Risk of Pavement Fracture due to Eigenstresses at Early Ages and Beyond

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Risk of Pavement Fracture due to Eigenstresses at Early Ages and Beyond

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2 authors, including:

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Some of the authors of this publication are also working on these related projects:

- A computational framework for the analysis of rain-induced erosion in wind turbine blades

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ABSTRACT

Tensile cracks significantly affects the durability of concrete pavements leading to an increase in the costs of maintenance and rehabilitation. A model is developed that relates thermal, chemical and hygral evolutions at small scales due to different distress mechanisms to the risk of fracture at structural scale. The method is based on application of linear elastic fracture mechanics (LEFM) to eigenstresses that develop in an infinite- and a finite-length beam on an elastic foundation that represents the subgrade. Axial and bending contributions to energy release rate is determined for a worst-case scenario of an entirely cracked pavement section in functions of material properties, structural dimensions and eigenstress forces and moments. By way of example, the model is used to study the risk of fracture of concrete pavements due to two different mechanisms; (i): autogeneous shrinkage at early ages of placing concrete and (ii): thermal cycles both at short term and long term after temperature change. In addition, scaling relationships are developed that provide insight into the improvement of different structural and material properties for minimizing the risk of fracture.

Keywords: Linear Elastic Fracture Mechanics, Beam on Elastic Foundation, Eigenstresses,
INTRODUCTION

Aging pavement infrastructure along with increasing demand for road transportation, requires large investments in pavement maintenance, rehabilitation and construction. According to the most recent data from the United States Department of Transportation (ARTBA; U.S. Department of Transportation, Federal Transit Administration 2013) the government needs to spend annually $95.5 to $109 billion during 2014 and 2020, to only maintain current pavement condition and performance; and this cost would increase to $161.7 and $184.2 billion to improve national highways. In addition, pavement deterioration changes pavement surface and structural properties, resulting in an increase in rolling resistance and thus fuel consumption (Pouget et al. 2011; Louhghalam et al. 2013; Louhghalam et al. 2015; Chatti and Zaabar 2012; Sandberg et al. 2011). There is thus an urgent need to seek innovative materials and structural solutions with long-term durability that require less resources for maintenance and improve the environmental footprint of our nation’s roadway network. A starting point for this endeavor is research into the governing material and structural parameters that delineate pavement durability.

One of the major sources of concrete pavement degradation are tensile cracks. They provide sites for penetration of water and adverse chemicals and accelerate their deterioration. Hence, the design of durable pavements should rely on a fracture-based approach that takes into account the risk of fracture. Fracture is often due to the accumulation of a self-balanced stress field, the so called ‘eigenstress’ within the concrete pavement, when the deformation of the pavement is restrained by subgrade action and neighboring sections. These stresses develop at early ages and beyond due to various thermo-, hygro-, and chemo-mechanical distress mechanisms such as shrinkage, creep, freeze-thaw, alkali-silica-reactions (ASR).

Much of the current practice (e.g., AASHTO Guide for Design of Pavement Structures (AASHTO 1993) and Mechanistic-Empirical Design of New and Rehabilitated Pavement Structures (NCHRP1-37A 2004)) is based on strength approaches for pavement design (Yo-
der and Witczak 1975; Delatte 2014) that utilizes a linear elastic Westergaard-type model for stress analysis (Westergaard 1926; Westergaard 1927; Ioannides et al. 1985). While these approaches account for pavement fatigue using the stress-to-strength ratio and by using large safety factors, they do not consider fracture criteria based on surface energy dissipation. The model developed herein thus aims at shifting the paradigm from existing strength approaches to a fracture-based design. To this end, a model is developed that expresses pavement fracture criteria in terms of the eigenstresses within pavement. The model is based on the application of linear elastic fracture mechanics (LEFM) to a beam on elastic foundation. By way of illustration, the risk of fracture due to two distress mechanisms are investigated. The first problem studies the risk of fracture due to shrinkage-induced eigenstresses at early stages of concrete pavement placement. In the second example the impact of thermal eigenstresses on the risk of fracture is examined both right after applying a surface temperature gradient and thereafter.

MECHANICS-BASED MODEL FOR PAVEMENT FRACTURE

We choose the simplest model of a pavement: as an elastic beam on an elastic foundation representing the stiffness of the subgrade material. This elastic foundation opposes both the longitudinal and transversal deformations of the beam through line-spring actions in both horizontal and vertical directions. For long beams where thickness is small compared to the length, the moment caused by the horizontal spring is negligible, and the horizontal spring action can be applied at the neutral axis. When subjected to different thermal, hygral or chemical evolutions, eigenstresses \( \sigma^p \) build up within the infinitely long pavement that cannot deform. The eigenstresses can lead to fracture of the pavement if the energy release rate \( G \) reaches the fracture energy of concrete \( G_c \) (Griffith 1921; Irwin 1957). The worst case scenario is investigated here where the fully-cracked sections create a pavement of finite size \( \ell \), and the risk of fracture is determined using LEFM. Note that if the crack propagation were of interest one must consider the solution of a beam with partial cracks (Hong et al. 1997; Rice and Levy 1972; Roesler and Khazanovich 1997).
**Energy Release Rate**

Consider a continuous beam subjected to longitudinal eigenstress \( \sigma^0_{xx} (x) = \sigma^0 (x) \), with \( x \) the position vector. Fracture occurs when the energy release rate is equal to the fracture energy of the material:

\[
G = \frac{1}{2S} \int_{\Gamma} T^0 \cdot [[u]] \, da \leq G_c
\]  

with \( E_{pot} \) the potential energy and \( \Gamma \) the crack surface. The above equation for linear elastic fracture mechanics is obtained noting that the energy release rate \( G \) is the amount of stored potential energy that is released when the crack surface increases by \( \delta \Gamma \), i.e., \(-\delta E_{pot} = G \delta \Gamma\).

We consider Irwin’s argument (Irwin 1958), that the released energy is equal to the work needed to close the crack, with \( T^0 = \sigma^0_{xx} e_x \) the initial traction vector before fracture, and \( [[u]] \) the jump in displacement as a consequence of fracture when \( T^0 \) is applied to the fracture surface with opposite sign. Assuming a Navier-Bernoulli beam, the displacement jump is considered of the form,

\[
[[u]] = [[u_0 (x)]] + [[\omega (x)]] \times \mathbf{x}_S
\]

with \( [[u_0 (x)]] = [[u]] e_x \) the axial displacement jump at the neutral axis. Considering transversal cracking, the position vector is \( \mathbf{x}_S = (z - z_c) e_z + y e_y \) where \( z \) is the coordinate along the beam’s thickness, and \( z_c \) is the coordinate of the neutral axis of the pavement section \( S \). The rotation vector is \( [[\omega (x)]] = [[\omega_y (x)]] e_y \). The energy release rate is thus:

\[
G = \frac{1}{2S} \int_{\Gamma=S} \sigma^0_{xx} ([[u]] + (z - z_c) [[\omega_y]]) \, da
\]

For the break-through solution under consideration, the first order approximation of energy release rate is evaluated by obtaining the change of potential energy between two linear elastic equilibrium (minimum potential energy) states, i.e. before fracture when beam is subjected to eigenstress \( \sigma^0_{xx} \) and after fracture of the entire section:

\[
G = \frac{1}{2S} \left( N^0 [[u]] + M^0 [[\omega_y]] \right)
\]
where \( N^0 \) and \( M^0 \) are respectively the axial force and moment prior to fracture due to the initially self-balanced stress field \( \sigma_{xx}^0 \):

\[
N^0 = \int_S \sigma_{xx}^0 (\bar{x}) \, da \tag{5}
\]

\[
M^0 = \int_S (z - z_c) \sigma_{xx}^0 (\bar{x}) \, da \tag{6}
\]

In turn, \([u] = u^+ - u^-\) is the displacement jump in the beam’s reference axis when \( N^0 \) is imposed with opposite sign onto the crack surfaces; while \([\omega_y] = \omega_y^+ - \omega_y^-\) is the jump in rotation angle of the section when a moment \( M^0 \) is imposed with opposite sign onto the crack surfaces. Note that for the crack to be entirely open over the section, the following compatibility condition needs to be satisfied:

\[
[u] \geq [\omega_y] \frac{h}{2} \tag{7}
\]

otherwise decoupling the axial and bending contributions to the energy release rate is not possible. For an infinite beam, the initial axial force \( N^0 = N^p \) and moment \( M^0 = M^p \) are defined by Eqs. (5) and (6) with eigenstress \( \sigma_{xx}^0 (\bar{x}) = \sigma^p (\bar{x}) \).

