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Wildfire management with robotic fire extinguishers

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Controlling Stochastic Growth Processes on Lattices: Wildfire Management with Robotic Fire Extinguishers

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Abstract—Forest fires continue to cause considerable social and economic damage. Fortunately, the emergence of new robotics technologies, including capable autonomous unmanned aerial vehicles, may help improve wildfire management in the near future. In this paper, we characterize the number of vehicles required to combat wildfires, using a percolation-theoretic analysis that originated in the mathematical physics community. We model the wildfire as a stochastic growth process on a square lattice, where the local growth probabilities depend on the presence of robotic fire-extinguishing vehicles. We develop two control policies: First treats only a fraction of burning nodes at a given time, and the second treats burning nodes only at finite time intervals. We characterize the conditions under which these policies can stabilize a wildfire, i.e., ensure the fire stops eventually almost surely. We also provide computational results which demonstrate our theoretical analysis.

I. INTRODUCTION

Thousands of acres of forests are destroyed by wildfires annually. It is estimated that more than 3 million acres of land were affected by forest fires in 2012 alone [1]. Many ecologists believe that the number of such fires are expected to only grow in the near future, due to the adverse effects of climate change and rapid urbanization [2]. These wildfires not only burn forests but also endanger habitable regions around the forest while posing significant risk to fire-fighters. Even though major amount of fire fighting is still done on the ground, over the last decade, aerial fire-fighting is being seen as an important resource to fight wildfires [3]. As more air tankers, helicopters, and scoopers penetrate into fire-fighting, pilots are put in considerable risk when flying close to blazing fires and above uneven terrains, as evidenced by the recent incidents involving air-tankers.

In this paper, we characterize the efficacy of controlling forest fires by utilizing a fleet of Unmanned Aerial Vehicles (UAVs) as robotic fire extinguishers. In order to develop insights to control the growth of wildfire, we take a theoretical approach to the problem and begin by modeling the wildfire as a stochastic growth process on lattices. Although the use of UAVs for tracking, monitoring and patrolling environmental boundaries has been widely studied in the past (see, for example, [4]–[10]), to the best of our knowledge, a detailed study on the effectiveness and subsequent policies for the UAVs operating in stochastic environments, for instance, modeled as growth processes on lattices, has not been considered in the robotics literature. This type of micro modeling considers each tree individually, and it is often considered to be a natural model that better represents wildfires [11], for instance when compared to PDE-based models.

We further note two important points related to novelty of the topic considered. Firstly, although growth models on lattices forms the basis of percolation theory literature [11]–[13], designing and analyzing feedback polices to control growth process is novel. Secondly, it is important to note that in the related literature on information, rumor, and epidemic spreading on networks [14]–[17], optimal policies to control epidemics on general networks are not known [18].

The main contribution of the paper is twofold: (i) we develop a simple but rich model to capture stochastic phenomena prevalent in wildfire management, and (ii) we analyze two polices to control the growth of fire process on these models using UAVs. We first show that, for a large class of policies, it is not be possible to contain an unstable fire process using only a bounded number of vehicles. Thus, we ask relaxed question: If a growing number of vehicles are available, what is the policy that the vehicles should follow to contain the fire process and how quickly should we increase the number of vehicles? We provide one such policy that ensures boundedness of the growth process. Also in practical systems, the UAVs have limited fire-retardant carrying capacity and need finite time to refill the fire retardant, and as such the UAVs might not be available to fight fire at all times. Thus we analyze a second policy that ensures containment of the wildfire in the case when vehicles are available only after finite time intervals. The arguments made are supported with both theoretical results and computational experiments throughout the paper.

The paper is organized as follows. In section II we discuss the model of the forest fire process and the robotic agents. We also discuss the necessary background from percolation theory literature. In section III, we discuss the effect of using fire fighting vehicles on the forest fire process and discuss the closed loop behavior of our model. The computational experiments are presented in section IV and finally, a summary of our work and subsequent remarks are discussed in the following section V.

