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Classification of linearly compact simple Nambu-Poisson algebras

Nicoletta Cantarini∗ Victo r G. Kac†

Abstract

We introduce the notion of universal odd generalized Poisson superalgebra associated to an associative algebra \( A \), by generalizing a construction made in [5]. By making use of this notion we give a complete classification of simple linearly compact (generalized) \( n \)-Nambu-Poisson algebras over an algebraically closed field of characteristic zero.

Introduction

In 1973 Y. Nambu proposed a generalization of Hamiltonian mechanics, based on the notion of \( n \)-ary bracket in place of the usual binary Poisson bracket [9]. Nambu dynamics is described by the flow, given by a system of ordinary differential equations which involves \( n – 1 \) Hamiltonians:

\[
\frac{du}{dt} = \{ u, h_1, \ldots, h_{n-1} \}.
\]

The (only) example, proposed by Nambu is the following \( n \)-ary bracket on the space of functions in \( N \geq n \) variables:

\[
\{ f_1, \ldots, f_n \} = \det \left( \frac{\partial f_i}{\partial x_j} \right)_{i,j=1}^n.
\]

He pointed out that this \( n \)-ary bracket satisfies the following axioms, similar to that of a Poisson bracket:

(Leibniz rule) \( \{ f_1, \ldots, f_1, \ldots, f_n \} = f_i \{ f_1, \ldots, \hat{f}_i, \ldots, f_n \} + \hat{f}_i \{ f_1, \ldots, f_i, \ldots, f_n \} \);

(skewsymmetry) \( \{ f_{\sigma(1)}, \ldots, f_{\sigma(n)} \} = (\text{sign}\sigma) \{ f_1, \ldots, f_n \} \).

Twelve years later this example was rediscovered by F. T. Filippov in his theory of \( n \)-Lie algebras which is a natural generalization of ordinary (binary) Lie algebras [7]. Namely, an \( n \)-Lie algebra is a vector space with \( n \)-ary bracket \([ a_1, \ldots, a_n ]\), which is skewsymmetric (as above) and satisfies the following Filippov-Jacobi identity:

\[
[a_1, \ldots, a_{n-1}, [b_1, \ldots, b_n]] = [[a_1, \ldots, a_{n-1}, b_1], [b_2, \ldots, b_n]] + [[b_1, a_1, \ldots, a_{n-1}, b_2], b_3, \ldots, b_n] + \ldots + [b_1, \ldots, b_{n-1}, [a_1, \ldots, a_{n-1}, b_n]].
\]

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In particular, Filippov proved that the Nambu bracket (0.2) satisfies the Filippov-Jacobi identity.

Following Takhtajan [10], we call an \( n \)-Nambu-Poisson algebra a unital commutative associative algebra \( N \), endowed with an \( n \)-ary bracket, satisfying the Leibniz rule, skew-symmetry and Filippov-Jacobi identity. Of course for \( n = 2 \) this is the definition of a Poisson algebra.

In [4] we classified simple linearly compact \( n \)-Lie algebras with \( n > 2 \) over a field \( \mathbb{F} \) of characteristic 0. The classification is based on a bijective correspondence between \( n \)-Lie algebras and \( (L,\mu) \), where \( L \) is a \( \mathbb{Z} \)-graded Lie superalgebra of the form \( L = \bigoplus_{j=-1}^{n-1} L_j \) satisfying certain additional properties, and \( L_{n-1} = \mathbb{F}\mu \), thereby reducing it to the known classification of simple linearly compact Lie superalgebras and their \( \mathbb{Z} \)-gradings [8], [1]. For this construction we used the universal \( \mathbb{Z} \)-graded Lie superalgebra, associated to a vector superspace.

In the present paper we use an analogous correspondence between linearly compact \( n \)-Nambu-Poisson algebras and certain ”good” pairs \( (P,\mu) \), where \( P \) is a \( \mathbb{Z}_+ \)-graded odd Poisson superalgebra \( P = \bigoplus_{j=-1}^{\infty} P_j \) and \( \mu \in P_{n-1} \) is an element of parity \( n \mod 2 \). For this construction we use the universal \( \mathbb{Z} \)-graded odd Poisson superalgebra, associated to an associative algebra, considered in [5]. As a result, using the classification of simple linearly compact odd Poisson superalgebras [3], we obtain the following theorem.

**Theorem 0.1** For \( n > 2 \), any simple linearly compact \( n \)-Nambu-Poisson algebra is isomorphic to the algebra \( \mathbb{F}[[x_1,\ldots,x_n]] \) with the \( n \)-ary bracket (0.2).

Note the sharp difference with the Poisson case, when each algebra \( \mathbb{F}[[p_1,\ldots,p_n,q_1,\ldots,q_n]] \) carries a Poisson bracket

\[
\{f,g\}_P = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right),
\]

making it a simple linearly compact Poisson algebra (and these are all, up to isomorphism [2]).

In the present paper we treat also the case of a generalized \( n \)-Nambu-Poisson bracket, which is an \( n \)-ary analogue of the generalized Poisson bracket, called also the Lagrange’s bracket. For the latter bracket the Leibniz rule is modified by adding an extra term:

\[
\{a, bc\} = \{a, b\}c + \{a, c\}b - \{a, 1\}bc.
\]

In order to treat this case along similar lines, we construct the universal \( \mathbb{Z} \)-graded generalized odd Poisson superalgebra, associated to an associative algebra, which is a generalization of the construction in [5]. Our main result in this direction is the following theorem, which uses the classification of simple linearly compact odd generalized Poisson superalgebras [3].

**Theorem 0.2** For \( n > 2 \), any simple linearly compact generalized \( n \)-Nambu-Poisson algebra is gauge equivalent (see Remark 1.4 for the definition) either to the Nambu \( n \)-algebra from Theorem 0.1 or to the Dzhumadildaev \( n \)-algebra [2], which is \( \mathbb{F}[[x_1,\ldots,x_{n-1}]] \) with the \( n \)-ary bracket

\[
\{f_1,\ldots,f_n\} = \text{det} \begin{pmatrix} f_1 & \cdots & f_n \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_1}{\partial x_{n-1}} & \cdots & \frac{\partial f_n}{\partial x_{n-1}} \end{pmatrix}.
\]
Note again the sharp difference with the generalized Poisson case, when each algebra \( F[[p_1, \ldots, p_n, q_1, \ldots, q_n, t]] \) carries a Lagrange bracket

\[
\{f, g\}_L = \{f, g\}_P + (2 - E) \frac{\partial q}{\partial t} - \frac{\partial f}{\partial t} (2 - E) g,
\]

where \( \{f, g\}_P \) is given by \( 0.4 \) and \( E = \sum_{i=1}^{n} (p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i}) \), making it a simple linearly compact generalized Poisson algebra (and those, along with \( 0.4 \), are all, up to gauge equivalence).

Throughout the paper our base field \( \mathbb{F} \) has characteristic 0 and is algebraically closed.

## 1 Nambu-Poisson algebras

**Definition 1.1** A generalized \( n \)-Nambu-Poisson algebra is a triple \((N, \{\cdot, \ldots, \cdot \}, \cdot)\) such that

- \((N, \cdot)\) is a unital associative commutative algebra;
- \((N, \{\cdot, \ldots, \cdot \})\) is an \( n \)-Lie algebra;
- the following generalized Leibniz rule holds:

\[
\{ a_1, \ldots, a_{n-1}, bc \} = \{ a_1, \ldots, a_{n-1}, b \} c + b \{ a_1, \ldots, a_{n-1}, c \} - \{ a_1, \ldots, a_{n-1}, 1 \} bc.
\]

If \( \{ a_1, \ldots, a_{n-1}, 1 \} = 0 \), then (1.1) is the usual Leibniz rule and \((N, \{\cdot, \ldots, \cdot \}, \cdot)\) is called simply \( n \)-Nambu-Poisson algebra.

For \( n = 2 \) Definition 1.1 is the definition of a generalized Poisson algebra. Simple linearly compact generalized Poisson (super)algebras were classified in [2, Corollary 7.1].

**Example 1.2** Let \( N = \mathbb{F}[[x_1, \ldots, x_n]] \) with the usual commutative associative product and \( n \)-ary bracket defined, for \( f_1, \ldots, f_n \in N \), by:

\[
\{ f_1, \ldots, f_n \} = \det \begin{pmatrix} D_1(f_1) & \cdots & D_1(f_n) \\ \vdots & \ddots & \vdots \\ D_n(f_1) & \cdots & D_n(f_n) \end{pmatrix}
\]

where \( D_i = \frac{\partial}{\partial x_i}, i = 1, \ldots, n \). Then \( N \) is an \( n \)-Nambu-Poisson algebra, introduced by Nambu [9], that we will call the \( n \)-Nambu algebra (cf. [9], [7], [4]).

**Example 1.3** Let \( N = \mathbb{F}[[x_1, \ldots, x_{n-1}]] \) with the usual commutative associative product and \( n \)-ary bracket defined, for \( f_1, \ldots, f_n \in N \), by

\[
\{ f_1, \ldots, f_n \} = \det \begin{pmatrix} f_1 & \cdots & f_n \\ D_1(f_1) & \cdots & D_1(f_n) \\ \vdots & \ddots & \vdots \\ D_{n-1}(f_1) & \cdots & D_{n-1}(f_n) \end{pmatrix}
\]

where \( D_i = \frac{\partial}{\partial x_i}, i = 1, \ldots, n - 1 \). Then \( N \) is a generalized Nambu-Poisson algebra that we will call the \( n \)-Dzhumadildaev algebra (cf. [6], [4]).
We shall say that the generalized Nambu-Poisson algebras $N = \{\cdot, \cdot, \cdot\}$ is odd derivation of the associative product and of the Lie superalgebra bracket. 

\[
\{f_1, \ldots, f_n\}^\varphi = \varphi^{-1}\{\varphi f_1, \ldots, \varphi f_n\}. 
\]

Then $N^\varphi = (N, \{\cdot, \cdot, \cdot\}^\varphi, \cdot)$ is another generalized $n$-Nambu-Poisson algebra. Indeed, the skew-symmetry of the bracket is straightforward and the Filippov-Jacobi identity for the bracket $\{\cdot, \cdot, \cdot\}^\varphi$ easily follows from the Filippov-Jacobi identity for the bracket $\{\cdot, \cdot, \cdot\}$. Let us check that $\{\cdot, \cdot, \cdot\}^\varphi$ satisfies the generalized Leibniz rule. We have:

\[
\begin{align*}
\{f_1, \ldots, f_{n-1}, gh\}^\varphi &= \varphi^{-1}\{\varphi f_1, \ldots, \varphi f_{n-1}, \varphi gh\} = \varphi^{-1}\{\varphi f_1, \ldots, \varphi f_{n-1}, \varphi g\}h \\
+ &\varphi g\{\varphi f_1, \ldots, \varphi f_{n-1}, h\} - \{\varphi f_1, \ldots, \varphi f_{n-1}, 1\}\varphi gh = \{f_1, \ldots, f_{n-1}, g\}^\varphi h \\
+ &g\{\varphi f_1, \ldots, \varphi f_{n-1}, h\} - \{\varphi f_1, \ldots, \varphi f_{n-1}, 1\}gh = \{f_1, \ldots, f_{n-1}, g\}^\varphi h \\
+ &g\{\varphi f_1, \ldots, \varphi f_{n-1}, h\} - \{\varphi f_1, \ldots, \varphi f_{n-1}, 1\}\varphi + g\{f_1, \ldots, f_{n-1}, h\}^\varphi \\
- &\varphi^{-1}g\{\varphi f_1, \ldots, \varphi f_{n-1}, \varphi h\} = \{f_1, \ldots, f_{n-1}, g\}^\varphi h + g\varphi f_1, \ldots, \varphi f_{n-1}, h\} \\
- &\{\varphi f_1, \ldots, \varphi f_{n-1}, 1\}gh + g\{f_1, \ldots, f_{n-1}, h\}^\varphi - g\varphi f_1, \ldots, \varphi f_{n-1}, h\} \\
- &\varphi^{-1}g\{\varphi f_1, \ldots, \varphi f_{n-1}, \varphi h\} = \{f_1, \ldots, f_{n-1}, g\}^\varphi h + g\{f_1, \ldots, f_{n-1}, h\}^\varphi - \{f_1, \ldots, f_{n-1}, 1\}^\varphi gh.
\end{align*}
\]

We shall say that the generalized Nambu-Poisson algebras $N$ and $N^\varphi$ are gauge equivalent.