The above analysis provides a first order approximation of the risk of fracture by using a beam response assumption. More refined results can be obtained by considering partial cracks that allow for gradual variation of stress distribution as well as taking into account the nonlinear fracture phenomena around the crack tip.

Separating the axial and bending actions in Eq. (3), in the following the energy required to close a crack is evaluated for both infinite and finite-length beam. To this end, the problem is expressed in terms of two boundary value problems using the principle of superposition (see Figure 1); i: a beam with no crack subjected to eigenstress forces \( N^0 \) or moments \( M^0 \) (Figures 1 a, d); ii: a cracked beam under an applied traction force \( N^0 \) or moment \( M^0 \) with the opposite sign at the crack surface (Figure 1 b, e). The total deformation is the
sum of the deformation of the two problems \((u = u_1 + u_2 \text{ and } \omega = \omega_1 + \omega_2)\). However, the
deformations of the first problem is zero due to symmetry. Thus one only needs to solve
the second problem shown in Figures 1(b) and (e) to determine the crack opening, and to
evaluate the corresponding energy release.

**Axial Contribution**

Consider a beam with cross sectional area \(S\) and elastic modulus \(E\) on an elastic foun-
dation under uniform axial traction. The differential equation for axial deformation \(u\) of the
neutral axis (i.e. at \(z = z_c\)) is well known:

\[
\frac{d^2 u}{dx^2} - \frac{k_H}{ES} u = 0 \quad (8)
\]

where \(k_H\) is horizontal stiffness of the spring. Let the midspan of the finite-length pavement
between two cracks be the origin of the coordinate system. The solution to the above
differential equation classically is \(u = C_1 \exp(-\xi \beta) + C_2 \exp(\xi \beta)\) with \(\xi = x/(\ell/2)\) and
\(\beta = \ell \sqrt{k_H/4ES}\), and \(C_1\) and \(C_2\) two constants to be determined from boundary conditions.

**Infinite Pavement**

To solve for the constants \(C_1\) and \(C_2\), we make use of the symmetry condition at the
origin \((u(\xi = 0) = 0)\), and the boundary condition \(N = ESu' = -N^p\), where \((\cdot)'\) denotes
the derivative with respect to \(x\), at the two crack surfaces at \(x = \pm \ell/2\) (see Figure 1(b)).
The axial deformation of the beam at the crack location is readily obtained:

\[
u (x = \ell/2) = -\frac{(N^p)}{2ES} \ell \times U_\infty (\beta) \quad (9)
\]

with \(U_\infty (\beta)\) given by (and plotted in Figure 2):

\[
U_\infty (\beta) = \frac{\tanh (\beta)}{\beta} \quad (10)
\]

For small values of \(\beta\), the above approaches unity in the form of \(U_\infty (\beta) \to 1 - \beta^2/3 + O (\beta^4)\).
In the absence of rotation of the crack surface, the crack opening is \( [u] = -2u \frac{\ell}{2} \), and the energy release rate corresponding to the axial eigenstress force, \( N^0 = N^p \) is obtained:

\[
G_{[u]} = \frac{1}{2S}N^p [u] = \frac{1}{2E} \left( \frac{\langle N^p \rangle}{S} \right) ^2 \ell \times U_\infty (\beta)
\] (11)

where \( \langle x \rangle = (x + |x|)/2 \). The stress intensity factor then reads:

\[
K_{[u]} = \sqrt{E G_{[u]}} = \frac{\langle N^p \rangle}{S} \sqrt{\frac{\ell}{2}} \times U_\infty (\beta)
\] (12)

The stress intensity factor is given in connection with the global energy release rate and does not represent any local stress singularity.

**Finite-Length Pavement**

For a finite-length pavement of length \( \ell = \mathcal{L} \), the initial state of stress is determined by solving differential equation (8) with symmetry condition at midspan, and traction free surface boundary condition, \( N = ESu' + N^p = 0 \), at \( x = \pm \ell/2 \). The eigenstress axial force before fracture is then:

\[
N^0 = N^p \left( 1 - \frac{\cosh (\beta \xi)}{\cosh (\beta)} \right)
\] (13)

The pavement is likely to crack in the middle, i.e. at \( \xi = 0 \), where the initial force has a maximum value equal to \( \alpha_N N^p \), where \( \alpha_N = 1 - 1/\cosh \beta \). Once the initial stress state in the finite beam is known, using a similar argument given in the previous section one needs to find the displacement \( u_2 \) in Figure 1(b) such that it satisfies \( N = ESu' = -\alpha_N N^p \) at the crack surface \( (x = 0) \). In addition, a zero resultant force, \( N = ESu' = 0 \), at the two edges \( (x = \pm \ell/2) \) of the finite-length pavement must be maintained. Solving Eq. (8) with these boundary conditions, the displacement at the crack surface is:

\[
u(0) = \alpha_N \frac{N^p}{ES} \frac{\ell}{2\beta} \coth \beta
\] (14)
Noting that the displacement jump at the crack-joint is \([u(0)] = +2u(0)\) and the initial eigenstress force in a finite-length beam is \(N^0 = \alpha_N N^p\), the energy release rate and the stress intensity factor read, respectively, as:

\[
G_{[u]} = \frac{1}{2E} \left( \frac{\langle N^p \rangle}{S} \right)^2 \ell \times U(\beta)
\]  
(15)

and

\[
K_{[u]} = \sqrt{EG_{[u]}} = \frac{\langle N^p \rangle}{S} \sqrt{\frac{\ell}{2}} \times U(\beta)
\]  
(16)

with \(U(\beta)\) given by (and illustrated in Figure 2):

\[
U(\beta) = \frac{(\cosh \beta - 1)^2}{\beta \sinh \beta \cosh \beta}
\]  
(17)

Function \(U(\beta)\) has a maximum value of \(U_{\text{max}}(\beta) = 0.2850\) at \(\beta = 2.3354570468\). It can be readily shown that for small values of \(\beta\), function \(U(\beta) \rightarrow \beta^2/4 - \beta^4/8 + O(\beta^6)\). The fact that function \(U(\beta)\) starts at zero when there is no horizontal bedding (\(\beta = 0\)), is in agreement with a zero axial force in an unconstrained finite-length pavement. Otherwise said, the risk of fracture of a finite-length pavement depends on the horizontal stiffness of the subgrade. Compared to the infinite pavement (Eq. (11)), the energy release rate of a finite-length beam (Eq. (15)) decreases by a factor of:

\[
\frac{U(\beta)}{U_\infty(\beta)} = \frac{(e^\beta - 1)^2}{(e^\beta + 1)^2} < 1
\]  
(18)

