\textbf{A$^1$-HOMOTOPY INVARIANTS OF CORNER SKEW LAURENT POLYNOMIAL ALGEBRAS}

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\textbf{Abstract.} In this note we prove some structural properties of all the A$^1$-homotopy invariants of corner skew Laurent polynomial algebras. As an application, we compute the mod-$l$ algebraic $K$-theory of Leavitt path algebras using solely the kernel/cokernel of the incidence matrix. This leads naturally to some vanishing and divisibility properties of the $K$-theory of these algebras.

1. Corner skew Laurent polynomial algebras

Let $k$ be a field, $A$ a unital $k$-algebra, $e$ an idempotent of $A$, and $\phi: A \xrightarrow{\sim} eAe$ a “corner” isomorphism. Following Ara-Barroso-Goodearl-Pardo \cite{3, §2}, the associated \textit{corner skew Laurent polynomial algebra} $A[t_+, t_-; \phi]$ is defined as follows: the elements are formal expressions $t^m a_m + \cdots + t a_1 + a_0 + a_1 t_+ \cdots + a_n t^n_-$ with $a_i \in \phi(1)A$ and $a_i \in A\phi(1)$ for every $i \geq 0$; the addition is defined component-wise; the multiplication is determined by the distributive law and by the relations $t_- t_+ = 1$, $t_+ t_- = e$, $a t_- = t_- \phi(a)$ for every $a \in A$, and $t_+ a = \phi(a)t_+$ for every $a \in A$. Note that $A[t_+, t_-; \phi]$ admits a canonical $\mathbb{Z}$-grading with $\deg(t_+) = \pm 1$.

As proved in \cite[Lem. 2.4]{3}, the corner skew Laurent polynomial algebras can be characterized as those $\mathbb{Z}$-graded algebras $C = \bigoplus_{n \in \mathbb{Z}} C_n$ containing elements $t_+ \in C_1$ and $t_- \in C_{-1}$ such that $t_- t_+ = 1$. Concretely, we have $C = A[t_+, t_-; \phi]$ with $A := C_0$, $e := t_+ t_-$, and $\phi: C_0 \to t_+ t_- C_0 t_+ t_-$ given by $c_0 \mapsto t_+ c_0 t_-$.\phantom{\text{Example 1.1 (Skew Laurent polynomial algebras). When } e = 1, A[t_+, t_-; \phi] \text{ reduces to the classical skew Laurent polynomial algebra } A \rtimes_{\phi} \mathbb{Z}. \text{ In the particular case where } \phi \text{ is the identity, } A \rtimes_{\phi} \mathbb{Z} \text{ reduces furthermore to } A[t, t^{-1}].}$

\textbf{Example 1.1 (Skew Laurent polynomial algebras).} When $e = 1$, $A[t_+, t_-; \phi]$ reduces to the classical skew Laurent polynomial algebra $A \rtimes_{\phi} \mathbb{Z}$. In the particular case where $\phi$ is the identity, $A \rtimes_{\phi} \mathbb{Z}$ reduces furthermore to $A[t, t^{-1}]$.

\textbf{Example 1.2 (Jacobson algebras).} Following \cite{8}, the \textit{Jacobson algebra} $J_n$, $n \geq 0$, is the $k$-algebra generated by elements $x_0, \ldots, x_n, y_0, \ldots, y_n$ subject to the relations $y_i x_j = \delta_{ij}$. Note that the canonical $\mathbb{Z}$-grading, with $\deg(x_i) = 1$ and $\deg(y_i) = -1$, makes $J_n$ into a corner skew Laurent polynomial algebra. The algebras $J_n$ are also usually called Cohn algebras (see \cite{1}), and $J_0$ the (algebraic) Toeplitz algebra.

\textbf{Example 1.3 (Leavitt algebras).} Following \cite{12}, the \textit{Leavitt algebra} $L_n$, $n \geq 0$, is the $k$-algebra generated by elements $x_0, \ldots, x_n, y_0, \ldots, y_n$ subject to the relations $y_i x_j = \delta_{ij}$ and $\sum_{i=0}^n x_i y_i = 1$. Note the canonical $\mathbb{Z}$-grading, with $\deg(x_i) = 1$ and $\deg(y_i) = -1$, makes $L_n$ into a corner skew Laurent polynomial algebra. Note also that $L_0 \simeq k[t, t^{-1}]$. In the remaining case $n \geq 1$, $L_n$ is the universal example of a $k$-algebra of module type $(1, n+1)$, i.e. $L_n \simeq L_{n+1}^{(n+1)}$ as right $L_n$-modules.

\textit{Date:} February 12, 2018.

2010 \textit{Mathematics Subject Classification.} 14A22, 16S36, 19D50, 19D55.

\textit{Key words and phrases.} Corner skew Laurent polynomial algebra, Leavitt path algebra, algebraic $K$-theory, noncommutative mixed motives, noncommutative algebraic geometry.