## 2 Odd generalized Poisson superalgebras

**Definition 2.1** An odd generalized Poisson superalgebra $(\mathcal{P}, [\cdot, \cdot], \wedge)$ is a triple such that

1. $(\mathcal{P}, \wedge)$ is a unital associative commutative superalgebra with parity $p$;
2. $(\Pi\mathcal{P}, [\cdot, \cdot])$ is a Lie superalgebra (here $\Pi\mathcal{P}$ denotes the space $\mathcal{P}$ with parity $\bar{p} = p + \bar{1}$);
3. the following generalized odd Leibniz rule holds:

\[
[a, b \wedge c] = [a, b] \wedge c + (-1)^{(p(a)+1)p(b)}b \wedge [a, c] + (-1)^{(p(a)+1)}D(a) \wedge b \wedge c,
\]

where $D(a) = [1, a]$. If $D = 0$, then relation (2.1) becomes the odd Leibniz rule; in this case $(\mathcal{P}, [\cdot, \cdot], \wedge)$ is called an odd Poisson superalgebra (or Gerstenhaber superalgebra). Note that $D$ is an odd derivation of the associative product and of the Lie superalgebra bracket.
Example 2.2 Consider the commutative associative superalgebra \( \mathcal{O}(m, n) = \Lambda(n)[[x_1, \ldots, x_m]] \), where \( \Lambda(n) \) denotes the Grassmann algebra over \( \mathbb{F} \) on \( n \) anti-commuting indeterminates \( \xi_1, \ldots, \xi_n \), and the superalgebra parity is defined by \( p(x_i) = 0, p(\xi_i) = \bar{1} \).

Set \( m = n \) and define the following bracket, known as the Buttin bracket, on \( \mathcal{O}(n, n) \) \((f, g \in \mathcal{O}(n, n))\):

\[
[f, g]_{HO} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i}.
\]

Then \( \mathcal{O}(n, n) \) with this bracket is an odd Poisson superalgebra, which we denote by \( PO(n, n) \).

Example 2.3 Consider the associative superalgebra \( \mathcal{O}(n, n+1) \) with even indeterminates \( x_1, \ldots, x_n \) and odd indeterminates \( \xi_1, \ldots, \xi_n, \xi_{n+1} = \tau \). Define on \( \mathcal{O}(n, n+1) \) the following bracket \((f, g \in \mathcal{O}(n, n+1))\):

\[
[f, g]_{KO} = [f, g]_{HO} + (E - 2)(f) \frac{\partial g}{\partial \tau} + (-1)^{p(f)} \frac{\partial f}{\partial \tau} (E - 2)(g),
\]

where \([\cdot, \cdot]_{HO}\) is the Buttin bracket \(2.2\) and \( E = \sum_{i=1}^{n} (x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i}) \) is the Euler operator. Then \( \mathcal{O}(n, n+1) \) with bracket \([\cdot, \cdot]_{KO}\) is an odd generalized Poisson superalgebra with \( D = -2 \frac{\partial}{\partial \tau} \) \[\text{Remark 4.1}\], which we denote by \( PO(n, n+1) \).

Remark 2.4 Let \( P = (\mathcal{P}, [\cdot, \cdot], \cdot) \) be an odd generalized Poisson superalgebra. For any invertible element \( \varphi \in \mathcal{P} \), such that \( p(\varphi) = 0 \) and \([\varphi, \varphi] = 0\), define the following bracket on \( P \):

\[
[a, b]^{\varphi} = \varphi^{-1}[\varphi a, \varphi b].
\]

Then \( P^{\varphi} = (\mathcal{P}, [\cdot, \cdot]^{\varphi}, \cdot) \) is another odd generalized Poisson superalgebra, with derivation

\[
D_{\varphi}(a) = [1, a]^{\varphi} = [\varphi, a] - D(\varphi)a.
\]

The odd generalized Poisson superalgebras \( P \) and \( P^{\varphi} \) are called gauge equivalent (cf. \[\text{3}, \text{Example 3.4}\]). Note that the associative products in \( P \) and \( P^{\varphi} \) are the same.

Theorem 2.5 \[\text{3}, \text{Corollary 9.2}\]

a) Any simple linearly compact odd generalized Poisson superalgebra is gauge equivalent to \( PO(n, n) \) or \( PO(n, n+1) \).

b) Any simple linearly compact odd Poisson superalgebra is isomorphic to \( PO(n, n) \).

Definition 2.6 A \( \mathbb{Z} \)-graded (resp. \( \mathbb{Z}_+ \)-graded) odd generalized Poisson superalgebra is an odd generalized Poisson superalgebra \((\mathcal{P}, [\cdot, \cdot], \wedge)\) such that \((\Pi \mathcal{P}, [\cdot, \cdot])\) is a \( \mathbb{Z} \)-graded Lie superalgebra: \( \Pi \mathcal{P} = \bigoplus_{j \in \mathbb{Z}} \mathcal{P}_j \) (resp. a \( \mathbb{Z}_+ \)-graded Lie superalgebra of depth 1: \( \Pi \mathcal{P} = \bigoplus_{j \geq -1} \mathcal{P}_j \)) and \((\mathcal{P}, \wedge)\) is a \( \mathbb{Z} \)-graded commutative associative superalgebra: \( \mathcal{P} = \bigoplus_{k \in \mathbb{Z}} \mathcal{Q}_k \) (resp. a \( \mathbb{Z}_+ \)-graded commutative associative superalgebra: \( \mathcal{P} = \bigoplus_{k \in \mathbb{Z}_+} \mathcal{Q}_k \)) such that \( \mathcal{P}_j = \Pi \mathcal{Q}_{j+1} \).
Example 2.7 Let us consider the odd Poisson superalgebra \( PO(n,n) \) (resp. \( PO(n,n+1) \)). Set \( \deg x_i = 0 \) and \( \deg \xi_i = 1 \) for every \( i = 1, \ldots, n \) (resp. \( \deg x_i = 0 \), \( \deg \xi_i = 1 \) for every \( i = 1, \ldots, n \) and \( \deg \tau = 1 \)). Then \( PO(n,n) \) (resp. \( PO(n,n+1) \)) becomes a \( \mathbb{Z}_+ \) graded odd (resp. generalized) Poisson superalgebra with

\[
Q_j = \{ f \in O(n,n) \mid \deg(f) = j \}
\]
and

\[
P_j = \{ f \in O(n,n) \mid \deg(f) = j + 1 \}.
\]
We will call this grading a grading of type \((0,\ldots,0|1,\ldots,1)\) (resp. \((0,\ldots,0|1,\ldots,1,1)\)). We thus have, for \( P = PO(n,n) \):

\[
\Pi P_{-1} = Q_0 = \mathbb{F}[[x_1,\ldots,x_n]]
\]
and, for \( j \geq 0 \),

\[
\Pi P_j = Q_{j+1} = \langle \xi_1 \cdots \xi_{i+1} \mid 1 \leq i_1 < \cdots < i_{j+1} \leq n \rangle \otimes \mathbb{F}[[x_1,\ldots,x_n]].
\]
Similarly, for \( P = PO(n,n+1) \), we have:

\[
P_{-1} = Q_0 = \mathbb{F}[[x_1,\ldots,x_n]]
\]
\[
P_j = Q_1 = \langle \xi_1 \cdots \xi_{i+1} \mid 1 \leq i_1 < \cdots < i_{j+1} \leq n+1 \rangle \otimes \mathbb{F}[[x_1,\ldots,x_n]].
\]

Remark 2.8 From the properties of the \( \mathbb{Z} \)-gradings of the Lie superalgebras \( HO(n,n) \) and \( KO(n,n+1) \) (see, for example, [5]), one can deduce that the grading of type \((0,\ldots,0|1,\ldots,1)\) (resp. \((0,\ldots,0|1,\ldots,1,1)\)) is, up to isomorphisms, the only \( \mathbb{Z}_+ \)-grading of \( P = PO(n,n) \) (resp. \( P = PO(n,n+1) \)) such that \( P_{-1} \) is completely odd.

Remark 2.9 Let \( P = PO(n,n) \) or \( P = PO(n,n+1) \) and let \( P^\varphi \) be an odd generalized Poisson superalgebra gauge equivalent to \( P \). Then the grading of type \((0,\ldots,0|1,\ldots,1)\) (resp. \((0,\ldots,0|1,\ldots,1,1)\)) is, up to isomorphisms, the only \( \mathbb{Z}_+ \)-grading of \( P^\varphi \) such that \( P^\varphi_{-1} \) is completely odd. Indeed, let \( P^\varphi = \oplus_{k \in \mathbb{Z}_+} Q_k^\varphi = \oplus_{j \geq 1} P_j^\varphi \), with \( P_j^\varphi = \Pi Q_{j+1}^\varphi \) a \( \mathbb{Z}_+ \)-grading of \( P^\varphi \). Suppose that \( x_i \in Q_k^\varphi \) and \( \xi_i \in Q_j^\varphi \) for some \( 1 \leq i \leq n \) and some \( k,j \in \mathbb{Z}_+ \). Then

\[
[\varphi x_i^\varphi, \xi_i^\varphi] = \Pi P_{k+j-2}^\varphi = Q_{k+j-1}^\varphi.
\]
On the other hand, by (2.3), we have:

\[
[x_i, \varphi x_i^\varphi] = \varphi^{-1} [\varphi x_i, \varphi \xi_i] = \varphi^{-1} ([\varphi x_i, \varphi] \xi_i + \varphi [\varphi x_i, \xi_i] - D(\varphi x_i) \varphi \xi_i) = \frac{\partial \varphi}{\partial \xi_i} x_i + \frac{1}{2} \frac{\partial \varphi}{\partial x_i} x_i + \varphi D(\varphi) x_i \xi_i - D(\varphi) x_i \varphi x_i^\varphi
\]
where \( D = 0 \) if \( P = PO(n,n) \) and \( D = -2 \frac{\partial \varphi}{\partial \xi_i} \) if \( P = PO(n,n+1) \). Note that \( [x_i, \xi_i^\varphi] \) is invertible since \( \varphi \) is invertible and, by (2.3), it is homogeneous, hence \( k = j = 1 \), i.e., either \( k = 0 \) and \( j = 1 \) or \( k = 1 \) and \( j = 0 \). It follows that the only \( \mathbb{Z}_+ \)-grading of \( P^\varphi \) such that \( P^\varphi_{-1} \) is completely odd is the grading of type \((0,\ldots,0|1,\ldots,1)\). We can thus simply denote the graded components of \( P^\varphi \) with respect to this grading by \( P_j = \Pi Q_{j+1}^\varphi \).

Now let \( a \in Q_i = \Pi P_{i-1} \) and \( b \in Q_k = \Pi P_{k-1} \). We have:

\[
[a, b] = [a, \varphi] b + [\varphi a, b] + (-1)^{p(a)+1} D(\varphi) ab + D(\varphi a) b.
\]
Suppose that \( \varphi = \sum_{j \geq 0} \varphi_j \) with \( \varphi_j \in Q_j \). Then one can show, using the fact that \( [a, b] = \Pi P_{i+k-2} = Q_{i+k-1} \), that \( [a, b] = [a, b]^{\varphi_0} \) holds. It follows that when dealing with the \( \mathbb{Z}_+ \)-graded odd generalized Poisson superalgebras \( P^\varphi \) we can always assume \( \varphi \in Q_0 \).
3 The universal odd generalized Poisson superalgebra

**Definition 3.1** Let $A$ be a unital commutative associative superalgebra with parity $p$. A linear map $X : A \to A$ is called a generalized derivation of $A$ if it satisfies the generalized Leibniz rule:

\begin{equation}
X(bc) = X(b)c + (-1)^{p(b)p(c)}X(c)b - X(1)bc.
\end{equation}

We denote by $GDer(A)$ the set of generalized derivations of $A$. If $X(1) = 0$, relation (3.1) becomes the usual Leibniz rule and $X$ is called a derivation. We denote by $Der(A)$ the set of derivations of $A$.