II. MODELING THE FOREST FIRE PROCESS

A. Forest fire model

We model the forest as a set of trees represented by nodes on a $d$ dimensional plain square lattice $\mathbb{Z}^d$ (for $d = 1$, the set $\mathbb{Z}$ is the set of integers; for $d = 2$, the set $\mathbb{Z}^2$ is the integer grid, and so on). The trees can be in three states: green (not burning), red (on fire) or black (burnt). If a green tree catches...
fire, it burns for a time that is an exponentially distributed random variable with parameter $\lambda$. After this time, the tree is considered burnt (black). During the time that a tree is burning, it can spread the fire to its neighbors, independently with a probability $\alpha$. Initially, the fire starts at the origin. Define $\beta = e^{-\lambda}$, which is the probability that a tree burns for at least one time step (due to the exponential distribution). Then, the probability that a red tree propagates the fire to its green neighbor is $p = \alpha \beta$. The propagation probability $p$ is an important parameter of the fire process. Intuitively, the parameter $\alpha$ models the spatial probability of spreading and is related to the package density of the trees (how close the trees are). The parameter $\beta$ models the temporal probability of propagation and is related to the thickness of trees (how long does the tree burn). In general, $\alpha(x,t)$ and $\beta(x,t)$ are functions of space and time, and the model can easily be extended to incorporate the effects of wind, terrain, and the fuel (dry grass, trees etc) by varying them accordingly. However, for the results discussed in this paper, we assume that $\alpha$ and $\beta$ are constants. In the discussions to follow, the stochastic model discussed in this section would be referred to as the fire process in $d$ dimensional forest.

**B. Preliminaries**

Our models for forest fires are closely related to bond percolation model used in percolation theory literature [11]. The bond percolation model consists of nodes on a $d$ dimensional lattice where a bond is formed between any two nodes with probability $p$ independent of other nodes. In bond percolation model, we are interested in the critical probability $p_c$ at which the entire lattice becomes connected - i.e. there exists a path between any two points on the lattice. A standard result from percolation theory literature is that the connectivity of the lattice for $d \geq 2$ shows phase transition - there exists a critical probability threshold $p_c$ below which the lattice is disconnected and above which the lattice is completely connected. Further it can be shown that this critical probability $p_c = 1/2$ on a 2-D grid ($d = 2$). This seemingly simple result almost took a decade to be established [19] and transition thresholds for percolation models in general are unknown [13]. As such researchers usually provide empirical evidence for results using Monte-Carlo simulations where strong theoretical results are not yet available. Percolation theory can be used to model many phenomena that show phase transitions as a parameter of the system is varied - for instance the state of water changes from liquid to solid instantly as the temperature is reduced below $0^\circ C$, or an insulator becomes a conductor when a critical dopant concentration is reached. Similarly for wildfire we know that a sufficiently small fire is self contained and only when the fire reaches a certain critical threshold, the fire cannot be contained. This is the motivation for using percolation like models for studying the forest fire process.

To understand the fire process model described in the previous section better, several definitions are in place.

**Definition 1** (Stability) Let $X_n = 0$ be the number of burning nodes (red nodes) at time instant $n$. A fire process is said to be stable or sub-critical if,

$$\limsup_{n \to \infty} X_n = 0 \quad \text{almost surely.}$$

A fire process is said to be unstable or super-critical if,

$$\liminf_{n \to \infty} X_n = \infty \quad \text{almost surely.}$$

**Definition 2** (Critical probability) Let $Y_n$ be the number of non-green nodes (total of burning and already burnt nodes) at time instant $n$. There exists a critical probability $p_c$ such that,

$$\lim_{n \to \infty} Y_n \begin{cases} < \infty & \text{almost surely, } p \leq p_c \smallskip \leq \infty & \text{almost surely, } p > p_c \end{cases}$$