The author was partially supported by a NSF CAREER Award.
Example 1.4 (Leavitt path algebras). Let $Q = (Q_0, Q_1, s, r)$ be a finite quiver; $Q_0$ and $Q_1$ stand for the sets of vertex and arrows, respectively, and $s$ and $r$ for the source and target maps, respectively. We assume that $Q$ has no sources, i.e. vertices $i \in Q_0$ such that $\{ \alpha \mid r(\alpha) = i \} = \emptyset$. Consider the double quiver $Q' = (Q_0, Q_1 \cup Q_1^*, s, r)$ obtained from $Q$ by adding an arrow $\alpha^*$, in the converse direction, for each arrow $\alpha \in Q_1$. Following Abrams-Pino [2] and Ara-Moreno-Pardo [5], the Leavitt path algebra $L_Q$ of $Q$ is the quotient of the quiver algebra $kQ$ (which is generated by elements $\alpha \in Q_1 \cup Q_1^*$ and $e_i$ with $i \in Q_0$) by the Cuntz-Krieger's relations: $\alpha^* \beta = \delta_{\alpha \beta} e_{r(\alpha)}$ for every $\alpha, \beta \in Q_1$; $\sum_{\alpha \in Q_1, s(\alpha) = i} \alpha \alpha^* = e_i$ for every non-sink $i \in Q_0$. Note that $L_Q$ admits a canonical $\mathbb{Z}$-grading with $\deg(\alpha) = 1$ and $\deg(\alpha^*) = -1$. For every vertex $i \in Q_0$ choose an arrow $\alpha_i$ such that $r(\alpha_i) = i$ and consider the associated elements $t_+ := \sum_{i \in Q_1} \alpha_i$ and $t_- := t_+^*$. Since $\deg(t_\pm) = \pm 1$ and $t_- t_+ = 1$, $L_Q$ is also an example of a corner skew Laurent polynomial algebra.

In the particular case where $Q$ is the quiver with one vertex and $n + 1$ arrows, $L_Q$ is isomorphic to $L_n$. Similarly, when $Q$ is the quiver with two vertices $\{1, 2\}$ and $2(n + 1)$ arrows $(n + 1)$ from 1 to 1 and $n + 1$ from 1 to 2), we have $L_Q \simeq J_n$.

2. $\mathbb{A}^1$-homotopy invariants

A dg category $\mathcal{A}$, over a base field $k$, is a category enriched over cochain complexes of $k$-vector spaces; see §4.1. Every (dg) $k$-algebra $A$ gives rise to a dg category with a single object. Another source of examples is provided by schemes since the category of perfect complexes $\text{perf}(X)$ of every quasi-compact quasi-separated $k$-scheme $X$ admits a canonical $\mathbb{Z}$-grading with $\deg(\alpha) = 1$ and $\deg(\alpha^*) = -1$. For every vertex $i \in Q_0$ choose an arrow $\alpha_i$ such that $r(\alpha_i) = i$ and consider the associated elements $t_+ := \sum_{i \in Q_1} \alpha_i$ and $t_- := t_+^*$. Since $\deg(t_\pm) = \pm 1$ and $t_- t_+ = 1$, $L_Q$ is also an example of a corner skew Laurent polynomial algebra.

A functor $E : \text{dgcat}(k) \to \mathcal{T}$, with values in a triangulated category, is called a localizing invariant if it satisfies the following three conditions:

- it inverts the derived Morita equivalences (see §4.1);
- it sends1 sequential (homotopy) colimits to sequential homotopy colimits;
- it sends1 short exact sequences of dg categories, in the sense of Drinfeld [6] and Keller [11] (see [9, §4.6]), to distinguished triangles

$$0 \to A \to B \to C \to 0 \to E(A) \to E(B) \to E(C) \to \Sigma E(A).$$

When $E$ inverts moreover the dg functors $\mathcal{A} \to \mathcal{A}[t]$, where $\mathcal{A}[t]$ stands for the tensor product $\mathcal{A} \otimes k[t]$, we call it an $\mathbb{A}^1$-homotopy invariant. Examples of localizing invariants include Hochschild homology $HH$, topological Hochschild homology $THH$, cyclic homology $HC$, the mixed complex $C$, noncommutative algebraic $K$-theory $\mathcal{K}$, mod-$l^n$ noncommutative algebraic $K$-theory $\mathcal{K}(\_; \mathbb{Z}/l^n)$ (with $l^n$ a prime power) and its variant $\mathcal{K}(\_; \mathbb{Z}[1/l])$, homotopy $K$-theory $KH$, mod-$l^n$ homotopy $K$-theory $KH(\_; \mathbb{Z}/l^n)$, and étale $K$-theory $K^{et}(\_; \mathbb{Z}/l^n)$; see [14, §8.2]. Among those, $\mathcal{K}(\_; \mathbb{Z}/l^n)$ (with $l \nmid \text{char}(k)$), $\mathcal{K}(\_; \mathbb{Z}[1/l])$ (with $l = \text{char}(k)$), $KH$, $KH(\_; \mathbb{Z}/l^n)$, and $K^{et}(\_; \mathbb{Z}/l^n)$ are $\mathbb{A}^1$-homotopy invariants; see [14, §8.5]. When applied to $A$, resp. to $\text{perf}_d_{dg}(X)$, the preceding invariants reduce to the corresponding invariants of the (dg) $k$-algebra $A$, resp. of the $k$-scheme $X$.

Example 2.1 (Noncommutative mixed motives). Let $\text{Mot}_{loc}(k)$, resp. $\text{Mot}^+_{loc}(k)$, be the category of noncommutative mixed motives constructed in [14, §8.2], resp.
in [14, §8.5.1]. By construction, this triangulated category comes equipped with a localizing invariant \( U_{\text{loc}}: \text{dgcat}(k) \to \text{Mot}_{\text{loc}}(k) \), resp. with an \( A^1 \)-homotopy invariant \( U^A_{\text{loc}}: \text{dgcat}(k) \to \text{Mot}^A_{\text{loc}}(k) \), which is initial among all localizing invariants, resp. among all \( A^1 \)-homotopy invariants; consult [14, §8.9] for further details.

