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THE SMITH NORMAL FORM DISTRIBUTION OF A RANDOM INTEGER MATRIX

YINGHUI WANG† AND RICHARD P. STANLEY†

Abstract. We show that the density \( \mu \) of the Smith normal form (SNF) of a random integer matrix exists and equals a product of densities \( \mu_{p^s} \) of SNF over \( \mathbb{Z}/p^s\mathbb{Z} \) with \( p \) a prime and \( s \) some positive integer. Our approach is to connect the SNF of a matrix with the greatest common divisors (gcds) of certain polynomials of matrix entries and develop the theory of multi-gcd distribution of polynomial values at a random integer vector. We also derive a formula for \( \mu_{p^s} \) and compute the density \( \mu \) for several interesting types of sets. As an application, we determine the probability that the cokernel of a random integer square matrix has at most \( \ell \) generators for a positive integer \( \ell \), and establish its asymptotics as \( \ell \to \infty \), which extends a result of Ekedahl (1991) on the case \( \ell = 1 \).

Key words. Smith normal form, random integer matrix, greatest common divisor distribution

AMS subject classifications. Primary, 05A05, 11C20, 15A21, 60B20; Secondary, 11C08, 15B33

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1. Introduction. Let \( M \) be a nonzero \( n \times m \) matrix over a commutative ring \( R \) (with identity) and \( r \) be the rank of \( M \). If there exist invertible \( n \times n \) and \( m \times m \) matrices \( P \) and \( Q \) such that the product \( PMQ \) is a diagonal matrix (i.e., a matrix that vanishes off the main diagonal) whose main diagonal \( (d_1, d_2, \ldots, d_r, 0, 0, \ldots, 0) \) satisfies that \( d_i | d_{i+1} \) for all \( 1 \leq i \leq r-1 \), then \( PMQ \) is the Smith normal form (SNF) of \( M \) (named for H. J. S. Smith [19]; see [11, 12, 21] for references on this topic). In general, the SNF does not exist. It does exist when \( R \) is a principal ideal ring, i.e., a ring (not necessarily an integral domain) for which every ideal is principal. This class of rings includes the integers \( \mathbb{Z} \) and their quotients \( \mathbb{Z}/q\mathbb{Z} \), which are the rings of interest to us here. In fact, for the rings \( \mathbb{Z}/q\mathbb{Z} \) we will be particularly concerned with the case \( q = p^s \), a prime power. For principal ideal rings, the diagonal entries (also known as elementary divisors or invariants) are uniquely determined—up to multiplication by a unit—by \( g_i d_i = g_{i+1} \) \( (1 \leq i \leq r) \), where \( g_0 = 1 \) and \( g_i \) is the greatest common divisor (gcd) of all \( i \times i \) minors of \( M \). We have the following algebraic correspondence between the SNF and the cokernel of \( M \):

\[
coker M \cong R/d_1R \oplus R/d_2R \oplus \cdots \oplus R/d_rR \oplus R^{n-r}.
\]

There has been a huge amount of research on eigenvalues and eigenvectors of a random matrix (see, e.g., [1, 2, 10, 15, 22, 23]). Much less attention has been paid to the SNF of a random matrix. Some basic results in this area are known, but they appear in papers not focused on SNF per se, in particular, some applications to communication theory [8]. We develop the theory in a systematic way, collecting previous work in this area, and providing some new results.

We shall define the density \( \mu \) of SNF of a random \( n \times m \) integer matrix as the limit (if it exists) as \( k \to \infty \) of \( \mu^{(k)} \), the density of SNF of a random \( n \times m \) matrix with...
entries independent and uniformly distributed over \{−k, −k+1, \ldots, k\} (see Definition 3.1 below for a precise definition).

As a motivating example, the probability that \(d_1 = 1\) for a random \(n \times m\) integer matrix is the probability that the \(nm\) matrix entries are relatively prime, or equivalently, that \(nm\) random integers are relatively prime, and thus equals \(1/\zeta(nm)\), where \(\zeta(\cdot)\) is the Riemann zeta function.

If we regard the minors of an \(n \times m\) matrix as polynomials of the \(nm\) matrix entries with integer coefficients, then the SNF of a matrix is uniquely determined by the gcds of the values of these polynomials (recall the definition of SNF from the beginning). This motivates us to study the theory of multi-gcd distribution of polynomial values.

Given a collection of relatively prime polynomials in \(\mathbb{Z}[x_1, x_2, \ldots, x_d]\), let \(g(x)\) be the gcd of the values of these polynomials at \(x = (x_1, x_2, \ldots, x_d)\). We shall define the \emph{density} \(\lambda\) of \(g(x)\) of a random \(d\)-dimensional integer vector \(x\) as the limit (if it exists) as \(k \to \infty\) of \(\lambda^{(k)}\), the density of \(g(x)\) with \(x\) uniformly distributed over \{−\(k\), −\(k+1\), \ldots, \(k\)\} \(d\) (see Definition 2.1 for a precise definition).

In the spirit of previous work in number theory such as [7, 17, 18] and the Cohen–Lenstra heuristics [4, 5], one might conjecture that \(\lambda\) exists and equals the product of density \(\lambda_p\) of \(g(x)\) over \((\mathbb{Z}/p\mathbb{Z})^d\) over all primes \(p\). In fact, we will prove this conjecture with the more general density \(\lambda_{p^r}\) of \(g(x)\) over \(\mathbb{Z}/p^r\mathbb{Z}\) for sets of form (2.4) (see Theorem 2.6), with the aid of a result in number theory [18, Lemma 21]. Note that the special case \(s = 0\) or \(1\) follows from [7, Theorem 2.3] directly. In particular, this result applies to the probability that \(g(x) = 1\), in other words, that the polynomial values are relatively prime. Furthermore, all these results hold for the multi-gcd distribution of polynomial values, namely, when \(g(x)\) is a vector whose components are the gcds of the values of given collections of polynomials at \(x\).

Then we apply this theory to the SNF distribution of a random integer matrix to show that the density \(\mu\) (of SNF of a random \(n \times m\) integer matrix) equals a product of some densities \(\mu_{p^r}\) of SNF over \(\mathbb{Z}/p^r\mathbb{Z}\) for sets of form (3.3) (Theorem 3.6). We also derive a formula for \(\mu_{p^r}\) (Theorem 3.2), which allows us to compute \(\mu_{p^r}\) and hence \(\mu\) explicitly (Theorem 4.3). Some special cases of this formula are consistent with [20, Exercise 1.192(b)] and [9, pp. 233, 236]. Other papers related to our work are [14] (on rings \(\mathbb{Z}_p\) of \(p\)-adic integers and \(\mathbb{Z}/q\mathbb{Z}\)) and [24] (on Laplacian matrices) (see [21, section 4] for a survey on this topic).

On the strength of these results, we determine the value of \(\mu\) for some interesting types of sets, specifically, matrices with the first few diagonal entries given, matrices with diagonal entries all equal to 1, and square matrices with at most \(\ell(= 1, 2, \ldots)\) diagonal entries not equal to 1, i.e., whose corresponding cokernel has at most \(\ell\) generators; further, for the last set we establish the asymptotics of \(\mu\) as \(\ell \to \infty\). In the case \(\ell = 1\), which is equivalent to the matrix having a cyclic cokernel, our results echo those of Ekedahl [7, section 3] via a different approach, and the density coincides with that in [16]. We also show that \(\mu\) of a finite set is 0 and that the asymptotic probability that a random integer matrix is full rank is 1, which agrees with [13, Theorem 1] in the case of square matrices.

The remainder of this paper is organized as follows. Section 2 develops the theory of multi-gcd distribution of polynomial values. Section 3 applies this theory to the SNF distribution and derives a formula for \(\mu_{p^r}\). Finally, section 4 computes the density \(\mu\) for several types of sets.

We shall assume that throughout this paper, \(p\) represents a prime, \(p_j\) is the \(j\)th smallest prime, \(\prod_p\) means a product over all primes \(p\), and \(\mathbb{Z}_{(k)}^d\) denotes \{−\(k\), −\(k+1\), \ldots, \(k\)\} \(d\).
2. Multi-gcd distribution of polynomial values. Suppose that \( d \) and \( h \) are positive integers and \( F_1, F_2, \ldots, F_h \in \mathbb{Z}[x_1, x_2, \ldots, x_d] \) are nonzero polynomials. Let 
\[
g(x) := \gcd (F_1(x), F_2(x), \ldots, F_h(x)), \quad x \in \mathbb{Z}^d,
\]
and \( g(x) = 0 \) if \( F_j(x) = 0 \) for all \( 1 \leq j \leq h \).

We shall define the density of \( g(x) \) of a random \( d \)-dimensional integer vector \( x \) as the limit (if it exists) of the density of \( g(x) \) with \( x \) uniformly distributed over \( \{-k, -k + 1, \ldots, k\}^d := \mathbb{Z}_k^d \) as \( k \to \infty \), precisely as follows.

**Definition 2.1.**

(i) For \( Z \subseteq \mathbb{Z} \), we denote by \( \lambda^{(k)}(Z) \) the probability that \( g(x) \in Z \) with \( x \) uniformly distributed over \( \mathbb{Z}_k^d \). If \( \lim_{k \to \infty} \lambda^{(k)}(Z) = \lambda(Z) \) exists, then we say that the probability that \( g(x) \in Z \) with \( x \) a random \( d \)-dimensional integer vector is \( \lambda(Z) \). If this is the case, then \( \lambda(Z) \in [0,1] \) since \( \lambda^{(k)}(Z) \in [0,1] \) for all \( k \).

(ii) We define similarly the gcd distribution over the ring of integers mod \( p^s \) for prime \( p \) and positive integer \( s \); more generally, for a finite set \( P \) of prime and positive integer pairs \((p,s)\), we denote
\[
P_P := \prod_{(p,s) \in P} p^s
\]
and by \( \lambda^{(k)}_{P_P}(Z) \) (resp., \( \lambda_{P_P}(Z) \)) the probability that \( g(x) \in Z \pmod{P_P} \)—up to multiplication by a unit—with \( x \) uniformly distributed over \( \mathbb{Z}_k^d \) (resp., \( \mathbb{Z}/P_P\mathbb{Z}^d \)). Note that \( \lambda_{P_P}(Z) \) is the number of solutions to \( g(x) \in Z \pmod{P_P} \) (up to multiplication by a unit) divided by \( P_P^d \). In the special case \( P = \{(p,s)\} \), we have \( P_P = p^s \).