Alternatively, on a finite lattice, $Y_n$ can be replaced by the fraction of non-green nodes $F_n$,

$$\lim_{n \to \infty} F_n = \begin{cases} 0 & \text{almost surely, } p \leq p_c \smallskip 1 & \text{almost surely, } p > p_c \end{cases}$$

Intuitively, the definitions mean the following. There exists a critical probability threshold $p_c$ below which the fire process is self-stabilizing. However, if the spreading probability $p > p_c$, then the fire process is super-critical, i.e., the fire does not die out. From a practical perspective, this fire is one that may result in the large-scale burning of a forest.
C. Fire process in a 1D forest

It is known that the bond percolation model behaves much differently for $d \geq 2$ when compared to the same percolation model in $d = 1$. For $d = 1$, it is known that phase transitions do not occur, and the critical probability is $p_c = 1$. In this section, we analyze the fire process in $d = 1$ mostly for completeness, although the model has no practical meaning.

The one dimensional model of the forest consists of trees lined up on a straight line. Although unrealistic, the one dimensional model is analytically tractable. As the fire growth is symmetric about origin, we study the process in only one direction. Figure 1 shows a Markov chain representation of the 1-D forest fire process. The state represents where the fire has currently reached. When the process in state $k$, one of the following three can happen: The fire might propagate to state $k+1$ with probability $p = \alpha \beta$; it might remain burning without propagating with probability $(1 - \alpha)\beta$; or it might burn out with probability $1 - \beta$. It can be easily shown that the (steady state) probability of being in state $k$ ($\pi_k$) and probability of being absorbed in state $ak$ ($\pi_{ak}$) is given by a geometric distribution,

$$\pi_k = \left( \frac{\alpha \beta}{1 - \beta + \alpha \beta} \right)^{k-1},$$

$$\pi_{ak} = \left( \frac{\alpha \beta}{1 - \beta + \alpha \beta} \right)^{k-1} \frac{1 - \beta}{1 - \beta + \alpha \beta}.$$

Further it can be shown that the number of trees burnt is a geometric random variable with parameter $1 - \beta$ when $\alpha \beta < 1$ ($p < 1$). The forest fire process in 1-D is always stable for $\beta < 1$ ($p < 1$).

D. Fire process in 2D forest

The fire process in a 2D forest shows several interesting phenomena that are not observed in the 1D case. There are multiple paths and times available for the fire to propagate in the 2D forest. Figure 2 shows the Markov chain representation of the fire process. It can also be seen that the boundary of the growing fire in 2D is not a regular polygon but an irregular shaped crystal with a fractal dimension. The fire process shows a phase transition in propagation probability $p$ - there exists a critical probability $p_c$ beyond which there is a wide spread forest fire and a large part of forest gets burnt. Below this critical probability, the forest fire is self-contained. This critical probability threshold is a function of forest parameters $\lambda$ and $\alpha$ and can be shown to be $p = \alpha e^{-\lambda}$. Figure 3 shows how the critical probability $p_c$ varies as a function propagation probability $\alpha$ and $1/\lambda$. It is convenient to think about $p_c$ with regards to the average time it takes for a tree on fire to burn out. For the exponential distribution, this value is $1/\lambda$. In Figure 3 (b) the red zone indicates the super-critical or the unstable regime of the fire process. In the blue zone the process is self-stabilizing.

E. Robotic agent model

We now assume that fire fighting UAVs carrying fire retardants are available. UAVs control fire by reducing the propagation probability $p$. Specifically, if a particular node is treated by a UAV, the fire propagation probability $p$ of the node is reduced by an amount $\Delta p$. We define $\Delta p$ as the fire-fighting power of the UAVs. Further we assume that the UAVs have a finite retardant carrying capacity and as such, if a UAV is utilized, it is reusable only after a finite time interval $\tau$ which is the time required to refill the retardant. For mathematical simplicity, we assume that one vehicle can treat one node at a given time. It should be noted that a model in which the number of nodes that can be treated by a UAV at a given time are more than one only results in scaling the problem differently. We also assume that the vehicle speeds of the UAVs are much faster than that of the fire growth process, and thus the vehicles can act instantly on the nodes if available with the delay being only due to the time taken for refilling the retardant.