3. Statement of results and applications

Let \( A \) be a unital \( k \)-algebra, \( e \) an idempotent of \( A \), and \( \phi: A \to eAe \) a “corner” isomorphism. Out of this data, we can construct the associated corner skew Laurent polynomial algebra \( A[t_+, t_-; \phi] \). In the same vein, let us write \( A[t_+, \phi] \) for the algebra defined by the formal expressions \( a_0 + a_1 t_+ + a_2 t_+^2 + \cdots + a_n t_+^n \) and by the relations \( t_+ a = \phi(a)t_+ \). Consider the \( A \)-\( A \)-bimodule \( eA \) associated to \( \phi \) (see Notation 4.1) and the \textit{square-zero extension} \( A[e] := A \ltimes (\phi A[1]) \) of \( A \) by the suspension \( \phi A[1] \) of \( \phi A \). Concretely, \( A[e] = A \oplus (eA[1]) \) with multiplication law \( (a, eb) \cdot (a', eb') := (aa', e(ab'+ ba')) \). Consider also the \( \text{dg} \) \( A[e]-A[t_+, \phi] \)-bimodule \( B \) whose restriction to \( A[t_+, \phi] \) is the projective resolution \( t_+ \cdots \to A[t_+, \phi] \to eA[t_+, \phi] \) of the trivial right \( A[t_+, \phi] \)-module \( eA \). The left \( \text{dg} \) action is given by the canonical identification of \( A[e] \) with the Ext-algebra \( \text{Ext}^{A[t_+, \phi]}_A(eA) \). As explained in §4.1, the \( \text{dg} \) bimodule \( B \) corresponds to a morphism from \( A[e] \) to \( A[t_+, \phi] \) in the localization of \( \text{dgcat}(k) \) with respect to derived Morita equivalences. Therefore, given any functor \( E \) which inverts derived Morita equivalences, we obtain an induced morphism \( E(B): E(A[e]) \to E(A[t_+, \phi]) \).

**Theorem 3.1.** For every localizing invariant \( E \), we have the homotopy colimit
\[
\text{hocolim} E(B) \to \cdots \to E(A[t_+, t_-; \phi]),
\]
where the transition morphism(s) is induced by the corner isomorphism \( \phi \).

When \( E \) is an \( A^1 \)-homotopy invariant, \( E(B) \) reduces to \( \text{id} - E(A) \): \( E(A) \to E(A) \) and the composition \( \text{(3.2)} \) is an isomorphism. Consequently, we obtain a triangle
\[
\text{hocolim} E(B) \to \cdots \to E(A[t_+, t_-; \phi]) \to \Sigma E(A).
\]

**Remark 3.4.** (Generalization). Given a \( \text{dg} \) category \( A \), Theorem 3.1 holds more generally with \( A \) and \( A[t_+, t_-; \phi] \) replaced by \( A \otimes A \) and \( A \otimes A[t_+, t_-; \phi] \); see §7.

**Corollary 3.5.** Given a \( \text{dg} \) category \( A \), we have a distinguished triangle of spectra:
\[
KH(A \otimes A) \to KH(A \otimes A[t_+, t_-; \phi]) \to \Sigma KH(A \otimes A).
\]

**Remark 3.6.** (Related work). Given a (not necessarily unital) \( k \)-algebra \( C \), Ara-Brustenga-Cortiñas proved in [4, Thms. 3.6 and 8.4] the analogue of Corollary 3.5 with \( A \) replaced by \( C \). Their proof, which is inspired from operator \( K \)-theory, makes essential use of non-unital algebras. Since these latter objects don’t belong to the realm of \( \text{dg} \) categories, our proof of Corollary 3.5 is necessarily different. Among other ideas used in the proof, we show that every corner skew Laurent polynomial algebra is derived Morita equivalent to a \( \text{dg} \) orbit category in the sense of Keller; see Proposition 6.6. Finally, note that in contrast with Ara-Brustenga-Cortiñas’ result, Corollary 3.5 (and more generally Theorem 3.1) holds for \( \text{dg} \) algebras and schemes.

Let \( Q = (Q_0, Q_1, s, r) \) be a finite quiver, without sources, with \( v \) vertices and \( v' \) sinks. We assume that the set \( Q_0 \) is ordered with the first \( v' \) elements corresponding to the sinks. Let \( I_Q \) be the incidence matrix of \( Q \), \( I^t_Q \) the matrix obtained from \( I_Q \) by removing the first \( v' \) rows (which are zero), and \( I^t_Q \) the transpose of \( I_Q \).
Theorem 3.7. Let $\mathcal{A}$ be a dg category and $Q$ a finite quiver without sources. For every $A^1$-homotopy invariant $E$, we have a distinguished triangle

$$\oplus_{i=1}^{n-1} E(\mathcal{A}) \xrightarrow{(\partial_i)^{-1} J_{Q}^{n+1}} \oplus_{i=1}^{n} E(\mathcal{A}) \rightarrow E(\mathcal{A} \otimes L_{Q}) \xrightarrow{\partial} \oplus_{i=1}^{n-1} \Sigma E(\mathcal{A}).$$

Theorem 3.7 shows that all the information concerning $A^1$-homotopy invariants of Leavitt path algebras $L_{Q}$ is encoded in the incidence matrix of the quiver $Q$. 

Example 3.8 (Jacobson algebras). Let $Q$ be the quiver with two vertices $\{1, 2\}$ and $2(n+1)$ arrows ($n+1$ from $1$ to $1$ and $n+1$ from $1$ to $2$). In this particular case, the distinguished triangle of Theorem 3.7 reduces to

$$E(\mathcal{A}) \xrightarrow{n \id} E(\mathcal{A}) \rightarrow E(\mathcal{A} \otimes L_{Q}) \xrightarrow{\partial} \Sigma E(\mathcal{A}).$$

Since $(n, n+1) = 1$, we conclude that $E(\mathcal{A} \otimes J_{n}) \simeq E(\mathcal{A})$. This shows that the $A^1$-homotopy invariants don’t distinguish the Jacobson algebras $J_{n}$ from $k$. Note that $J_{n}$ is much bigger than $k$; for instance, it contains the path algebra $kQ$.