(iii) The above definitions also extend to the distribution of multi-gcds. Suppose that \( U = \{U_i\}_{i=1}^w \) is a collection of \( w \) nonempty subsets \( U_i \) of \( \{F_1, F_2, \ldots, F_h\} \). Let
\[
(2.1) \quad g_i(x) := \gcd (F(x) : F \in U_i) \quad \text{and} \quad g(x) := (g_1, g_2, \ldots, g_w)(x) \in \mathbb{Z}^w, \quad x \in \mathbb{Z}^d;
\]
then we adopt the above definitions of functions \( \lambda^{(k)}, \lambda, \lambda^{(k)}_{P_P}, \) and \( \lambda_{P_P} \) for \( Z \subseteq \mathbb{Z}^w \) with only one slight modification: replace “up to multiplication by a unit” with “up to multiplication of the components of \( g \) by units.”

For convenience, we shall always assume that the notion \( g(x) \in Z \) or \( Z = \mathbb{Z}' \pmod{P_P} \) for some \( Z, Z' \subseteq \mathbb{Z}^w \) implies the equivalence of multiplication of components by units and that the random vector \( x \) is uniformly distributed on its range (if known, e.g., \( \mathbb{Z}_k^d \) or \( \mathbb{Z}/P_P\mathbb{Z}^d \)).

**Remark 2.2.** The density \( \lambda_p \) defined above in Definition 2.1(ii) is consistent with the normalized Haar measure on \( \mathbb{Z}/P_P\mathbb{Z}^d \), as in [18].

In this section, we establish the properties of \( \lambda_{P_P} \) and \( \lambda \), the existence of \( \lambda \), and a connection between \( \lambda \) and the \( \lambda_p \)'s.

2.1. Multi-gcd distribution over \( \mathbb{Z}/P_P\mathbb{Z}^d \). We show that the density \( \lambda^{(k)}_{P_P} \rightarrow \lambda_{P_P} \) as \( k \to \infty \) and that \( \lambda_{P_P} = \prod_{(p,s) \in P} \lambda_p \).
Theorem 2.3. For any \( Z \subseteq \mathbb{Z}^w \), we have

\[
\lim_{k \to \infty} \lambda_{P^d_p}(Z) = \lambda_{P_p}(Z) = \prod_{(p,s) \in P} \lambda_{p,s}(Z).
\]

Proof. For the second equality, we let \( N_{P_p}(Z) \) be the number of \( x \in (\mathbb{Z}/P_p\mathbb{Z})^d \) for which \( g(x) \in Z \) (mod \( P_p \)). The Chinese remainder theorem then gives

\[
P^d_p \lambda_{P_p}(Z) = N_{P_p}(Z) = \prod_{(p,s) \in P} p^{sd} \lambda_{p,s}(Z) = P^d_p \prod_{(p,s) \in P} \lambda_{p,s}(Z).
\]

Dividing both sides by \( P^d_p \) leads to the desired equality.

Let \( k \in \mathbb{Z} \) such that \( K := 2k + 1 \geq P_p \); then there exists \( q \in \mathbb{Z}_+ \) such that

\[
q \cdot P_p \leq K < (q + 1) \cdot P_p.
\]

It follows that for any integer \( y \), there are either \( q \) or \( q + 1 \) numbers among \( Z_{(k)} \) that equal \( y \) mod \( P_p \). Thus the number of \( x \in Z_{(k)} \) for which \( g(x) \in Z \) (mod \( P_p \)) is between \( q^d N' \) and \( (q + 1)^d N' \), where \( N' := N_{P_p}(Z) \). Coupling with (2.3) yields

\[
\lambda_{P_p}(Z) \in \left[ \frac{q^d N'}{K^d}, \frac{(q + 1)^d N'}{K^d} \right] = \left[ \left( \frac{q}{q + 1} \right)^d \frac{N'}{P_p^d}, \left( \frac{q + 1}{q} \right)^d \frac{N'}{P_p^d} \right],
\]

whose left and right endpoints both converge to \( N'/P_p^d = \lambda_{P_p}(Z) \) as \( q \to \infty \) or, equivalently, as \( k \to \infty \), as desired.

2.2. Multi-gcd distribution over \( \mathbb{Z} \). We start with some properties of the density \( \lambda \) of set unions, subtractions, and complements, which are simple consequences of Definition 2.1 and useful in determining the value of \( \lambda \) for specific sets (such as in the explanation below Remark 2.7).

Theorem 2.4.

(i) Suppose that \( \{ Z_{\alpha} \}_{\alpha \in A} \) are pairwise disjoint subsets of \( \mathbb{Z}^w \) such that \( \lambda(Z_{\alpha}) \) exists for all \( \alpha \in A \). If \( A \) is a finite set, then \( \lambda(\cup_{\alpha \in A} Z_{\alpha}) = \sum_{\alpha \in A} \lambda(Z_{\alpha}) \).

(ii) Suppose that \( Z' \subseteq Z \subseteq \mathbb{Z}^w \) such that \( \lambda(Z') \) and \( \lambda(Z) \) both exist; then \( \lambda(Z \setminus Z') = \lambda(Z) - \lambda(Z') \). In particular, for the complement \( Z^c \) of \( Z \) in \( \mathbb{Z}^w \), we have \( \lambda(Z^c) = 1 - \lambda(Z) \).

(iii) Suppose that \( Y \in \mathbb{Z}^w \) such that \( \lambda(Y) = 0 \); then for any \( Y' \subseteq Y \), we have \( \lambda(Y') = 0 \) as well.

Now we show that the density \( \lambda \) exists and, in fact, equals the product of some \( \lambda_{p,s} \)’s.

Assumption 2.5. For all \( 1 \leq i \leq w \), we have

\[
gcd(F_1, F_2, \ldots, F_h) = \gcd(F : F \in U_i) = 1 \quad \text{in} \quad \mathbb{Q}[x_1, x_2, \ldots, x_d].
\]

Theorem 2.6. Suppose that Assumption 2.5 holds. Given positive integers \( r \leq w \) and \( y_i \) (\( 1 \leq i \leq r \)), let \( y = \prod_{j=1}^{\infty} p_j^{y_j} \) with \( p_j \) the \( j \)th smallest prime and \( s_j \) nonnegative.
Then Theorem 2.3 gives that the right-hand side of (2.5) is
\[
\integers, j = 1, 2, \ldots, \text{such that } y_i \mid y \text{ for all } 1 \leq i \leq r; \text{then the probability } \lambda(Z) \text{ exists}
\]
for the set
\[
Z = \{(z_1, z_2, \ldots, z_w) \in \integers_+^w : z_i = y_i \ \forall \ i \leq r\},
\]
and in fact
\[
\lambda(Z) = \prod_{j=1}^{\infty} \lambda_{p_j^i}(Z).
\]

Note that the right-hand side of (2.5) is well-defined since \( \lambda_{p^s} \in [0, 1] \) for all \( p \) and \( s \).

Remark 2.7. The special case that all \( s_j \)'s are either 0 or 1 follows from [7, Theorem 2.3]; in particular, the case \( Z = \{1\} \) gives the probability of relatively prime polynomial values.

We have assumed in Theorem 2.6 that the \( y_i \)'s are positive. In fact, if \( y_i = 0 \) for some \( i \), then we have \( \lambda(Z) = 0 \) on the strength of Theorem 2.4(iii) and the well-known result [17, Lemma 4.1].

**Theorem 2.8.** Let \( G \in \integers[x_1, x_2, \ldots, x_d] \) be a nonzero polynomial and \( \sigma^{(k)} \) the probability that \( G(x) = 0 \) with \( x \) uniformly distributed over \( \integers^d \). Then \( \sigma^{(k)} \to 0 \) as \( k \to \infty \); in words, the probability that a nonzero polynomial at a random integer vector equals zero is 0. As a consequence, for any given integer \( c \), the probability that \( G(x) = c \) is either 0 or 1 (consider the polynomial \( G(x) - c \)).

To prove Theorem 2.6, we need Theorem 2.3 and the following two lemmas.

**Lemma 2.9** (see [17, Lemma 5.1] or [18, Lemma 21]). Suppose that \( F, G \in \integers[x_1, \ldots, x_d] \) are relatively prime as elements of \( \integers[x_1, \ldots, x_d] \). Let \( \nu^{(k)}_\ell \) be the probability that \( p \mid F(x), G(x) \) for some prime \( p > \ell \) with \( x \) uniformly distributed over \( \integers^d \), i.e.,
\[
\nu^{(k)}_\ell := \#\{x \in \integers^d: \exists \text{ prime } p > \ell \text{ s.t. } p \mid F(x), G(x)\}/(2k + 1)^d.
\]
Then
\[
\lim_{\ell \to \infty} \limsup_{k \to \infty} \nu^{(k)}_\ell = 0.
\]

**Lemma 2.10.** Suppose that \( G_1, \ldots, G_h \in \integers[x_1, \ldots, x_d] \) (\( h \geq 2 \)) are relatively prime. Then there exists \( v = (v_3, \ldots, v_h) \in \integers^{h-2} \) such that \( \gcd(G_1, G_2 + \sum_{i=3}^{h} v_i G_i) = 1 \).

Proof. We prove by induction on \( h \). See Appendix A for details.

Now we are ready to prove Theorem 2.6.