It should be noted that in the 1D model, the optimal
policy for controlling the fire process is to use the UAVs at the edge of the fire front. It is important to note that the optimal policy is in fact a greedy policy. Though the 2D fire process is different from 1D, we conjecture that a greedy policy of using vehicles at the boundary remains optimal even in 2D and study the same. Thus, all our discussions in the subsequent sections are for UAVs operating on the boundary of fire process in a 2D forest. We also assume that all the UAVs are identical having the same fire-fighting power \( \Delta p \).

**III. CONTROLLING FOREST FIRE PROCESS**

In the discussion to follow, we assume that the 2D fire process is super-critical i.e. \( p > p_c \). As the performance of the UAVs is determined by the policies which are used to control the fire process, in this section we first analyze the following policy,

**Definition 3** Random node treatment (RNT) policy: Given \( N(t) \) burning (red) boundary nodes of the fire process at time \( t \), treat a fraction \( f \) of the nodes by selecting them uniformly at random.

The effect of using a stabilizing control policy is shown in Figure 4. It can be seen that a stabilizing policy results in increasing the critical probability threshold \( p_c \). The amount by which this threshold moves to the right is a function of fire-fighting power \( \Delta p \) and the available number of the UAVs \( n \). This observation seems to suggest we might need as many vehicles as the boundary nodes of the fire process. As such a large number of vehicles might not be available in practice, we ask if a finite number of vehicles can stabilize a super-critical fire process? The result turns out to be negative as indicated by the following theorem.

**Theorem 1** A constant number of vehicles cannot stabilize a super-critical fire process using the Random Node Treatment (RNT) policy. Let \( n(t) \) be the number of available vehicles and let \( N(t) \) be the number of burning boundary nodes at time \( t \). A fire process is unstable under RNT if

\[
\lim_{t \to \infty} \frac{n(t)}{N(t)} = 0
\]

**Proof:** For a fire process in 2D reaching any particular state \( s \) has non-zero probability. For instance, the probability for a fire starting from origin and propagating along a straight line to \( m \) nodes is at least \( p^m > 0 \). Therefore \( \mathbb{P}(s) > 0 \). Now let the state \( s(t = 0) \) have \( N \) burning nodes on the boundary and let \( N > n \). Here \( N \) is sufficiently larger than \( n \) such that even if all \( n \) vehicles continue to act at every time step \( N(t) - n(t) > 0, \forall t > 0 \). The nodes that are not treated by UAVs continue to and the fire process becomes unstable. In particular if \( \lim_{t \to \infty} \frac{n(t)}{N(t)} = 0 \), the fire process is unstable.

The above theorem leads to the natural question: Suppose we can supply a growing number of vehicles; what is the rate of supply of vehicles that stabilizes the fire process? Alternatively how much does the critical probability threshold \( p_c \) increase when an increasing number of fire-fighting vehicles are available? The following theorem answers this question.

**Theorem 2** A random node treatment policy applied \( \forall t \) can stabilize a super-critical fire if \( q < p_c \), where \( q = p - f \Delta p \).

**Proof:** At any given time \( t \), let the number of vehicles be \( n(t) \) and let the number of burning (red) boundary nodes be \( N(t) \). Let \( A \) denote the event that a particular node is treated by an UAV. Then, \( \mathbb{P}(A) = \frac{\binom{N-1}{f}}{\binom{N}{f}} = \frac{n}{N} = f \). Further, the propagation probability of any node is \( q = (p - \Delta p) \mathbb{P}(A) + p \mathbb{P}(A^c) = p - f \Delta p \). The result follows from the fact that the fire process is stable if \( q < p_c \).

**Remark:** It is important to note that by treating only a small fraction of boundary nodes \( f < 1 \) the RNT policy stabilizes a super-critical fire process. Figure 6 shows the transition diagram as a function of fraction \( f \).