**Proposition 3.2** The set $GDer(A)$ is a subalgebra of the Lie superalgebra $End(A)$.

**Proof.** This follows by direct computations. \hfill \Box

Our construction of the universal odd generalized Poisson superalgebra is inspired by the one of the universal odd Poisson superalgebra explained in [5]. The universal odd Poisson superalgebra associated to $A$ is the full prolongation of the subalgebra $Der(A)$ of the Lie superalgebra $End(A)$ (the definitions will be given below). In this section we generalize this construction when $Der(A)$ is replaced by the subalgebra $GDer(A)$.

Consider the universal Lie superalgebra $W(\Pi A)$ associated to the vector superspace $\Pi A$: this is the $\mathbb{Z}_+$-graded Lie superalgebra:

$$W(\Pi A) = \bigoplus_{k=-1}^\infty W_k(\Pi A)$$

where $W_{-1} = \Pi A$ and for all $k \geq 0$, $W_k(V) = \text{Hom}(S^{k+1}(\Pi A), \Pi A)$ is the vector superspace of $(k+1)$-linear supersymmetric functions on $\Pi A$ with values in $\Pi A$. The Lie superalgebra structure on $W(\Pi A)$ is defined as follows: for $X \in W_p(\Pi A)$ and $Y \in W_q(\Pi A)$ with $p, q \geq -1$, we define $X \square Y \in W_{p+q}(\Pi A)$ by:

\begin{equation}
X \square Y(a_0, \ldots, a_{p+q}) = \sum \epsilon_a(i_0, \ldots, i_{p+q})X(Y(a_{i_0}, \ldots, a_{i_q}), a_{i_{q+1}}, \ldots, a_{i_{q+p}}).
\end{equation}

Here $\epsilon_a(i_0, \ldots, i_{p+q}) = (-1)^N$ where $N$ is the number of interchanges of indices of odd $a_i$’s in the permutation $\sigma(s) = i_s$, $s = 0, \ldots, p + q$. Then the bracket on $W(\Pi A)$ is given by:

$$[X, Y] = X \square Y - (-1)^{\beta(X)\beta(Y)}Y \square X.$$

As $GDer(A)$ is a subalgebra of the Lie superalgebra $W_0(\Pi A) = End(\Pi A)$, we can consider its full prolongation $GW^{as}(\Pi A)$: this is the $\mathbb{Z}_+$-graded subalgebra $GW^{as}(\Pi A) = \bigoplus_{k=-1}^\infty GW^{as}_k(\Pi A)$ of the Lie superalgebra $W(\Pi A)$ defined by setting $GW^{as}_{-1}(\Pi A) = \Pi A$, $GW^{as}_{0}(\Pi A) = GDer(\Pi A)$, and inductively for $k \geq 1$,

$$GW^{as}_k(\Pi A) = \{ X \in W_k(\Pi A) | [X, W_{-1}(\Pi A)] \subset GW^{as}_{k-1}(\Pi A) \}.$$
Proposition 3.3 For $k \geq 0$, the superspace $GW^{as}(\Pi A)$ consists of linear maps $X : S^{k+1}(\Pi A) \to \Pi A$ satisfying the following generalized Leibniz rule:

$(3.3)$
\[ X(a_0, \ldots, a_{k-1}, bc) = X(a_0, \ldots, a_{k-1}, b)c + (-1)^{p(b)p(c)} X(a_0, \ldots, a_{k-1}, c)b - X(a_0, \ldots, a_{k-1}, 1)bc \]

for $a_0, \ldots, a_{k-1}, b, c \in \Pi A$.

Proof. According to formula $(3.2)$, for all $X \in W_p(\Pi A)$ and $Y \in W_{-1}(\Pi A) = \Pi A$, we have:

$(3.4)$
\[ [X, Y](a_1, \ldots, a_p) = X(Y, a_1, \ldots, a_p) \]

with $a_1, \ldots, a_p \in \Pi A$. Now we proceed by induction on $k \geq 0$: for $k = 0$, $GW^{as}_0(\Pi A) = GDer(A)$ and equality $(3.3)$ holds by definition of generalized derivation. Assume property $(3.3)$ for elements in $GW^{as}_{k-1}(\Pi A)$, and let $X$ in $GW^{as}_k(\Pi A)$. For any $a_0, a_1, \ldots, a_{k-1}, b, c \in \Pi A$, we have by $(3.4)$:

\[ X(a_0, a_1, \ldots, a_{k-1}, bc) = [X, a_0](a_1, \ldots, a_{k-1}, bc). \]

By definition of $GW^{as}(\Pi A)$, we have $[X, a_0] \in GW^{as}_{k-1}(\Pi A)$. Using the inductive hypothesis on $[X, a_0]$, we get:

\[ [X, a_0](a_1, \ldots, a_{k-1}, bc) = [X, a_0](a_1, \ldots, a_{k-1}, b)c + (-1)^{p(b)p(c)} [X, a_0](a_1, \ldots, a_{k-1}, c)b \]

\[ - [X, a_0](a_1, \ldots, a_{k-1}, 1)bc \]

which is exactly formula $(3.3)$ for $X$. $\square$

For $X \in \Pi W_{h-1}(\Pi A)$ and $Y \in \Pi W_{k-1}(\Pi A)$ with $h, k \geq 0$, we define their concatenation product $X \wedge Y \in \Pi W_{h+k-1}(\Pi A)$ by

$(3.5)$
\[ X \wedge Y(a_1, \ldots, a_{h+k}) = \sum_{i_1 < \cdots < i_h} \epsilon_a(i_1, \ldots, i_{h+k}) (-1)^{p(Y)(\bar{p}(a_{i_1}) + \cdots + \bar{p}(a_{i_h}))} \times X(a_{i_1}, \ldots, a_{i_h}) Y(a_{i_{h+1}}, \ldots, a_{i_{h+k}}) \]

where $\epsilon_a$ is defined as in $(3.2)$ with $a_1, \ldots, a_{h+k} \in \Pi A$.

Proposition 3.4 $(\Pi GW^{as}(\Pi A), [\cdot, \cdot], \wedge)$ is a $\mathbb{Z}_+$-graded odd generalized Poisson superalgebra.

We will denote $(\Pi GW^{as}(\Pi A), [\cdot, \cdot], \wedge)$ by $\mathcal{G}(A)$ and call it the universal odd generalized Poisson superalgebra associated to $A$. The rest of this section is devoted to the proof of Proposition 3.4.

Lemma 3.5 $(\Pi GW^{as}(\Pi A), \wedge)$ is a unital $\mathbb{Z}_+$-graded associative commutative superalgebra with parity $p$.

Proof. It is already proved in [5] that $(\Pi W(\Pi A), \wedge)$ is a unital $\mathbb{Z}_+$-graded associative commutative superalgebra with parity $p$, therefore we only need to prove that for $X \in \Pi GW^{as}_{h-1}(\Pi A)$ and

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$Y \in \Pi GW_{k-1}^{as}(\Pi A)$ with $h, k \geq 0$, $X \wedge Y \in \Pi W_{h+k-1}(\Pi A)$ satisfies the generalized Leibniz rule (3.3). We have:

$$X \wedge Y(a_1, \ldots, a_{h+k-1}, bc) =$$

$$= \sum_{i_1 < \cdots < i_h} \epsilon_{a_1, \ldots, a_{h+k-1}, bc}(i_1, \ldots, i_{h+k})(-1)^p(Y)(\bar{p}(a_{i_1}) + \cdots + \bar{p}(a_{i_k}))$$

$$\times X(a_{i_1}, \ldots, a_{i_h})Y(a_{i_{h+1}}, \ldots, a_{i_{h+k-1}}, bc)$$

$$+ \sum_{i_1 < \cdots < i_h} \epsilon_{a_1, \ldots, a_{h+k-1}, bc}(i_1, \ldots, i_{h+k})(-1)^p(Y)(\bar{p}(a_{i_1}) + \cdots + \bar{p}(a_{i_{h-k}}) + \bar{p}(bc))$$

$$\times X(a_{i_1}, \ldots, a_{i_{h-k}}, bc)Y(a_{i_{h+1}}, \ldots, a_{i_{h+k}})$$

(3.6)

For the first summand in the right hand side, since $i_{h+k} = h + k$, we have:

$$\epsilon_{a_1, \ldots, a_{h+k-1}, bc}(i_1, \ldots, i_{h+k}) = \epsilon_{a_1, \ldots, a_{h+k-1}, c}(i_1, \ldots, i_{h+k})$$

and

$$Y(a_{i_{h+1}}, \ldots, a_{i_{h+k-1}}, bc) = Y(a_{i_{h+1}}, \ldots, a_{i_{h+k-1}}, 1)c + (-1)^p(b)p(c)Y(a_{i_{h+1}}, \ldots, a_{i_{h+k-1}}, c)b$$

$$- Y(a_{i_{h+1}}, \ldots, a_{i_{h+k-1}}, 1)bc.$$

In the second summand, since $i_{h+k} = h$, we have:

$$\epsilon_{a_1, \ldots, a_{h+k-1}, bc}(i_1, \ldots, i_{h+k}) = \epsilon_{a_1, \ldots, a_{h+k-1}, c}(i_1, \ldots, i_{h+k})$$

and

$$X(a_{i_1}, \ldots, a_{i_h}, bc)Y(a_{i_{h+1}}, \ldots, a_{i_{h+k}}) =$$

$$(-1)^p(c)p(Y)(\bar{p}(a_{i_{h+1}}) + \cdots + \bar{p}(a_{i_{h+k}}))X(a_{i_1}, \ldots, a_{i_{h-1}}, b)Y(a_{i_{h+1}}, \ldots, a_{i_{h+k}})c$$

$$+ (-1)^p(b)p(c)p(Y)(\bar{p}(a_{i_{h+1}}) + \cdots + \bar{p}(a_{i_{h+k}}))X(a_{i_1}, \ldots, a_{i_{h-1}}, c)Y(a_{i_{h+1}}, \ldots, a_{i_{h+k}})b$$

$$- (-1)^p(b)c(Y)(\bar{p}(a_{i_{h+1}}) + \cdots + \bar{p}(a_{i_{h+k}}))X(a_{i_1}, \ldots, a_{i_{h-1}}, 1)Y(a_{i_{h+1}}, \ldots, a_{i_{h+k}})bc$$

The generalized Leibniz rule for $X \wedge Y$ then follows by replacing these equalities in (3.6). □

It remains to prove that the Lie bracket on $\Pi GW^{as}(\Pi A)$ satisfies the generalized odd Leibniz rule (2.1). This follows from the following lemma.
Lemma 3.6 The following equalities hold for $X, Y, Z \in \Pi Gw^{as}(\Pi A)$:

$$X \Box (Y \wedge Z) = (X \Box Y) \wedge Z + (-1)^{p(X)p(Y)}(X \Box Z) - (X \Box 1) \wedge Y \wedge Z,$$

$$(X \wedge Y) \Box Z = X \wedge (Y \Box Z) + (-1)^{p(Y)p(Z)}(X \Box Z) \wedge Y.$$