That is, the energy release rate of the finite-length beam is always less than that of the infinite beam, provided that the length of the beam \(L\) is the same as the spacing between cracks \(\ell\) generated in the infinite beam. This implies that the distance between the cracks in an infinite beam due to non-increasing axial forces is such that a second crack will not occur, leading to a stable fracture process.
**Bending Contribution**

We proceed in a similar fashion for the bending contribution, by solving the equation for transverse deformation \( w \) of an elastic beam on a Winkler foundation with stiffness \( k_V \):

\[
\frac{d^4w}{dx^4} + \frac{k_V}{EI} w = 0 \tag{19}
\]

where \( I \) is the beam’s moment of inertia. The general solution of the above is well known (Ugural and Fenster 2003):

\[
w = e^{-\frac{\sqrt{2}\gamma}{2}\xi} \left( C_1 \sin \frac{\sqrt{2}\gamma\xi}{2} + C_2 \cos \frac{\sqrt{2}\gamma\xi}{2} \right) + e^{\frac{\sqrt{2}\gamma}{2}\xi} \left( C_3 \sin \frac{\sqrt{2}\gamma\xi}{2} + C_4 \cos \frac{\sqrt{2}\gamma\xi}{2} \right) \tag{20}
\]

where \( \gamma = \ell \sqrt{k_V/16EI} \).

**Infinite Pavement**

In the case of an infinite beam, the coefficients \( C_1 \) to \( C_4 \) in equation (20) are determined by using symmetry condition at the origin, i.e. both the shear force \( Q_z \) and the beam’s rotation \( \omega_y \) are zero. Furthermore as observed in Figure 1(e), at the two crack surfaces, \( x = \pm \ell/2 \), the resultant shear force \( Q_z \) is zero \( (EIw''' = 0) \), and the resultant bending moment \( M \) is equal with opposite sign to the initial eigenstress moment \( M_p \) \( (EIw'' + M_p = 0) \). The beam’s displacement at the crack front is then readily found in the form:

\[
w = -\frac{M_p \ell^2}{EI} F_1 (\gamma) \left( \sin \frac{\sqrt{2}\gamma\xi}{2} \sinh \frac{\sqrt{2}\gamma\xi}{2} + F_2 (\gamma) \cos \frac{\sqrt{2}\gamma\xi}{2} \cosh \frac{\sqrt{2}\gamma\xi}{2} \right) \tag{21}
\]

with:

\[
F_1 (\gamma) = \frac{1}{4\gamma^2} \left( \frac{\cos \sqrt{2}\gamma}{2} \cosh \frac{\sqrt{2}\gamma}{2} - F_2 (\gamma) \sin \frac{\sqrt{2}\gamma}{2} \sinh \frac{\sqrt{2}\gamma}{2} \right)^{-1} \tag{22}
\]

\[
F_2 (\gamma) = \frac{\sinh \frac{\sqrt{2}\gamma}{2} \cos \frac{\sqrt{2}\gamma}{2} - \cosh \frac{\sqrt{2}\gamma}{2} \sin \frac{\sqrt{2}\gamma}{2}}{\sinh \frac{\sqrt{2}\gamma}{2} \cos \frac{\sqrt{2}\gamma}{2} + \cosh \frac{\sqrt{2}\gamma}{2} \sin \frac{\sqrt{2}\gamma}{2}} \tag{23}
\]
Note that doweling effects are neglected at the cracks. The angular rotation at the crack front ($\xi = 1$) thus reads:

$$\omega_y = - w'|_{\ell/2} = \frac{M_p}{2EI} \ell \times W_\infty (\gamma)$$

with $W_\infty (\gamma)$ given by (and plotted in Figure 3):

$$W_\infty (\gamma) = \frac{\sqrt{2} \cosh \sqrt{2}\gamma - \cos \sqrt{2}\gamma}{\gamma \sinh \sqrt{2}\gamma + \sin \sqrt{2}\gamma}$$

For small values of $\gamma$, the above approaches unity according to $W_\infty (\gamma) \rightarrow 1 - \gamma^4/45 + O(\gamma^6)$, while it decays to zero for large values of $\gamma$. Noting that crack opening due to bending is $[[\omega_y]] = 2\omega_y$, the energy release rate associated with bending action of an infinite pavement is:

$$G[[\omega]] = \frac{1}{2S} (M_p [[\omega]]) = \frac{(M_p)^2}{2ESI} \ell \times W_\infty (\gamma)$$

and the stress intensity factor reads:

$$K[[\omega]] = \sqrt{E G[[\omega]]} = M_p \sqrt{\frac{\ell}{2SI}} \times W_\infty = \frac{M_p}{W} \sqrt{\frac{1}{6} \ell \times W_\infty (\gamma)}$$

with $W = 2I/h$. Note that $\ell$ herein is the crack spacing.

**Finite-Length Pavement**

For a finite beam of length $\ell = L$, one needs to determine the initial state of eigenstresses before fracture. To this end, the coefficients $C_1$ to $C_4$ in Eq. (20) are determined by using symmetry condition ($w' = 0$ and $Q_z = 0$) at the beam’s midspan ($x = 0$), as well as traction free boundary condition ($Q_z = 0$ and $M^0 = EIw'' + M^p = 0$) at the two ends ($x = \ell/2$). Solving for the unknown coefficients in Eq. (20), the beam’s normalized displacement before cracking is expressed as:

$$\frac{w^0(x)}{w(0)} = \frac{1}{F_2(\gamma)} \sin \frac{\sqrt{2}\gamma \xi}{2} \sinh \frac{\sqrt{2}\gamma \xi}{2} + \cos \frac{\sqrt{2}\gamma \xi}{2} \cosh \frac{\sqrt{2}\gamma \xi}{2}$$
with the initial beam’s displacement at midspan:

\[ w(0) = \frac{(-M_p)}{EI} t^2 \times F_1(\gamma) F_2(\gamma) \]  

(29)

Similarly the normalized eigenstress moment before fracture reads:

\[ \frac{M^0(x)}{M_p} = 1 - 4\gamma^2 F_1(\gamma) \left( \cosh \frac{\sqrt{2} \gamma \xi}{2} \cos \frac{\sqrt{2} \gamma \xi}{2} - F_2(\gamma) \sinh \frac{\sqrt{2} \gamma \xi}{2} \sin \frac{\sqrt{2} \gamma \xi}{2} \right) \]  

(30)

For different \( \gamma \) values, Figure 4(a) and (b) respectively illustrate the beam’s displacements and moments before fracture, normalized by their corresponding values at midspan. The maximum moment occurs in midspan if the shear force has only one root along \( \xi \in [-1, 1] \).