Example 3.9 (Leavitt algebras). Let $Q$ be the quiver with one vertex and $n+1$ arrows. In this particular case, the distinguished triangle of Theorem 3.7 reduces to

$$(3.10) \quad E(\mathcal{A}) \xrightarrow{n \id} E(\mathcal{A}) \rightarrow E(\mathcal{A} \otimes L_{n}) \xrightarrow{\partial} \Sigma E(\mathcal{A}).$$

When $n = 0$, the distinguished triangle (3.10) splits and gives rise to the “fundamental” isomorphism $E(\mathcal{A} \otimes L_{0}) \simeq E(\mathcal{A}) \oplus \Sigma E(\mathcal{A})$. When $n = 1$, we have $E(\mathcal{A} \otimes L_{1}) = 0$. In the remaining case $n \geq 2$, $E(\mathcal{A} \otimes L_{n})$ identifies with the mod-$n$ Moore object of $E(\mathcal{A})$. Intuitively speaking, this shows that the functor $\mathcal{A} \rightarrow \mathcal{A} \otimes L_{n}$, with $n \geq 2$, is a model of the mod-$n$ Moore construction.

Proposition 3.11. Let $l_{1}^{\nu_{1}} \times \cdots \times l_{\nu_{r}}^{\nu}$ be the prime decomposition of an integer $n \geq 2$. For every dg category $\mathcal{A}$ and $A^1$-homotopy invariant $E$, we have a direct sum decomposition $E(\mathcal{A} \otimes L_{n}) \simeq E(\mathcal{A} \otimes L_{1}^{\nu_{1}}) \oplus \cdots \oplus E(\mathcal{A} \otimes L_{1}^{\nu_{r}})$.

Roughly speaking, Proposition 3.11 shows that all the $A^1$-homotopy invariants of Leavitt algebras are “$l$-local”. Note that $L_{n} \neq L_{1}^{\nu_{1}} \times \cdots \times L_{1}^{\nu_{r}}$.

Remark 3.12 (Homotopy $K$-theory). By taking $E = KH$ in Theorem 3.7, we obtain a distinguished triangle of spectra

$$\oplus_{i=1}^{n-1} KH(\mathcal{A}) \xrightarrow{(\partial_i)^{-1} J_{Q}^{n+1}} \oplus_{i=1}^{n} KH(\mathcal{A}) \rightarrow KH(\mathcal{A} \otimes L_{Q}) \xrightarrow{\partial} \oplus_{i=1}^{n-1} \Sigma KH(\mathcal{A}).$$

Given a (not necessarily unital) $k$-algebra $C$, Ara-Brustenga-Cortiñas constructed in [4, Thm. 8.6] the analogue of the preceding distinguished triangle with $\mathcal{A}$ replaced by $C$. Our construction is different and applies also to dg categories and schemes.

Remark 3.13. By taking $E = KH$ and $n = l^{\nu}$ in Example 3.9, we obtain an isomorphism between $KH(\mathcal{A} \otimes L_{n})$ and the mod-$l^{\nu}$ homotopy $K$-theory spectrum $KH(\mathcal{A}; \mathbb{Z}/l^{\nu})$. When $l \nmid \text{char}(k)$, the latter spectrum is isomorphic to $KH(\mathcal{A}; \mathbb{Z}/l^{\nu})$.

Mod-$l^{\nu}$ algebraic $K$-theory of Leavitt path algebras. Let $l^{\nu}$ be a prime power such that $l \neq \text{char}(k)$ and $Q$ a finite quiver without sources. By taking $\mathcal{A} = k$ and $E = KH(\mathcal{A}; \mathbb{Z}/l^{\nu})$ in Theorem 3.7, we obtain a distinguished triangle of spectra

$$\oplus_{i=1}^{n-1} KH(k; \mathbb{Z}/l^{\nu}) \xrightarrow{(\partial_i)^{-1} J_{Q}^{n+1}} \oplus_{i=1}^{n} KH(k; \mathbb{Z}/l^{\nu}) \rightarrow KH(L_{Q}; \mathbb{Z}/l^{\nu}) \xrightarrow{\partial} \oplus_{i=1}^{n-1} \Sigma KH(k; \mathbb{Z}/l^{\nu}).$$

Remark 3.14. The preceding triangle follows also from the work of Ara-Brustenga-Cortiñas [4]. Indeed, since by hypothesis \( l \nmid \text{char}(k) \), the functors \( KH(-; \mathbb{Z}/l^n) \) and \( KH(-; \mathbb{Z}/l^n) \) are isomorphic. Moreover, as explained in Remark 3.13, the latter functor identifies with \( KH(- \otimes \mathbb{L}_{l^n}) \). Therefore, if in Remark 3.12 we take for \( C \) the \( k \)-algebra \( L_{l^n} \), we obtain the preceding distinguished triangle of spectra.

Assume that \( k \) is algebraically closed. As proved by Suslin\(^2\) in [13, Cor. 3.13], we have \( KH_n(k; \mathbb{Z}/l^n) \simeq \mathbb{Z}/l^n \) if \( n \geq 0 \) is even and \( KH_n(k; \mathbb{Z}/l^n) = 0 \) otherwise. Consequently, making use of the long exact sequence of algebraic \( K \)-theory groups associated to the preceding triangle of spectra, we obtain the following result:

Corollary 3.15. We have the following computation

\[
KH_n(L_Q; \mathbb{Z}/l^n) \simeq \begin{cases} 
\text{cokernel of } M & \text{if } n \geq 0 \text{ even} \\
\text{kernel of } M & \text{if } n \geq 0 \text{ odd} \\
0 & \text{if } n < 0 ,
\end{cases}
\]

where \( M \) stands for the homomorphism \( \bigoplus_{i=1}^{n-v} \mathbb{Z}/l^n(\nu_i - l_0) \bigoplus_{i=1}^{v} \mathbb{Z}/l^n. \)