**Proof.** An analogue result is proved in [5, Lemma 3.5]. For $X \in \Pi Gw^{as}_{l-k}(\Pi A)$, $Y \in \Pi Gw^{as}_{h-1}(\Pi A)$ and $Z \in \Pi Gw^{as}_{k-l-1}(\Pi A)$ with $h, k - h, l - k + 1 \geq 0$, we have:

$$X \Box (Y \wedge Z)(a_1, \ldots, a_l) = \sum_{i_1 \leq \cdots \leq i_h, i_{h+1} \leq \cdots \leq i_k, i_{k+1} \leq \cdots \leq i_l} \varepsilon_a(i_1, \ldots, i_l)(-1)^{p(Z)(\varepsilon(a_{i_1}) + \cdots + \varepsilon(a_{i_h}))}
	imes X(Y(a_{i_1}, \ldots, a_{i_h}), a_{i_{h+1}}, \ldots, a_{i_k})Z(a_{i_{k+1}}, \ldots, a_{i_l})$$

The following equalities hold for $(3.7)$

$$ \sum_{i_1 \leq \cdots \leq i_k} \varepsilon_a(i_1, \ldots, i_l)(-1)^{p(Z)(\varepsilon(a_{i_1}) + \cdots + \varepsilon(a_{i_k}))}
	imes X(Y(a_{i_1}, \ldots, a_{i_k})) = (X \Box Y) \wedge (Z(a_{i_{k+1}}, \ldots, a_{i_l})) = (X \Box Y) \wedge Z.$$

Using this equality in $(3.7)$, $X \Box (Y \wedge Z)(a_1, \ldots, a_l)$ is then of the form:

$$X \Box (Y \wedge Z)(a_1, \ldots, a_l) = A + B - C.$$

The first term $A$ is equal to

$$\sum_{i_1 \leq \cdots \leq i_k} \varepsilon_a(i_1, \ldots, i_l)(-1)^{p(Z)(\varepsilon(a_{i_1}) + \cdots + \varepsilon(a_{i_k}))}
	imes X(Y(a_{i_1}, \ldots, a_{i_k}), a_{i_{k+1}}, \ldots, a_{i_l})Z(a_{i_{k+1}}, \ldots, a_{i_l}) =$$

$$= \sum_{i_1 \leq \cdots \leq i_k} \varepsilon_a(i_1, \ldots, i_l)(-1)^{p(Z)(\varepsilon(a_{i_1}) + \cdots + \varepsilon(a_{i_k}))}
	imes (X \Box Y) \wedge Z.$$
since \( p(X \square Z) = \bar{p}(X) + p(Z) \).

Finally, the third term \( C \) is equal to
\[
\sum_{i_1 < \cdots < i_k \atop i_{k+1} < \cdots < i_l} \epsilon_a(i_1, \ldots, i_l) (-1)^l p(Z) (\bar{p}(a_{i_1}) + \cdots + \bar{p}(a_{i_k})) (-1)^p (p(Y) + p(Z) + \bar{p}(a_{i_1}) + \cdots + \bar{p}(a_{i_k})) (-1)^p (p(Y) + p(Z) + \bar{p}(a_{i_1}) + \cdots + \bar{p}(a_{i_k}))
\]
\[
\times X(1, a_{i_{k+1}}, \ldots, a_{i_l}) Y(a_{i_1}, \ldots, a_{i_k}) Z(a_{i_{k+1}}, \ldots, a_{i_l})
\]
\[
= \sum_{i_1 < \cdots < i_k \atop i_{k+1} < \cdots < i_l} \epsilon_a(i_1, \ldots, i_l) (-1)^l p(Z) (\bar{p}(a_{i_1}) + \cdots + \bar{p}(a_{i_k})) (-1)^p (p(Y) + p(Z) + \bar{p}(a_{i_1}) + \cdots + \bar{p}(a_{i_k}))
\]
\[
\times (X \square 1)(a_{i_1}, \ldots, a_{i_{k-1}}) Y(a_{i_{k+1}}, \ldots, a_{i_{k+l}}) Z(a_{i_{k+1}}, \ldots, a_{i_l}) = (X \square 1) \cdot Y \cdot Z(a_1, \ldots, a_l).
\]

This proves the first equality. The second equality can be proved in the same way, using the definition of the box product (3.2) and the concatenation product (3.3).

\[
4 \text{ The main construction}
\]

Let \((\mathcal{N}, \{\cdot, \cdot, \cdot\}, \cdot)\) be a generalized \( n \)-Nambu-Poisson algebra and denote by \( \Pi \mathcal{N} \) the space \( \mathcal{N} \) with reversed parity. Define
\[
\mu : \Pi \mathcal{N} \otimes \cdots \otimes \Pi \mathcal{N} \to \Pi \mathcal{N}
\]
\[
\mu(f_1, \ldots, f_n) = \{ f_1, \ldots, f_n \}.
\]

Then \( \mu \) is a supersymmetric function on \( (\Pi \mathcal{N})^\otimes n \) (Lemma 1.2). Furthermore \( \mu \) satisfies the generalized Leibniz rule
\[
\mu(f_1, \ldots, f_{n-1}, gh) = \mu(f_1, \ldots, f_{n-1}, g) h + g \mu(f_1, \ldots, f_{n-1}, h) - \mu(f_1, \ldots, f_{n-1}, 1) gh,
\]
hence \( \mu \) lies in \( \mathcal{G}W_{n-1}^{as}(\Pi \mathcal{N}) \).

Let \( OP(\mathcal{N}) \) be the odd Poisson subalgebra of \( \mathcal{G}(\mathcal{N}) \) generated by \( \Pi \mathcal{N} \) and \( \mu \). Then, by construction, \( OP(\mathcal{N}) \) is a transitive Lie subalgebra of \( \mathcal{G}W_{n-1}^{as}(\Pi \mathcal{N}) \), hence it is a transitive subalgebra of \( W(\Pi \mathcal{N}) \). Furthermore \( OP(\mathcal{N}) \) is a \( \mathbb{Z}_+ \)-graded odd Poisson subalgebra of \( \mathcal{G}(\mathcal{N}) \). Let us denote by \( OP(\mathcal{N}) = \oplus_{j \geq 1} P_j(\mathcal{N}) \) its depth 1 \( \mathbb{Z} \)-grading as a Lie superalgebra.

**Proposition 4.1** If \( \mathcal{N} \) is a simple generalized \( n \)-Nambu-Poisson algebra then \( OP(\mathcal{N}) \) is a simple generalized odd Poisson superalgebra.
Proof. Let $I$ be a non-zero ideal of $OP(N)$. Then, by transitivity, $I \cap P_{-1}(N) = I \cap N \neq 0$. Note that $I \cap N$ is a Nambu-Poisson ideal of $N$. Indeed, $(I \cap N) \cdot N = (I \cap N) \wedge N \subset I \cap N$ and $[I \cap N, N] \subset [N, N] = 0$. Since $N$ is simple, $I \cap N = N$, hence $1 \in I$, hence $I = OP(N)$. □

Remark 4.2 We recall that since $(N, \{\cdot, \ldots, \cdot\})$ is an $n$-Lie algebra, the Filippov-Jacobi identity holds, i.e., for every $a_1, \ldots, a_n \in N$, the map $D_{a_1, \ldots, a_n} : N \to N$, $D_{a_1, \ldots, a_n}(a) = \{a_1, \ldots, a_n, a\}$ is a derivation of $(N, \{\cdot, \ldots, \cdot\})$. By [48 Lemma 2.1(b)], this is equivalent to the condition $[\mu, D_{a_1, \ldots, a_n}] = 0$ in $OP(N)$. By (4.1), we have: $D_{a_1, \ldots, a_n} = [[\mu, a_1], \ldots, a_n]$, therefore $\mu$ satisfies the following condition:

$$[[\mu, [\mu, a_1], \ldots, a_n]] = 0$$

for every $a_1, \ldots, a_n \in N$.

Definition 4.3 We say that a pair $(\mathcal{P}, \mu)$, consisting of a $\mathbb{Z}_+\text{-graded}$ generalized odd Poisson superalgebra $\mathcal{P}$ and an element $\mu \in \mathcal{P}_{-1}$ of parity $p(\mu) \equiv n \pmod{2}$, is a good $n$-pair if it satisfies the following properties:

G1) $\mathcal{P} = \bigoplus_{j \geq -1} \mathcal{P}_j$ is a transitive $\mathbb{Z}$-graded Lie superalgebra of depth 1 such that $\mathcal{P}_{-1}$ is completely odd;

G2) $\mu$ and $\mathcal{P}_{-1}$ generate $\mathcal{P}$ as a (generalized) odd Poisson superalgebra;

G3) $[\mu, [[\mu, a_1], \ldots, a_n]] = 0$ for every $a_1, \ldots, a_n \in \mathcal{P}_{-1}$.

Example 4.4 Let $\mathcal{P} = PO(2h, 2h)$, $h \geq 1$, with the grading of type $(0, \ldots, 0|1, \ldots, 1)$, and let $\mu = \sum_{i=1}^{h} \xi_i \xi_i + h$. Then $(\mathcal{P}, \mu)$ is a good 2-pair. Indeed, for $1 \leq i \leq h$, $[x_i, \mu]_{HO} = \xi_i$ and $[x_i, x_j, \mu]_{HO} = -\xi_i \cdot \xi_j$, therefore $\mathcal{P}_{-1}$ and $\mu$ generate $\mathcal{P}$. Furthermore, for $f \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \ldots, x_n]]$, we have: $[f, \mu]_{HO} = \sum_{i=1}^{h} \left( \frac{\partial f}{\partial x_i} \xi_i + h - \frac{\partial f}{\partial \xi_i} \xi_i \right)$, hence

$$[\mu, [f, \mu]_{HO}]_{HO} = \sum_{i,j=1}^{h} [\xi_j \xi_j + h, \frac{\partial f}{\partial x_i} \xi_i - \frac{\partial f}{\partial \xi_i} \xi_i]_{HO} =$$

$$= \sum_{i,j=1}^{h} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \xi_j + h \xi_i + h - \frac{\partial^2 f}{\partial x_i \partial x_i} \xi_i - h \xi_i + \frac{\partial^2 f}{\partial x_j \partial x_i} \xi_j + h \xi_i \right) = 0.$$ 

Therefore $(\mathcal{P}, \mu)$ satisfies property G3.