This condition is essential in application of the model developed hereafter. It can be shown that if \( \gamma \leq 5.57 \) this condition is satisfied (see Figure 5). For this range of \( \gamma \) values, the ratio of the maximum initial moment to the initial eigenstress moment of an infinite beam \( M_p \) is:

\[ \max_{x=0} \frac{M^0(x)}{M_p} = \alpha_M = 1 - 4\gamma^2 F_1(\gamma) \]  

(31)

Once the initial state of the beam before fracture is known, one obtains the displacement solution of Eq. (19) that satisfies \( M = EI w'' = -\alpha_M M_p \) (see Figure 1(e)) at the crack surface \( (x = 0) \). In addition, the moments at the two ends of the beam \( (x = \pm \ell/2) \) must be zero, that is \( M = EI w'' = 0 \). Furthermore, shear forces are zero both at midspan and at the ends of the beam due to symmetry and traction free condition leading to \( w''' = 0 \). The beam’s rotation at crack location is thus obtained:

\[ \omega_y(0) = -\alpha_M \frac{M_p}{2EI} \ell \times F_4(\gamma) \]  

(32)
with $F_4(\gamma)$ given by:

$$F_4(\gamma) = -\frac{\sqrt{2}}{\gamma}\left(1 - \frac{\sinh\left(\frac{\sqrt{2}}{2}\gamma\right)\sin\left(\frac{\sqrt{2}}{2}\gamma\right)}{\cosh\left(\frac{\sqrt{2}}{2}\gamma\right) + \cos\left(\frac{\sqrt{2}}{2}\gamma\right) - 2}\right) \left(e^{\frac{\sqrt{2}}{2}\gamma} - \cos\left(\frac{\sqrt{2}}{2}\gamma\right)\sin\left(\frac{\sqrt{2}}{2}\gamma\right) + \sinh\left(\frac{\sqrt{2}}{2}\gamma\right)\right)$$

$$+ \frac{e^{\frac{\sqrt{2}}{2}\gamma}\cos\left(\frac{\sqrt{2}}{2}\gamma\right) - e^{\frac{\sqrt{2}}{2}\gamma}\sin\left(\frac{\sqrt{2}}{2}\gamma\right)}{\cosh\left(\frac{\sqrt{2}}{2}\gamma\right) - \cos\left(\frac{\sqrt{2}}{2}\gamma\right)\sin\left(\frac{\sqrt{2}}{2}\gamma\right)}\right)$$

(33)

Noting that the crack opening is $[\omega_y] = -2\omega_y(0)$, and the initial eigenstress moment is $M^0 = \alpha_M M^p$, the energy release rate and the stress intensity factor are obtained as:

$$G_{[\omega]} = -\frac{1}{S} \alpha_M M^p \omega_y(0) = \frac{(M^p)^2}{2ESI} \ell \times W(\gamma)$$

(34)

and:

$$K_{[\omega]} = \sqrt{E^2 G_{[\omega]}} = M^p \sqrt{\frac{\ell}{2SI}} \times W(\gamma) = \frac{M^p}{W} \sqrt{\frac{1}{6} \ell \times W(\gamma)}$$

(35)

with $W(\gamma)$ given by (and plotted in Figure 3 along with the ratio $W(\gamma)/W_\infty(\gamma)$):

$$W(\gamma) = \alpha_M^2 F_4 = (1 - 4\gamma^2 F_1(\gamma))^2 F_4(\gamma)$$

(36)

For small values of $\gamma$, function $W(\gamma)$ approaches zero according to $W(\gamma) \to \gamma^4/48 + O(\gamma^8)$. As observed in Figure 3, for $\gamma < 3.3446$, the infinite-length beam exhibiting transversal cracks with a spacing of $\ell$ has a greater energy release rate; meaning that the fracture process is stable. A higher energy release rate is observed for $3.3446 < \gamma < 5.5531$, where $\alpha_M > 1$ meaning that the initial moment in the finite beam is greater than that of the infinite beam. Thus, as long as $\gamma < 3.3446$, the ratio $\alpha_M$ is less than unity. This means that the maximum initial moment in a finite-length beam is less than that of an infinite beam, and, as a consequence, the fracture process is stable. In contrast, when an infinite beam subjected to an eigenstress moment generates a crack spacing of $\ell$, the next fracture at half-length will be generated; –that is, the crack process is unstable. We therefore shall restrict
the use of this model to values of $\gamma < 3.3446$. For a typical jointed concrete pavement, the joints spacing is 15 ft (4.572 m). In addition, the typical value for the Winkler length $\sqrt{EI/kV}$ varies between 1-3 m (Louhghalam et al. 2014); thus $\gamma$ ranges from 0.762 to 2.286 below the critical value of $\gamma_{cr} = 3.3446$. This ensures that the maximum moment occurs in midspan of the beam and the fracture process is stable, which is in agreement with the model assumptions.

The compatibility condition in (7) indicates that for the crack to open over the entire section, the ratio of moment to axial force must be less than or equal to $U_\infty(\beta)/W_\infty(\gamma) \times h/6$.

In the above analysis only the eigenstresses in the pavement are considered while the stresses within the subgrade are disregarded. However, a constant uplift or downlift of the foundation due to homogenous displacement of the subgrade does not generate eigenstresses, and will thus not affect the risk of fracture.

**APPLICATIONS**

By way of application, we show that the above developments provide a convenient framework for a fracture-based design of concrete pavements, which should be the main focus when durability is a criterion. The first example examines concrete pavement at early ages. The above fracture criterion is used to determine the critical hydration degree and critical time before which the joints must be cut to avoid fracture. The second example investigates concrete pavements under temperature cycles with a focus on identifying the important structural and material properties that result in pavement fracture.

**Concrete at Early Ages: Criterion for Joint Cutting**

As concrete hardens through the hydration process, its mechanical properties evolve from liquid to stone. The hydration of cement also results in shrinkage which induces eigenstresses of thermal and chemical origin within concrete (Ulm and Coussy 1995; Ulm and Coussy 1996; Ulm and Coussy 1997). Given the typical thickness of pavements, thermal eigenstresses related to the exothermic hydration can be neglected (Ulm and Coussy 2001), whereas autogenous shrinkage related to volume changes associated with the hydration is
the dominating factor for risk of fracture evaluations. These chemical eigenstresses can lead
to fracture and significantly reduce concrete durability. Situated within the framework of
continuum microporomechanics (Dormieux et al. 2006) the eigenstresses can be expressed in
an incremental form in function of hydration degree $\alpha$ (Ulm et al. 2014):

$$d\sigma^p (x) = (1 - 2\nu (\alpha)) [(1 - b (\alpha)) d\sigma^* (x) - b (\alpha) \ dp (x)]$$

(37)

where $\nu (\alpha)$ is the Poisson’s ratio in function of the hydration degree $\alpha$, and $b (\alpha) = 1 - K (\alpha)/K^s$ is the Biot coefficient, with $K$ and $K^s$ the bulk moduli of the composite and solid
phase. Finally, $\sigma^*$ and $p$ stand respectively for the eigenstress generated in the solid phase and the pressure in the assumed saturated porosity.

After placing concrete pavements, these chemo-mechanical eigenstresses start accumu-
lating. To release eigenstresses and avoid uncontrolled and random cracks, joints are cut to
a depth of $1/4 - 1/3$ of pavement thickness after some time $t$ of placing concrete (Okamoto
et al. 1994). The cutting of joints can be viewed as a transition from an infinite beam to a
finite-length beam with lower energy release rate as discussed in the previous section.

To simplify the calculations we assume a homogeneous development of the incremental
eigenstress defined by (37); so that the eigenstress only depends on the hydration degree;
i.e. $\sigma^p (\alpha) = \int d\sigma^p (\alpha)$, and is constant over the pavement thickness. Thus, according to Eq.
(5) and (6) only an axial eigenstress force, $N^0 = S\sigma^p (\alpha)$, is generated, while the eigenstress
moment, $M^0$, is zero. Applying the LEFM expression (12), the stress intensity must be
smaller or equal the fracture toughness, $K_c = K_c (\alpha)$, that is:

$$K_c [u] = \langle \sigma^p (\alpha) \rangle \sqrt{\frac{\ell}{2} \times U_\infty (\beta)} \leq K_c (\alpha)$$

(38)

Noting that the stress intensity has a maximum as $\beta$ goes to zero (see functional relation of
$U_\infty (\beta)$ in Figure 2) and decreases with increasing $\beta$, the smallest crack spacing for a specific
level of eigenstresses in an infinite beam is obtained when $\beta$ approaches zero. In addition,
the risk of secondary cracks are low for small $\beta$. However for large values of $\beta$ (e.g., for pavements with large horizontal subgrade stiffness), a small increase in eigenstresses may cause secondary cracking.