**Proof of Theorem 2.6.** Let
\[
\mathcal{P}_\ell := \{(p_j, s_j + 1)\}_{j=1}^\ell, \quad \ell \in \integers_+.
\]
Then Theorem 2.3 gives that the right-hand side of (2.5) is
\[
\lim_{\ell \to \infty} \prod_{j=1}^\ell \lambda_{p_j^{s_j+1}}(Z) = \lim_{\ell \to \infty} \lambda_{\mathcal{P}_\ell}(Z).
\]
Therefore it remains to show that
\begin{equation}
\lim_{k \to \infty} \lambda^{(k)}(Z) = \lim_{k \to \infty} \lambda_{P_{p_{j}}}^{(k)}(Z).
\end{equation}

Since \( y \) is finite, there exists \( j^* \in \mathbb{Z}_+ \) such that \( s_j = 0 \) for all \( j > j^* \). Then
\[ Z = \mathcal{I} := \{(z_1, z_2, \ldots, z_w) : z_1 = z_2 = \cdots = z_r = 1 \} \mod p_j \quad \forall j > j^*.
\]

We define for \( \ell > j^* \),
\[ A(\ell) := \{ x \in \mathbb{Z}^d : g(x) \in \mathcal{Z} \mod P_{p_{j}} \}, \quad A^{(k)}(\ell) := \{ x \in \mathbb{Z}^d : g(x) \in \mathcal{Z} \mod P_{p_{k}} \}, \]
\[ A^{(k)} := \{ x \in \mathbb{Z}^d : g(x) \in \mathcal{Z} \} \subseteq A^{(k)}(\ell), \quad \text{and} \quad B^{(k)} := A^{(k)} \setminus A^{(k)}.
\]
Then we have
\begin{equation}
\lambda^{(k)}(Z) = \frac{\#A^{(k)}}{K^d} \quad \text{and} \quad \lambda_{P_{p_{j}}}^{(k)}(Z) = \frac{\#A^{(k)}}{K^d} - \frac{\#B^{(k)}}{K^d}
\end{equation}
with \( K := 2k + 1 \). Therefore
\[ \lambda_{P_{p_{j}}}^{(k)}(Z) = \lim_{k \to \infty} \lambda_{P_{p_{j}}}^{(k)}(Z) = \lim_{k \to \infty} \frac{\#A^{(k)}}{K^d} - \frac{\#B^{(k)}}{K^d}.
\]
Combining with the first equation in (2.7) leads to
\begin{equation}
\lim_{k \to \infty} \lambda^{(k)}(Z) \leq \lim_{k \to \infty} \frac{\#A^{(k)} + \#B^{(k)}}{K^d} = \lambda_{P_{p_{j}}}^{(k)}(Z)
\end{equation}
and
\begin{equation}
\lim_{k \to \infty} \lambda^{(k)}(Z) \geq \lim_{k \to \infty} \frac{\#A^{(k)} + \#B^{(k)}}{K^d} - \lim_{k \to \infty} \frac{\#B^{(k)}}{K^d} = \lambda_{P_{p_{j}}}^{(k)}(Z) - \lim_{k \to \infty} \frac{\#B^{(k)}}{K^d}.
\end{equation}

Once we show that
\begin{equation}
\lim_{\ell \to \infty} \sup_{k \to \infty} \frac{\#B^{(k)}(\ell)}{K^d} = 0,
\end{equation}
taking \( \ell \to \infty \) in (2.8) and (2.9) will yield (2.6).

Now let us prove (2.10). For any \( x \in B^{(k)}(\ell) \), by definition there exists \( j > \ell (> j^*) \) such that \( g(x) \notin \mathcal{Z} (= \mathcal{I}) \mod p_j \). Hence \( p_j | g_\eta(x) \) for some \( \eta \leq r \).

Recall that \( g_\eta \) is the gcd of some relatively prime \( F_i \)'s. If two or more \( F_i \)'s are involved, then applying Lemma 2.10 to these \( F_i \)'s leads to two relatively prime linear combinations \( G_\eta \) and \( H_\eta \) of these \( F_i \)'s with integer coefficients. If there is only one \( F_1 \) involved, then it must be a constant since the gcd of itself is 1 in \( \mathbb{Q}[x_1, x_2, \ldots, x_d] \); in this case, we take \( G_\eta = H_\eta = F_1 \) so that \( \gcd(G_\eta, H_\eta) = 1 \) still holds.

Since \( p_j | g_\eta(x) \), we have \( p_j | G_\eta(x), H_\eta(x) \). Hence
\begin{equation}
B^{(k)}(\ell) \subseteq \bigcup_{\eta=1}^r \overline{B}^{(k)}(\ell) \text{ with } \overline{B}^{(k)}(\ell) := \{ x \in \mathbb{Z}^d : \exists j > \ell \text{ s.t. } p_j \mid G_\eta(x), H_\eta(x) \}.
\end{equation}
Applying Lemma 2.9 to $G_n$ and $H_n$ gives
\[
\lim_{k \to \infty} \limsup_{\ell \to \infty} \frac{\#B_{\ell}^{(k)}(\ell)}{K^d} = 0 \quad \forall \eta.
\]
Combining with (2.11) and the fact that $\#B^{(k)}(\ell) \geq 0$, we obtain
\[
0 \leq \limsup_{\ell \to \infty} \limsup_{k \to \infty} \frac{\#B^{(k)}(\ell)}{K^d} \leq \sum_{\eta=1}^{r} \limsup_{\ell \to \infty} \limsup_{k \to \infty} \frac{\#B_{\ell}^{(k)}(\ell)}{K^d} = 0.
\]
Hence (2.10) indeed holds. \hfill \Box

3. **SNF distribution.** Let $m \leq n$ be two positive integers. We shall define the density of SNF of a random $n \times m$ integer matrix as the limit (if it exists) of the density of SNF of a random $n \times m$ matrix with entries independent and uniformly distributed over $\mathbb{Z}(k)$ as $k \to \infty$ (see Definition 3.1 below for a precise definition).

If we regard the minors of an $n \times m$ matrix as polynomials of the $nm$ matrix entries with integer coefficients, then the SNF of a matrix is uniquely determined by the values of these polynomials. Specifically, let $x_1, x_2, \ldots, x_{nm}$ be the $nm$ entries of an $n \times m$ matrix, $F_j$’s be the minors of an $n \times m$ matrix as elements in $\mathbb{Z}[x_1, x_2, \ldots, x_{nm}]$, and $U_i$ be the set of $i \times i$ minors ($1 \leq i \leq m$). Then the SNF of this matrix is the diagonal matrix whose $i$th diagonal entry is 0 if $g_i(x) = 0$ and $g_i(x)/g_{i-1}(x)$ otherwise, where $x = (x_1, x_2, \ldots, x_{nm})$ and $g_i(x)$ is defined in (2.1).

In this spirit, the multi-gcd distribution as well as the results in sections 2.1 and 2.2 have analogues for the SNF distribution of a random integer matrix. This section presents these analogues and the next section will use them to compute the density $\mu$ for some interesting types of sets.

Conventionally, the SNF is only defined for a nonzero matrix; however, for convenience, we shall define the SNF of a zero matrix to be itself, so that SNF is well-defined for all matrices. Clearly this definition does not change the density (if it exists) of SNF of a random $n \times m$ integer matrix.

We denote the SNF of an $n \times m$ matrix $M$ by $\text{SNF}(M) = (\text{SNF}(M)_{i,j})_{n \times m}$ and let $S$ be the set of all candidates for SNF of an $n \times m$ integer matrix, i.e., the set of $n \times m$ diagonal matrices whose diagonal entries $d_1, d_2, \ldots, d_m$ are nonnegative integers such that $d_{i+1}$ is a multiple of $d_i$ ($i = 1, 2, \ldots, m-1$).

For convenience, we shall always assume that the matrix entries are independent and uniformly distributed on their range (if known, e.g., $\mathbb{Z}(k)$ or $\mathbb{Z}/P\mathbb{Z}$) and that the notion $\text{SNF}(M) \in S$ or $\text{SNF}(M) = D \pmod{P}$ for some $S \subseteq S$ and $D \in S$ implies the equivalence of multiplication of the entries of $M$ by units in $\mathbb{Z}/P\mathbb{Z}$. Thus we can assume that the entries of $\text{SNF}(M) \pmod{P}$ are zero or divisors of $P$.

**Definition 3.1.**

(i) For $S \subseteq S$, we denote by $\mu^{(k)}(S)$ the probability that $\text{SNF}(M) \in S$ with entries of $M$ from $\mathbb{Z}(k)$. If $\lim_{k \to \infty} \mu^{(k)}(S) = \mu(S)$ exists, then we say that the probability that $\text{SNF}(M) \in S$ with $M$ a random $n \times m$ integer matrix is $\mu(S)$. If this is the case, then $\mu(S) \in [0, 1]$ since $\mu^{(k)}(S) \in [0, 1]$ for all $k$.

(ii) We define similarly the SNF distribution over the ring of integers mod $p^s$ for prime $p$ and positive integer $s$; more generally, for a finite set $P$ of prime and positive integer pairs $(p, s)$, we denote by $\mu^{(k)}_{P_p}(S)$ (resp., $\mu_{P_p}(S)$) the probability that $\text{SNF}(M) \in S \pmod{P}$ with entries of $M$ from $\mathbb{Z}(k)$ (resp., $\mathbb{Z}/P\mathbb{Z}$). Note that $\mu_{P_p}(S)$ is the number of matrices $M$ over $P$ such that
SNF(M) ∈ $S \pmod {P_p}$ divided by $P_p^{nm}$. In the special case $P = \{(p, s)\}$, we have $P_p = p^s$.

In this section, we establish a formula for $\mu_{p^s}$, discuss the properties of $\mu_{P}p$ and $\mu$, show the existence of $\mu$, and represent it as a product of $\mu_{p^s}$'s.

3.1. SNF distribution over $\mathbb{Z}/Pp\mathbb{Z}$. We have the following formula for $\mu_{p^s}$ and analogue of Theorem 2.3 for SNFs.

**Theorem 3.2.**

(i) Given a prime $p$, a positive integer $s$, and integers $0 = a_0 ≤ a_1 ≤ ... ≤ a_s ≤ a_{s+1} = m$, let $a = (a_1, a_2, ..., a_s)$ and $D_a \in S$ be the diagonal matrix with exactly $(a_i - a_{i-1})p^{i-1}s$, i.e., $a_i$ non-$p$-multiples, $1 ≤ i ≤ s$ on its diagonal. Then

$$\mu_{p^s}(\{D_a\}) = p^{-\sum_{i=1}^{s}(a_i-1)} \frac{[p,n][p,m]}{[p,n-a_s][p,m-a_s] \prod_{i=1}^{s}[p,a_i-a_{i-1}]}$$

where

$$[p,0] = 1, \quad [p,\ell] := \prod_{j=1}^{\ell} (1 - p^{-j}), \quad \ell \in \mathbb{Z}^+.$$

(ii) For any $S \subseteq S$, we have

$$\lim_{k \to \infty} \mu_{p^s}^{(k)}(S) = \mu_{P}p^s(S) = \prod_{(p, s) \in P} \mu_{p^s}(S).$$

**Proof.** Part (ii) is a direct application of Theorem 2.3 to SNFs. For (i), we compute the number, denoted by $N$, of $n \times m$ matrices over $\mathbb{Z}/p^s\mathbb{Z}$ whose SNF is $D_a$ by [8, Theorem 2] and simplify it to

$$N = p^{\sum_{i=1}^{s}(n+m)a_i-a_i^2} \cdot \frac{\prod_{i=0}^{a_s-1}(1 - p^{n+j})(1 - p^{m+j})}{\prod_{i=0}^{s-1} \prod_{j=1}^{a_{i+1}-a_i}(1 - p^{-j})}.$$ 

Notice that $\mu_{p^s}(\{D_a\}) = p^{-snm}N$, and thus (3.1) follows.