Example 4.5 Let $\mathcal{P} = PO(n, n)$ with the grading of type $(0, \ldots, 0|1, \ldots, 1)$, and let $\mu = \xi_1 \ldots \xi_n$. Then $(\mathcal{P}, \mu)$ is a good $n$-pair. Indeed, $[x_n-1, [x_2, [x_1, \mu]]]_{HO} = \xi_n$, and similarly all the $\xi_i$’s can be obtained by commuting $\mu$ with different $x_j$’s. Therefore $\mathcal{P}_{-1}$ and $\mu$ generate $\mathcal{P}$. Furthermore, let $f = \sum_{i=1}^{n} f_i \xi_i \in \mathcal{P}_0$, with $f_i \in \mathbb{F}[[x_1, \ldots, x_n]]$, such that

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} = 0.$$ 

Then $[f, \mu]_{HO} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \xi_1 \ldots \xi_n = 0$. Notice that all elements of the form $[[\mu, a_1], \ldots, a_n]_{-1}$ with $a_1, \ldots, a_n \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \ldots, x_n]]$ satisfy property (4.2), hence $(\mathcal{P}, \mu)$ satisfies property G3.
Example 4.6 Let $\mathcal{P} = PO(2h + 1, 2h + 2)$, $h \geq 1$, with the grading of type $(0, \ldots, 0|1, \ldots, 1)$, and let $\mu = \sum_{i=1}^{h+1} \xi_i \epsilon_{i+h+1}$ (recall that $\epsilon_{2h+2} = \tau$). Then $(\mathcal{P}, \mu)$ is a good 2-pair. Indeed, we have: $[1, \mu]_{KO} = 2\xi_{h+1}$ and $[x_i, \mu]_{KO} = \xi_{i+h+1} - x_i \xi_{h+1}$ for $1 \leq i \leq h+1, [x_{i+h+1}, \mu]_{KO} = -\xi_i - x_{i+h+1} \xi_{h+1}$ for $1 \leq i \leq h$. Hence $\mathcal{P}_{-1}$ and $\mu$ generate $\mathcal{P}$. Furthermore, if $f \in \mathcal{P}_{-1} = \mathbb{F}[x_1, \ldots, x_n]$, we have:

$$[f, \mu]_{KO} = \sum_{i=1}^{h} \left( \frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} - (E - 2)(f) \xi_{h+1},$$

hence

$$[\mu, [f, \mu]_{KO}]_{KO} = \sum_{j=1}^{h+1} \xi_j \xi_{h+1+j}, \sum_{i=1}^{h} \left( \frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} - (E - 2)(f) \xi_{h+1} \right)_{KO}$$

$$- \frac{\partial ((E - 2)(f))}{\partial x_j} \xi_{h+1} - \sum_{i=1}^{h} \xi_i \left( \frac{\partial f}{\partial x_{i+h+1}} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) - \sum_{j=1}^{h+1} \xi_{h+1+j} \left( \frac{\partial f}{\partial x_{j+1+h}} \right) \xi_{2h+2}$$

$$- \frac{\partial ((E - 2)(f))}{\partial x_{j+1+h}} \xi_{j+1} - \sum_{j=1}^{h+1} \xi_{j+1} \left( \frac{\partial f}{\partial x_{j+h+1}} \xi_{j+h+1} - \frac{\partial f}{\partial x_{j+h+1}} \xi_j \right) - \sum_{j=1}^{h} \xi_{j} \left( \frac{\partial f}{\partial x_{j+h+1}} \xi_{j+h+1} - \frac{\partial f}{\partial x_{j+h+1}} \xi_j \right)$$

$$- \sum_{j=1}^{h} \frac{\partial ((E - 2)(f))}{\partial x_{j+h+1}} \xi_{j} - (E - 2)(f) \sum_{i=1}^{h} \left( \frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} = 0.$$

Example 4.7 Let $\mathcal{P} = PO(n, n + 1)$, with the grading of type $(0, \ldots, 0|1, \ldots, 1)$, and let $\mu = \xi_1 \ldots \xi_n \tau$ (recall that $\tau = \xi_{n+1}$). Then $(\mathcal{P}, \mu)$ is a good $(n+1)$-pair. Indeed, we have: $[1, \mu]_{KO} = 2(-1)^{n+1} \xi_1 \ldots \xi_n$, $[x_i, [\ldots [\xi_{i-1}, \xi_i \ldots \xi_n]_{KO}]_{KO}]_{KO} = \pm \xi_i$ for $i_1 \neq \cdots \neq i_{n-1} \neq i_n$, $[x_i, \xi_1 \ldots \xi_n \tau]_{KO} = \xi_{i+1} \ldots \xi_n \tau + (-1)^{n-i} x_i \xi_1 \ldots \xi_n$ for $1 \leq i \leq n$. Hence $\mathcal{P}_{-1}$ and $\mu$ generate $\mathcal{P}$. Now let $\text{div}_1 = \Delta + (E - n) \frac{\partial}{\partial x_j}$ where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i} \partial_{x_i}$ is the odd Laplacian, and let $f = \sum f_i \xi_i \in \mathcal{P}_0$, $f_i \in \mathbb{F}[x_1, \ldots, x_n]$, such that $0 = \text{div}_1(f) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + (E - n)(f_{n+1})$. Then we have:

$$\sum_{i=1}^{n} f_i \xi_i [\mu]_{KO} = \sum_{i=1}^{n} f_i [\xi_i, \mu]_{KO} + \sum_{i=1}^{n+1} (E - 2)(f_i \xi_i)(-1)^n \xi_1 \ldots \xi_n - f_{n+1}(n-2)\xi_1 \ldots \xi_n \tau$$

$$= \sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j} \xi_1 \ldots \xi_n \tau + (-1)^n (E - 2)(f_{n+1} \xi_n \xi_{n+1} \xi_1 \ldots \xi_{n-1}) - (n-2) f_{n+1} \xi_1 \ldots \xi_n \tau$$

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\[
\sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} + (E - 2)(f_{n+1}) - (n - 2)f_{n+1})\xi_1 \ldots \xi_n = 0.
\]

Notice that, since \( \text{div}_1(\mu) = 0 \) and \( \text{div}_1(f) = 0 \) for every \( f \in \mathcal{P}_- \), then \( \text{div}_1([[[\mu, a_1], \ldots, a_{n-1}]] = 0 \) for every \( a_1, \ldots, a_{n-1} \in \mathcal{P}_- \). Hence property G3) is satisfied.

**Remark 4.8** Let us consider \( \mathcal{P} = PO(k, k) \) (resp. \( \mathcal{P} = PO(k, k + 1) \)) with the grading of type \((0, \ldots, 0|1, \ldots, 1)\) (resp. \((0, \ldots, 0|1, \ldots, 1, 1)\)). Let \( \varphi \in \mathcal{P}_- \) be an invertible element. By Remark 2.3, the grading of type \((0, \ldots, 0|1, \ldots, 1)\) (resp. \((0, \ldots, 0|1, \ldots, 1, 1)\)) defines a \( \mathbb{Z}_+ \)-graded structure on the odd generalized Poisson superalgebra \( \mathcal{P}^\varphi \), such that \( \mathcal{P} = \mathcal{P}^\varphi \). Then, by (2.4), \( (\mathcal{P}, \mu) \) is a good \( n \)-pair with respect to this grading if and only if \((\mathcal{P}^\varphi, \varphi^{-1} \mu) \) is.

The map \( \mathcal{N} \mapsto (OP(\mathcal{N}), \mu) \) establishes a correspondence between (simple) generalized \( n \)-Nambu-Poisson algebras \( \mathcal{N} \) and good \( n \)-pairs \((OP(\mathcal{N}), \mu) \). We now want to show that this correspondence is bijective.

**Lemma 4.9** Let \( \mathcal{N} \) be a generalized \( n \)-Nambu-Poisson algebra. Then the 0-th graded component \( \mathcal{P}_0(\mathcal{N}) \) of \( OP(\mathcal{N}) \) is generated, as a Lie superalgebra, by elements of the form

\[
[a_1, [a_2, \ldots, [a_{n-1}, \mu]b]
\]

with \( a_i, b \in \Pi \mathcal{N} \).

**Proof.** Let \( L_{-1} := \Pi \mathcal{N} \) and let \( L_0 \) be the Lie subsuperalgebra of \( GW^{as}_0(\Pi \mathcal{N}) = G\text{Der}(\Pi \mathcal{N}) \) generated by the elements of the form \([a_1, [a_2, \ldots, [a_{n-1}, \mu]b]]\) with \( a_1, \ldots, a_{n-1}, b \in \Pi \mathcal{N} \). Note that, since \( GW^{as}_0(\Pi \mathcal{N}) \) is \( \mathbb{Z} \)-graded of depth 1, and \( 1 \in \mathcal{N} \), the restriction to \( \mathcal{N} \) of the derivation \( D \) of \( G(\mathcal{N}) \) is zero, hence

\[
[a_1, [a_2, \ldots, [a_{n-1}, \mu]b]] = [a_1, [a_2, \ldots, [a_{n-1}, \mu]b]b.
\]

An induction argument on the length of the commutators of the generating elements of \( L_0 \) shows that \( L_0 \) is stable with respect to the concatenation product by elements of \( \Pi \mathcal{N} \).

Let \( L \) be the full prolongation of \( L_{-1} \oplus L_0 \), i.e., \( L = L_{-1} \oplus L_0 \oplus (\oplus_{j \geq 1} L_j) \), where \( L_j = \{ \varphi \in GW^{as}_0(\Pi \mathcal{N}) | [\varphi, L_{-1}] \subset L_{j-1} \} \). Note that \( L_j \), for \( j \geq 1 \), is stable with respect to the concatenation product by elements of \( \Pi \mathcal{N} \). Indeed, if \( \varphi \in L_j \), then

\[
[\varphi \Pi \mathcal{N}, L_{-1}] = [\varphi, L_{-1}] \Pi \mathcal{N} \subset L_{j-1} \Pi \mathcal{N},
\]

hence one can conclude by induction on \( j \) since \( L_0 \Pi \mathcal{N} \subset L_0 \). It follows that \( L \) is closed under the concatenation product, hence it is an odd generalized Poisson subsuperalgebra of \( GW^{as}(\Pi \mathcal{N}) \). Indeed, using induction on \( i + j \geq 0 \), one shows that \( L_i L_j \subset L \) for every \( i, j \geq 0 \).

It follows that \( OP(\mathcal{N}) \) is an odd generalized subsuperalgebra of \( L \), since \( L \) is an odd generalized Poisson superalgebra containing \( \Pi \mathcal{N} \) and \( \mu \). As a consequence, the 0-th graded component \( \mathcal{P}_0(\mathcal{N}) \) of \( OP(\mathcal{N}) \) is generated, as a Lie superalgebra, by elements of the form

\[
[a_1, [a_2, \ldots, [a_{n-1}, \mu]b]
\]

with \( a_i, b \in \Pi \mathcal{N} \). \( \square \)
Proposition 4.10 Let \((\mathcal{P}, \mu)\) be a good \(n\)-pair, and define on \(\mathcal{N} := \Pi \mathcal{P}_-\) the following product:

\[
\{x_1, \ldots, x_n\} = [\ldots [[\mu, x_1], \ldots, x_n]].
\]

Then:

(a) \((\mathcal{N}, \{\ldots, \cdot, \ldots\}, \wedge)\) is a generalized Nambu-Poisson algebra, \(\wedge\) being the restriction to \(\mathcal{N}\) of the commutative associative product \(\wedge\) defined on \(\mathcal{P}\).

(b) If \(\mathcal{P}\) is a simple odd generalized Poisson superalgebra, then \((\mathcal{N}, \{\ldots, \cdot, \ldots\}, \wedge)\) is a simple generalized Nambu-Poisson algebra.

Proof. (a) By Definitions 2.1 and 2.6 \(\mathcal{N} = \mathcal{Q}_0\) is a commutative associative subalgebra of \(\mathcal{P}\). Furthermore \(\{\ldots, \cdot, \ldots\}\) is an \(n\)-Lie bracket due to \([\mathcal{I}, \mathcal{P}] = \odot \mathcal{Q}_j\) and property G3). Finally, for \(f_1, \ldots, f_{n-1}, g, h \in \Pi \mathcal{P}_-\), we have:

\[
\{f_1, \ldots, f_{n-1}, gh\} = [[\ldots [\mu, f_1], \ldots, f_{n-1}], gh] = [[\ldots [\mu, f_1], \ldots, f_{n-1}], g]h + g[[\ldots [\mu, f_1], \ldots, f_{n-1}], h] + (-1)^p[[\ldots [\mu, f_1], \ldots, f_{n-1}], 1][1, \ldots [\mu, f_1], \ldots, f_{n-1}, 1]]gh = \{f_1, \ldots, f_{n-1}, g\}h + g\{f_1, \ldots, f_{n-1}, h\} + (-1)^p[[\ldots [\mu, f_1], \ldots, f_{n-1}], 1](-1)^p[[\ldots [\mu, f_1], \ldots, f_{n-1}], 1]\{f_1, \ldots, f_{n-1}, 1\}gh
\]

\[
= \{f_1, \ldots, f_{n-1}, g\}h + g\{f_1, \ldots, f_{n-1}, h\} - \{f_1, \ldots, f_{n-1}, 1\}gh.
\]

