KIMURA-FINITENESS OF
QUADRIC FIBRATIONS OVER SMOOTH CURVES

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Abstract. In this short note, making use of the recent theory of noncommutative mixed motives, we prove that the Voevodsky’s mixed motive of a quadric fibration over a smooth curve is Kimura-finite.

1. Introduction

Let \((\mathcal{C}, \otimes, 1)\) be a \(\mathbb{Q}\)-linear, idempotent complete, symmetric monoidal category. Given a partition \(\lambda\) of an integer \(n \geq 1\), consider the corresponding irreducible \(\mathbb{Q}\)-linear representation \(V_\lambda\) of the symmetric group \(S_n\) and the associated idempotent \(e_\lambda \in \mathbb{Q}[S_n]\). Under these notations, the Schur-functor \(S_\lambda : \mathcal{C} \to \mathcal{C}\) sends an object \(a\) to the direct summand of \(a \otimes a\) determined by \(e_\lambda\). In the particular case of the partition \(\lambda = (1, \ldots, 1)\), resp. \(\lambda = (n)\), the associated Schur-functor \(\wedge^n := S((1, \ldots, 1))\), resp. \(\text{Sym}^n := S((n))\), is called the \(n\)th wedge product, resp. the \(n\)th symmetric product. Following Kimura [9], an object \(a \in \mathcal{C}\) is called even-dimensional, resp. odd-dimensional, if \(\wedge^n(a)\), resp. \(\text{Sym}^n(a) = 0\), for some \(n \gg 0\). The biggest integer \(\text{kim}_+(a)\), resp. \(\text{kim}_-(a)\), for which \(\wedge^{\text{kim}_+(a)}(a) \neq 0\), resp. \(\text{Sym}^{\text{kim}_-(a)}(a) \neq 0\), is called the even, resp. odd, Kimura-dimension of \(a\). An object \(a \in \mathcal{C}\) is called Kimura-finite if \(a \simeq a_+ \oplus a_-\), with \(a_+\) even-dimensional and \(a_-\) odd-dimensional. The integer \(\text{kim}(a) = \text{kim}_+(a_+) + \text{kim}_-(a_-)\) is called the Kimura-dimension of \(a\).

Voevodsky introduced in [18] an important triangulated category of geometric mixed motives \(\text{DM}_{gm}(k)_{\mathbb{Q}}\) (over a perfect base field \(k\)). By construction, this category is \(\mathbb{Q}\)-linear, idempotent complete, rigid symmetric monoidal, and comes equipped with a symmetric monoidal functor \(M(-)_{\mathbb{Q}} : \text{Sm}(k) \to \text{DM}_{gm}(k)_{\mathbb{Q}}\), defined on smooth \(k\)-schemes. An important open problem\(^1\) is the classification of all the Kimura-finite mixed motives and the computation of the corresponding Kimura-dimensions. On the negative side, O’Sullivan constructed a certain smooth surface \(S\) whose mixed motive \(M(S)_{\mathbb{Q}}\) is not Kimura-finite; consult [12, §5.1] for details. On the positive side, Guletskii [6] and Mazza [12] proved, independently, that the mixed motive \(M(C)_{\mathbb{Q}}\) of every smooth curve \(C\) is Kimura-finite.

The following result bootstraps Kimura-finiteness from smooth curves to families of quadrics over smooth curves:

**Theorem 1.1.** Let \(k\) be a field, \(C\) a smooth \(k\)-curve, and \(q : Q \to C\) a flat quadric fibration of relative dimension \(d - 2\). Assume that \(Q\) is smooth and that \(q\) has only simple degenerations, i.e. that all the fibers of \(q\) have corank \(\leq 1\).

(i) When \(d\) is even, the mixed motive \(M(Q)_{\mathbb{Q}}\) is Kimura-finite. Moreover, we have

\[
\text{kim}(M(Q)_{\mathbb{Q}}) = \text{kim}(M(\tilde{C})_{\mathbb{Q}}) + (d - 2)\text{kim}(M(C)_{\mathbb{Q}}),
\]

\(^1\) Among other consequences, Kimura-finiteness implies rationality of the motivic zeta function.
where $D \hookrightarrow C$ stands for the finite set of critical values of $q$ and $\tilde{C}$ for the discriminant double cover of $C$ (ramified over $D$).