**Remark 3.3.** When $s = 1$, formula (3.2) gives the number of $n \times m$ matrices over $\mathbb{Z}/p\mathbb{Z}$ of rank $a_1$ and is consistent with [20, Exercise 1.192(b)], whereas in the case $n = m$, formula (3.2) is consistent with [9, pp. 233, 236] (their $|\text{Aut} H|$ is our $N$).

3.2. SNF distribution over $\mathbb{Z}$. The properties of $\lambda$ of set unions, subtractions, and complements in section 2.2 also carry over to SNFs. They will be useful in determining the value of $\mu$ for some specific sets (for instance, in section 4.2).

**Theorem 3.4.**

(i) Suppose that $\{S_\alpha\}_{\alpha \in A}$ are pairwise disjoint subsets of $S$ such that $\mu(S_\alpha)$ exists for all $\alpha \in A$. If $A$ is a finite set, then $\mu(\cup_{\alpha \in A} S_\alpha) = \sum_{\alpha \in A} \mu(S_\alpha)$.

(ii) Suppose that $S' \subseteq S \subseteq S$ such that $\mu(S')$ and $\mu(S)$ both exist. Then $\mu(S \setminus S') = \mu(S) - \mu(S')$. In particular for the complement $S' \subseteq S$ in $S$, we have $\mu(S') = 1 - \mu(S)$.

(iii) Suppose that $T \subseteq S$ such that $\mu(T) = 0$. Then for any $T' \subseteq T$, we have $\mu(T') = 0$ as well.

Theorem 2.6 has an analogue for SNFs as well, by virtue of the following well-known lemma (see [3, Theorem 61.1] for an easy proof).
LEMMA 3.5. Fix a positive integer \( r \). The determinant of an \( r \times r \) matrix as a polynomial of its \( r^2 \) entries \( x_1, x_2, \ldots, x_{r^2} \) is irreducible in \( \mathbb{Q}[x_1, x_2, \ldots, x_{r^2}] \).

For any \( i \leq m \wedge (n-1) \) (i.e., \( \min \{m, n-1\} \)), the set \( U_i \) contains at least two different minors, which are both irreducible as polynomials of the entries Lemma 3.5 and therefore relatively prime. Hence Assumption 2.5 holds with \( w = m \wedge (n-1) \). This allows us to apply Theorem 2.6 to SNFs and obtain the following analogue. In addition, we will compute the density \( \mu(S) \) explicitly later in section 4.1.

**Theorem 3.6.** Given positive integers \( r \leq m \wedge (n-1) \) and \( d_1 \mid d_2 \mid \cdots \mid d_r \), let
\[
z = \prod_{j=1}^{\infty} p_j^{s_j}
\]
with \( p_j \) the \( j \)th smallest prime and \( s_j \) nonnegative integers \( (j = 1, 2, \ldots) \) such that \( d_r \mid z \), then the probability \( \mu(S) \) exists for the set
\[
S = \{D := (D_{i,j})_{n \times m} \in S : D_{i,i} = d_i \ \forall \ i \leq r\},
\]
and in fact
\[
\mu(S) = \prod_{j=1}^{\infty} \mu_{p_j^{s_j+1}}(S).
\]
Note that the right-hand side of (3.4) is well-defined since \( \mu_{p^r} \in [0, 1] \) for all \( p \) and \( s \).

We have assumed that \( r \leq m \wedge (n-1) \); otherwise, we must have \( r = m = n \) (note that \( r \leq m \leq n \)) and \( \mu(S) = 0 \). In fact, any matrix \( M \) with SNF(\( M \)) \( \in S \) satisfies \( |M| = \pm d_1 d_2 \cdots d_n \). We will show later (Theorem 4.5) that the probability that the determinant of a random \( n \times n \) integer matrix equals \( c \) is 0 for all constant \( c \). It then follows from Theorem 3.4(iii) that \( \mu(S) = 0 \).

We have also assumed that the \( d_i \)'s are positive; in fact, we have \( \mu(S) = 0 \) otherwise. If \( d_i = 0 \) for some \( i \), then all \( i \times i \) minors of any matrix \( M \) with SNF(\( M \)) \( \in S \) are zero. Thus Theorems 3.4(iii) and 4.5 with \( c = 0 \) lead to \( \mu(S) = 0 \).

4. **Applications.** Now we apply Theorems 3.2 and 3.6 to compute the density \( \mu \) explicitly for the following subsets of \( S \): matrices with first few diagonal entries given (i.e., with the form of (3.3)), full rank matrices, a finite subset (in particular, matrices with diagonal entries all equal to 1), and square matrices with at most \( \ell (= 1, 2, \ldots, n) \) diagonal entries not equal to 1.

4.1. **Density of the set** (3.3). For the set \( S \) of (3.3), i.e., of matrices with first \( r \) diagonal entries given, we take \( z = d_r \) in Theorem 3.6 and compute \( \mu_{p^{r+1}}(S) \) for each \((p, s) = (p_j, s_j)\). Working modulo \( p^{r+1} \), the set \( S \) has \( m-r+1 \) elements (see (4.12) below), and formula (3.1) gives the density \( \mu_{p^{r+1}} \) of each element of \( S \). Thus one can take the sum over \( S \) to get an expression for \( \mu_{p^{r+1}}(S) \), and compute this sum explicitly when \( m-r \) is small, such as in Theorems 4.8 and 4.9 below. However, this sum is hard to compute when \( m-r \) is large, for example, when \( m \) is large and \( r \) is fixed; in this case, we recast \( S \) as the difference between a subset of \( S \) and the union of other \( r-1 \) subsets such that for each of these \( r \) sets, its density \( \mu_{p^{r+1}} \) is given directly by (3.1).

We work out two examples to illustrate this idea and then deal with the general case.

4.1.1. **The first example: Relatively prime entries.** Our approach reproduces the following result mentioned at the beginning of this paper.
**Theorem 4.1.** Let \( S \) be the set of \((3.3)\) with \( r = 1 \) and \( d_1 = 1 \). Then we have

\[
\mu(S) = \frac{1}{\zeta(nm)},
\]

where \( \zeta(\cdot) \) is the Riemann zeta function.

**Proof.** Applying Theorem 3.6 with \( r = 1 \), \( d_1 = 1 \), and \( s_j = 0 \), \( j = 1, 2, \ldots \), gives

\[
\mu(S) = \prod_p \mu_p(S),
\]

so it reduces to computing \( \mu_p(S) \) for each \( p \).

Recalling the equivalence of multiplication by units, we see that the matrix entries modulo \( p \) can only be 1 or 0. Thus the set \( S \) (mod \( p \)) consists of all the matrices in \( S \) whose first diagonal entry is 1, i.e., \( S = \{D(a_1) : a_1 \geq 1\} \) (mod \( p \)) (recall the notation \( D(a_1,a_2,\ldots,a_s) \) from Theorem 3.2). Therefore

\[
\mu_p(S) = 1 - \mu_p(\{D(0)\}) = 1 - p^{-nm}
\]

by applying (3.1) to \( \mu_p(\{D(0)\}) \). Plugging into (4.2) with the Euler product formula

\[
\prod_p (1 - p^{-i}) = \frac{1}{\zeta(i)} \in (0, 1), \quad i \geq 2,
\]

yields (4.1).

**4.1.2. Another example.**

**Theorem 4.2.** Let \( S \) be the set of \((3.3)\) with \( r = 2 \), \( d_1 = 2 \), and \( d_2 = 6 \). Then

\[
\mu(S) = \mu_2^z(S) \mu_3^z(S) \prod_{p > 3} \mu_p(S),
\]

where

\[
\mu_2^z(S) = 2^{-nm} \left( 1 - 2^{-nm} - 2^{-(n-1)(m-1)} \cdot \frac{(1 - 2^{-n})(1 - 2^{-m})}{1 - 2^{-1}} \right),
\]

\[
\mu_3^z(S) = 3^{-(n-1)(m-1)} \left( 1 - 3^{-(n-1)(m-1)} \cdot \frac{(1 - 3^{-n})(1 - 3^{-m})}{1 - 3^{-1}} \right),
\]

\[
\mu_p(S) = 1 - p^{-nm} - p^{-(n-1)(m-1)} \cdot \frac{(1 - p^{-n})(1 - p^{-m})}{1 - p^{-1}}
\]

\[
= 1 - \sum_{i=(n-1)(m-1)}^{(n-1)m} p^{-i} + \sum_{i=n(m-1)+1}^{nm-1} p^{-i}.
\]

**Proof.** The first equation (4.4) follows directly from Theorem 3.6 with \( r = 2 \), \( d_1 = 2 \), \( d_2 = 6 \), \( s_1 = s_2 = 1 \), \( s_j = 0 \), \( j \geq 3 \). Therefore it reduces to calculating \( \mu_p(S) \) for \((p,s) = (2,2), (3,2), \) and \((p,1) \) with \( p > 3 \).
Case 1. $p > 3$ and $s = 1$. The matrix entries modulo $p$ (up to multiplication by units) can only be 1 or 0. Thus the set $\mathcal{S}$ (mod $p$) consists of all the matrices in $\mathbb{S}$ whose first two diagonal entries are 1, i.e., $\mathcal{S} = \{D(a_1) : a_1 \geq 2\}$ (mod $p$) (recall the notation $D(a_1,a_2,...,a_r)$ from Theorem 3.2). Therefore

$$
\mu_p(\mathcal{S}) = 1 - \mu_p(\{D(0)\}) - \mu_p(\{D(1)\}).
$$

Then we apply (3.1) to $\mu_p(\{D(0)\})$ and $\mu_p(\{D(1)\})$, and (4.7) follows.