(b) Now we want to show that if \(\mathcal{P}\) is simple, then \(\mathcal{N}\) is simple. Suppose that \(I\) is a non zero ideal of \(\mathcal{N}\), and let \(\bar{I}\) be the ideal of \(\mathcal{P}\) generated by \(\Pi I\) and \(\mu\): \(\bar{I} = \oplus_{j=1}^\infty \bar{I}_j\), with \(\bar{I}_j \subset \mathcal{P}_j\). We want to show that \(\bar{I}_j = \bar{I} \cap \mathcal{P}_- = \Pi I\). In fact, the concatenation product by elements in \(\oplus_{j=0}^\infty \mathcal{Q}_j\) maps \(\mathcal{Q}_0\) to \(\oplus_{j=1}^\infty \mathcal{Q}_j\) hence it does not produce any element in \(\mathcal{P}_- = \mathcal{Q}_0\). On the other hand, \(I \cap \mathcal{Q}_0 = I \cap \mathcal{N} \subset I\) since \(I\) is an ideal of \(\mathcal{N}\). The bracket between elements in \(\oplus_{j=0}^\infty \mathcal{P}_j\) and the bracket between \(I\) and elements in \(\oplus_{j=0}^\infty \mathcal{Q}_j\) lies in \(\oplus_{j=1}^\infty \mathcal{P}_j\). Therefore we just need to consider the brackets between elements in \(I\) and elements in \(\mathcal{P}_0\). By hypothesis, \(\mathcal{P}\) is generated by \(\mathcal{P}_-\) and \(\mu\), hence, by the same argument as in Lemma 1.9 \(\mathcal{P}_0\) is generated by elements of the form \([a_1, [a_2, \ldots, [a_{n-1}, \mu]]]b\) with \(a_i, b \in \Pi \mathcal{P}_-\). We have:

\[
[I, [a_1, [a_2, \ldots, [a_{n-1}, \mu]]]]b = [I, [a_1, [a_2, \ldots, [a_{n-1}, \mu]]]]b
\]

since \([I, b] = 0\) and \(D|I = 0\). Since \([I, [a_1, [a_2, \ldots, [a_{n-1}, \mu]]]] = [I, a_1, \ldots, a_{n-1}]\) and \(I\) is an ideal of \(\mathcal{N}\), \([I, \mathcal{P}_0] \subset I\). \(\square\)

Definition 4.11 Two good \(n\)-pairs \((\mathcal{P}, \mu)\) and \((\mathcal{P}', \mu')\) are called isomorphic if there exists an odd Poisson superalgebras isomorphism \(\Phi : \mathcal{P} \to \mathcal{P}'\) such that \(\Phi(\mathcal{P}_j) = \mathcal{P}'_j\), \(\Phi(\mathcal{Q}_j) = \mathcal{Q}'_j\) for all \(j\) and \(\phi(\mu) \in \mathbb{F}^x \mu'\).

Theorem 4.12 The map

\[
\mathcal{N} \to (OP(\mathcal{N}), \mu)
\]

with \(\mu\) defined as in [4.4], establishes a bijection between isomorphism classes of generalized \(n\)-Nambu-Poisson algebras and isomorphism classes of good \(n\)-pairs. Moreover:

(i) \(\mathcal{N}\) is simple (linearly compact) if and only if \(OP(\mathcal{N})\) is;

(ii) \(\mathcal{N}\) is a Nambu-Poisson algebra if and only if \(OP(\mathcal{N})\) is an odd Poisson superalgebra.
Remark 4.13 One can check (see also [4]) that if \( N \) is the \( n \)-Nambu algebra, then \( (OP(N), \mu) = (PO(n, n), \xi_1 \ldots \xi_n) \) and if \( N \) is the \( n \)-Dzhumalidaev algebra, then \( (OP(N), \mu) = (PO(n - 1, n), \xi_1 \ldots \xi_{n-1}) \).

5 Classification of good pairs

In this section we will consider the odd Poisson (resp. generalized odd Poisson) superalgebra \( PO(n, n) \) (resp. \( PO(n, n+1) \)) with the grading of type \((0, \ldots, 0|1, \ldots, 1)\) (resp. \((0, \ldots, 0|1, \ldots, 1, 1)\)).

Proposition 5.1 Let \( \mathcal{P} = PO(n, n) \) or \( \mathcal{P} = PO(n, n + 1) \) and \((\mathcal{P}, \mu)\) be a good \( k \)-pair. Then the Lie subalgebra \( \mathcal{P}_0 \) of \( \mathcal{P} \) is spanned by elements of the form:

\[ [[\mu, a_1], \ldots, a_{k-1}]b \]

with \( a_1, \ldots, a_{k-1}, b \in \mathcal{P}_{-1} \).

Proof. By Theorem 4.12 \( \mathcal{P} = OP(N) \) for some \( k \)-Nambu-Poisson algebra \( N \). Hence, by Lemma 4.9 \( \mathcal{P}_0 \) is generated as a Lie algebra by elements of the form

\[ [[\mu, a_1], \ldots, a_{k-1}]b \]

with \( a_1, \ldots, a_{k-1}, b \in \mathcal{P}_{-1} \). Let \( S = \langle [[\mu, a_1], \ldots, a_{k-1}] | a_1, \ldots, a_{k-1} \in \mathcal{P}_{-1} \rangle \subset \mathcal{P}_0 \).

Let \( \mathcal{P} = PO(n, n) \). Then, for \( z_1, z_2 \in S, b_1, b_2 \in \mathcal{P}_{-1} \), we have:

\[ [z_1b_1, z_2b_2] = [z_1b_1, z_2]b_2 + (-1)^{\mu(z_2)(\mu(z_1)+\mu(b_1)+1)}z_2[z_1b_1, b_2] = (-1)^{\mu(b_1)(\mu(z_2)+1)}z_2[1, z_2]b_1b_2 + +z_1[1, z_2]b_2 + (-1)^{\mu(b_1)(\mu(b_2)+1)+\mu(z_2)(\mu(z_1)+\mu(b_1)+1)}z_2[z_1, b_2]b_1 \]

since \([b_1, b_2] = 0\). We recall that \([z_1, z_2] \in S \) by [4] Theorem 0.2. Finally, note that \([z_1, b_2] \) and \([b_1, z_2] \) lie in \( \mathcal{P}_{-1} \). It follows that \( \mathcal{P}_0 \subseteq \langle [[\mu, a_1], \ldots, a_{k-1}]b | a_1, b \in \mathcal{P}_{-1} \rangle \subseteq \mathcal{P}_0 \), hence the statement holds for \( \mathcal{P} = PO(n, n) \).

If \( \mathcal{P} = PO(n, n + 1) \), one uses exactly the same argument and the fact that \( D_{\mathcal{P}_{-1}} = 0 \), \( D(S) \subseteq \mathcal{P}_{-1} \).

For any element \( f \in \mathcal{P}_{k-1} = \mathbb{F}[[x_1, \ldots, x_n]] \otimes \wedge^k \mathbb{F}^n \), we let \( f_0 = f |_{x_1 = \ldots = x_n = 0} \in \wedge^k \mathbb{F}^n \). We shall say that \( f \) has positive order if \( f_0 = 0 \).

Corollary 5.2 Let \( \mathcal{P} = PO(n, n) \) (resp. \( PO(n, n + 1) \)) with the grading of type \((0, \ldots, 0|1, \ldots, 1)\) (resp. \((0, \ldots, 0|1, \ldots, 1, 1)\)). If \( \mu \in \mathcal{P}_{k-1} \) is such that \( \mu_0 \) lies in the Grassmann subalgebra of \( \wedge^k (\mathbb{F}^n) \) (resp. \( \wedge^k (\mathbb{F}^{n+1}) \)) generated by some variables \( \xi_{i_1}, \ldots, \xi_{i_h} \), for some \( h < n \) (resp. \( h < n + 1 \)), then \( \mu \) does not satisfy property G2). In particular, if \( \mu_0 = 0 \), then \( \mu \) does not satisfy property G2).

Proof. Suppose, on the contrary, that some \( \xi_i \) does not appear in the expression of \( \mu_0 \). Then, by Proposition 5.1 \( \mathcal{P}_0 \) does not contain \( \xi_i \) and this is a contradiction since if \( \mathcal{P} = PO(n, n) \) (resp. \( \mathcal{P} = PO(n, n + 1) \)), \( \mathcal{P}_0 = \langle \xi_1, \ldots, \xi_n \rangle \otimes \mathbb{F}[[x_1, \ldots, x_n]] \) (resp. \( \mathcal{P}_0 = \langle \xi_1, \ldots, \xi_{n+1} \rangle \otimes \mathbb{F}[[x_1, \ldots, x_n]] \)).
5.1 The case $PO(n, n)$

In this subsection we shall determine good $k$-pairs $(P, \mu)$ for $P = PO(n, n)$ with the $\mathbb{Z}_+$-grading of type $(0, \ldots, 0|1, \ldots, 1)$. We will denote the Lie superalgebra bracket in $PO(n, n)$ simply by $[\cdot, \cdot]$. Recall the corresponding description of the $\mathbb{Z}_+$-grading given in Example 2.7. When writing a monomial in $\xi_i$'s we will assume that the indices increase; elements from $\wedge^k \mathbb{F}^n$ will be written as linear combinations of such monomials.

**Lemma 5.3** Let $2 < k < n - 1$ and suppose that $\mu \in PO(n, n)_{k-1}$ can be written in the following form:

\[
\mu = \xi_1 \ldots \xi_k + \xi_1 \ldots \xi_h \xi_{k+1} \xi_{h+1} \ldots \xi_{k-2} + \phi + \psi,
\]

where:

\[
\mu_0 = \xi_1 \ldots \xi_k + \xi_1 \ldots \xi_h \xi_{k+1} \xi_{h+1} \ldots \xi_{k-2} + \phi, \quad \phi \in \wedge^k \mathbb{F}^n, \quad \psi_0 = 0,
\]

$h = \max\{0 \leq j \leq k - 2 \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_1 \ldots \partial \xi_{j+1}} \neq 0, \text{ for some } i_1 < \ldots < i_j \leq k, \text{ and some } r, s > k\}$,

\[
\frac{\partial^{k-1} \phi}{\partial \xi_1 \ldots \partial \xi_{k-1}} = 0, \quad \frac{\partial^k \phi}{\partial \xi_1 \ldots \partial \xi_{k+1} \partial \xi_{k+2} \partial \xi_{k+3} \partial \xi_{k+4} \ldots} = 0.
\]

Then $\mu$ does not satisfy property G3.

**Proof.** Let us first suppose that $h \geq 1$. We have:

\[
[x_{k+1}, \mu] = (-1)^h \xi_1 \ldots \xi_k \xi_k + 2 \xi_{h+1} \ldots \xi_{k-2} + \frac{\partial (\phi + \psi)}{\partial \xi_{k+1}};
\]

\[
[x_{i_1}, \ldots, x_{i_{k+1}}, x_h, \ldots, x_x, [x_1, [x_{k+1}, \mu]]]] = 2(-1)^{k-2} x_{k+2} + 2 x_{h+1} \frac{\partial^{k-1} (\phi + \psi)}{\partial \xi_{k+2} \ldots \partial \xi_{i_{h+1}} \partial \xi_{h} \ldots \partial \xi_{1} \partial \xi_{k+1}}.
\]

Therefore $[\mu, [x_{i_1}, \ldots, x_{i_{h+1}}, [x_h, \ldots, x_x, [x_1, [x_{k+1}, \mu]]]]] =

\[
= 2(-1)^k \xi_2 \ldots \xi_k ((-1)^{k-2} \xi_k + 2 \xi_{h+1} \frac{\partial^{k-1} \phi}{\partial \xi_{k+2} \ldots \partial \xi_{h+1} \partial \xi_{h} \ldots \partial \xi_{1} \partial \xi_{k+1}}) + \xi_2 \ldots \xi_h \xi_k + 2 \xi_{h+1} \ldots \xi_{k-2} \frac{\partial^{k-1} \phi}{\partial \xi_{k+2} \ldots \partial \xi_{h+1} \partial \xi_{h} \ldots \partial \xi_{1} \partial \xi_{k+1}} + \frac{\partial \phi}{\partial \xi_1} \frac{(-1)^{k-2} \xi_k + 2 \xi_{h+1} \frac{\partial^{k-1} \phi}{\partial \xi_{k+2} \ldots \partial \xi_{h+1} \partial \xi_{h} \ldots \partial \xi_{1} \partial \xi_{k+1}}}) + \omega,
\]

for some $\omega$ of positive order. Note that, the summand $2 \xi_2 \ldots \xi_k \xi_{k+2}$ in the expression of $[\mu, [x_{i_1}, \ldots, x_{i_{h+1}}, [x_h, \ldots, x_x, [x_1, [x_{k+1}, \mu]]]]]$ does not cancel out. Indeed, due to the hypotheses on $\phi$, the only possibility to cancel the summand $2 \xi_2 \ldots \xi_k \xi_{k+2}$ is that the expression of $\phi$ contains the sum $a \xi_1 \ldots \xi_h \xi_{k+1} \xi_{h+1} \ldots \xi_{k-2} \xi_t + b \xi_1 \ldots \xi_{t-1} \xi_{t+1} \ldots \xi_{k+2}$, for some $t, 2 \leq t \leq k$, and some suitable coefficients $a, b \in \mathbb{F}$. But this is impossible since it is in contradiction with the maximality of $h$ if $h = k - 2$, and with the hypotheses on $\phi$ if $h < k - 2$. It follows that $[\mu, [x_{i_1}, \ldots, x_{i_{h+1}}, [x_h, \ldots, x_x, [x_1, [x_{k+1}, \mu]]]]] \neq 0$ and property G3 is not satisfied.