(ii) When $d$ is odd, $k$ is algebraically closed, and $1/2 \in k$, the mixed motive $M(Q)_\mathbb{Q}$ is Kimura-finite. Moreover, we have the following equality:

$$\text{kim}(M(Q)_\mathbb{Q}) = \#D + (d - 1)\text{kim}(M(C)_\mathbb{Q}).$$

To the best of the authors’ knowledge, Theorem 1.1 is new in the literature. It not only provides new (families of) examples of Kimura-finite mixed motives but also computes the corresponding Kimura dimensions.

**Remark 1.2.** In the particular case where $k$ is algebraically closed and $Q, C$ are moreover projective, Vial proved in [17, Cor. 4.4] that the Chow motive $h(Q)_\mathbb{Q}$ is Kimura-finite. Since the category of Chow motives embeds fully-faithfully into $\text{DM}_{\text{gm}}(k)_\mathbb{Q}$ (see [18, §4]), we then obtain in this particular case an alternative “geometric” proof of the Kimura-finiteness of $M(Q)_\mathbb{Q}$. Moreover, when $k = \mathbb{C}$ and $d$ is odd, Bouali refined Vial’s work by showing that $h(Q)_\mathbb{Q}$ is isomorphic to $\mathbb{Q}(-\frac{d-1}{2})^\oplus \#D \oplus \bigoplus_{i=0}^{d-2} h(C)_\mathbb{Q}(-i)$; see [4, Rk. 1.10(i)]. In this particular case, this leads to an alternative “geometric” computation of the Kimura-dimension of $M(Q)_\mathbb{Q}$.

2. **Preliminaries**

In what follows, $k$ denotes a base field.

**Dg categories.** For a survey on dg categories consult Keller’s ICM talk [8]. In what follows, we write $\text{dgcat}(k)$ for the category of (small) dg categories and dg functors. Every (dg) $k$-algebra gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes/stacks since the category of perfect complexes $\text{perf}(X)$ of every $k$-scheme $X$ (or, more generally, algebraic stack $X$) admits a canonical dg enhancement $\text{perf}^{\text{dg}}(X)$; see [8, §4.6][11].

**Noncommutative mixed motives.** For a book, resp. survey, on noncommutative motives consult [13], resp. [14]. Recall from [13, §8.5.1] the construction of Kontsevich’s triangulated category of noncommutative mixed motives $\text{NMot}(k)$; denoted by $\text{NMot}^{\text{loc}}_{\text{nr}}(k)$ in loc. cit. By construction, this category is idempotent complete, closed symmetric monoidal, and comes equipped with a symmetric monoidal functor $U: \text{dgcat}(k) \to \text{NMot}(k)$.

**Root stacks.** Let $X$ be a $k$-scheme, $\mathcal{L}$ a line bundle on $X$, $\sigma \in \Gamma(X, \mathcal{L})$ a global section, and $r > 0$ an integer. In what follows, we write $D \hookrightarrow X$ for the zero locus of $\sigma$. Recall from [5, Def. 2.2.1] (see also [1, Appendix B]) that the associated root stack is defined as the following fiber-product of algebraic stacks

$$\sqrt{(\mathcal{L}, \sigma)/X} \rightarrow [\mathbb{A}^1/\mathbb{G}_m],$$

$$X \rightarrow [\mathbb{A}^1/\mathbb{G}_m],$$

where $\theta_r$ stands for the morphism induced by the $r^{th}$ power maps on $\mathbb{A}^1$ and $\mathbb{G}_m$.

**Proposition 2.1.** We have an isomorphism $U(\sqrt{(\mathcal{L}, \sigma)/X}) \simeq U(D)^\oplus (r-1) \oplus U(X)$ whenever $X$ and $D$ are $k$-smooth.
Recall from Proposition 3.1. Finally, the category $\pi$ has only simple degenerations, objects $C$ and morphisms $\hom_{\pi}(a, b) := \oplus_{n \in \mathbb{Z}} \hom_{\pi}(a, b \otimes \mathcal{O}^{\otimes n})$. Given objects $a, b, c$ and morphisms $f = \{f_n\}_{n \in \mathbb{Z}}$ and $g = \{g_n\}_{n \in \mathbb{Z}}$, the $i$th component of $g \circ f$ is defined as $\sum_n (g_{i-n} \otimes \mathcal{O}^{\otimes n}) \circ f_n$. The canonical functor $\pi: C \to \pi$, given by $a \mapsto a$ and $f \mapsto f$, where $f_0 = f$ and $f_n = 0$ if $n \neq 0$, is endowed with an isomorphism $\pi \circ (- \otimes \mathcal{O}) \Rightarrow \pi$ and is 2-universal among all such functors. Finally, the category $C_{\pi}$ is a symmetric monoidal structure making $\pi$ symmetric monoidal.