Case 2. $p = 2$ and $s = 2$. The matrix entries modulo $2^2$ can be 1, 2 or 0. Thus the set $\mathcal{S}$ (mod $2^2$) consists of all the matrices in $\mathbb{S}$ whose first two diagonal entries are 2, i.e., $\mathcal{S} = \{D(a_1,a_2) : a_1 = 0, a_2 \geq 2\}$ (mod $2^2$). Therefore

$$
(4.8) \quad \mu_{2^2}(\mathcal{S}) = \mu_{2^2}(\{D(a_1,a_2) : a_1 = 0\}) - \mu_{2^2}(\{D(0,0)\}) - \mu_{2^2}(\{D(0,1)\}).
$$

Notice that the set $\{D(a_1,a_2) : a_1 = 0\}$ (mod $2^2$) consists of all the matrices in $\mathbb{S}$ whose diagonal entries are all multiples of 2 (i.e., 2 or 0) or, equivalently, the zero matrix in mod 2. Hence

$$
\mu_{2^2}(\{D(a_1,a_2) : a_1 = 0\}) = \mu_2(\{D(0)\}).
$$

Plugging into (4.8) and applying (3.1) to $\mu_2(\{D(0)\})$, $\mu_{2^2}(\{D(0,0)\})$, and $\mu_{2^2}(\{D(0,1)\})$, we obtain (4.5).

Case 3. $p = 3$ and $s = 2$. The matrix entries in mod $3^2$ can be 1, 3, or 0. Thus the set $\mathcal{S}$ (mod $3^2$) consists of all the matrices in $\mathbb{S}$ whose first two diagonal entries are 1 and 3, respectively, i.e., $\mathcal{S} = \{D(a_1,a_2) : a_1 = 1, a_2 \geq 2\}$ (mod $3^2$). Therefore

$$
(4.9) \quad \mu_{3^2}(\mathcal{S}) = \mu_{3^2}(\{D(a_1,a_2) : a_1 = 1\}) - \mu_{3^2}(\{D(1,1)\}).
$$

Notice that the set $\{D(a_1,a_2) : a_1 = 1\}$ (mod $3^2$) consists of all the matrices in $\mathbb{S}$ whose first diagonal entry is 1 and all other diagonal entries are multiples of 3 (i.e., 3 or 0), equivalently, thediag$(1,0,0,...,0)$ in mod 3. Hence

$$
\mu_{3^2}(\{D(a_1,a_2) : a_1 = 1\}) = \mu_3(\{D(1)\}).
$$

Plugging into (4.9) and applying (3.1) to $\mu_3(\{D(1)\})$ and $\mu_{3^2}(\{D(1,1)\})$, we obtain (4.6).

\[\square\]

4.1.3. The general case.

**Theorem 4.3.** Let $\mathcal{S}$ be the set of (3.3) in Theorem 3.6 with $d_r = \prod_{j=1}^{\infty} p_j^{s_j}$. Then for $(p,s) = (p_j,s_j)$, $j = 1,2,...$, we have

$$
(4.10) \quad \mu_{p^{s_j}}(\mathcal{S}) = \frac{p^{-\sum_{i=1}^{s_j}(n-\tilde{a}_i)(m-\tilde{a}_i)} \cdot [p,n][p,m]}{[p,n-\tilde{a}_s][p,m-\tilde{a}_s] \prod_{i=1}^{s_j}[p,\tilde{a}_i-\tilde{a}_{i-1}]} - \sum_{\ell=\tilde{a}_s}^{r-1} p^{-\sum_{i=1}^{s_j}(n-\tilde{a}_i)(m-\tilde{a}_i)} \cdot [p,n][p,m] \cdot \frac{[p,n-\ell][p,m-\ell]}{[p,\ell-\tilde{a}_s] \prod_{i=1}^{s_j}[p,\tilde{a}_i-\tilde{a}_{i-1}]).}
$$

where $\tilde{a}_i$ $(0 \leq i \leq s)$ is the number of non-$p^j$-multiples among $d_1,d_2,...,d_r$ (thus $\tilde{a}_s \leq r-1$). In particular, when $s = 0$ (which holds for all but finitely many $j$’s), we have

$$
(4.11) \quad \mu_p(\mathcal{S}) = 1 - \sum_{\ell=0}^{r-1} p^{-\sum_{i=1}^{s_j}(n-\tilde{a}_i)(m-\tilde{a}_i)} \cdot [p,n][p,m] \cdot \frac{[p,n-\ell][p,m-\ell]}{[p,\ell][p,\ell-\tilde{a}_s] \prod_{i=1}^{s_j}[p,\tilde{a}_i-\tilde{a}_{i-1}]).}
$$

The value of $\mu(\mathcal{S})$ is then given by Theorem 3.6 with $z = d_r$. 

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Proof. Recalling from Theorem 3.2 the notation of \( D(a_1, a_2, \ldots, a_{s+1}) \), we recast \( S \) as

\[
S = \{ D(a_1, a_2, \ldots, a_{s+1}) : a_i = \tilde{a}_i, \ 1 \leq i \leq s, \ a_{s+1} \geq r \} \mod p^{s+1},
\]

and therefore

\[
\mu_p^{s+1}(S) = \mu_p^{s+1}(\{ D(a_1, a_2, \ldots, a_{s+1}) : a_i = \tilde{a}_i, \ 1 \leq i \leq s \})
\]

\[
- \sum_{\ell=\tilde{a}_s}^{r-1} \mu_p^{s+1}(\{ D(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_s, \ell) \}).
\]

Notice that the set \( \{ D(a_1, a_2, \ldots, a_{s+1}) : a_i = \tilde{a}_i, \ 1 \leq i \leq s \} \mod p^{s+1} \) in the first term on the right-hand side of (4.13) consists of all the matrices in \( S \) with exactly \( \tilde{a}_i \) (\( 1 \leq i \leq s \)) non-\( p^s \)-multiples on its diagonal, equivalently, the diagonal matrix in \( S \) with exactly \( \tilde{a}_i \) non-\( p^s \)-multiples (\( 1 \leq i \leq s \)) in mod \( p^s \). Hence

\[
\mu_p^{s+1}(\{ D(a_1, a_2, \ldots, a_{s+1}) : a_i = \tilde{a}_i, \ 1 \leq i \leq s \}) = \mu_p^r(\{ D(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_s) \}).
\]

Plugging into (4.13) and applying (3.1) to get formulas for \( \mu_p^r(\{ D(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_s) \}) \) and \( \mu_p^{s+1}(\{ D(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_s, \ell) \}) \) for \( 1 \leq \ell \leq r-1 \), we obtain (4.10). \( \square \)

Remark 4.4. We notice that the density \( \mu_p^r(\{ D(a) \}) \) of (3.1) is a polynomial in \( p^{-1} \) with integer coefficients since \( m - a_s + \sum_{i=1}^{s}(a_i - a_{i-1}) = m \). The \( \mu_p(S) \) of (4.11) is also a polynomial in \( p^{-1} \) with integer coefficients and with constant term 1 (see the \( \mu_p(S) \) of (4.7) as an example). If we replace each occurrence of \( p \) by \( p^z \), where \( z \) is a complex variable, and plug into (3.4) of Theorem 3.6, we get an Euler product for some kind of generalized zeta function.

For instance, when \( m = n = 3 \), for the set \( S \) in Theorem 4.2, we apply (4.7),

\[
\mu_p(S) = 1 - p^{-4} - p^{-5} - p^{-6} + p^{-7} + p^{-8} = (1 - p^{-2})(1 - p^{-3})(1 + p^{-2} + p^{-3}),
\]

take the product over all primes \( p \), and apply the Euler product formula (4.3) to get

\[
\prod_p \mu_p(S) = \frac{1}{\zeta(2)\zeta(3)} \prod_p \left( 1 + p^{-2} + p^{-3} \right).
\]

Plugging into (4.4), we see that to obtain the density \( \mu(S) \), it reduces to computing \( \prod_p (1 + p^{-2} + p^{-3}) \), or to understanding the Euler product \( \prod_p (1 + p^{-2z} + p^{-3z}) \).

It would be interesting to study whether such an Euler product for some generalized zeta function, first, has any interesting properties relevant to SNF, second, extends to a meromorphic function on all of \( \mathbb{C} \), and third, satisfies a functional equation.

4.2. The determinantal. The determinant of an \( m \times m \) matrix can be regarded as a polynomial \( G \) of its \( m^2 \) entries. Since \( G \) is not a constant we can apply Theorem 2.8 to \( G \) and obtain the following result.

**Theorem 4.5.** Let \( c \) be an integer. The probability that the determinant equals \( c \) for an \( m \times m \) matrix with entries from \( \mathbb{Z}(k) \) goes to 0 as \( k \to \infty \); in other words, the density of the determinant of a random \( m \times m \) integer matrix is always 0.

This result agrees with [13, Theorem 1] for the case \( c = 0 \) and [6, Theorem 1.2 and Example 1.6] for the case \( c \neq 0 \). It also leads to the following two corollaries. The first of them shows that the probability of a full rank integer matrix is 1.
Corollary 4.6. If $S \subseteq \mathbb{S}$ satisfies $D_{m,m} = 0$ for all $D = (D_{i,j})_{n \times m} \in S$, then we have $\mu(S) = 0$; in other words, the probability of a full rank $n \times m$ matrix with entries from $Z_{(k)}$ goes to 1 as $k \to \infty$.

Proof. If SNF$(M) \in S$, then all $m \times m$ minors of $M$ are zero. The result then follows from Theorem 4.5 with $c = 0$ and Theorem 3.4(iii).

When $m = n$, we can generalize Corollary 4.6 to any finite subset $S \subseteq \mathbb{S}$.

Corollary 4.7. Suppose that $m = n$ and $S \subseteq \mathbb{S}$. Then we have $\mu(S) = 0$ if the set $\{D_{n,n} : D = (D_{i,j})_{n \times n} \in S\}$ is finite; in particular, this holds for any finite subset $S \subseteq \mathbb{S}$.

Proof. For any $M$ such that $D = $ SNF$(M) \in S$, we have $|M| = \pm D_{1,1} \cdots D_{n,n}$. If $D_{n,n} = 0$, then $|M| = 0$; if $D_{n,n} \neq 0$, then the $D_{i,i}$s are divisors of $D_{n,n}$ and therefore $|M|$ has finitely many choices. We then conclude by Theorems 4.5 and 3.4(iii).

4.3. Probability that all diagonal entries of an SNF are 1. Corollary 4.7 with Theorem 3.4(iii) implies that the probability that all diagonal entries of an SNF are 1 is 0 if $m = n$; however, this probability is positive if $m < n$.