If $h = 0$, then one can use the same argument by showing that the commutator $[\mu, [x_1 x_{k+1}, [x_{i_1}, \ldots, [x_{i_{h-2}}, \mu]]]]$ is different from zero.

**Theorem 5.4** Let $P = PO(n, n)$. Suppose that $2 < k < n - 1$ and that $\mu \in PO(n, n)_{k-1}$. Then $(P, \mu)$ is not a good $k$-pair.

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Proof. By Corollary 5.2 if $\mu_0 = 0$ then $\mu$ does not satisfy property G2). Now suppose $\mu_0 \neq 0$. Since $\mu_0$ lies in $\Lambda^k(\mathbb{P}^n)$, we can assume, up to a linear change of indeterminates, that $\mu_0 = \xi_1 \ldots \xi_k + f$ for some $f \in \Lambda^k(\mathbb{P}^n)$ such that $\frac{\partial^k f}{\partial \xi_1 \ldots \partial \xi_k} = 0$. Then, either $\mu$ does not satisfy property G2) and $(\mathcal{P}, \mu)$ is not a good $k$-pair, or, again by Corollary 5.2, all $\xi_i$'s appear in the expression of $\mu_0$. Let us thus assume to be in the latter case. Then, since $k < n - 1$, either there exist some $r, s > k$ such that the indeterminates $\xi_r$ and $\xi_s$ both appear in the expression of $\mu_0$ in at least one monomial (case A), or all the indeterminates $\xi_i$ and $\xi_s$ with $r, s > k$ appear in distinct monomials (case B).

Suppose we are in case A), and let $h = \max\{0 \leq j \leq k - 2 \mid \frac{\partial^{i+2} \mu_0}{\partial \xi_1 \ldots \partial \xi_i \partial \xi_r \partial \xi_s} \neq 0, i_1 < \ldots < i_j \leq k; r, s > k\}$. Then we can write

$$\mu_0 = \xi_1 \ldots \xi_k + \xi_{i_1} \ldots \xi_{i_h} \xi_i \xi_{i_{h+1}} \ldots \xi_{i_{k-2}} + \varphi$$

for some $r, s, i_{h+1}, \ldots, i_{k-2} > k$, $i_1, \ldots, i_h \leq k$ and some $\varphi \in \Lambda^k(\mathbb{P}^n)$ such that $\frac{\partial^k \varphi}{\partial \xi_1 \ldots \partial \xi_k} = 0$ and $\frac{\partial^k \varphi}{\partial \xi_{i_1} \ldots \partial \xi_{i_h} \partial \xi_r \partial \xi_s} = 0$. Up to a permutation of indices we can assume $r = k + 1$, $s = k + 2$, $\{i_1, \ldots, i_h\} = \{1, \ldots, h\}$ and up to a linear change of indeterminates we can assume $\frac{\partial^{k-1} \varphi}{\partial \xi_1 \ldots \partial \xi_{k-1}} = 0$. Therefore $\mu$ satisfies the hypotheses of Lemma 5.3 (hence it does not satisfy property G3).

Now suppose we are in case B). Then

$$\mu_0 = \xi_1 \ldots \xi_k + \xi_{i_1} \ldots \xi_{i_{k-1}} \xi_{k+1} + \xi_{j_1} \ldots \xi_{j_{k-1}} \xi_{k+2} + \psi$$

for some $i_1 < \ldots < i_{k-1} \leq k$, $j_1 < \ldots < j_{k-1} \leq k$ and $\psi \in \Lambda^k(\mathbb{P}^n)$ such that $\frac{\partial^k \psi}{\partial \xi_1 \ldots \partial \xi_k} = 0$, $\frac{\partial^k \psi}{\partial \xi_{i_1} \ldots \partial \xi_{i_{k-1}} \partial \xi_{k+1}} = 0$, $\frac{\partial^k \psi}{\partial \xi_{r} \partial \xi_{s}} = 0$ for every $r, s > k$. Again by Corollary 5.2 we can assume that $\{i_1, \ldots, i_{k-1}\} \neq \{j_1, \ldots, j_{k-1}\} \neq \{1, \ldots, k - 1\}$. Therefore there exists an index $\bar{j} \in \{1, \ldots, k\} \cap \{j_1, \ldots, j_{k-1}\}$ such that $\bar{j} \notin \{i_1, \ldots, i_{k-1}\}$.

Now consider the following change of indeterminates:

$$\xi'_{\bar{j}} = \xi_{\bar{j}} + \xi_{k+1}; \quad \xi'_j = \xi_j \forall j \neq \bar{j}.$$ 

Then

$$\mu_0 = \xi'_1 \ldots \xi'_k + \xi'_{i_1} \ldots \xi'_{i_{k-1}} \xi_{k+2} + \xi'_{j_1} \ldots \xi'_{j_{k-1}} \xi_{k+1} + \xi'_{j_{k-1}} \xi_{k+1} \xi_{k+2} + \rho$$

for some $\rho \in \Lambda^k(\mathbb{P}^n)$ such that $\frac{\partial^k \rho}{\partial \xi_1 \ldots \partial \xi_k} = 0$, $\frac{\partial^k \rho}{\partial \xi_{i_1} \ldots \partial \xi_{i_{k-1}} \partial \xi_{k+1}} = 0$, $\frac{\partial^k \rho}{\partial \xi_{r} \partial \xi_{s}} = 0$. We are now again in case A) hence the proof is concluded. \hfill \Box

**Theorem 5.5** Let $\mathcal{P} = PO(n, n)$. If $(\mathcal{P}, \mu)$ is a good $k$-pair, then, up to isomorphisms, one of the following possibilities may occur:

a) If $n = 2h$:

a1) $k = 2$ and $\mu_0 = \sum_{i=1}^{h} \xi_i \xi_{i+h}$;
a2) $k = n$ and $\mu_0 = \xi_1 \ldots \xi_n$.

b) If $n = 2h + 1$:

b1) $k = n$ and $\mu_0 = \xi_1 \ldots \xi_n$. 

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Proof. By Theorem 5.3, the only possibilities for \( k \) are \( k = 2 \), \( k = n - 1 \) or \( k = n \).

By Corollary 5.2, \( \frac{\partial^{n-2}\mu_0}{\partial \xi_{i_1} \ldots \partial \xi_{i_{n-2}}} \neq 0 \) for every \( i = 1, \ldots, n \). Using the classification of non-degenerate skew-symmetric bilinear forms, it thus follows that the case \( k = 2 \) can occur only if \( n = 2h \) and, up to equivalence, \( \mu_0 = \sum_{i=1}^{h} \xi_i \xi_{i+h} \), hence we get (a1).

If \( k = n \) then, up to rescaling the odd indeterminates, \( \mu_0 = \xi_1 \ldots \xi_n \) and we get cases (a2) and (b1).

Now assume \( k = n - 1 \). Assume that \( \frac{\partial^{n-2}\mu_0}{\partial \xi_{i_1} \ldots \partial \xi_{i_{n-2}}} = \alpha \xi_{i_{n-1}} + \beta \xi_{i_n} \) for some \( i_1 < \cdots < i_{n-2}, i_{n-1} < i_n \), and some \( \alpha, \beta \in \mathbb{F}^* \). Consider the following change of indeterminates:

\[
\xi'_{i_{n-1}} = \alpha \xi_{i_{n-1}} + \beta \xi_{i_n} \quad \xi'_{i_j} = \xi_{i_j} \quad \forall j \neq n - 1.
\]

Then \( \frac{\partial^{n-2}\mu_0}{\partial \xi'_{i_1} \ldots \partial \xi'_{i_{n-2}}} = \xi'_{i_{n-1}} \). By using induction on the lexicographic order of the indices \( i_1 < \cdots < i_{n-2} \), one can thus show that, up to a linear change of indeterminates, \( \mu_0 = \xi_1 \ldots \xi_{n-1} \), hence \((\mathcal{P}, \mu)\) is not a good \( k \)-pair due to Corollary 5.2.

\[\square\]

5.2 The case \( PO(n, n+1) \)

In this subsection we shall determine good pairs \((\mathcal{P}, \mu)\) for \( \mathcal{P} = PO(n, n+1) \) with the \( \mathbb{Z} \)-grading of type \((0, \ldots, 0|1, \ldots, 1, 1)\). We shall adopt the same notation as in the previous subsection.

Lemma 5.6 Let \( 2 \leq k < n - 1 \), \( \mu \in PO(n, n+1)_k \) and suppose that \( \mu_0 \) can be written in one of the following forms:

1. \begin{equation}
\mu_0 = \xi_1 \ldots \xi_k \tau + \xi_1 \ldots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \ldots \xi_{i_{k-1}} + \varphi
\end{equation}

where:

(a) \( h = \max\{0 \leq j \leq k \mid \frac{\partial^{j+2}\mu_0}{\partial \xi_{i_1} \ldots \partial \xi_{i_j} \partial \varphi} \neq 0 \), for some \( i_1 < \cdots < i_j \leq k \), and \( r, s > k \} \);

(b) \( \varphi \in \Lambda^{k+1} \mathbb{F}^{n+1} \) is such that \( \frac{\partial^{k+2}\varphi}{\partial \xi_{i_1} \ldots \partial \xi_{i_k} \partial \tau} = 0 \);

2. \begin{equation}
\mu_0 = \xi_1 \ldots \xi_k \tau + \xi_1 \ldots \xi_h \xi_{k+1} \tau \xi_{i_{h+1}} \ldots \xi_{i_{k-1}} + \varphi
\end{equation}

where:

(a) \( h = \max\{0 \leq j < k \mid \frac{\partial^{j+1}\mu_0}{\partial \xi_{i_1} \ldots \partial \xi_{i_j} \partial \tau} \neq 0 \), for some \( i_1 < \cdots < i_j \leq k \} \);

(b) \( \varphi \in \Lambda^{k+1} \mathbb{F}^{n+1} \) is such that \( \frac{\partial^{k+1}\varphi}{\partial \xi_{i_1} \ldots \partial \xi_{i_k} \partial \tau} = 0 \) and \( \frac{\partial^{k+1}\varphi}{\partial \xi_{i_1} \ldots \partial \xi_{i_{k-1}} \partial \tau} = 0 \).

Then \( \mu \) does not satisfy property G3).