The distance between two consecutive cracks generated by early-age eigenstresses can be determined by inverting Eq. (38) at the limit case $K_{[\mu]} = K_c$:

$$\ell_c (\alpha) = 2 \sqrt{\frac{E (\alpha) S}{k_H}} \arctanh \left( \sqrt{\frac{k_H}{E (\alpha) S}} \left( \frac{K_c (\alpha)}{\langle \sigma^p (\alpha) \rangle} \right)^2 \right)$$  \hspace{1cm} (39)$$

From a design perspective, however, one needs to use the highest stress intensity corresponding to $\beta$ values approaching zero:

$$\ell_c (\alpha) = \left( \frac{\sqrt{2} K_c (\alpha)}{\langle \sigma^p (\alpha) \rangle} \right)^2$$  \hspace{1cm} (40)$$

The above equation implies that if a chosen joint spacing $\mathcal{L}$, is smaller than the critical crack spacing $\ell_c (\alpha)$ generated in an infinite pavement, the fracture will not occur. Alternatively, for a given joint distance $\mathcal{L}$, one obtains a critical hydration degree, $\alpha_c$, for which $\mathcal{L} = \ell_c (\alpha_c)$.

Therefore if the joints are cut at a hydration degree smaller than $\alpha_c$, no crack will occur due to shrinkage-induced eigenstresses, that is:

$$\alpha \leq \alpha_c : \alpha_c = \text{Inv}_{\mathcal{L} = \ell_c (\alpha_c)} \ell_c (\alpha)$$  \hspace{1cm} (41)$$

For purpose of illustration, we assume a linear chemo-elastic relationship for these incremental eigenstresses as $d\sigma^p = -E (\alpha) \beta_s d\alpha$, where $\beta_s < 0$ is the shrinkage coefficient, i.e. a chemical dilation coefficient relating the degree of hydration to the volumetric strain (Ulm and Coussy 1995; Ulm and Coussy 1996; Wittmann 1976). For example, consider a linear chemo-mechanical model for evolution of elastic modulus and fracture toughness as $E (\alpha) = E (\infty) \langle (\alpha - \alpha_0)/(1 - \alpha_0) \rangle$ and $K_c (\alpha) = K_c (\infty) \langle (\alpha - \alpha_0)/(1 - \alpha_0) \rangle$, where $E (\infty)$ and $K_c (\infty)$ are asymptotic values of the Young’s modulus and the fracture toughness,
whereas $\alpha_0$ is the degree of hydration at the solid percolation threshold. Substituting the shrinkage-induced eigenstress into Eq. (40), the fracture criterion reads:

$$L \leq \ell_c (\alpha) = \left( \frac{2\sqrt{2}K_c (\infty)}{E (\infty) \langle -\beta_s \rangle (\alpha - \alpha_0)} \right)^2 \Leftrightarrow \alpha \leq \alpha_c = \alpha_0 + \frac{2\sqrt{2}K_c (\infty)}{E (\infty) \langle -\beta_s \rangle \sqrt{L}} \quad (42)$$

A more realistic chemo-mechanical model is a logarithmic model for evolution of mechanical properties in the form of $E (\alpha) = E (\infty) \langle \ln (\alpha/\alpha_0) \rangle / \ln (1/\alpha_0)$ for the Young’s modulus and $K_c (\alpha) = K_c (\infty) \langle \ln (\alpha/\alpha_0) \rangle / \ln (1/\alpha_0)$ for fracture toughness (Hoover and Ulm 2015). Thus the eigenstress is $\sigma^p = E (\infty) \langle -\beta_s \rangle / \ln (1/\alpha_0) \times (\alpha \langle \ln (\alpha/\alpha_0) \rangle - (\alpha - \alpha_0))$ and the spacing between two cracks reads:

$$L \leq \ell_c (\alpha) = \left( \frac{\sqrt{2}K_c (\infty)}{E (\infty) \langle -\beta_s \rangle} \times \frac{\langle \ln \alpha/\alpha_0 \rangle}{\alpha \langle \ln \alpha/\alpha_0 \rangle - \alpha_0 (\alpha/\alpha_0 - 1)} \right) \quad (43)$$

For hydration degrees around the percolation threshold $\alpha_0$, the critical hydration degree can then be developed in the form:

$$\alpha \leq \alpha_c \approx \alpha_0 + \frac{\sqrt{2}K_c (\infty)}{E (\infty) \langle -\beta_s \rangle \sqrt{L}} \quad (44)$$

For instance, for typical values of concrete fracture toughness $K_c (\infty)=0.6\text{MPa}\sqrt{\text{m}}$ (Akono et al. 2011; Hu and Wittmann 2000); Young’s modulus $E (\infty) =30,000 \text{ MPa}$; shrinkage coefficient $\langle -\beta_s \rangle =50 \times 10^{-6}$ (Ulm and Coussy 1996; Acker 2004), and pavement joint spacing $L =15\text{ft} = 4.572\text{m}$, the critical degree of hydration is estimated as $\alpha_c = 0.36$, where a typical value of $\alpha_0 = 0.1$ was used for the degree of hydration at percolation threshold. Eq. (44) also provides scaling relationship for the critical degree of hydration. That is, one can change pavement joint spacing $L$ by either increasing the toughness-to-stiffness ratio (Qomi et al. 2014) or reducing shrinkage coefficient $\beta_s$, for instance by using admixtures (Pease 2005; Weiss et al. 2008). Specifically, to increase the joint distance by a factor of $\lambda$, one would need to increase the ratio of fracture toughness to elastic modulus by a factor of $\lambda^{1/2}$.
In return, an increase in shrinkage would need to be compensated by the same increase in the toughness-to-stiffness ratio.

Finally relating the hydration degree to time \( t \) and temperature history (Ulm and Coussy 1997), the critical time for cutting pavement joints can be determined. To this end, we use the kinetics of hydration and solve the differential equation:

\[
\frac{d\alpha}{dt} = A(\alpha) \exp\left(-\frac{E_a}{RT}\right) \tag{45}
\]

where \( A(\alpha) \) is the normalized affinity of the chemical reaction (Ulm and Coussy 2001), which is obtained from a calorimetric test, whereas \( E_a \) is the activation energy, \( R \) is the universal gas constant and \( T \) is the absolute temperature. The critical time is then obtained from:

\[
t_c = \int_0^{\alpha_c} \frac{d\alpha}{A(\alpha) \exp\left(\frac{E_a}{RT}\right)} \tag{46}
\]

If the joints are cut later than \( t_c \), the cracks may not occur under the saw cuts. Hence, while cutting joints before \( t_c \) prevents random uncontrolled crack formation due shrinkage-induced eigenstresses, it cannot guarantee proper joint formation.

It is worth noting that in the above analysis the risk of fracture due to chemo-mechanical eigenstresses corresponding to chemical shrinkage is investigated while the effect of drying shrinkage, that induces nonuniform hygro-mechanical eigenstress profile over the thickness of the pavement is not considered.

**Temperature Cycles**

As second example we investigate the risk of fracture of a pavement section with initial temperature \( T_0 \), subjected to an external surface temperature \( T_{ext} \) at time zero. As shown later in this section at small times right after the temperature change, there is a gradient of temperature over the pavement thickness, while the temperature gradient disappears when approaching the steady-state condition. In the following, we examine the pavement fracture criteria due to thermal eigenstresses in both steady-state and transitory conditions.
respectively at large and small times.