Theorem 4.8. Let $E$ be the $n \times m$ matrix diag$(1,1,\ldots,1)$. If $m < n$, then

$$\mu(\{E\}) = \prod_{i=n-m+1}^{n} \frac{1}{\zeta(i)} \to \begin{cases} 1 & \text{if } m \text{ is fixed} \\ \frac{1}{\prod_{i=n-m+1}^{n} \zeta(i)} & \text{if } n-m \text{ is fixed} \end{cases} \text{ as } n \to \infty.$$

Proof. Apply Theorem 3.6 with $S = \{E\}$, $r = m$, $d_i = z = 1$, $s_j = 0$ for all $i,j$, Theorem 3.2 with $s = 1$, $a_1 = m$, and finally the Euler product formula (4.3):

$$\mu(\{E\}) = \prod_{p} \mu_p(\{E\}) = \prod_{p} \left[ \prod_{i=n-m+1}^{n} \frac{1}{\zeta(i)} \right] = \prod_{i=n-m+1}^{n} \frac{1}{\zeta(i)};$$

thanks to $n > m$. Then we derive the limits from the fact that $\zeta(i) \downarrow 1$ as $i \to \infty$.

4.4. Probability that at most $\ell$ diagonal entries of an SNF are not 1. In this section, we assume that $m = n$. We provide a formula for the probability that an SNF has at most $\ell$ diagonal entries not equal to 1 and a formula for the limit of this probability as $n \to \infty$. In particular, when $\ell = 1$, this limit is the reciprocal of a product of values of the Riemann zeta function at positive integers and equals $0.846936\ldots$. For bigger $\ell$, we prove that this limit converges to 1 as $\ell \to \infty$ and find its asymptotics (see Theorem 4.13).

4.4.1. Cyclic SNFs ($\ell = 1$). We shall say that an SNF is cyclic if it has at most one diagonal entry not equal to 1, i.e., if the corresponding cokernel is cyclic. Denote the set of $n \times n$ cyclic SNFs by $\mathcal{T}_n$. We will compute the probability $\mu(\mathcal{T}_n)$ of having a cyclic SNF and show that this probability strictly decreases to $1/(\zeta(6) \prod_{i=4}^{\infty} \zeta(i)) \approx 0.846936$ as $n \to \infty$. As mentioned above, this result was first obtained by Ekedahl [7, section 3]. We present a complete and more detailed proof via a slightly different approach (in particular, for the second equality of (4.16) below).

Theorem 4.9. Let $Z_n = \mu(\mathcal{T}_n)$. Then we have

$$(4.14) \quad Z_n = \frac{\prod_{p} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^\ell} \right)}{\prod_{i=2}^{\infty} \zeta(i)} \approx 0.846936 \text{ as } n \to \infty.$$

In particular, when $n = 2$, we have $Z_2 = 1/\zeta(4) = 90/\pi^4 \approx 0.923938$. 

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Proof.

(i) For the first equality, we apply Theorem 3.6 with \( S = T_n, \ r = n - 1, \ d_i = z = 1, \ s_j = 0 \) for all \( i, j \), and then Theorem 3.2 with \( s = 1, \ a_1 = n, \ n - 1 \):

\[
Z_n = \prod_p \mu_p(T_n) = \prod_p \frac{[p, n]}{[p, 1]} \left( 1 - p^{-1} + \frac{p^{-1}(1 - p^{-n})}{1 - p^{-1}} \right) = \frac{\prod_p \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^r} \right)}{\prod_{i=2}^n \zeta(i)}.
\]

Here in the last equality we used the Euler product formula (4.3):

\[
\prod_p \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^r} \right) = \frac{1}{(1 - p^{-1})}
\]

(ii) For the monotonicity of \( Z_n \), we consider the ratio \( Z_{n+1}/Z_n \) and see that

\[
\frac{Z_{n+1}}{Z_n} = \prod_p \left( 1 - p^{-(n+1)} \right) \frac{1 + p^{-2} + p^{-3} + \cdots + p^{-n}}{1 + p^{-2} + p^{-3} + \cdots + p^{-n}} < 1,
\]

since for each \( p \), the factor is at most \( (1 - p^{-(n+1)})(1 + p^{-(n+1)}) = 1 - p^{-(2n+1)} < 1 \).

(iii) To find \( \lim_{n \to \infty} Z_n \), we assume that \( n \geq 3 \). It suffices to show that

\[
\frac{\zeta(2) \zeta(3)}{\zeta(6)} = \prod_p \left( 1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots \right) = \lim_{n \to \infty} \prod_p \left( 1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n} \right).
\]

For ease of notation, let \( t := p^{-1} \). For the first equality of (4.16), we observe that

\[
1 + t^2 + t^3 + \cdots = 1 + \frac{t^2}{1-t} = \frac{1 + t^3}{1 - t} = \frac{1}{(1-t)(1-t^3)}.
\]

Taking the product of this equation over all reciprocals \( t \) of primes and applying the Euler product formula (4.3) yields the desired equality.

For the second equality of (4.16), we see that

\[
1 > \prod_t \frac{1 + t^2 + t^3 + \cdots + t^n}{1 + t^2 + t^3 + \cdots} = \prod_t \left( 1 - \frac{t^{n+1} + t^{n+2} + \cdots}{1 + t^2 + t^3 + \cdots} \right) > \prod_t (1 - t^{n-1}) = \frac{1}{\zeta(n-1)}
\]

which converges to 1 as \( n \to \infty \). Here \( \prod_t \) represents a product over all reciprocals \( t \) of primes.

One can also show the second equality of (4.16) using the fact that

\[
1 < 1 + p^{-2} + p^{-3} + \cdots + p^{-n} \uparrow 1 + p^{-2} + p^{-3} + \cdots \quad \text{as} \quad n \to \infty
\]

and the following version of the monotone convergence theorem (which will also be useful in Theorem 4.14(iii)) with \( x_{i,j} = 1 + p^{-2} + p^{-3} + \cdots + p^{-j} \) and \( x_i = 1 + p_i^{-2} + p_i^{-3} + \cdots \).

**Lemma 4.10.** If real numbers \( x_{i,j} (i, j = 1, 2, \ldots) \) satisfy \( 1 \leq x_{i,j} \uparrow x_i \) as \( j \to \infty \) for all \( i \), then we have

\[
\lim_{j \to \infty} \prod_{i=1}^{\infty} x_{i,j} = \prod_{i=1}^{\infty} x_i.
\]

*Here we allow the products and the limit to be infinity.*
Proof. Apply the monotone convergence theorem to $\log x_{i,j} \geq 0$.

(iv) When $n = 2$, we derive from the first equality of (4.14) along with (4.3) that

$$Z_2 = \prod_p \left(1 - p^{-2}\right) \left(1 + p^{-2}\right) = \prod_p \left(1 - p^{-4}\right) = \frac{1}{\zeta(4)}. \quad \Box$$

Remark 4.11.

(1) The proof of the convergence result in Theorem 4.9 is reminiscent of (though not directly related to) [20, Exercise 1.186(c)].

(2) Later Nguyen and Shparlinski [16, (1.2)] showed that if we take a subgroup of $\mathbb{Z}^n$ uniformly among all subgroups of index at most $V$ and let $V \to \infty$, then the probability that the quotient group is cyclic is also $\mu(\mathcal{T}_n)$. This result is equivalent to computing the probability that an $n \times n$ integer matrix has a cyclic cokernel using a certain probability distribution different from $\mu$. We do not know a simple reason why these two probability distributions yield the same probability of a cyclic cokernel. Perhaps there is a universality result which gives the same conclusion for a wide class of probability distributions.

4.4.2. More generators (general $\ell$). Now we consider the SNFs with at most $\ell \leq n$ diagonal entries not equal to 1, i.e., whose corresponding cokernel has at most $\ell$ generators. Denote the set of such $n \times n$ SNFs by $\mathcal{T}_n(\ell)$. In particular, when $\ell = n$, we have $\mu(\mathcal{T}_n(n)) = 1$. The above discussion on cyclic SNFs is for the case $\ell = 1$. We will compute the density $\mu(\mathcal{T}_n(\ell))$ and its limit as $n \to \infty$, show that this limit increases to 1 as $\ell \to \infty$, and establish its asymptotics.

We start with a lemma which will play an important role in our proof.

Lemma 4.12. For any real number $x \in (0, 1/2]$, the positive sequence $\{(1/x, k) := \prod_{j=1}^k (1-x^j)\}_{k=1}^\infty$ is decreasing and thus has a limit as $k \to \infty$:

$$C(x) := (1 - x) \left(1 - x^2\right) \cdots \in \left[ e^{-2x/(1-x)}, 1 \right].$$

This also implies that $C(x) \to 1$ as $x \to 0$ and that $[1/x, k] \in \left[ e^{-2x/(1-x)}, 1 \right)$ for all $x \in (0, 1/2]$ and $k \geq 1$.

In particular, when $x = 1/p$, we have

$$[p, k] \downarrow C_p := C(1/p) \in \left[ e^{-2/(p-1)}, 1 \right] \subseteq \left[ e^{-2}, 1 \right] \quad \text{as } k \to \infty,$$

$C_p \to 1$ as $p \to \infty$, and $[p, k] \in \left[ e^{-2/(p-1)}, 1 \right)$ for all $p$ and $k \geq 1$.

Proof. The sequence $[1/x, k]$ is decreasing in $k$ because $0 < 1 - x^j < 1$ for all $j$. To get the lower bound for $C(x)$, we will use the following inequality:

$$\ln y \geq -\frac{1 - y}{y}, \quad y \in (0, 1].$$

To see this, let $\psi(y) := \ln y + (1 - y)/y$. Then $\psi'(y) = 1/y - 1/y^2 \leq 0$. Hence $\psi(y) \leq \psi(1) = 0$.

Applying (4.20) with $y = 1 - x^j$ ($j = 1, 2, \ldots, k$) yields

$$\ln \left[1/x, k\right] = \sum_{j=1}^k \ln \left(1 - x^j\right) \geq -\sum_{j=1}^k \frac{x^j}{1-x^j} \geq -\sum_{j=1}^k \frac{2x^j}{1-x} = -\frac{2x}{1-x},$$
since $x^j \leq x \leq 1/2$. Hence $C(x) = \lim_{k \to \infty} [1/x, k] \geq e^{-2x/(1-x)}. \quad \Box$
The main results of this subsection are as follows.