Proof. Let us first suppose that \( \mu_0 \) is of the form 5.2. Then, using the same arguments as in the proof of Lemma 5.3, one can show that \([\mu, [x_{i_{k-1}}, \ldots, [x_{i_{k+1}}, [x_h, \ldots, [x_2, [x_1, [x_k+1, \mu]]]]]]] \neq 0 \), since in its expression the summand \( \xi_2 \ldots \xi_k \xi_{k+2} \tau \) does not cancel out.

Similarly, if \( \mu_0 \) is of the form 5.3, then one can show that \([\mu, [x_{i_{k-1}}, \ldots, [x_{i_{k+1}}, [x_h, \ldots, [x_2, [x_1, [1, \mu]]]]]]] \neq 0 \), since in its expression the summand \( \xi_2 \ldots \xi_{k+1} \tau \) does not cancel out. \[\square\]
Theorem 5.7 Let \( \mathcal{P} = PO(n, n+1) \). Suppose that \( 2 \leq k < n - 1 \) and that \( \mu \in \mathcal{P}_k \). Then \((\mathcal{P}, \mu)\) is not a good \((k+1)\)-pair.

Proof. Let us fix a set of odd indeterminates \( \xi_1, \ldots, \xi_n, \xi_{n+1} = \tau \) and the corresponding basis of monomials of \( \wedge(\mathbb{F}^{n+1}) \). By Corollary 5.2, if \( \mu_0 = 0 \) or \( \frac{\partial \mu_0}{\partial \tau} = 0 \), then \( \mu \) does not satisfy property \( G_2 \). Hence suppose that \( \frac{\partial \mu_0}{\partial \tau} \neq 0 \). Then we may assume, up to a linear change of indeterminates, that \( \mu_0 = \xi_1 \cdots \xi_k \tau + \varphi \) for some \( \varphi \in \wedge^{k+1}(\mathbb{F}^{n+1}) \) such that \( \frac{\partial^{k+1} \varphi}{\partial \xi_1 \cdots \partial \xi_k \partial \tau} = 0 \). Then, either \( \frac{\partial \varphi}{\partial \tau} = 0 \) or \( \frac{\partial \varphi}{\partial \tau} \neq 0 \).

Suppose first \( \frac{\partial \varphi}{\partial \tau} = 0 \). Then, either for every \( r, s > k \) the indeterminates \( \xi_r, \xi_s \) appear in different monomials in the expression of \( \varphi \), or there exist some \( r, s > k \) such that \( \xi_r, \xi_s \) appear in the same monomial.

In the first case \( \mu_0 = \xi_1 \cdots \xi_k \tau + \xi_1 \cdots \xi_k (\xi_{k+1} + \xi_{k+2}) + \rho \) for some \( \rho \in \wedge^{k+1}(\mathbb{F}^{n+1}) \) such that \( \frac{\partial^{k+1} \rho}{\partial \xi_1 \cdots \partial \xi_k \partial \xi_{k+1} \partial \xi_{k+2}} = 0 \). By Corollary 5.2 such an element does not satisfy property \( G_2 \). Therefore we may assume that there exist some \( r, s > k \) such that \( \xi_r, \xi_s \) appear in the same monomial, i.e., that, up to a linear change of indeterminates, \( \mu_0 \) is of the following form:

\[
\mu_0 = \xi_1 \cdots \xi_k \tau + \varphi' \]

for some \( \varphi' \in \wedge^{k+1}(\mathbb{F}^{n+1}) \) such that \( \frac{\partial^{k+1} \varphi'}{\partial \xi_1 \cdots \partial \xi_k \partial \xi_{k+1} \cdots \partial \xi_{k+1} \partial \xi_{k+2}} = 0 \) and \( \frac{\partial^{k+1} \varphi'}{\partial \xi_1 \cdots \partial \xi_k \partial \xi_{k+1} \partial \xi_{k+2}} = 0 \), where \( h = \max\{0 \leq j \leq k \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \cdots \partial \xi_{i_j} \partial \tau} \neq 0, \text{ for some } i_1 < \cdots < i_j \leq k \text{ and } r, s > k \} \). Therefore \( \mu \) satisfies hypothesis 1. of Lemma 5.6 hence it does not satisfy property \( G_3 \).

Now suppose \( \frac{\partial \varphi}{\partial \tau} \neq 0 \). Then

\[
\mu_0 = \xi_1 \cdots \xi_k \tau + \xi_{i_1} \cdots \xi_{i_h} \tau \xi_{i+h+1} \cdots \xi_{i_k} + \psi
\]

for some \( i_1 < \cdots < i_h \leq k < i_{h+1} < \cdots < i_k \), for some \( \psi \in \wedge^{k+1}(\mathbb{F}^{n+1}) \) such that \( \frac{\partial^{k+1} \psi}{\partial \xi_{i_1} \cdots \partial \xi_{i_h} \partial \tau} = 0 \) and \( \frac{\partial^{k+1} \psi}{\partial \xi_{i_1} \cdots \partial \xi_{i_h} \partial \xi_{i+1} \partial \xi_{i+h+1}} = 0 \), where \( h = \max\{0 \leq j \leq k \mid \frac{\partial^{j+1} \mu_0}{\partial \xi_{i_1} \cdots \partial \xi_{i_j} \partial \tau} \neq 0, \text{ for some } i_1 < \cdots < i_j \leq k \} \). Now, up to a permutation of indices, we may assume that \( \{i_1, \ldots, i_h\} = \{1, \ldots, h\} \) and \( i_{h+1} = k + 1 \). Then, either \( \mu \) does not satisfy property \( G_2 \), or we may also assume that \( \frac{\partial^{k+1} \psi}{\partial \xi_{i+1} \cdots \partial \xi_{i+h+1} \partial \tau} = 0 \). Therefore \( \mu \) satisfies hypothesis 2. of Lemma 5.6 hence it does not satisfy property \( G_3 \). \( \square \)

Theorem 5.8 Let \( \mathcal{P} = PO(n, n+1) \). If \((\mathcal{P}, \mu)\) is a good \((k+1)\)-pair, then, up to isomorphisms, one of the following possibilities occur:

a) If \( n = 2h + 1 \):

\[a1) \ k = 1 \text{ and } \mu_0 = \sum_{i=1}^{h+1} \xi_i \xi_{i+h+1}; \]
\[a2) \ k = n \text{ and } \mu_0 = \xi_1 \cdots \xi_{n+1}.\]

b) If \( n = 2h \):

\[b1) \ k = n \text{ and } \mu_0 = \xi_1 \cdots \xi_{n+1}.\]
Theorem 6.3

Let $\psi$ we may assume $\psi \in \mathbb{F}(P^N+1)$ such that $\frac{\partial f}{\partial \xi_{n+1}} = 0$. If $f = 0$ then $\mu$ does not satisfy property G2) by Corollary 5.2. If $f \neq 0$, then, up to a linear change of indeterminates, $\mu_0 = \xi_1 \ldots \xi_{n-1} \xi_{n+1} + \xi_1 \ldots \xi_n = \xi_1 \ldots \xi_{n-1}(\xi_{n+1} + \xi_n)$. Then, by Proposition 5.1 $\mu$ does not satisfy property G2).

6 The classification theorem

Remark 6.1 For every invertible element $\varphi \in \mathbb{F}[[x_1, \ldots, x_n]]$, the following change of indeterminates preserves the odd symplectic form, i.e., the bracket in $HO(n,n)$, and maps $\varphi \xi_1 \ldots \xi_n$ to $\xi'_1 \ldots \xi'_n$:

$$x'_1 = \int_0^t \varphi^{-1}(t, x_2, \ldots, x_n)dt =: \Phi,$$

$$x'_i = x_i \ \forall \ i \neq 1,$$

$$\xi'_1 = \varphi \xi_1,$$

$$\xi'_i = \varphi - \varphi \frac{\partial \varphi}{\partial x_i} \xi_1 \ \forall \ i \neq 1.$$

Indeed one can check that $\{x'_1, x'_j\}_{HO} = 0 = \{\xi'_1, \xi'_j\}_{HO}$ and $\{x'_1, \xi'_j\}_{HO} = \delta_{ij}$ for every $i, j = 1, \ldots, n$.

Note that the same change of variables, with the extra condition $\tau' = \tau$, preserves the bracket in the Lie superalgebra $KO(n,n+1)$, and maps $\varphi \xi_1 \ldots \xi_n \tau$ to $\xi'_1 \ldots \xi'_n \tau$.

Theorem 6.2 A complete list, up to isomorphisms, of good k-pairs with $k > 2$, is the following:

i) $(\mathcal{P}^\varphi, \varphi^{-1} \mu)$ with $\mathcal{P} = PO(n,n)$, $n > 2$, $k = n$, $\mu = \xi_1 \ldots \xi_n$, $\varphi \in \mathbb{F}[[x_1, \ldots, x_n]]$;

ii) $(\mathcal{P}^\varphi, \varphi^{-1} \mu)$ with $\mathcal{P} = PO(n,n+1)$, $n > 1$, $k = n+1$, $\mu = \xi_1 \ldots \xi_n \tau$, $\varphi \in \mathbb{F}[[x_1, \ldots, x_n]]$.

Proof. Let $\mathcal{P} = PO(n,n)$ with the grading of type $(0, \ldots, 0|1, \ldots, 1)$, and let $(\mathcal{P}, \mu)$ be a good $k$-pair for $k > 2$. Then, by Theorem 5.3 we have necessarily $n > 2$, $k = n$, and $\mu_0 = \xi_1 \ldots \xi_n \psi$ for some invertible element $\psi \in \mathbb{F}[[x_1, \ldots, x_n]]$. By Remark 6.1 up to a change of variables, we may assume $\psi = 1$. In Example 4.5 we showed that the pair $(\mathcal{P}, \xi_1 \ldots \xi_n)$ is a good n-pair. Statement i) then follows from Theorem 2.5, Remark 2.7, and Remark 4.8.

Likewise, if $\mathcal{P} = PO(n,n+1)$ with the grading of type $(0, \ldots, 0|1, \ldots, 1)$ and $(\mathcal{P}, \mu)$ is a good $k$-pair for $k > 2$, by Theorem 5.8 we have necessarily $n > 1$, $k = n+1$ and $\mu_0 = \xi_1 \ldots \xi_n \tau \psi$ for some invertible element $\psi \in \mathbb{F}[[x_1, \ldots, x_n]]$. Again by Remark 6.1 we may assume $\psi = 1$. Furthermore in Example 4.7 we showed that $(\mathcal{P}, \xi_1 \ldots \xi_n \tau)$ is a good n-pair. Statement ii) then follows from Theorem 2.5, Remark 2.7, and Remark 4.8.

Theorem 6.3 Let $n > 2$.

a) Any simple linearly compact generalized n-Nambu-Poisson algebra is gauge equivalent either to the n-Nambu algebra or to the n-Dzhumadildaev algebra.

b) Any simple linearly compact n-Nambu-Poisson algebra is isomorphic to the n-Nambu algebra.
Proof. By Theorems 4.12 and 2.5, we first need to consider good $n$-pairs $(\mathcal{P}, \mu)$ where $\mathcal{P} = PO(k, k)$ or $\mathcal{P} = PO(k, k + 1)$ and $n > 2$. A complete list, up to isomorphisms, of such pairs is given in Theorem 6.2. The statement then follows from the construction described in Proposition 4.10. We point out that the pair $(\mathcal{P}, \varphi^{-1}\xi_1 \cdots \xi_n)$, with $\mathcal{P} = PO(n, n)$, corresponds to $\mathcal{N}^\varphi$ where $\mathcal{N}$ is the $n$-Nambu algebra; similarly, the pair $(\mathcal{P}, \varphi^{-1}\xi_1 \cdots \xi_n \tau)$, with $\mathcal{P} = PO(n, n + 1)$, corresponds to $\mathcal{N}^\varphi$, where $\mathcal{N}$ is the $n$-Dzhumadildaev algebra (see also Remark 4.13).

□

References


