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VANISHING OF THE NEGATIVE HOMOTOPY $K$-THEORY OF QUOTIENT SINGULARITIES

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Abstract. Making use of Gruson-Raynaud’s technique of “platification par éclatement”, Kerz and Strunk proved that the negative homotopy $K$-theory groups of a Noetherian scheme $X$ of Krull dimension $d$ vanish below $-d$. In this article, making use of noncommutative algebraic geometry, we improve this result in the case of quotient singularities by proving that the negative homotopy $K$-theory groups vanish below $-1$. Furthermore, in the case of cyclic quotient singularities, we provide an explicit “upper bound” for the first negative homotopy $K$-theory group.

1. Introduction and statement of results

Given a Noetherian scheme $X$ of Krull dimension $d$, Kerz and Strunk proved in [11, Thm. 1] that $KH_n(X) = 0$ for every $n < -d$. The first goal of this article is to improve this vanishing result in the case where $X$ is a quotient singularity.

Let $k$ be a field of characteristic zero, $d \geq 2$ an integer, and $G$ a finite subgroup of $\text{SL}_d(k)$. The group $G$ acts naturally on the polynomial ring $S := k[t_1, \ldots, t_d]$ and the associated invariant ring $R := S^G$ is Gorenstein and of Krull dimension $d$.

Throughout the article we will assume that the local ring $R$ is an isolated singularity.

Example 1.1 (Cyclic quotient singularities). Let $k$ be an algebraically closed field of characteristic zero, $m \geq 2$ an integer, $\zeta$ a primitive $m$th root of unit, and $a_1, \ldots, a_d$ integers satisfying the following three conditions: $0 < a_j < m$, $\gcd(a_j, m) = 1$ for every $1 \leq j \leq d$, and $a_1 + \cdots + a_d = m$. When $G$ is the cyclic subgroup of $\text{SL}_d(k)$ generated by $\text{diag}(\zeta^{a_1}, \ldots, \zeta^{a_d})$, the ring $R$ is an isolated quotient singularity of Krull dimension $d$. For example, when $d = 2$, $a_1 = 1$, and $a_2 = m - 1$, the ring $R$ identifies with the Kleinian singularity $k[u, v, w]/(u^m + vw)$ of type $A_{m-1}$.

Remark 1.2. As proved by Kurano-Nishi in [12], all Gorenstein isolated quotient singularities of odd prime (Krull) dimension are cyclic. Moreover, in all the other (Krull) dimensions there exist non-cyclic Gorenstein isolated quotient singularities. For example, in dimension two all Gorenstein quotient singularities are isolated.

Let us write $X$ for the affine (singular) $k$-scheme $\text{Spec}(R)$.

Theorem 1.3. We have $KH_n(X) = 0$ for every $n < -1$.

Intuitively speaking, Theorem 1.3 shows that in the case of quotient singularities the vanishing of the negative homotopy $K$-theory groups is independent of the Krull dimension! To the best of the author’s knowledge, this result is new in the literature. Such a vanishing result does not hold in general because, for every
integer $d \geq 2$, there exist Gorenstein isolated singularities $X$ of Krull dimension $d$ with $KH_{-d}(X) \neq 0$; consult Reid [15] for details. Theorem 1.3 leads naturally to the following divisibility properties of nonconnective algebraic $K$-theory:

**Proposition 1.4.** (i) The abelian group $\mathbb{K}_{-2}(X)$ is divisible;
(ii) The abelian groups $\mathbb{K}_n(X), n < -2,$ are uniquely divisible.

**Remark 1.5.** Since the base field $k$ is of characteristic zero and the scheme $X$ is of finite type over $k$, it follows from the work of Cortiñas-Haesemeyer-Schlichting-Weibel [4] that $\mathbb{K}_n(X) = 0$ for every $n < -d$. Moreover, $\mathbb{K}_{-d}(X) \simeq KH_{-d}(X)$. Consequently, making use of Theorem 1.3, we conclude that $\mathbb{K}_{-d}(X) = 0$.

The second goal of this article is to provide some information about $KH_{-1}(X)$ in the case of cyclic quotient singularities. Let $k$ be an algebraically closed field of characteristic zero and $(Q, \rho)$ the quiver with relations defined as follows:

(s1) consider the quiver with vertices $\mathbb{Z}/m$ and with arrows $x^i_j: i \to i + a_j$, where $i \in \mathbb{Z}/m$ and $1 \leq j \leq d$. The relations $\rho$ are given by $x^{i+a_j}_j x^i_j = x^{i+a'_j}_j x^i_j$ for every $i \in \mathbb{Z}/m$ and $1 \leq j, j' \leq d$. 