3. Proof of Theorem 1.1

Following Kuznetsov [10, §3] (see also Auel-Bernardara-Bolognesi [3, §1.2]), let $E$ be a vector bundle of rank $d$ on $C$, $p: \mathbb{P}(E) \to C$ the projectivization of $E$ on $C$. $\mathcal{O}_{\mathbb{P}(E)}(1)$ the Grothendieck line bundle on $\mathbb{P}(E)$, $\mathcal{L}$ a line bundle on $C$, and finally $\rho \in \Gamma(C, S^2(E^\vee) \otimes \mathcal{L}^\vee) = \Gamma(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \mathcal{L}^\vee)$ a global section. Given this data, $Q \subset \mathbb{P}(E)$ is defined as the zero locus of $\rho$ on $\mathbb{P}(E)$ and $q: Q \to C$ as the restriction of $p$ to $Q$; the relative dimension of $q$ is equal to $d - 2$. Consider also the discriminant global section $\text{disc}(q) \in \Gamma(C, \det(E^\vee)^{\otimes 2} \otimes \mathcal{L}^{\otimes d})$ and the associated zero locus $D \hookrightarrow C$. Note that $D$ agrees with the finite set of critical values of $q$. Recall from [10, §3.5] (see also [3, §1.6]) that, when $d$ is even we have a discriminant double cover $\tilde{C}$ of $C$ ramified over $D$. Moreover, since by hypothesis $q$ has only simple degenerations, $\tilde{C}$ is $k$-smooth. Under the above notations, we have the following computation:

**Proposition 3.1.** Let $q: Q \to C$ be a flat quadric fibration as above.

1. When $d$ is even, we have an isomorphism $U(Q)_{\tilde{2}/1/2} \simeq U(\tilde{C})_{\tilde{2}/1/2} \oplus U(C)_{\tilde{2}/1/2}^{(d-2)}$.
2. When $d$ is odd, $k$ is algebraically closed, and $1/2 \in k$, we have an isomorphism $U(Q) \simeq U(D) \oplus U(C)^{\oplus (d-1)}$.

**Proof.** Recall from [10, §3] (see also [3, §1.5]) the construction of the sheaf $\mathcal{C}_0$ of even parts of the Clifford algebra associated to $q$. As proved in [10, Thm. 4.2] (see also [3, Thm. 2.2.1]), we have a semi-orthogonal decomposition

$$\text{perf}(Q) = \langle \text{perf}(\mathcal{C}_0), \text{perf}(C)_1, \ldots, \text{perf}(C)_{d-2} \rangle,$$

where $\text{perf}(\mathcal{C}_0)$ stands for the category of perfect $\mathcal{C}_0$-modules and $\text{perf}(C)_i := q^*(\text{perf}(C)) \otimes \mathcal{O}_{Q/C}(i)$. Note that all the categories $\text{perf}(C)_i$ are equivalent (via a Fourier-Mukai type functor) to $\text{perf}(C)$. Since the functor $U$ sends semi-orthogonal decompositions to direct sums, we then obtain a direct sum decomposition

$$U(Q) \simeq U(\text{perf}^d(C; \mathcal{C}_0)) \oplus U(C)^{\oplus (d-2)},$$

(3.2)
where perf$_{dg}(C; C_0)$ stands for the dg enhancement of perf$(C; C_0)$ induced from perf$_{dg}(Q)$. As explained in [10, Prop. 4.9] (see also [3, §2.2]), the inclusion of categories perf$(C; C_0) \hookrightarrow$ perf$(Q)$ is of Fourier-Mukai type. Therefore, the associated kernel leads to a Fourier-Mukai Morita equivalence between perf$_{dg}(C; C_0)$ and perf$_{dg}(C; C_0)$. Consequently, we can replace the dg category perf$_{dg}(C; C_0)$ by perf$_{dg}(C; C_0)$ in the above decomposition (3.2).