**Theorem 4.13.** We have

\[
\mu(\mathcal{T}_n(\ell)) = \frac{1}{\prod_{i=2}^{n} \zeta(i)} \prod_{p} [p, 1] \sum_{i=0}^{\ell} \frac{p^{-i^2} [p, n]}{[p, i]^2 [p, n-i]} \to \prod_{p} C_p \sum_{i=0}^{\ell} \frac{p^{-i^2}}{[p, i]^2} \text{ as } n \to \infty
\]

and

\[
\lim_{n \to \infty} \mu(\mathcal{T}_n(\ell)) \uparrow 1 \quad \text{and} \quad \lim_{n \to \infty} \mu(\mathcal{T}_n(\ell)) = 1 - C_2^{-1} \cdot 2^{-(\ell+1)^2} [1 - 2^{-\ell} + O(4^{-\ell})] \text{ as } \ell \to \infty,
\]

where \(C_2^{-1} = \frac{1}{((1 - 2^{-1})(1 - 2^{-2}) \cdots)} \approx 3.46275\).

We split the proof into the following five parts.

**Theorem 4.14.** Let

\[
Z_n(\ell) = \mu(\mathcal{T}_n(\ell)), \quad Z(p, \ell) = \mu_p(\mathcal{T}_n(\ell)) = [p, n] \sum_{i=0}^{\ell} \frac{p^{-i^2} [p, n]}{[p, i]^2 [p, n-i]}
\]

and

\[
Y_n(p, \ell) = [p, 1] \frac{Z_n(p, \ell)}{Z_n(p, \ell)} = [p, 1] \sum_{i=0}^{\ell} \frac{p^{-i^2} [p, n]}{[p, i]^2 [p, n-i]}.
\]

Then we have the following results:

(i)

\[
Z_n(\ell) = \prod_{p} Z_n(p, \ell) = \frac{1}{\prod_{i=2}^{n} \zeta(i)} \prod_{p} Y_n(p, \ell);
\]

(ii)

\[
Y_n(p, \ell) \uparrow [p, 1] \sum_{i=0}^{\ell} \frac{p^{-i^2}}{[p, i]^2} =: Y(p, \ell) \text{ as } n \to \infty
\]

and

\[
Y(p, \ell) \uparrow \frac{[p, 1]}{C_p} \text{ as } \ell \to \infty,
\]

where \(C_p = (1 - p^{-1})(1 - p^{-2}) \cdots \) as defined in (4.19) and (4.18); further, combining with definition (4.22) and Lemma 4.12, it follows that

\[
Z_n(p, \ell) = [p, 1] Y_n(p, \ell) \to \frac{C_p}{[p, 1]} Y(p, \ell) = C_p \sum_{i=0}^{\ell} \frac{p^{-i^2}}{[p, i]^2} =: Z(p, \ell) \text{ as } n \to \infty
\]

and

\[
Z(p, \ell) \uparrow 1 \text{ as } \ell \to \infty;
\]
(iii) \[ Z_n(\ell) \to \prod_p \frac{\zeta \left( \ell \right)}{\zeta \left( \ell \right)} = \prod_p \frac{\zeta \left( \ell \right)}{\zeta \left( \ell \right)} = \prod_p \frac{\zeta \left( \ell \right)}{\zeta \left( \ell \right)} = \prod_\infty \frac{\zeta \left( \ell \right)}{\zeta \left( \ell \right)} = \prod_\infty \frac{\zeta \left( \ell \right)}{\zeta \left( \ell \right)} = \prod_\infty \frac{\zeta \left( \ell \right)}{\zeta \left( \ell \right)} \text{ as } n \to \infty \]

and \[ Z(\ell) \uparrow 1 \text{ as } \ell \to \infty; \]

(iv) \[ Z(\ell) = 1 - C_2^{-1} \cdot 2^{-\ell(\ell+1)} \left[ 1 - \frac{2}{p^2-p} \cdot p^{-\ell} + O \left( p^{-2\ell} \right) \right] \text{ as } \ell \to \infty, \]

and more precisely, this \( O(p^{-2\ell}) \in (0, 2p^{-2\ell}) \);

(v) \[ Z(\ell) = 1 - C_2^{-1} \cdot 2^{-\ell(\ell+1)} \left[ 1 - 2^{-\ell} + O \left( 4^{-\ell} \right) \right] \text{ as } \ell \to \infty, \]

where \( C_2^{-1} = 1/((1-2^{-1})(1-2^{-2})\cdots) \approx 3.46275. \)

Figure 1 and Table 1 illustrate the asymptotics (4.29) of \( Z(\ell) \) and the fast rate of convergence.

![Graph showing asymptotics of Z(ell)](image)

**Fig. 1. Asymptotics of \( Z(\ell) \).**

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>( Z(\ell) )</th>
<th>( 1 - Z(\ell) )</th>
<th>( 2^{(\ell+1)}(1 - Z(\ell)) )</th>
<th>( f(\ell) := -\ln \left[ 1 - C_2 \cdot 2^{(\ell+1)}(1 - Z(\ell)) \right] / \ln 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.846935901735</td>
<td>1.5306498265 \times 10^{-1}</td>
<td>2.41490257244 \times 10^{-1}</td>
<td>1.77225611430</td>
</tr>
<tr>
<td>2</td>
<td>0.994626865443</td>
<td>5.3741145734 \times 10^{-3}</td>
<td>2.75103562616 \times 10^{-3}</td>
<td>2.2825439912</td>
</tr>
<tr>
<td>3</td>
<td>0.999953203075</td>
<td>4.6704924839 \times 10^{-5}</td>
<td>3.0606834423 \times 10^{-5}</td>
<td>3.1079348192</td>
</tr>
<tr>
<td>4</td>
<td>0.999999043035</td>
<td>9.6946459361 \times 10^{-8}</td>
<td>2.2359308764 \times 10^{-8}</td>
<td>4.9026383651</td>
</tr>
<tr>
<td>5</td>
<td>0.999999955184</td>
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<td>3.5635178214 \times 10^{-11}</td>
<td>5.0441603986</td>
</tr>
<tr>
<td>6</td>
<td>1.000000000000</td>
<td>6.0557286766 \times 10^{-15}</td>
<td>3.4090970537 \times 10^{-15}</td>
<td>6.0120665280</td>
</tr>
<tr>
<td>7</td>
<td>1.000000000000</td>
<td>1.8625553206 \times 10^{-19}</td>
<td>3.3458081323 \times 10^{-19}</td>
<td>7.00610418193</td>
</tr>
<tr>
<td>8</td>
<td>1.000000000000</td>
<td>1.42657588960 \times 10^{-24}</td>
<td>3.4492488316 \times 10^{-24}</td>
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<tr>
<td>9</td>
<td>1.000000000000</td>
<td>2.72192686798 \times 10^{-30}</td>
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<td>9.0052622794</td>
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<tr>
<td>10</td>
<td>1.000000000000</td>
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<td>3.45936681921 \times 10^{-36}</td>
<td>10.0007631292</td>
</tr>
</tbody>
</table>
Proof.

(i) The first equality follows from Theorem 3.6 with \( S = T_n(\ell) \), \( r = n - \ell \), \( d_i = z = 1 \), \( s_j = 0 \) for all \( i, j \), and Theorem 3.2 with \( s = 1 \), \( a_1 = n, n - 1, \ldots, n - \ell \).

The second equality follows from definition (4.22) and (4.15).

(ii) The first result (4.23) follows from the observation

\[
\frac{[p, n]}{[p, n-i]} = (1 - p^{-n}) \left(1 - p^{-(n-i)}\right) \cdots \left(1 - p^{-(n-(n-1)+1)}\right) \uparrow 1 \quad \text{as} \quad n \to \infty .
\]

For the second result (4.24), by definition we see that

\[
Y_n(p, n) = \frac{[p, 1]}{[p, n]} \mu_p(T_n(n)) = \frac{[p, 1]}{[p, n]} \quad \text{and, similarly,} \quad Y_\ell(p, \ell) = \frac{[p, 1]}{[p, \ell]} .
\]

Since \( Y_n(p, \ell) \) is also increasing in \( \ell \) by definition (4.22), so is \( Y(p, \ell) \). Thus we have

\[
\frac{[p, 1]}{[p, \ell]} = Y_\ell(p, \ell) \leq Y_n(p, \ell) \leq Y_n(p, n) = \frac{[p, 1]}{[p, n]} < \frac{[p, 1]}{C_p} \quad \forall \ell \leq n .
\]

Taking \( n \to \infty \) yields

\[
\frac{[p, 1]}{[p, \ell]} \leq Y(p, \ell) \leq \frac{[p, 1]}{C_p} .
\]

Then taking \( \ell \to \infty \) and applying Lemma 4.12 leads to (4.24).

(iii) It follows from definitions (4.21) and (4.22) that

\[
Z_n(\ell) = \prod_p Z_n(p, \ell) = \left(\prod_p \frac{[p, n]}{[p, 1]}\right) \prod_p Y_n(p, \ell) .
\]

By virtue of (4.15), the first product on the right-hand side of (4.30) becomes

\[
\prod_p \frac{[p, n]}{[p, 1]} = \frac{1}{\prod_{i=2}^{\infty} \zeta(i)} \to \frac{1}{\prod_{i=2}^{\infty} \zeta(i)} = 0.435757 \quad \text{as} \quad n \to \infty ,
\]

and further, by (4.3) this limit is given by

\[
\frac{1}{\prod_{i=2}^{\infty} \zeta(i)} = \prod_{i=2}^{\infty} \prod_p (1 - p^{-i}) = \prod_{i=2}^{\infty} \prod_p (1 - p^{-i}) = \prod_p \frac{C_p}{[p, 1]} .
\]

One can also deduce \( \prod_p \frac{[p, n]}{[p, 1]} \to \prod_p \frac{C_p}{[p, 1]} \) as \( n \to \infty \) from Lemma 4.10 with \( x_{i,j} = [p_i, 1]/[p_i, j] \) and \( x_i = [p, 1]/C_p \), on the strength of Lemma 4.12.