**Steady-state conditions**

Consider a pavement section with thickness \( h \) subjected to thermal cycles. Under steady-state conditions, the solution to the one-dimensional heat equation, \( \partial^2 T / \partial Z^2 = 0 \), provides a linear temperature profile over the pavement thickness. Here, we consider a linear thermal exchange condition at the two boundaries, i.e. \( q(Z) = \kappa_a (T(Z) - T_{ext}) \) at \( Z = 0 \) and \( q(Z) = \kappa_s (T(Z) - T_{soil}) \) at \( Z = h \); where \( q = -k dT/dZ \) is the heat flux according to Fourier’s law with \( k \) the heat conductivity of the material. The thermal exchange coefficients \( \kappa_a \) and \( \kappa_s \) account for the flow of heat respectively from the atmosphere with external temperature \( T_{ext} \) and from the subgrade with Temperature \( T_{soil} \) into the pavement structure. Thus the temperature profile that satisfies the above boundary conditions reads \( T = C_1 + C_2 \frac{Z}{h} \) with:

\[
C_1 = \frac{(k + h\kappa_s) \kappa_a T_{ext} - k\kappa_s T_{soil}}{k (\kappa_a - \kappa_s) + h \kappa_a \kappa_s} \tag{47}
\]

\[
C_2 = -\frac{\kappa_a \kappa_s h}{2k (\kappa_s - \kappa_a) + 2h \kappa_a \kappa_s} (T_{ext} - T_{soil}) \tag{48}
\]

We also note that for a subgrade with no or little heat exchange when \( \kappa_s \to 0 \) the constant \( C_2 = 0 \), and the temperature profile \( T(z) = T_{ext} \) is constant over the pavement thickness. In other words, under steady-state conditions, \( t \to \infty \), only a heat exchange with the subgrade will entail a temperature gradient. In the absence of such an exchange, only axial eigenstress forces will develop in the pavement section:

\[
N^p = -\alpha_T E S (T_{ext} - T_0) \tag{49}
\]

with \( \alpha_T \) the thermal expansion coefficient. Using Eq. (12), the LEFM fracture criterion then reads here:

\[
K_{[\{u\}]} = \alpha_T E \langle T_0 - T_{ext} \rangle \sqrt{\frac{\ell}{2}} \times U(\beta) \leq K_c \tag{50}
\]
with $U(\beta)$ given by Eqs. (10) and (17) for respectively infinite and finite-length pavements. Thus to avoid pavement fracture, the maximum temperature change applied to the pavement must be limited to:

$$\langle T_0 - T_{ext} \rangle \leq \frac{\sqrt{2K_c}}{\alpha_T E \sqrt{U(\beta)}} \times \ell^{-1/2}$$  \hspace{1cm} (51)

For finite-length pavements, the above equation expresses the allowable temperature change in function of joint spacing $\ell = \mathcal{L}$, whereas for infinite pavements, the relation estimates the distance between cracks as a function of temperature variations. Also note that only when $T_0 > T_{ext}$ one needs to account for a risk of fracture. For a finite-length pavement at small values of $\beta$, Eq. (51) becomes:

$$\langle T_0 - T_{ext} \rangle \leq \frac{4\sqrt{2K_c}}{\alpha_T} \times \left( \frac{S}{E k_H} \right)^{1/2} \times \mathcal{L}^{-3/2}$$ \hspace{1cm} (52)

The above provides a scaling relationship for the allowable temperature change. For instance to increase the allowable temperature variation by a factor of $\lambda$ one can decrease the joint spacing of pavement by factor of $\lambda^{3/2}$ or increase the thickness by $\lambda^{1/2}$. The same can be achieved by decreasing the horizontal subgrade stiffness by a factor of $\lambda^{1/2}$. From a material design perspective, the maximum allowable temperature change can be improved by increasing the ratio of fracture toughness of concrete to the square root of its Young’s modulus.

**Transitory conditions**

To study the effect of temperature gradient, we consider a pavement section with initial temperature $T_0$ ($T(Z,t < 0) = T_0$) subjected to a constant surface temperature at time $t = 0$: $T_{ext}$ ($T(Z = 0, t > 0) = T_{ext}$). Assuming there is no heat exchange between the pavement and subgrade, $q(Z = h, t) = -k \partial T/\partial Z = 0$, the temperature profile is determined by solving the heat equation:

$$\frac{\partial T}{\partial t} = D_h \frac{\partial^2 T}{\partial Z^2}$$  \hspace{1cm} (53)
where $D_\theta = k/C$ is the thermal diffusivity of the material (with typical value of $D_\theta = 4 \times 10^{-3} \text{m}^2/\text{h}$ for concrete (Ulm and Coussy 2001)) and $C$ is volume heat capacity. We define the characteristic time of heat diffusion as $\tau = h^2/D_\theta$, which takes values around 10 hours for a typical concrete pavement of thickness $h = 0.2$ m. Using the following linear transformations,

$$
\bar{z} = \frac{Z}{h}, \quad \bar{t} = \frac{t}{\tau}, \quad \bar{T} = \frac{(T - T_{\text{ext}})}{(T_0 - T_{\text{ext}})}
$$

Eq. (53) and the initial and boundary conditions are expressed in dimensionless forms:

$$
\frac{\partial \bar{T}}{\partial \bar{t}} = \frac{\partial^2 \bar{T}}{\partial \bar{z}^2} \quad \quad (55)
$$

$$
\bar{T} (\bar{z}, 0) = 1 \quad \quad (56)
$$

$$
\bar{T} (\bar{z} = 0, \bar{t} > 0) = 0 \quad \quad (57)
$$

$$
\frac{\partial \bar{T}}{\partial \bar{z}} (\bar{z} = 1, \bar{t}) = 0 \quad \quad (58)
$$

The solution of the above set of equations for the case where $\bar{T} (\bar{z} = 1, \bar{t}) \leq 1$ reads:

$$
\bar{T} (\bar{z}, \bar{t}) = \frac{T - T_{\text{ext}}}{T_0 - T_{\text{ext}}} = \sum_{n=0}^{n=\infty} \frac{4}{\pi (2n + 1)} \sin \left( \frac{(2n + 1) \pi \bar{z}}{2} \right) \exp \left( -\frac{(2n + 1)^2 \pi^2 \bar{t}}{4} \right) \quad \quad (59)
$$

The so obtained temperature profile is plotted in Figure 6 over pavement thickness for different values of $\bar{t}$. As $\bar{t}$ increases the temperature profile approaches the one of steady-state condition. To find the eigenstress forces and moments due to temperature change $T - T_0$ we rewrite Eq. (5) and Eq. (6), that were originally expressed in the beam coordinate system $z = h/2 - Z$, in the $\bar{z}$ coordinate system:

$$
N^p (\bar{t}) = N^p (\infty) \int_0^1 (1 - \bar{T} (\bar{z}, \bar{t})) \ d\bar{z} \quad \quad (60)
$$

$$
M^p (\bar{t}) = N^p (\infty) \frac{h}{2} \int_0^1 (1 - 2\bar{z}) (1 - \bar{T} (\bar{z}, \bar{t})) \ d\bar{z} \quad \quad (61)
$$
where \( N_p(\infty) \) is the axial eigenstress force according to Eq. (49) under steady-state conditions. Note that in the above analysis only axial and bending stresses are considered and the effect of nonlinear self-equilibrating stresses (Ioannides and Khazanovich 1998) is disregarded. Substituting the dimensionless temperature from Eq. (59) into Eq. (60) and dividing by \( N_p(\infty) \), the temporal variation of the normalized axial eigenstress force is obtained:

\[
\frac{N_p(t)}{N_p(\infty)} = 1 - \sum_{n=0}^{\infty} \frac{8 \exp \left( -\frac{(2n+1)^2 \pi^2}{4 t} \right)}{\pi^2 (2n+1)^2} \left( 1 - \pi \right)^2
\]

(62)

The above expression (also illustrated in Figure 7) indicates that as time increases, the axial force approaches its corresponding value at steady-state condition. Similarly, the normalized eigenstress moment due to the temperature gradient is also obtained by dividing Eq. (61) by \( N_p(\infty)h/2 \):

\[
\frac{M_p(t)}{N_p(\infty)(h/2)} = \sum_{n=0}^{\infty} \frac{4 (-1)^n}{\pi (2n+1) - 1} \frac{8 \exp \left( -\frac{(2n+1)^2 \pi^2}{4 t} \right)}{\pi^2 (2n+1)^2} \left( 1 + \pi \right)^2
\]

(63)

To study pavement fracture in transitory condition, one needs to consider the combined effect of axial forces and bending moments in the energy release rate, that is \( G = G_{[u]} + G_{[\omega]} \).