Corollary 3.15 provides a complete and explicit computation of the mod-\( l^n \) (non-connective) algebraic \( K \)-theory of Leavitt path algebras. To the best of the author’s knowledge, these computations are new in the literature. In particular, they yield a complete answer to the “mod-\( l^n \) version” of Question 2 raised by Gabe-Ruiz-Tomforde-Whalen in [7, page 38]. These computations lead also naturally to the following vanishing and divisibility properties of algebraic \( K \)-theory:

Proposition 3.16. (i) If there exists a prime power \( l^n \) and an even (resp. odd) integer \( n \geq 0 \) such that \( KH_n(L_Q; \mathbb{Z}/l^n) \neq 0 \), then for every even (resp. odd) integer \( n \geq 0 \) at least one of the groups \( KH_n(L_Q), KH_n-1(L_Q) \) is non-zero.

(ii) If there exists a prime power \( l^n \) such that \( KH_n(L_Q; \mathbb{Z}/l^n) = 0 \) for every \( n \geq 0 \), then the groups \( KH_n(L_Q), n \geq 0 \), are uniquely \( l^n \)-divisible, i.e. \( \mathbb{Z}[1/l^n] \)-modules.

Proof. Combine the universal coefficients sequence (see [14, §2.2.2])

\[
0 \rightarrow KH_n(L_Q) \otimes \mathbb{Z}/l^n \rightarrow KH_n(L_Q; \mathbb{Z}/l^n) \rightarrow l^nKH_{n-1}(L_Q) \rightarrow 0
\]

with the computation of Corollary 3.15. \( \Box \)

Example 3.17 (Quivers without sinks). Let \( Q \) be a quiver without sinks. In this case, \( (i_0^Q)^{0} - I_Q^0 \) is a square matrix. If \( l \) is a prime such that \( l \nmid \det((i_0^Q)^{0} - I_Q^0) \), then the homomorphism \( M \) of Corollary 3.15 is invertible. Consequently, \( KH_n(L_Q; \mathbb{Z}/l^n) = 0 \) for every \( n \geq 0 \). Making use of Proposition 3.16(ii), we then conclude that the algebraic \( K \)-theory groups \( KH_n(L_Q), n \geq 0 \), are uniquely \( l^n \)-divisible.

Schemes and stacks. Let \( X \) be a quasi-compact quasi-separated \( k \)-scheme. By applying the results/examples/remarks of §3 to the dg category \( A = \text{perf}_{dg}(X) \), we obtain corresponding results/examples/remarks concerning the scheme \( X \). For instance, Remark 3.13 yields an isomorphism between \( KH(\text{perf}_{dg}(X) \otimes L_{l^n}) \) and \( KH(X; \mathbb{Z}/l^n) \). When \( l \nmid \text{char}(k) \), the latter spectrum is isomorphic to \( KH(X; \mathbb{Z}/l^n) \).

Roughly speaking, the dg category \( \text{perf}_{dg}(X) \otimes L_{l^n} \) may be understood as the “noncommutative mod-\( l^n \) Moore object of \( X \)”. More generally, we can consider the

\(^2\) Given a quiver \( Q \), let \( C^*_0(Q) \) be the associated Cuntz-Krieger \( C^* \)-algebra. Cortiñas kindly informed the author that the work of Suslin was also used in [4, Thm. 9.4] in order to prove that \( KH_n(\mathbb{C} \otimes L_Q) \simeq KH_n(\mathbb{C}^*_0(Q)), n \geq 0 \), for every quiver \( Q \) without sinks such that \( \det((i_0^Q)^{0} - I_Q^0) \neq 0 \).
dg category $\mathcal{A} = \text{perf}_d(X)$ of perfect complexes of an algebraic stack $X$. In the particular case of a quotient stack $X = [X/G]$, with $G$ an algebraic group scheme acting on $X$, Remark 3.13 yields an isomorphism between $KH(\text{perf}_d([X/G]) \otimes L^{l''})$ and the mod-$l''$ $G$-equivariant homotopy $K$-theory spectrum $KH^G(X; \mathbb{Z}/l'')$. When $l \nmid \text{char}(k)$, the latter spectrum is isomorphic to $K^G(X; \mathbb{Z}/l'')$.

4. Preliminaries

4.1. Dg categories. Let $(\mathcal{C}(k), \otimes, k)$ be the category of cochain complexes of $k$-vector spaces. A dg category $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ and a dg functor $F : \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller’s ICM survey [9].

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\text{op}}$ has the same objects as $\mathcal{A}$ and $\mathcal{A}^{\text{op}}(x, y) := \mathcal{A}(y, x)$. A right dg $\mathcal{A}$-module is a dg functor $\mathcal{A}^{\text{op}} \to \mathcal{C}_{dg}(k)$ with values in the dg category $\mathcal{C}_{dg}(k)$ of complexes of $k$-vector spaces. Let us denote by $\mathcal{C}(\mathcal{A})$ the category of right dg $\mathcal{A}$-modules. Following [9, §3.2], the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms. We write $\mathcal{D}_c(\mathcal{A})$ for the subcategory of compact objects.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a derived Morita equivalence if it induces an equivalence of categories $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B})$; see [9, §4.6]. As proved in [14, Thm. 1.37], $\mathcal{Dcat}(k)$ admits a Quillen model structure whose weak equivalences are the derived Morita equivalences. Let $\text{Hmo}(k)$ be the associated homotopy category.