**Item (i).** As explained in [10, §3.5] (see also [3, §1.6]), the category perf$(C; C_0)$ is equivalent (via a Fourier-Mukai type functor) to perf$(C; B_0)$, where $B_0$ is a certain sheaf of Azumaya algebras over $\tilde{C}$ of rank $2^{(d/2)-1}$. Therefore, the associated kernel leads to a Fourier-Mukai equivalence between perf$_{dg}(C; C_0)$ and perf$_{dg}(C; B_0)$. As proved in [16, Thm. 2.1], since $B_0$ is a sheaf of Azumaya algebras of rank $2^{(d/2)-1}$, the noncommutative mixed motive $U((\text{perf}_{dg}(C; B_0)))$ is Morita equivalent to perf$_{dg}(\tilde{C})$. Consequently, the $\mathbb{Z}[1/2]$-linearization of the right-hand side of (3.2) reduces to $U(\tilde{C})$.

**Item (ii).** As explained in [10, Cor. 3.16] (see also [3, §1.7]), since by assumption $k$ is algebraically closed and $1/2 \in k$, the category perf$(C; C_0)$ is equivalent (via a Fourier-Mukai type functor) to perf$(X)$. This implies that the dg category perf$_{dg}(C; C_0)$ is Morita equivalent to perf$_{dg}(X)$. Consequently, since $C$ and $D$ are $k$-smooth, we conclude from the above Proposition 2.1 that the right-hand side of (3.2) reduces to $U(D)$. \hfill \Box

**Item (i).** As proved in [15, Thm. 2.8], there exists a $\mathbb{Q}$-linear, fully-faithful, symmetric monoidal functor $\Phi$ making the following diagram commute

\[
\begin{array}{ccc}
\text{Sm}(k) & \xleftarrow{X \mapsto \text{perf}_{dg}(X)} & \text{dgcat}(k) \\
M(-)_Q & & U(-)_Q \\
DM_{gm}(k)\mathbb{Q} & \xleftarrow{\pi} & \text{NMot}(k)\mathbb{Q} \\
\downarrow & & \downarrow \text{Hom}(\cdot, U(k)\mathbb{Q}) \\
DM_{gm}(k)\mathbb{Q}/-\otimes \mathbb{Q}(1)_{[2]} & \xrightarrow{\Phi} & \text{NMot}(k)\mathbb{Q},
\end{array}
\]

where $\text{Hom}(\cdot, \cdot)$ stands for the internal Hom of the closed symmetric monoidal structure and $\mathbb{Q}(1)_{[2]}$ for the Tate object. Since the functor $\pi$, resp. $\Phi$, is additive, resp. fully-faithful and additive, we hence conclude from the combination of Proposition 3.1 with the above commutative diagram (3.3) that

\[
\pi(M(Q)\mathbb{Q}) \simeq \pi(M(\tilde{C})\mathbb{Q}) \oplus M(C)_{\mathbb{Q}}^{\otimes (d-2)}.
\]

By definition of the orbit category, there exist then morphisms $f = \{f_n\}_{n \in \mathbb{Z}} \in \text{Hom}_{DM_{gm}(k)\mathbb{Q}}(M(Q)\mathbb{Q}, (M(\tilde{C})\mathbb{Q}) \oplus M(C)_{\mathbb{Q}}^{\otimes (d-1)}(n)[2n])$ and $g = \{g_n\}_{n \in \mathbb{Z}} \in \text{Hom}_{DM_{gm}(k)\mathbb{Q}}(M(\tilde{C})\mathbb{Q}) \oplus M(C)_{\mathbb{Q}}^{\otimes (d-1)}, M(Q)\mathbb{Q}(n)[2n])$ verifying the equalities $g \circ f = \text{id} = f \circ g$; in order to simplify the exposition, we write $-\otimes \mathbb{Q}(1)_{[2]}$ instead of $- \otimes \mathbb{Q}(1)_{[2]}^{\otimes n}$. Moreover, only finitely many of these morphisms are non-zero. Let us choose an integer $N \gg 0$ such that $f_n = g_n = 0$ for $n \geq N$. \hfill \Box
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