(s2) remove from (s1) all arrows $x^i_j: i \to i'$ with $i > i'$;

(s3) remove from (s2) the vertex 0.

Consider the $(m - 1) \times (m - 1)$ matrix $C$ such that $C_{ij}$ equals the number of arrows in $Q$ from $j$ to $i$ (counted modulo the relations). Let $M := (-1)^{d-1}C(C^{-1})^T - \text{id}$ and $M: \oplus_{r=1}^{m-1} \mathbb{Z} \to \oplus_{r=1}^{m-1} \mathbb{Z}$ the associated (matrix) homomorphism.

**Theorem 1.6.** The abelian group $KH_{-1}(X)$ is a quotient of the cokernel of $M$.

Intuitively speaking, Theorem 1.6 provides an explicit “upper bound” for the first negative homotopy $K$-theory group of cyclic quotient singularities.

**Example 1.7 (Kleinian singularities of type $A$).** When $d = 2$, $a_1 = 1$, and $a_2 = m-1$, the three steps (s1)-(s3) lead to the quiver $Q: 1 \to 2 \to 3 \to \cdots \to m-2 \to m-1$ (without relations). Consequently, we obtain the following matrix:

$$M = \begin{bmatrix}
-2 & 1 & 0 & \cdots & 0 \\
-1 & -1 & \ddots & \ddots & \vdots \\
-1 & 0 & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 1 \\
-1 & 0 & \cdots & 0 & -1
\end{bmatrix}_{(m-1) \times (m-1)}$$

The cokernel of $M$ is isomorphic to $\mathbb{Z}/m$; a generator is given by the image of the vector $(0, \ldots, 0, -1)$. Thanks to Theorem 1.6, we hence conclude that $KH_{-1}(\text{Spec}(k[u,v,w]/(u^m + vw)))$ is a quotient of $\mathbb{Z}/m$.

**Example 1.8 (A three dimensional singularity).** When $d = m = 3$ and $a_1 = a_2 = a_3 = 1$, the three steps (s1)-(s3) lead to the quiver $Q: 1 \xrightarrow{3} 2$ (without relations). Consequently, we obtain the matrix $M = \begin{bmatrix} 0 & -3 \\ 3 & -9 \end{bmatrix}$. The cokernel of $M$ is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$; generators are given by the images of $(1, 0)$ and $(-1, -3)$. Thanks to Theorem 1.6, we hence conclude that $KH_{-1}(X)$ is a quotient of $\mathbb{Z}/3 \times \mathbb{Z}/3$. 
Preliminaries. We will assume the reader is familiar with the language of differential graded (=dg) categories; consult Keller’s ICM survey [8]. Let dgcat(\(k\)) be the category of dg categories and dg functors. Given a (dg) \(k\)-algebra \(A\), we will still write \(A\) for the associated dg category with a single object. Consider the simplicial \(k\)-algebra \(\Delta_n := k[t_0, \ldots, t_n]/(\sum t_i - 1), n \geq 0, \) with faces and degeneracies:

\[
\partial_j(t_i) := \begin{cases} 
  t_i & \text{if } i < j \\
  0 & \text{if } i = j \\
  t_{i-1} & \text{if } i > j 
\end{cases} \quad \delta_j(t_i) := \begin{cases} 
  t_i & \text{if } i < j \\
  t_i + t_{i+1} & \text{if } i = j \\
  t_{i+1} & \text{if } i > j 
\end{cases}.
\]

Following [19, §2.2.5], the homotopy \(K\)-theory functor is defined by the formula

\[
KH : \text{dgcat}(k) \to \text{Spt} \quad A \mapsto \operatorname{hocolim}_{n \geq 0} \mathcal{K}(A \otimes \Delta_n),
\]

where Spt stands for the homotopy category of spectra.

