{y} - \frac{x}{x - y} - \frac{2}{y} - \frac{x}{y - 1}\right) \ln(y - x) - \left(\frac{1}{2 y} + \frac{1}{2(x - y)}\right) \ln \frac{\Lambda(0, x)}{\Lambda(x, 0)}$$
$$+ \left(\frac{x}{2(1 - y) - \frac{1}{2(1 - y)} - \frac{1}{x - y}}\right) \ln \frac{\Lambda(1, x)}{\Lambda(x, 1)} + \left(\frac{x}{2(1 - y) - \frac{1}{2 y} - \frac{1}{x - y}}\right) \ln \frac{\Lambda(x, y)}{\Lambda(y, x)},$$

$$G_3(x, y, \Lambda, \mu, P_z) = G_2(1 - x, 1 - y, \Lambda, \mu, P_z),$$
$$G_4(x, y, \Lambda, \mu, P_z) = G_1(1 - x, 1 - y, \Lambda, \mu, P_z),$$

where $\Lambda(x, y) = \sqrt{\Lambda^2 + (x - y)^2 P_z^2} + (x - y) P_z$. 

094514-7
Zhang, Chen, Ji, Jin, and Lin

In this appendix, we derive the meson-mass corrections to the quasi-DA of the pion. For the pion DA, we need to calculate the same series sum as for the unpolarized parton distribution in Ref. [45],

$$K_n = \frac{\langle (1 - 2x)^{\alpha - 1} \rangle}{\langle (1 - 2x)^{\alpha - 1} \rangle} = \sum_{i=0}^{\infty} C_{n-i}^{(1)} = \frac{\lambda_{(\mu_1 \cdots \mu_n) P_{\mu_1} \cdots P_{\mu_n}}}{\Lambda P_{\xi}(x-y)^2}. \quad (B1)$$

where $c = m_0^2/4P_x^2$ and (...) means the indices enclosed are symmetric and traceless. The result for even $n = 2k$ is

$$\sum_{j=0}^{k} C_{n-j}^{(1)} = \frac{1}{\sqrt{1 + 4c}} \left[ \left( \frac{f_+}{2} \right)^{2k+1} + \left( \frac{f_-}{2} \right)^{2k+1} \right]. \quad (B2)$$

while for odd $n = 2k + 1$, it is

$$\sum_{j=0}^{k} C_{n-j}^{(1)} = \frac{1}{\sqrt{1 + 4c}} \left[ -\left( \frac{f_-}{2} \right)^{2k+2} + \left( \frac{f_+}{2} \right)^{2k+2} \right], \quad (B3)$$

where $f_+ = \sqrt{1 + 4c} \pm 1$.

With Eqs. (B2) and (B3), we perform an inverse Mellin transform on the moment relation of Eq. (B1)

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds x^n/(1-2x)^n = (1 - 2x)^{\alpha - 1}.$$ \quad (B4)

To extract $\phi(x)$ from $\hat{\phi}(x)$, let us rewrite Eq. (B1) for an even $n = 2k$ as

$$\langle (1 - 2x)^{2k-1} \rangle = \langle (1 - 2x)^{2k-1} \rangle \frac{\sqrt{1 + 4c}}{(f_+)^{2k+1} + (f_-)^{2k+1}}$$

$$= \langle (1 - 2x)^{2k-1} \rangle \frac{\sqrt{1 + 4c}}{(f_+)^{2k+1}} \times \sum_{n=0}^{\infty} \left( -1 \right)^n \left( \frac{f_-}{f_+} \right)^{(2k+1)n}. \quad (B5)$$

The inverse Mellin transform then leads to

$$\phi(x) - \phi(1-x) = 2\sqrt{1 + 4c} \sum_{n=0}^{\infty} \frac{(-f_-)^n}{f_+^{n+1}} \left[ \frac{1}{2} - \frac{f_-^{n+1}(1 - 2x)}{4f_+^{n}} \right]. \quad (B6)$$

Similarly, we have

$$\phi(x) + \phi(1-x) = 2\sqrt{1 + 4c} \sum_{n=0}^{\infty} \frac{f_-^{n+1}}{f_+^{n+1}} \left[ \frac{1}{2} - \frac{f_-^{n+1}(1 - 2x)}{4f_+^{n}} \right]. \quad (B7)$$

Therefore,

$$\phi(x) = \sqrt{1 + 4c} \sum_{n=0}^{\infty} \frac{f_-^{n+1}}{f_+^{n+1}} \left[ (1 + (-1)^n)\hat{\phi} \left( \frac{1}{2} - \frac{f_-^{n+1}(1 - 2x)}{4f_+^{n}} \right) + (1 - (-1)^n)\hat{\phi} \left( \frac{1}{2} + \frac{f_-^{n+1}(1 - 2x)}{4f_+^{n}} \right) \right]$$

$$= \sqrt{1 + 4c} \sum_{n=0}^{\infty} \frac{f_-^{n+1}}{f_+^{n+1}} \left[ (1 + (-1)^n)\hat{\phi} \left( \frac{1}{2} - \frac{f_-^{n+1}(1 - 2x)}{4f_+^{n}} \right) + (1 - (-1)^n)\hat{\phi} \left( \frac{1}{2} + \frac{f_-^{n+1}(1 - 2x)}{4f_+^{n}} \right) \right], \quad (B8)$$

where in the last line we have used $f_+ f_- = 4c$. Since $f_+ \gg f_-$ or $c$ and the quasi-DA $\hat{\phi}(x)$ vanishes asymptotically for large $x$, the above sum is dominated by the first term with $n = 0$. In practical calculations, we can reach reasonable accuracy by taking only the first few terms in the sum. In Refs. [58,59], it was argued that for hadron-to-vacuum matrix elements, the mass corrections also receive contributions from higher-twist operators that can be reduced to total derivatives of twist-two ones. We do not explicitly consider such terms, since they will anyway be part of the higher-twist corrections that are parametrized with a specific form in the present work.
PION DISTRIBUTION AMPLITUDE FROM LATTICE QCD