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of (small) dg categories is defined as follows: the set of objects is the cartesian product and $(\mathcal{A} \otimes \mathcal{B})(x, y, (w, z)) := \mathcal{A}(x, y) \otimes \mathcal{B}(w, z)$. As explained in [9, §2.3], this construction gives rise to a symmetric monoidal structure on $\mathcal{Dcat}(k)$, which descends to the homotopy category $\text{Hmo}(k)$.

A dg $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{B}$ is a dg functor $\mathcal{A} \otimes \mathcal{B}^{\text{op}} \to \mathcal{C}_{dg}(k)$ or equivalently a right dg $(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$-module. Associated to a dg functor $F : \mathcal{A} \to \mathcal{B}$, we have the dg $\mathcal{A}$-$\mathcal{B}$-bimodule $F \mathcal{B} : \mathcal{A} \otimes \mathcal{B}^{\text{op}} \to \mathcal{C}_{dg}(k), (x, z) \mapsto \mathcal{B}(z, F(x))$. Let us write $\text{rep}(\mathcal{A}, \mathcal{B})$ for the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-bimodules $\mathcal{B}$ such that for every object $x \in \mathcal{A}$ the associated right dg $\mathcal{B}$-module $\mathcal{B}(x, -)$ belongs to $\mathcal{D}_c(\mathcal{B})$. Clearly, the dg $\mathcal{A}$-$\mathcal{B}$-bimodules $F \mathcal{B}$ belongs to $\text{rep}(\mathcal{A}, \mathcal{B})$.

As explained in [14, §1.6.3], there is a natural bijection between $\text{Hom}_{\text{Hmot}(k)}(\mathcal{A}, \mathcal{B})$ and the set of isomorphism classes of $\text{rep}(\mathcal{A}, \mathcal{B})$. Under this bijection, the composition law corresponds to the tensor product of dg bimodules.

Notation 4.1. Given a non-unital homomorphism $\phi : A \to B$ between unital $k$-algebras, let us denote by $\phi B$ the $A$-$B$-bimodule $\phi(1)B$ equipped with the $A$-$B$-action $a \cdot \phi(1)B \cdot b := \phi(a)Bb$. Note that $\phi B$ belongs to $\text{rep}(A, B)$.

Square-zero extensions. Let $\mathcal{A}$ be a dg category and $\mathcal{B}$ a dg $\mathcal{A}$-$\mathcal{A}$-bimodule. The square-zero extension $\mathcal{A} \ltimes \mathcal{B}$ of $\mathcal{A}$ by $\mathcal{B}$ is the dg category with the same objects as $\mathcal{A}$ and complexes of $k$-vector spaces $(\mathcal{A} \ltimes \mathcal{B})(x, y) := \mathcal{A}(x, y) \oplus B(y, x)$. Given morphisms $(f, f') \in (\mathcal{A} \ltimes \mathcal{B})(x, y)$ and $(g, g') \in (\mathcal{A} \ltimes \mathcal{B})(y, z)$, the composition $(g, g') \circ (f, f')$ is defined as $(g \circ f, g'f + g'f')$.

Dg orbit categories. Let $F : \mathcal{A} \to \mathcal{A}$ be an equivalence of dg categories. Following Keller [10, §5.1], the associated dg orbit category $\mathcal{A}/F^\mathbb{Z}$ has the same objects as $\mathcal{A}$ and complexes of $k$-vector spaces $(\mathcal{A}/F^\mathbb{Z})(x, y) := \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(x, F^n(y))$. Given objects $x, y, z$ and morphisms $f = \{f_n\}_{n \in \mathbb{Z}} \in \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(x, F^n(y))$ and $g = \{g_n\}_{n \in \mathbb{Z}} \in \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(y, F^n(z))$, the $m$th-component of $g \circ f$ is defined as $\sum_n F^n(g_{m-n}) \circ f_n$. 


When $A$ is a $k$-algebra $A$, the dg functor $F$ reduces to an isomorphism $\phi: A \xrightarrow{\sim} A$ and the dg orbit category $A/F^Z$ to the skew Laurent polynomial algebra $A \rtimes_\phi Z$.

Let us write $A/F^N$ for the dg category with the same objects as $A$ and complexes of $k$-vector spaces $(A/F^N)(x, y) := \bigoplus_{n \geq 0} A(x, F^n(y))$. The composition law is defined as above. By construction, we have a canonical dg functor $A/F^N \to A/F^Z$.

5. DG categories of idempotents

Let $A$ be a (not necessarily unital) $k$-algebra.

**Definition 5.1.** The dg category of idempotents of $A$, denoted by $\underline{A}$, is defined as follows: the objects are the symbols $e$ with $e$ an idempotent of $A$; the (complexes of) $k$-vector spaces $A(e, e')$ are given by $e A e'$; the composition law is induced by the multiplication in $A$; the identity of the object $e$ is the idempotent $e$.

**Notation 5.2.** Let $\text{alg}(k)$ be the category of (not necessarily unital) $k$-algebras and (not necessarily unital) $k$-algebra homomorphisms.

Note that the preceding construction gives rise to the following functor:

$$\text{alg}(k) \to \text{dgcat}(k) \quad A \mapsto \underline{A}, \; \phi \mapsto \underline{\phi}.$$  

(5.3)

**Lemma 5.4.** The functor (5.3) preserves filtered colimits.

**Proof.** Consider a filtered diagram $\{A_i\}_{i \in I}$ in $\text{alg}(k)$ with colimit $A$. Given an idempotent element $e$ of $A$, there exists an index $i' \in I$ and an idempotent $e' \in A_{i'}$ such that $e$ is the image of $e'$ under $A_{i'} \to A$. This implies that the induced dg functor $\text{colim}_i A_i \to \underline{A}$ is not only (essentially) surjective but also fully-faithful. □

Given a unital $k$-algebra $A$ with unit 1, let us write $\iota: A \to \underline{A}$ for the (unique) dg functor sending the single object of $A$ to the symbol 1.