2. Proof of Theorem 1.3

Let us write \(\mathcal{D}^b(\text{Mod}(X))\) for the bounded derived category of \(O_X\)-modules, \(\mathcal{D}^b(\text{coh}(X))\) for the bounded derived category of coherent \(O_X\)-modules, and \(\text{perf}(X)\) for the category of perfect complexes of \(O_X\)-modules. These categories admit canonical dg enhancements \(\mathcal{D}^b_{\text{dg}}(\text{Mod}(X)), \mathcal{D}^b_{\text{dg}}(\text{coh}(X)), \) and \(\text{perf}_{\text{dg}}(X)\), respectively; see [8, §4.4-§4.6][13]. Recall from Orlov [14] that the dg category of singularities \(\mathcal{D}^b_{\text{dg}}(X)\) is defined as the Drinfeld’s dg quotient \(\mathcal{D}^b_{\text{dg}}(\text{coh}(X))/\text{perf}_{\text{dg}}(X)\). Consequently, we have the following short exact sequence of dg categories (see [8, §4.6]):

\[
0 \to \text{perf}_{\text{dg}}(X) \to \mathcal{D}^b_{\text{dg}}(\text{coh}(X)) \to \mathcal{D}^b_{\text{dg}}(X) \to 0.
\]

Since homotopy \(K\)-theory \(KH\) is a localizing invariant of dg categories (see [19, §8.2.2]) and \(KH(\text{perf}_{\text{dg}}(X))\) is isomorphic to \(KH(X)\) (see [19, §2.2.5]), we obtain an induced distinguished triangle of spectra:

\[
(2.1) \quad KH(X) \to KH(D^b_{\text{dg}}(\text{coh}(X))) \to KH(D^b_{\text{dg}}(X)) \to \Sigma KH(X).
\]

Proposition 2.2. We have \(KH_n(D^b_{\text{dg}}(\text{coh}(X))) = 0\) for every \(n < 0\).

Proof. Given a dg category \(A\) and an integer \(p \geq 1\), let us write \(A[t_1, \ldots, t_p]\) for the tensor product \(A \otimes k[t_1, \ldots, t_p]\). Since homotopy \(K\)-theory is defined as the “realization” of a simplicial spectrum, we have the following standard convergent right half-plane spectral sequence:

\[
E^1_{pq} = N^p \mathcal{K}_q(A) \Rightarrow KH_{p+q}(A).
\]

Here, \(N^p \mathcal{K}_q(A) := \mathcal{K}_q(A)\) and \(N^p \mathcal{K}_q(A), p \geq 1,\) is defined as the intersection

\[
\bigcap_{i=1}^p \ker \left( \mathcal{K}_q(A[t_1, \ldots, t_p]) \xrightarrow{\delta_1 \otimes (t_i = 0)} \mathcal{K}_q(A[t_1, \ldots, t_i]), \ldots, t_p] \right) \bigg).
\]

Let us treat now the case where \(A\) is the dg category \(D^b_{\text{dg}}(\text{coh}(X))\). Using the fact that the dg category \(k[t_1, \ldots, t_p]\) is Morita equivalent to \(\text{perf}(\text{Spec}(k[t_1, \ldots, t_p])) \simeq \text{perf}(\text{Spec}(k[t_1])) \otimes \cdots \otimes \text{perf}(\text{Spec}(k[t_p]))\), Lemma 2.4 below (applied inductively) implies that \(D^b_{\text{dg}}(\text{coh}(X))[t_1, \ldots, t_p]\) is Morita equivalent to \(D^b_{\text{dg}}(\text{coh}(X[t_1, \ldots, t_p]))\). Consequently, since nonconnective algebraic \(K\)-theory is invariant under Morita equivalences, we obtain the following identifications

\[
\mathcal{K}(D^b_{\text{dg}}(\text{coh}(X))[t_1, \ldots, t_p]) \simeq \mathcal{K}(D^b_{\text{dg}}(\text{coh}(X[t_1, \ldots, t_p]))) = G(X[t_1, \ldots, t_p]),
\]