Thus, using Eq. (3), (15) and (26) the energy release rate becomes:

\[
G(t) = \frac{1}{2E} \left( \left( \frac{N_p}{S} \right)^2 U(\beta) + \frac{(M_p)^2}{SI} W(\gamma) \right) \times \ell
\]

(64)

The normalized value is then obtained by dividing the above energy release rate by the energy release rate corresponding to steady-state condition: \( G(\infty) = \frac{1}{2E} (\langle N_p(\infty) \rangle / S)^2 U(\beta) \times \ell \), for \( T_0 > T_{ext} \), for which the beam is under tensile axial force, i.e. \( N_p(\infty) > 0 \):

\[
\frac{G(t)}{G(\infty)} = \left( \frac{N_p(t)}{N_p(\infty)} \right)^2 + 3 \left( \frac{M_p(t)}{N_p(\infty) h/2 \times \omega} \right)^2 \times \omega
\]

(65)

with \( \omega = W(\gamma) / U(\beta) \). The variation of the normalized energy release rate with time
illustrated in Figure 9 peaks at small $t$ for large quantities of $\varpi$. Noting that in common practice the primary focus is to design for the maximum admissible temperature at steady-state condition (for example according to Eq. (51)), if the ratio $G(t)/G(\infty)$ is greater than one, special attention must be paid to the transitory condition as the bending moment can increase the energy release rate. To further investigate this ratio at different times and to develop scaling relationships it is useful to obtain approximate closed form relations between the normalized bending moment and the axial force. To this end, we examine the problem at two time-scales.

**Short time-scale**

We first consider the solution for short time-scales where $\bar{t} = \delta t^*$ and $z = \sqrt{\delta} z^*$, with $\sqrt{\delta} << 1$. For this case the absence of heat exchange at subgrade $\partial T/\partial z^* (z^* = 1/\sqrt{\delta}, t^*) = 0$ translates into a zero heat flux as $z^* \to \infty$, and the temperature profile is characterized by the solution of the one dimensional heat equation for an infinite half-space:

$$\bar{t} << 1; \ T = \text{erf} \left( \frac{z^*}{2\sqrt{\bar{t}^*}} \right) = \text{erf} \left( \frac{z}{2\sqrt{\bar{t}}} \right)$$

(66)

with $\text{erf}(x)$ the error function. Substituting Eq. (66) into respectively Eqs. (60) and (61) the normalized axial force and bending moment are obtained:

$$\frac{N_p (\bar{t})}{N_p (\infty)} = \text{erfc} \left( \frac{1}{2\sqrt{\bar{t}}} \right) + \sqrt{\frac{4}{\pi}} \left( 1 - \exp \left( -\frac{1}{4\bar{t}} \right) \right) \approx \sqrt{\frac{4}{\pi}} \bar{t} \ (0 < \bar{t} < 1/16)$$

(67)

$$\frac{M_p (\bar{t})}{N_p (\infty)(h/2)} = \sqrt{\frac{4}{\pi}} \bar{t} - 2\bar{t} \times \text{erf} \left( \frac{1}{2\sqrt{\bar{t}}} \right) \approx \sqrt{\frac{4}{\pi}} \bar{t} - 2\bar{t}$$

(68)

where $\text{erfc}(x) = 1 - \text{erf}(x)$ is the complementary error function. Thus the following relationship holds between bending moment and axial force:

$$\frac{M_p (\bar{t})}{N_p (\infty)(h/2)} \approx \frac{N_p (\bar{t})}{N_p (\infty)} - \frac{\pi}{2} \left( \frac{N_p (\bar{t})}{N_p (\infty)} \right)^2 = \sqrt{\frac{4\bar{t}}{\pi}} \left( 1 - \sqrt{\frac{\pi}{\bar{t}}} \right)$$

(69)
The above quadratic relation is illustrated in Figure 8 and compared with the exact functional
relation. The normalized moment has a maximum equal to $1/2\pi$ at $N^p(\bar{t})/N^p(\infty) = 1/\pi$,
which corresponds to the dimensionless time $\bar{t} = 1/4\pi$.

**Large time-scale**

For larger times, i.e. $\bar{t} > 0.1$, the zeroth-order approximation of Eq. (62),

$$\frac{N^p(\bar{t})}{N^p(\infty)} = 1 - \frac{8}{\pi^2} \exp \left(-\frac{\pi^2}{4}\bar{t}\right)$$  \hspace{1cm} (70)

provides an accurate estimate for the axial eigenstress force (see Figure 7). Using a zeroth-
order approximation of Eq. (63) for the bending moment, a functional relation similar to
Eq. (69) between bending moment and axial force is obtained for large times (Figure 8):

$$\frac{M^p(\bar{t})}{N^p(\infty)(h/2)} \approx \left(\frac{4}{\pi} - 1\right) \left(1 - \frac{N^p(\bar{t})}{N^p(\infty)}\right)$$ \hspace{1cm} (71)

It is observed that for $N^p/N^p(\infty) > 0.5$, which corresponds to $\bar{t} > 0.2$ the zeroth-order
approximation provides an accurate relationship, while at the intermediate range of $0.1 <
\bar{t} < 0.2$, the first-order approximation of Eq. (63) yields a more accurate estimate (see Figure
8):

$$\frac{M^p(\bar{t})}{N^p(\infty)(h/2)} \approx \left(1 - \frac{N^p(\bar{t})}{N^p(\infty)}\right)^2 \left(1 - \frac{N^p(\bar{t})}{N^p(\infty)}\right)^2 \left(1 - \frac{62}{73}\right)$$ \hspace{1cm} (72)

The two curves corresponding to small and large times coincide at $N^p(\bar{t})/N^p(\infty) = 0.4491$
corresponding to a dimensionless time $\bar{t} = 0.16$.

**Energy release rate**

For short time-scales, using the approximate relation in (69), the normalized energy
release rate is obtained (Figure 9):

$$\frac{G(\bar{t})}{G(\infty)} = \frac{4\bar{t}}{\pi} \left(1 + 3 \left(1 - \sqrt{\pi\bar{t}}\right)^2 \times \omega\right)$$ \hspace{1cm} (73)
where we remind ourselves that $\varpi = \mathcal{W}(\gamma)/\mathcal{U}(\beta)$. It is observed that for $\bar{t} < 0.1$, the transient risk of fracture due to both bending moment and axial force can surpass that of the steady-state condition ($\mathcal{G} \bar{t} / \mathcal{G} \infty > 1$) if $\varpi$ is large.