We now analyze the case when due to finite capacity, the UAVs are available only after finite time intervals. In this context, our next policy is the following.

**Definition 4** Finite time interval (FTI) policy: Treat ‘all’ burning boundary nodes of the fire process at finite time intervals \( \tau \).

It turns out that the FTI policy helps us to gain insight into the problem and is closely related to the RNT policy as stated by the following theorem,

**Theorem 3** A finite time interval policy is equivalent to the random node policy with \( f = \frac{1}{\tau} \). In particular if the RNT policy is stable for some \( f \), then the FTI policy is also stable if \( \tau \leq 1/f \).
IV. COMPUTATIONAL EXPERIMENTS

In this section we report computational experiments that have been performed to verify the policies. All the simulations have been performed on a 2GHz Intel machine with 4GB RAM. Each data point has been obtained by averaging 200 Monte-Carlo runs. As our results are applicable to infinite lattices, care has been taken while performing experiments on a finite grid. The simulation time has been adjusted so that the data is not corrupted by finite edge effects. Specific runs have been performed with \( \alpha = 0.2763, \beta = e^{-0.1} \) which yields \( p = 0.25 \). It should be noted from the phase transition diagram (Fig 3) that for these values the fire process is super-critical. Further \( \Delta p = 0.15 \) and the fire process is stable if \( f = 1 \).

Figure 4 shows the effect of closing the loop with vehicles for an FTI policy with \( \tau = 3 \). As expected the phase transi-

Corollary 4 A super-critical process can be stabilized by a random node treatment policy in conjunction with finite time interval policy if \( q = p - \frac{\tau}{\tau} \Delta p < p_c \)
tion diagram shifts to the right which results in increasing the transition threshold. As discussed in theorem 2, the amount by which the threshold shifts is a function of \( p, \Delta p, f, \) and \( \tau \). For a fixed value of \( p \) and \( \Delta p \), figure 5 shows the change in stability as a function of time interval between arrival of UAVs \( \tau \). Figure 6 shows the effect of RNT policy as a function of \( f \). It can be seen that the stability of the fire process shows a sharp transition as a function of \( f \).

Figure 7 depicts the relation between RNT and FTI policies graphically. The growth of boundary nodes generated for an FTI policy is compared against an RNT policy when \( \tau = 1/f \). Let \( N(t) \) is the number of burning boundary nodes at time \( t \). As the number of available vehicles \( n = N(t)/\tau \) need not always be an integer, we use the integer part \( \lfloor N(t)/\tau \rfloor \) of the real number. In particular, it should be noted that the FTI policy is clearly upper bounded by an RNT policy when \( n = \lfloor N(t)/\tau \rfloor \) vehicles are used. The FTI policy curve is also lower bounded by the RNT policy when \( n = \lfloor N(t)/\tau \rfloor + 1 \) vehicles are used. Clearly if the RNT policy is stabilizing then the FTI policy is also stabilizing.

V. CONCLUSION

In the paper, we analyzed the effect of using robotic vehicles to control a stochastic growth process such as a forest fire. We developed a simple lattice model of fire process and studied its open loop behavior on 1D and 2D lattices. We found that the 2D fire process model shows phase transition in propagation probability \( p \) when it is above a certain critical threshold \( p_c \). We estimated this threshold through numerical simulations. Further we showed that the UAVs control the fire process by increasing this critical threshold. We developed and analyzed two policies for stabilizing the fire process - (i) the RNT policy that treats a fraction of burning nodes randomly and (ii) the FTI policy that treats all burning nodes of fire process at finite time intervals. We also showed that for the RNT policy a constant number of vehicles is unable to stabilize a super-critical fire process.

There are several interesting questions that remain to be answered. What is the optimal policy for vehicles with capacity constraints? What is the minimum rate at which vehicles should be supplied to stabilize a growth process? How do the results change in presence of wind? Answering these questions would be the direction for our future work.

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