where \(G\) is the Greenlees–May localization at the first \(\mathcal{K}\).
The case $\dim(X) = 0$ is clear. Assume then that the claim holds for every affine $k$-scheme of finite type of dimension $< \dim(X)$. Let us write $T$ for the smallest thick triangulated subcategory of $D^b(\text{coh}(X[t]))$ containing the object $G[t]$. Given any object $\mathcal{H} \in D^b(\text{coh}(X[t]))$, we need to show that $\mathcal{H} \in T$. Let us denote by $i: X^* \to X$ the singular locus of $X$ and by $D^b_{X^*[t]}(\text{coh}(X[t]))$ the full triangulated subcategory of $D^b(\text{coh}(X[t]))$ consisting of those bounded complexes of coherent $O_{X[t]}$-modules that are supported on the closed subscheme $X^*[t]$. Similarly to the proof of [16, Thm. 7.38], we have a distinguished triangle
\[
\mathcal{H}_1 \to \mathcal{H} \oplus \Sigma \mathcal{H} \to \mathcal{H}_2 \to \Sigma \mathcal{H}_1
\]
with $\mathcal{H}_1 \in \text{perf}(X[t])$ and $\mathcal{H}_2 \in D^b_{X^*[t]}(\text{coh}(X[t]))$. Therefore, in order to prove our claim, it suffices to show that $\mathcal{H}_1$ and $\mathcal{H}_2$ belong to $T$. Since $G$ is a generator of $D^b(\text{coh}(X))$, the object $O_{X[t]} = O_X \boxtimes O_{\text{Spec}(k[t])}$ belongs to $T$. Consequently, since $O_{X[t]}$ is a generator of $\text{perf}(X[t])$, the object $\mathcal{H}_1$ also belongs to $T$. In what concerns the object $\mathcal{H}_2$, let us start by choosing a generator $G'$ of the triangulated category $D^b(\text{coh}(X^*)$. Since $\dim(X^*) < \dim(X)$, the induction assumption implies that $G'[t] := G' \boxtimes O_{\text{Spec}(k[t])}$ is a generator of the triangulated category $D^b(\text{coh}(X^*[t])).$

Now, a proof similar to the one of [21, Thm. 6.8(i)], with [6, Prop. (19.1.1)] replaced by the finite type assumption of $X$ and with [2, Prop. 6.1] replaced by [17, Prop. 6.6], shows that $i_*(O_{X^*}) \boxtimes O_{\text{Spec}(k[t])}$ is a generator of the triangulated category $D^b_{X^*[t]}(\text{coh}(X[t]))$. This implies that $i_*(G'[t]) := i_*(G') \boxtimes O_{\text{Spec}(k[t])}$ is also a generator of $D^b_{X^*[t]}(\text{coh}(X[t]))$. Since $i_*(G')$ belongs to the category $D^b(\text{coh}(X))$ and $G$ is a generator of $D^b(\text{coh}(X))$, the object $i_*(G'[t])$ belongs to $T$. Consequently, $\mathcal{H}_2$ also belongs to $T$. This concludes the proof of our claim.

Since $G$ is a generator of $D^b(\text{coh}(X))$ and $G[t]$ is a generator of $D^b(\text{coh}(X[t]))$, we have induced Morita equivalences between dg categories
\[
D^b_{dga}(\text{coh}(X)) \simeq \text{perf}_{dga}(\text{REnd}(G)) \quad D^b_{dga}(\text{coh}(X[t])) \simeq \text{perf}_{dga}(\text{REnd}(G[t]))
\]

\footnote{Lemma 2.4 is a particular case of a general result of Raphaël Rouquier [17]. The author is very grateful to Raphaël for kindly explaining him his ideas.}
where $\mathbf R \text{End}(\mathcal G)$ and $\mathbf R \text{End}(\mathcal G[t])$ stand for the (derived) dg $k$-algebra of endomorphisms of $\mathcal G$ and $\mathcal G[t]$, respectively. Consider the following classical adjunction

\begin{equation}
\begin{array}{rcl}
\mathcal{D}^b(\text{Mod}(X[t])) & \xrightarrow{\mathcal{F} \mapsto \mathcal{F}[t]} & \text{Res} \\
\mathcal{D}^b(\text{Mod}(X)) & \xrightarrow{} & \mathcal{D}^b(\text{Mod}(X[t])).
\end{array}
\end{equation}

By assumption, the affine $k$-scheme $X$ is of finite type and hence Noetherian. Therefore, since $\mathcal G$ belongs to the triangulated subcategory $\mathcal{D}^b(\text{coh}(X)) \subset \mathcal{D}^b(\text{mod}(X))$ and $\text{Res}(\mathcal G[t]) \simeq \bigoplus_{i=1}^{\infty} \mathcal G$, the combination of [16, Cor. 6.17] with (2.5) implies that the dg $k$-algebra $\mathbf R \text{End}(\mathcal G[t])$ is quasi-isomorphic to $\mathbf R \text{End}(\mathcal G)[t]$. This concludes the proof of Lemma 2.4 because $\text{perf}_{dg}(\text{Spec}(k[t]))$ is Morita equivalent to $k[t]$. \qed