**Lemma 5.5.** The dg functor $\iota$ is a derived Morita equivalence.

**Proof.** Note first that the dg functor $\iota$ is fully-faithful. Given an idempotent element $e$ of $A$, the morphisms $1 \xrightarrow{\sim} e$ and $e \xrightarrow{\sim} 1$ present the object $e$ as a direct summand of 1. This allows us to conclude that $\iota$ is a derived Morita equivalence. □

**Remark 5.6.** Given a non-unital homomorphism $\phi: A \to B$ between unital $k$-algebras, note that $\underline{B} \circ \underline{\phi} B = \underline{\phi} B \circ \underline{A}$ in the homotopy category $\text{Hmo}(k)$.

Let $A$ be a unital $k$-algebra and $M_2(A)$ the associated $k$-algebra of $2 \times 2$ matrices. Consider the following non-unital homomorphisms

$$j_1, j_2: A \to M_2(A) \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad a \mapsto \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$  

Note that if there exist elements $t_+, t_-$ of $A$ such that $t_+ t_+ = 1$, then we can also consider the non-unital homomorphism $\phi^\pm: A \to A, a \mapsto t_+ a t_-.$

**Proposition 5.7.** (i) The dg functors $j_1$ and $j_2$ are derived Morita equivalences. Moreover, their images in the homotopy category $\text{Hmo}(k)$ are the same.

(ii) The dg functor $\phi^\pm$ is a derived Morita equivalence. Moreover, its image in the homotopy category $\text{Hmo}(k)$ is the identity morphism.
Proof. (i) Recall first that a dg functor $F: A \to B$ is a derived Morita equivalence if and only if its image $F B$ in the homotopy category $\mathrm{Hmo}(k)$ is invertible. Thanks to Lemma 5.5 and Remark 5.6, it suffices then to show that the $A\text{-}M_2(A)$-bimodules $j_1 M_2(A)$ and $j_2 M_2(A)$ are invertible in $Hmo(k)$. Note that their inverses are given by the $M_2(A)$-$A$-bimodules $M_2(A) j_1(1)$ and $M_2(A) j_2(1)$, respectively. This shows the first claim. The second claim follows from the isomorphism $j_1 M_2(A) \xrightarrow{\sim} j_2 M_2(A)$ of $A\text{-}M_2(A)$-bimodules given by $egin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$.

(ii) Consider the following non-unital homomorphism
\[
\varphi: M_2(A) \to M_2(A) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} t_+ & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t_- & 0 \\ 0 & 1 \end{pmatrix},
\]
Note that $j_1 \circ \varphi = \varphi \circ j_1$ and $\varphi \circ j_2 = j_2$ in the category $\text{alg}(k)$. By applying the functor (5.3), we hence conclude from item (i) that the dg functor $\varphi$ is not only a derived Morita equivalence but moreover that its image in the homotopy category $\text{Hmo}(k)$ is the identity morphism. \qed

6. PROOF OF THEOREM 3.1

Consider the sequential colimit diagram $A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots \to C$ in the category $\text{alg}(k)$. Note that the $k$-algebra $C$ is non-unital and that the homomorphism $\varphi$ gives rise to an isomorphism $\varphi: C \xrightarrow{\sim} C$. Let us denote by $C \times_{\hat{\varphi}} N$, resp. $C \times_{\hat{\varphi}} Z$, the associated skew polynomial algebra, resp. skew Laurent polynomial algebra. Note also that $\varphi$ extends to (non-unital) homomorphisms $\varphi: A[t_+; \varphi] \to A[t_+; \varphi], a \mapsto t_+ a t_-$, and $\varphi: A[t_+, t_-; \varphi] \to A[t_+, t_-; \varphi], a \mapsto t_+ a t_-$. Under these notations, we have the following sequential colimit diagrams:
\[
(6.1) \quad A[t_+; \varphi] \xrightarrow{\varphi} A[t_+; \varphi] \xrightarrow{\varphi} \cdots \to C \times_{\hat{\varphi}} N
\]
\[
(6.2) \quad A[t_+, t_-; \varphi] \xrightarrow{\varphi} A[t_+, t_-; \varphi] \xrightarrow{\varphi} \cdots \to C \times_{\hat{\varphi}} Z.
\]

Lemma 6.3. The dg category of idempotents of $C \times_{\hat{\varphi}} N$, resp. $C \times_{\hat{\varphi}} Z$, is derived Morita equivalent to the dg category $C/\hat{\varphi}^N$, resp. $C/\hat{\varphi}^Z$.

Proof. We focus ourselves in the algebra $C \times_{\hat{\varphi}} Z$ and in the dg orbit category $C/\hat{\varphi}^Z$; the proof of the other case is similar. Let us denote by $1_n$ the unit of the $n$th copy of $A[t_+, t_-; \varphi]$ and by $e_n$ the image of $1_n$ under the induced homomorphism $A[t_+, t_-; \varphi] \to C \times_{\hat{\varphi}} Z$. Given an idempotent element $e$ of $C \times_{\hat{\varphi}} Z$, there exists an integer $n \gg 0$ such that $e$ is a direct summand of $e_n$. Since $e_n$ belongs to $C \subset C \times_{\hat{\varphi}} Z$, this allows us to conclude, in particular, that the dg category of idempotents of $C \times_{\hat{\varphi}} Z$ is derived Morita equivalent to its full dg category of symbols $e$ with $e$ an idempotent element of $C \subset C \times_{\hat{\varphi}} Z$. Given any two such symbols $e$ and $e'$, note that we have the following equalities of (complexes of) $k$-vector spaces:
\[
(C \times_{\hat{\varphi}} Z)(e, e') := e(C \times_{\hat{\varphi}} Z)e' = \bigoplus_{n \in \mathbb{Z}} e C \hat{\varphi}^n(e') = \bigoplus_{n \in \mathbb{Z}} C(e, \hat{\varphi}^n(e')) =: C/\hat{\varphi}^Z(e, e').
\]
Under these equalities, the composition law of the dg category of idempotents of $C \ltimes \hat{\phi} \mathbb{Z}$ corresponds to the composition law of the dg orbit category $C/\hat{\phi} \mathbb{Z}$. □