For the second product on the right-hand side of (4.30), by definition \( Y_n(p, \ell) \geq Y_n(p, 1) = 1 + p^{-2} + p^{-3} + \ldots + p^{-n} > 1 \) (see proof (i) of Theorem 4.9). Thanks to (4.23), we can apply Lemma 4.10 with \( x_{i,j} = Y_j(p_i, \ell) \) and \( x_i = Y(p_i, \ell) \):

\[
\prod_p Y_n(p, \ell) \uparrow \prod_p Y(p, \ell) \quad \text{as} \quad n \to \infty .
\]

Plugging (4.31), (4.32), and (4.33) into (4.30) yields (4.27).
Since $Y_n(p, \ell) > 1$ and is increasing in $\ell$, so is its limit $Y(p, \ell)$. Thus we can apply Lemma 4.10 with $x_{ij} = Y(p, j)$ and $x_i = [p_i, 1]/C_{p_i}$:

$$\prod_p Y(p, \ell) \uparrow \prod_p \frac{[p_i, 1]}{C_{p_i}} \quad \text{as} \quad \ell \to \infty.$$ 

Plugging into the second expression of $Z(\ell)$ in (4.27) leads to $Z(\ell) \uparrow 1$ as $\ell \to \infty$.

(iv) We prove the more general case $p = 1/x$ with $x \in (0, 1/2]$. Note that the proof of (ii) also holds for this case; in particular, (4.25) and (4.26) extend to

$$V(x, \ell) := Z(1/x, \ell) = C(x) \sum_{i=0}^{\ell} \frac{x^2}{[1/x, i]^2} \uparrow 1 \quad \text{as} \quad \ell \to \infty.$$ 

Recall that $C(x) = (1-x)(1-x^2)\cdots$ and $[1/x, i] = (1-x)\cdots(1-x^i)$. Thus

$$\frac{1}{C(x)} = \sum_{i=0}^{\infty} \frac{x^2}{[1/x, i]^2} = \sum_{i=0}^{\ell} \frac{x^2}{[1/x, i]^2} + \sum_{i=\ell+1}^{\infty} \frac{x^2}{[1/x, i]^2} = \frac{V(x, \ell)}{C(x)} + \sum_{i=\ell+1}^{\infty} \frac{x^2}{[1/x, i]^2}.$$ 

After rearrangements, we obtain

$$\frac{C(x)[1-V(x, \ell)]}{x^{\ell+1}} = \sum_{i=\ell+1}^{\infty} \frac{C^2(x)x^{2-i-(\ell+1)^2}}{[1/x, i]^2} = \prod_{j=\ell+2}^{\infty} (1-x^j)^2 + \sum_{i=\ell+2}^{\infty} x^{2-i-(\ell+1)^2} \prod_{j=i+1}^{\infty} (1-x^j)^2$$

(4.34)

$$\left(1 - 2 \sum_{j=\ell+2}^{\infty} x^j + \Delta_1\right) + \Delta_2 = 1 - \frac{2x^{\ell+2}}{1-x} + \Delta_1 + \Delta_2,$$

where

$$\Delta_1 := \prod_{j=\ell+2}^{\infty} (1-x^j)^2 - \left(1 - 2 \sum_{j=\ell+2}^{\infty} x^j\right) \quad \text{and} \quad \Delta_2 := \sum_{i=\ell+2}^{\infty} x^{2-i-(\ell+1)^2} \prod_{j=i+1}^{\infty} (1-x^j)^2.$$ 

Let us show that $0 \leq \Delta_1 \leq x^{2\ell}$ and $0 < \Delta_2 < x^{2\ell}$ for $x \in (0, 1/2]$. Then combining with (4.34) will lead to (4.28) with $p = 1/x$.

For $\Delta_1$, thanks to the inequality (proved easily by induction on $u$)

(4.35)

$$0 \leq \prod_{i=1}^{u} (1-\delta_i) - \left(1 - \sum_{i=1}^{u} \delta_i \right) \leq \prod_{1 \leq i < j \leq u} \delta_i \delta_j \quad \text{for} \quad \delta_1, \delta_2, \ldots, \delta_u \in [0, 1],$$

applying to $x^{\ell+2}, x^{\ell+2}, x^{\ell+3}, x^{\ell+3}, \ldots, x^{\ell}, x^{\ell}$ and letting $\ell \to \infty$, we see that

$$0 \leq \Delta_1 \leq 4 \sum_{j,j' \geq \ell+2} x^{j+j'} = \frac{4x^{2\ell+4}}{(1-x)^2} \leq x^{2\ell} \quad \text{for} \quad 0 < x \leq \frac{1}{2}.$$ 

For $\Delta_2$, we have

$$0 < \Delta_2 < \sum_{i=\ell+2}^{\infty} x^{2-i-(\ell+1)^2} < \sum_{i=2\ell+3}^{\infty} x^i = \frac{x^{2\ell+3}}{1-x} < x^{2\ell} \quad \text{for} \quad 0 < x \leq \frac{1}{2}.$$ 

Thus it follows from (4.34) that (4.28) holds for $p = 1/x$ with $x \in (0, 1/2]$. 

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(v) By definition $0 \leq Z(p, \ell) \leq 1$ for all $p$, and $Z(\ell) = \prod_p Z(p, \ell) \leq Z(2, \ell)$. On the other hand, the $O(p^{-2\ell})$ in (4.28) is at most $2p^{-2\ell} \leq \frac{2}{p^{2p}} p^{-\ell}$ when $\ell \geq 2$, thus

$$Z(p, \ell) > 1 - C_p^{-1}p^{-(\ell+1)^2}.$$  

Take the product of this inequality over $p \geq 3$ and apply the left inequality of (4.35):

$$Z(\ell) = \prod \sum_{p \geq 3} C_p^{-1}p^{-(\ell+1)^2} = Z(2, \ell) - 2^{-(\ell+1)^2} o(4^{-\ell}).$$

Here we used the following estimate. Since $C_p^{-1} \leq e^2$ by (4.19), the positive sum

$$2^{(\ell+1)^2} \sum_{p \geq 3} C_p^{-1}p^{-(\ell+1)^2} \leq e^2 \sum_{p \geq 3} \left( \frac{2}{p} \right)^{(\ell+1)^2} < e^2 \sum_{p \geq 3} \left( \frac{2}{p} \right)^2 = o(4^{-\ell}).$$

Finally, we apply (iv) to $Z(2, \ell)$ and arrive at (4.29).

\[ \square \]

\textbf{Remark 4.15.}

(1) The proof of (ii) and (iv) holds with $p = \frac{1}{x}$ for any $x \in (0, 1/2]$. The convergence result (4.26) in (ii) with $p = \frac{1}{x}$ implies Euler’s identity:

$$\sum_{i=0}^{\infty} (1-x)^2(1-x^2)^2\cdots(1-x^i)^2 = \frac{1}{(1-x)(1-x^2)\cdots}.$$  

(2) When $\ell = 1$, in the proof of Theorem 4.9 we wrote $Y(1/x, 1)$ as $(1-x^6)/(1-x^3)(1-x^3)$ (see (4.17)) in order to represent $Z(1) = \prod Y(p, 1)/\prod_{p=2}^{\infty} \zeta(i)$ as the reciprocal of a product of values of the Riemann zeta function at positive integers. However, this is not the case when $\ell > 1$; in fact, in general $Y(1/x, \ell)$ is not even a symmetric function in $x$, for instance,

$$Y\left(\frac{1}{x}, 2\right) = \frac{1-x-x^2+2x^3-x^5+x^6}{(1-x)(1+x)^2},$$  

$$Y\left(\frac{1}{x}, 3\right) = \frac{1-x-x^2+2x^4+x^5-2x^6-x^7+x^8+x^9-x^{11}+x^{12}}{(1-x)^3(1+x+x^2)^2}.$$  

\textbf{Appendix A. Proof of Lemma 2.10.}

\textbf{Proof.} We prove by induction on $h$. The case $h = 2$ is trivial as $\gcd(G_1, G_2) = 1$.

We prove the base case $h = 3$ by contradiction. Assume to the contrary that $\gcd(G_1, G_2 + zG_3) \neq 1$ for all $z \in \mathbb{Z}$. Suppose that the polynomial factorization of $G_1$ is $\phi_1 \cdot \phi_2 \cdots \phi_u$. Then each $G_2 + zG_3$ is a multiple of some nonconstant factor $\phi_{u(z)}$ of $G_1$ ($1 \leq u(z) \leq u$). Since there are infinitely many $z$’s, by the pigeonhole principle, there exist distinct $z$ and $z'$ such that $u(z) = u(z')$. Then $\phi_{u(z)} \mid (G_2 + zG_3) - (G_2 + z'G_3) = (z - z')G_3$. Thus $\phi_{u(z)} \mid G_3$, and hence $\phi_{u(z)} \mid (G_2 + zG_3) = zG_3 = G_2$. Recall that $\phi_{u(z)} \mid G_1$ as well. This contradicts the condition that $\gcd(G_1, G_2, G_3) = 1$.

Assume that the statement holds for $h - 1 (\geq 3)$. Let $H := (G_2, G_3, \ldots, G_h)$ and $H_i := H_i/H (2 \leq i \leq h)$. Then

\[ \gcd(G_1, H) = \gcd(G_1, G_2, \ldots, G_h) = 1 = \gcd(H_2, H_3, \ldots, H_h). \]
According to the induction hypothesis for $H_2, H_3, \ldots, H_h$, there exists $v = (v_4, \ldots, v_h) \in \mathbb{Z}^{h-3}$ such that
\begin{equation}
(A.2) \quad \gcd(H_2, H_3') = 1 \quad \text{for} \quad H_3' := H_3 + \sum_{i=4}^h v_i H_i.
\end{equation}
Combining with (A.1) gives
\[ \gcd(G_1, G_2, H_3' H) = \gcd(G_1, H_2 H, H_3' H) = \gcd(G_1, \gcd(H_2 H, H_3' H)) = \gcd(G_1, H) = 1. \]
Thus we can apply the base case $h = 3$ to $G_1, G_2, H_3' H$ to get an integer $z$ such that $\gcd(G_1, G_2 + z H_3' H) = 1$. Finally, we represent $z H_3' H$ back to a linear combination of the $G_i$’s with integer coefficients by definitions (A.2) and by $H_i = G_i / H$ ($2 \leq i \leq h$):
\[ z H_3' H = z H_3 H + z \sum_{i=4}^h v_i H_i H = z G_3 + \sum_{i=4}^h z v_i G_i. \]
Therefore, the statement holds for $h$ with the new $v = (z, z v_4, \ldots, z v_h)$.

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