By combining Proposition 2.2 with the long exact sequence of (stable) homotopy groups associated to the distinguished triangle of spectra (2.1), we conclude that $KH_n(\mathcal{D}^\text{rs}_{dg}(X)) \simeq KH_{n-1}(X)$ for every $n < 0$. Consequently, the proof of Theorem 1.3 follows from the following result:

**Proposition 2.6.** We have $KH_n(\mathcal{D}^\text{rs}_{dg}(X)) = 0$ for every $n < 0$.

**Proof.** Consider the polynomial algebra $S := k[t_1, \ldots, t_d]$ as a $\mathbb Z$-graded $k$-algebra with $\text{deg}(t_i) = 1$. Note that since the $G$-action on $S$ preserves the $\mathbb Z$-grading, $R := S^G$ is a $\mathbb Z$-graded $k$-subalgebra of $S$. Following Orlov [14], in addition to the dg category of singularities $\mathcal{D}^\text{rs}_{dg}(X)$, we can also consider the dg category of graded singularities $\mathcal{D}^\text{rs-gr}_{dg}(X)$. By construction, this latter dg category comes equipped with a degree shift dg functor $1) : \mathcal{D}^\text{rs-gr}_{dg}(X) \rightarrow \mathcal{D}^\text{rs-gr}_{dg}(X)$. Following Keller [9, §7.2], let us write $\mathcal{D}^\text{rs-gr}_{dg}(X)/(1) \simeq \mathcal{D}^\text{rs}_{dg}(X)$ for the associated dg orbit category. As proved by Keller-Murfet-Van den Bergh in [10, Props. A.2 and A.8], the forgetful dg functor $\mathcal{D}^\text{rs-gr}_{dg}(X) \rightarrow \mathcal{D}^\text{rs}_{dg}(X)$ induces a Morita equivalence $\mathcal{D}^\text{rs-gr}_{dg}(X)/(1) \simeq \mathcal{D}^\text{rs}_{dg}(X)$. Consequently, since homotopy $K$-theory is an $\mathbb A^1$-homotopy invariant of dg categories (see [19, §8.5]), [20, Thm. 1.5] yields a distinguished triangle of spectra:

$$KH(\mathcal{D}^\text{rs-gr}_{dg}(X)) \xrightarrow{KH(1)} KH(\mathcal{D}^\text{rs-gr}_{dg}(X)) \rightarrow KH(\mathcal{D}^\text{rs}_{dg}(X)) \rightarrow \Sigma KH(\mathcal{D}^\text{rs-gr}_{dg}(X)).$$

Let us denote by $\mathbf{MCM}^{gr}(R)$ the stable category of $\mathbb Z$-graded maximal Cohen-Macaulay $R$-modules. As proved by Iyama-Takahashi in [7, Cor. 2.12], the triangulated category $\mathbf{MCM}^{gr}(R)$ admits a full exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ with $D_r := \text{End}(\mathcal{E}_r)$, $1 \leq r \leq n$, finite dimensional division $k$-algebras. Making use of the classical equivalence of categories $\text{H}^0(\mathcal{D}^\text{rs-gr}_{dg}(X)) = \mathcal{D}^\text{rs-gr}_{dg}(X) \simeq \mathbf{MCM}^{gr}(R)$ (see Buchweitz [3]), and of the fact that homotopy $K$-theory is an additive invariant of dg categories (see [19, §2]), we hence conclude that $KH(\mathcal{D}^\text{rs-gr}_{dg}(X)) \simeq \bigoplus_{r=1}^{n} KH(D_r)$. Since $D_r$ is a (right) regular noetherian ring, we have $KH(D_r) \simeq GKH(D_r)$ (see Gersten [5, Prop. 3.14]) and $KH_n(D_r) = 0$ for every $n < 0$ (see Schlichting [18, Thm. 7]). Therefore, the proof follows now from the long exact sequence of (stable) homotopy groups associated to the above distinguished triangle of spectra. \qed