Similarly the following expression is obtained for the large-time solution, when considering the first-order approximation (72):

$$
\frac{\mathcal{G} \bar{t}}{\mathcal{G} \infty} = \left( 1 - \frac{8}{\pi^2} \exp \left( -\frac{\pi^2}{4} \right) \right)^2 + 3 \frac{33}{149} \exp \left( -\frac{\pi^2}{4} \bar{t} \right) - \frac{39}{304} \exp \left( -\frac{9\pi^2}{4} \bar{t} \right) \times \varpi \quad (74)
$$

As observed in Figure 9, the combined effect of bending moment and axial forces plays a role at small $\bar{t}$ values, while for larger values the steady-state situation prevails in defining the risk of fracture. Specifically, transient effects can induce a higher risk of fracture at short times if $\varpi \gg 1$. For an infinite pavement and at small values of $\gamma$ and $\beta$ this condition translates to:

$$
\frac{\mathcal{W}(\gamma)}{\mathcal{U}(\beta)} \gg 1 \iff \frac{\beta^2}{\gamma^4} \gg \frac{3}{45} \iff \frac{k_H}{k_V} \gg \frac{9}{45} \left( \frac{\ell}{h} \right)^2 \quad (75)
$$

with $\ell$ the distance between cracks, whereas for a finite-length pavement with joint spacing $L$, one can write:

$$
\frac{\mathcal{W}(\gamma)}{\mathcal{U}(\beta)} = \frac{k_V}{k_H} \left( \frac{L}{2h} \right)^2 \gg 1 \iff \frac{\beta^2}{\gamma^4} \ll \frac{1}{12} \iff \frac{k_H}{k_V} \ll \frac{1}{4} \left( \frac{L}{h} \right)^2 \quad (76)
$$

In other words, for finite-length pavements, the absence of a horizontal subgrade stiffness induces a higher risk of fracture during the transient condition than the steady-state condition, while the inverse is true for the infinite pavement. Thus, for small values of $\beta$ and $\gamma$, the scaling relationship of the maximum allowable temperature change in steady-state condition in Eq. (51),

$$
\langle T_0 - T_e \rangle \leq \frac{4\sqrt{6}Ke\sqrt{\varpi}}{\alpha T E^2 \sqrt{L}} \quad (77)
$$

is valid for $\varpi < 10$ so that the risk of fracture is not dominated by transitory effects.
CONCLUSIONS

The proposed mechanics-based model relates the risk of fracture of concrete pavements, subjected to different distress mechanisms, to their material and structural properties. Besides the classical design prescriptions, such as increasing pavement thickness and reducing joint spacing which both reduce the energy release rate, the results allow the following conclusions:

- For a fixed pavement structure, increasing the fracture toughness and decreasing the stiffness of the material reduces the risk of fracture.

- Increasing the horizontal stiffness of the subgrade (by e.g. compaction and consolidation) will improve the performance of concrete pavement subjected to autogenous shrinkage at early ages by reducing the energy release rate of the structure. However, such an increase has an inverse impact on the hardened pavement performance, when the pavement is subjected to thermal cycles during its lifetime.

- For the case of pavement subjected to thermal cycles, special attention must be paid to the ratio of dimensionless energy release rates due to bending and axial contribution to ensure that fracture will not occur during the transient condition right after application of a sudden temperature change. This ratio can be adjusted by choosing appropriate quantities for horizontal and vertical stiffnesses of subgrade as well as pavement thickness and joint spacing such that the normalized bending-to-axial energy release ratio is $\varpi \leq 10$.

The two problems herein investigated (i.e. autogenous shrinkage at early-ages and thermal gradient eigenstresses of the hardened state) are used as examples to illustrate the application of the proposed framework for designing pavements against fracture. Once the distributions of eigenstresses for any other distress mechanism such as creep, ASR and freeze-thaw are known, they can be similarly used for studying the risk of fracture and establishing relevant material and structural scaling relationships. In this regard the eigenstresses are
the key input to the developed model translating thermal, chemical and hygral evolutions at small scales into performance and durability assessments at structural scale.

The model herein derived is based on LEFM assumptions applied to the fracture of the entire section. While the model thus provides first-order estimates of the risk of fracture, and relevant scaling relations which engineers can fine tune in a design process, it would benefit from further refinements considering: (i) The actual crack propagation through the multilayer pavement for cases when the cracks reach the interface. Mechanics of thin film materials can provide guidance in this regard (see, for instance, Beuth 1992; Freund and Suresh 2004). (ii) The effect of non-linear fracture phenomena around the crack tip (Bažant 1984; Bazant and Planas 1997; Hoover and Bažant 2014) that results in a reduction of $d$ in the scaling relationship $\sigma \propto \ell^{-d}$ from its maximum value, i.e. $1/2$. Not considering the finite-sized fracture process zone (in concrete) and other dissipative mechanisms leads to a lower bound estimate of the actual load bearing capacity of pavement systems. As such, the model herein presented is but a first-order approach toward a shift of paradigm from the current strength-based design to a consistent fracture-based design for enhancing the durability of pavements against different distress mechanisms; which ultimately aims at reducing maintenance costs and at improving the environmental footprint of our Nation’s aging infrastructure.

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1. Determination of crack opening of a beam subjected to left panel: eigenstress axial force $N^0$; and right panel: eigenstress bending moment $M^0$.

2. Dimensionless axial contribution of energy release rate for infinite $U_\infty$ and finite-length beam $U$ and the ratio $U/U_\infty$ in function of $\beta$.

3. Dimensionless bending contribution of energy release rate for infinite $W_\infty$ and finite-length beam $W$ and the ratio $W/W_\infty$ in function $\gamma$.

4. Finite-length beam’s initial (a): displacement, and (b): eigenstress moment normalized by their corresponding values at midspan ($\xi = 0$).

5. Dimensionless shear force $Q_z\ell/M_p$ of a finite-length beam in function of $\gamma$. For $\gamma \leq 5.5531$ maximum moment occurs at midspan, since shear force has only one root at $\xi = 0$.

6. Dimensionless temperature profile over pavement thickness at different times. The dashed lines superposed on curves at $\ell = 0.1$, $10^{-2}$ and $10^{-4}$ correspond to the short time solution.

7. Normalized eigenstress force in function of dimensionless time.

8. Normalized eigenstress moment in function normalized axial force.

9. The ratio of energy release rate in transitory and steady-state conditions in function of dimensionless time for different quantities of $\varpi$. 

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FIG. 1. Determination of crack opening of a beam subjected to left panel: eigenstress axial force $N^0$; and right panel: eigenstress bending moment $M^0$
FIG. 2. Dimensionless axial contribution of energy release rate for infinite $U_\infty$ and finite-length beam $U$ and the ratio $U/U_\infty$ in function of $\beta$. 
FIG. 3. Dimensionless bending contribution of energy release rate for infinite $W_\infty$ and finite-length beam $W$ and the ratio $W/W_\infty$ in function $\gamma$. 
FIG. 4. Finite-length beam’s initial (a): displacement, and (b): eigenstress moment normalized by their corresponding values at midspan ($\xi = 0$).
FIG. 5. Dimensionless shear force \( Q_z \ell / M_p \) of a finite-length beam in function of \( \gamma \). For \( \gamma \leq 5.5531 \) maximum moment occurs at midspan, since shear force has only one root at \( \xi = 0 \).
FIG. 6. Dimensionless temperature profile over pavement thickness at different times. The dashed lines superposed on curves at $\bar{t} = 0.1$, $10^{-2}$ and $10^{-4}$ correspond to the short time solution.
FIG. 7. Normalized eigenstress force in function of dimensionless time
FIG. 8. Normalized eigenstress moment in function normalized axial force
FIG. 9. The ratio of energy release rate in transitory and steady-state conditions in function of dimensionless time for different quantities of $\varpi$. 