Doceamus: The Core Ideas in Our Teaching

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What will our students remember? One answer comes quickly but it is a counsel of despair: nothing at all. At the other extreme is an impossible hope that we all cherish: everything we say. Let me look for an intermediate answer, closer to reality, possibly by changing the question.

I have come to believe that each course has a central core. We may not see it ourselves, when we teach a new topic every day. For the calculus course, I won’t even venture an answer—at least not here. My examples will be differential equations and linear algebra, because writing a textbook forced me to uncover (painfully slowly!) the underlying structure of the course.

May I begin with linear algebra. The ideas of a vector space and a basis for that space are central. It is a serious job to help students understand these words. The building blocks are “linear combinations” and “linear independence.” We certainly need good examples, and good bases for them. I think it is here that the course becomes coherent—or it can scatter into unconnected examples of isolated ideas.

I will start with a matrix $A$. A more abstract person would start from a linear transformation. But we are aiming for a basis; we are choosing coordinates; they bring us to a matrix. There are four fundamental subspaces associated with that matrix:

1. Its nullspace $N(A)$ (the kernel) dimension $n - r$
2. Its column space $C(A)$ (the range) $r$
3. Its row space, which is $C(A^T)$ $r$
4. The nullspace $N(A^T)$ of the transpose $m - r$

These are the spaces that we want students to remember. I draw them as often as possible (two in $\mathbb{R}^n$ and two in $\mathbb{R}^m$). I count their basis vectors to find their dimension: the first big theorems in linear algebra. The rank $r$ determines all dimensions. I propose multiple choices of $A$—the beauty of this subject is in the wonderful variety of matrices. And I connect the four subspaces to factorizations of $A$, which are really choices of bases that lie at the absolute center of pure and applied linear algebra. The bases in $U$ and $Q$ and $S$ and $V$ become increasingly perfect.

We are constantly constructing bases for the fundamental subspaces. Elimination and Gram-Schmidt orthogonalization end after finitely many steps. Diagonalization by eigenvectors is deeper and better, but $A$ must be square and nondefective. The Singular Value Decomposition produces perfect bases $v_i$ and $u_i$ for all four subspaces—orthonormal and also diagonalizing for every matrix $A$:

$Av_i = \sigma_i u_i$ $(i \leq r)$ $Av_i = 0$ and $A^T u_i = 0$ $(i > r)$

The success of the SVD comes from the spectral theorem for symmetric matrices: $A^T A$ has a full set of orthonormal eigenvectors $v_i$. Beautifully, the $u_i$ turn out to be orthonormal eigenvectors of $A A^T$. This can be a highlight for the last days of a linear algebra course.
For an earlier day, one idea is to ask students to “read” a few matrices:
\[
\begin{bmatrix}
\cos \theta - \sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\]
The rotation is familiar, the projection is almost too easy. The difference matrix is also the incidence matrix for a simple graph (three nodes in a line). Incidence matrices of a larger graph are terrific examples—all four subspaces have a meaning.

May I turn from subspaces to the basic course on differential equations. Part of this course is a collection of methods to solve separable equations, exact equations, logistic equations \( y' = ay - by^2 \), and more. We go forward to systems of equations, and test nonlinear equations for stability. But the coherent part (the central problem) is to solve **linear equations with constant coefficients**. How can we present their solutions?

I believe we have to answer this question. It is the ODE equivalent of solving \( Ax = 0 \) and \( Ax = b \) and \( Ax = \lambda x \). It certainly rests on the most important functions in this course: exponentials \( e^{st} \) and \( e^{st} \). By working with exponentials, we (almost) turn the differential equation into algebra.

Start with the simplest right-hand sides \( f(t) = 0 \) and \( e^{st} \):
\[
Ay'' + By' + Cy = 0 \quad Ay'' + By' + Cy = e^{st}
\]
The key idea is to expect solutions \( y = Ge^{st} : G(As^2 + Bs + C)e^{st} = 0 \).

On the left, two values of \( s \) are allowed: the roots \( s_1 \) and \( s_2 \) of \( As^2 + Bs + C = 0 \). On the right, any \( s \) is allowed (and the possibilities \( s_1 = s_2 \) and \( s = s_1 = s_2 \) need special attention). Normally we have
\[
y_n = y_{nullspace} = c_1 e^{s_1 t} + c_2 e^{s_2 t}
y_p = y_{particular} = G(s)e^{st} = \frac{1}{As^2 + Bs + C} e^{st}
\]

Those two parts of \( y(t) \) connect linear differential equations to linear algebra. The complete solution combines all \( y_n \) with one \( y_p \). Linearity is in control and the consequence is \( y = y_n + y_p \). I apologize for asking you to read what you know so well. The simplicity of \( y = Ge^{st} \) has to be recognized and remembered. This is where calculus meets algebra. \( G \) is the prime example of an undetermined coefficient (determined by the equation). An elementary course could continue as far as \( f(t) = e^{it} \) and \( \cos \omega t \) and \( \sin \omega t \) and stop. The serious question is to solve the differential equation for all \( f(t) \).

I see two instructive ways to reach \( y(t) \). Both begin with special right-hand sides, and combine the solutions. The combination has to be an integral and not just a finite sum: calculus is needed now. Here are the good options:

1. **Combine exponentials \( e^{st} \) with weights \( F(s) \) to get \( f(t) \). By linearity, the solution \( y(t) \) will combine \( e^{st} \).

2. **Combine impulses \( \delta(t-s) \) with weights \( f(s) \) to get \( f(t) \). By linearity, the solution \( y(t) \) will combine \( e^{st} \).