### 3. Proof of Proposition 1.4

Given any prime power $l^r$, we have the universal coefficient sequences:

\begin{equation}(3.1) \quad 0 \rightarrow KH_n(X) \otimes \mathbb Z/l^r \rightarrow KH_n(X; \mathbb Z/l^r) \rightarrow \{l^r\text{-torsion in } KH_{n-1}(X)\} \rightarrow 0
\end{equation}
and from the natural triangle of spectra constructed in the proof of Proposition 1.6, the degree shift functor (1) corresponds to the functor $S$ via the equivalence of categories $MCM(k)$. Since $K_{H_0}(D_{a_k}(X))$ is isomorphic to the cokernel of the induced homomorphism: 

\[(4.1) \quad K_0((1)) \to K_0(D_{a_k}(X)) \to K_0(D_{a_k}(\text{coh}(X))).\]

Thanks to Proposition 2.2, we have $K_{H_1}(D_{a_k}(\text{coh}(X))) = 0$. Therefore, it follows that $K_{H_1}(X)$ is a quotient of $K_{H_0}(D_{a_k}(X))$. Let us now compute this latter group. As explained in the proof of Proposition 2.6, $K_{H}(D_{a_k}(X))$ is isomorphic to the direct sum $\oplus_{p=1}^{\nu} K_{H}(D_r)$. Since by assumption $k$ is algebraically closed, all the finite dimensional division $k$-algebras $D_r$ are isomorphic to $k$. Therefore, since $K_{H_0}(k) \simeq K_{H_1}(k) \simeq K_{H_1}(k) = 0$, we conclude from the long exact sequence of (stable) homotopy groups associated to the distinguished triangle of spectra constructed in the proof of Proposition 2.6 and from the natural identification $K_0(D_{a_k}(X)) = K_0(D_{a_k}(\text{ coh}(X)))$, that the group $K_{H_0}(D_{a_k}(X))$ is isomorphic to the cokernel of the induced homomorphism:

\[(4.1) \quad K_0((1)) \to K_0(D_{a_k}(\text{ coh}(X))) \to K_0(D_{a_k}(\text{ coh}(X))).\]

Via the equivalence of categories $D_{a_k}(\text{ coh}(X)) \simeq MCM_{a_k}(R)$, (4.1) reduces to 

\[(4.2) \quad K_0((1)) \to K_0(MCM_{a_k}(R)) \to K_0(MCM_{a_k}(R)).\]

As proved by Amiot-Iyama-Reiten in [1, Thm. 4.1], the triangulated category $MCM_{a_k}(R)$ admits a tilting object $T$. Moreover, the associated $k$-algebra $A := \text{End}(T)$ is finite dimensional and of finite global dimension. Let us denote by $D^b(\text{mod}(A))$ the bounded derived category of finitely generated (right) $A$-modules. By construction, this latter category comes equipped with the Serre functor $S$ with the Auslander-Reiten translation functor $\tau$. As proved in loc. cit., via the equivalence of categories $MCM_{a_k}(R) \simeq D^b(\text{mod}(A))$ induced by the tilting object $T$, the degree shift functor (1) corresponds to the functor $S^{-1}\Sigma^{d-1}$. Since $S^{-1}\Sigma = \tau$, we hence conclude that the above homomorphism (4.2) reduces to 

\[(4.3) \quad (-1)^{d-2} \Phi_A \cdot \text{id} : K_0(D^b(\text{mod}(A))) \to K_0(D^b(\text{mod}(A))),\]

where $\Phi_A$ stands for the inverse of the Coxeter matrix of $A$. As proved in [1, §5], the $k$-algebra $A$ is isomorphic to the $k$-algebra $kQ/\langle \rho \rangle$ associated to the quiver with relations $(Q, \rho)$ introduced at §1. On the one hand, this implies that the number of simple (right) $A$-modules agrees with the number of vertices of $Q$, i.e. $v = m - 1$. On the other hand, this implies that the matrix $\Phi_A$ can be written as $-C(C^{-1})^T$, where $C_{ij}$ equals the number of arrows in $Q$ from $j$ to $i$ (counted modulo the relations). Consequently, the above homomorphism (4.3) reduces to the (matrix) homomorphism $M: \oplus_{r=1}^{m-1} \mathbb{Z} \to \oplus_{r=1}^{m-1} \mathbb{Z}$ introduced at §1. This concludes the proof.
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