Where \( e^{st} \) is localized at frequency \( s \), the delta function \( \delta(t-s) \) is completely localized at time \( s \).

- **Method 1** uses the Laplace transform. The transform of \( f(t) \) gives the right weights \( F(s)G(s) \) for \( y(t) \).

   \[
   F(s) = \text{transform of } f(t) \\
   y(t) = \text{inverse transform of } F(s)G(s) 
   \]

   The transform \( F(s) \) might be easy. The hard part is the inverse Laplace transform, to combine the solutions \( F(s)G(s)e^{st} \) into \( y(t) \).

   Realistically, we know a very limited number of transform pairs. Method 1 almost limits us to the same short list as before: \( f \) can combine \( e^{(a+iv)t} \), \( \cos \omega t \), \( \sin \omega t \), and their products. This is a space of functions whose derivatives stay in the space. You can guess that I am advocating Method 2, which begins with an impulse \( \delta(t) \):

\[
(1) \quad Ag'' + Bg' + Cg = \delta(t) \quad \text{with } g(0) = 0 \quad \text{and } g'(0) = 0.
\]

Introducing that delta function is a good thing! We are finding the fundamental solution \( g(t) \)—the Green’s function, the growth factor, the impulse response. This is a high point in the course. And it is easy to do, because this same \( g(t) \) also solves the homogeneous equation:

\[
(2) \quad Ag'' + Bg' + Cg = 0 \quad \text{with } g(0) = 0 \quad \text{and } g'(0) = 1/A.
\]

The solution must have the form \( g(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} \). The two initial conditions give \( c_1 \) and \( c_2 \) and a neat formula for \( g(t) \):

\[
(3) \quad g(t) = \frac{e^{s_1 t} - e^{s_2 t}}{A(s_1 - s_2)} \left( \text{or } g(t) = \frac{t e^{s_1 t}}{A} \text{ when } s_1 = s_2 \right)
\]

Then the original equation, with any right side \( f(t) \), is solved by

\[
(4) \quad y_{\text{particular}}(t) = \int_0^t g(t-s)f(s) \, ds.
\]

**Discussion.** In coming quickly to the formula for \( y(t) \), I have left multiple loose ends. Let me go backwards more slowly, as we would certainly do in a classroom. Methods 1 and 2 are closely connected. The Laplace transform of \( \delta(t) \) is 1. Then equation (1) transforms to

\[
(5) \quad (As^2 + Bs + C) \, G(s) = 1.
\]

The transfer function \( G(s) = 1/(As^2 + Bs + C) \) is the Laplace transform of the impulse response \( g(t) \).
These functions can be written in terms of $A, B, C$ or $s_1$ and $s_2$. A lot of effort has gone into choosing good parameters! The damping ratio $B/\sqrt{4AC}$ and the natural frequency $\sqrt{C/A}$ are two of the best.

We must also explain why equations (1) and (2) have the same solution $g(t)$. Mechanically, this comes from partial fractions:

$$\frac{1}{As^2 + Bs + C} = \frac{1}{A(s - s_1)(s - s_2)} = \frac{1}{A(s_1 - s_2)} \left( \frac{1}{s - s_1} - \frac{1}{s - s_2} \right).$$

The inverse Laplace transform confirms that $e^{s_1t}$ and $e^{s_2t}$ go into $g(t)$.

Here is a truly "mechanical" explanation of (1) = (2). A bat hits a ball at $t = 0$. The velocity jumps instantly to $g'(0) = 1/A$. This comes from integrating $Ag'' + Bg' + Cg = \delta(t)$ from $t = 0$ to $t = h$. The left side produces the jump in $Ag'$ and the integral of $\delta(t)$ is 1. The other terms disappear as $h \to 0$, leaving $Ag'(0) = 1$.

In working with $\delta(t)$, some faith is needed. It is worth developing and it is not misplaced. A delta function is an extremely useful model. So is its integral the step function, which turns on a switch at $t = 0$. By linearity, the step response is the integral of $g(t)$.

Finally, let me connect Method 1 directly to Method 2. In the first method, the Laplace transform of $y(t)$ is $F(s) G(s)$. In the second method, $y(t)$ is the convolution of $f(t)$ with $g(t)$. The connection is the Convolution Rule: The transform of a convolution $f(t) \ast g(t)$ is a multiplication $F(s) G(s)$.

In the language of signal processing, any constant coefficient linear equation can be solved in the "s-domain" or the "t-domain." The poles $s_1, s_2$ of the transfer function $G(s) = 1/(As^2 + Bs + C)$ control the behavior of $y(t)$: oscillation, decay, or instability. The whole course develops out of the quadratic formula for those roots $s_1$ and $s_2$.

Note. The actual course would start with first order equations:

$$y' - ay = 0 \quad y' - ay = e^{st}$$

The null solutions are $y_n = ce^{at}$. The particular solution is $y_p = e^{st}/(s - a)$. The transfer function is $G(s) = 1/(s - a)$. The fundamental solution (impulse response, growth factor, Green’s function) solves

$$g' - ag = \delta(t) \quad \text{with} \quad g(0) = 0$$
$$g' - ag = 0 \quad \text{with} \quad g(0) = 1$$

This function is simply $g = e^{at}$. At this early point it doesn’t need all those names! We recognize it as $1/(integrating \text{ factor})$. Its Laplace transform is $G(s) = 1/(s - a)$. For systems $y' = Ay$, we have the matrix exponential $g = e^{At}$. The solution $y_n + y_p$ for any right-hand side $f(t)$ and initial condition $y(0)$ is

$$y(t) = y(0)e^{at} + \int_0^t e^{a(t-s)} f(s) \, ds.$$