By combining Lemmas 5.4-5.5 with Remark 5.6, we conclude from Lemma 6.3 that (6.1)-(6.2) give rise to the following sequential (homotopy) colimit diagrams

\begin{equation}
A[t_+; \phi] \xrightarrow{\phi A[t_+; \phi]} A[t_+; \phi] \xrightarrow{\phi A[t_+; \phi]} \cdots \to C/\hat{\phi} \mathbb{Z}
\end{equation}

in the homotopy category $\text{Hmo}(k)$.

Proposition 6.6. The (transfinite) composition (6.5) is an isomorphism. Consequently, the dg categories $A[t_+, t_-; \phi]$ and $C/\hat{\phi} \mathbb{Z}$ are derived Morita equivalent.

Now, consider the square-zero extension $C[e] := C \ltimes (\hat{\phi} C[1])$ of $C$ by the suspension $\hat{\phi} C[1]$ of the dg $C$-$C$-bimodule $\hat{\phi} C$ associated to $\hat{\phi}$. Consider also the dg $C[e]$-$C/\hat{\phi} \mathbb{Z}$-bimodule $\hat{B}$ introduced in [10, §4]; denoted by $B'$ in loc. cit.

**Lemma 6.7.** We have the following sequential (homotopy) colimit diagram

\[ A[e] \xrightarrow{\phi A[e]} A[e] \xrightarrow{\phi A[e]} \cdots \to C[e] \]

in the homotopy category $\text{Hmo}(k)$.

**Proof.** Thanks to Lemma 5.4, we have $A \xrightarrow{\phi} A \xrightarrow{\phi} \cdots \to C$ in the category $\text{dgcat}(k)$. Consider the square-zero extension $A[e] := A \ltimes (\hat{\phi} A[1])$ of $A$ by the suspension $\hat{\phi} A[1]$ of the dg $A$-$A$-bimodule $\hat{\phi} A$ associated to $\hat{\phi}$. Similarly to the proof of Lemma 5.4, we have the following induced sequential colimit diagram

\[ A[e] \xrightarrow{\phi} A[e] \xrightarrow{\phi} \cdots \to C[e] \].

Note that the dg functor $A[e] \to A[e]$ sending the single object $A[e]$ to the symbol $1$, where 1 is the unit of $A$, is a derived Morita equivalence. Under such derived Morita equivalence the dg functor $\phi : A[e] \to A[e]$ corresponds to the morphism $\phi A[e] : A[e] \to A[e]$ in the homotopy category $\text{Hmo}(k)$. This concludes the proof. □

By combining Lemma 6.7 with the sequential (homotopy) colimit diagram (6.4), we obtain the following sequential (homotopy) colimit diagram

\begin{equation}
\begin{array}{c}
C[e] \xrightarrow{\hat{B}} C/\hat{\phi} \mathbb{Z} \\
\downarrow \\
\vdots \\
\phi A[e] \xrightarrow{\phi A[t_+; \phi]} A[t_+; \phi] \\
A[e] \xrightarrow{B} A[t_+; \phi] \\
\phi A[e] \xrightarrow{\phi A[t_+; \phi]} A[t_+; \phi]
\end{array}
\end{equation}
in the homotopy category $\text{Hmo}(k)$. As explained in [10, §4], given any localizing invariant $E$, we have a distinguished triangle

$$E(C[e]) \xrightarrow{E(B)} E(C/\hat{\phi}^N) \xrightarrow{\hat{\phi}} \Sigma E(C[e]).$$

Therefore, by combining the diagram (6.8) with Proposition 6.6, we obtain the searched homotopy colimit diagram (3.2). This concludes the proof of the first claim.

We now prove the second claim. Let $E$ be an $\mathbb{A}^1$-homotopy invariant. As explained in [10, Prop. 4.6], $E(B)$ reduces to $\text{id} - E(\phi) : E(A) \to E(A)$. Similarly, the morphism $E(B)$ reduces to $\text{id} - E(\phi) : E(A) \to E(A)$. Therefore, making use of Lemma 5.5 and of Remark 5.6, we observe that by applying the functor $E$ to (6.8) we obtain (up to isomorphism) the following sequential colimit diagram:

$$E(C) \xrightarrow{id - E(\phi)} E(C) \xrightarrow{\ddots} \xrightarrow{id - E(\phi)} E(C) \xrightarrow{\ddots} \xrightarrow{id - E(\phi)} E(C).$$

We now claim that the induced (transfinite) composition

$$E(\phi) \xrightarrow{id - E(\phi)} E(\phi) \xrightarrow{\ddots} \xrightarrow{id - E(\phi)} E(\phi) \xrightarrow{\ddots}$$

is an isomorphism. As explained in [14, Thm. 8.25], the functor $U_{\text{loc}}^\mathbb{A}^1$ is the initial $\mathbb{A}^1$-homotopy invariant. Therefore, it suffices to prove the latter claim in the particular case where $E = U_{\text{loc}}^\mathbb{A}^1$. By construction, we have a factorization

$$U_{\text{loc}}^\mathbb{A}^1 : \text{dgcat}(k) \xrightarrow{U_{\text{add}}} \text{Mot}_{\text{add}}(k) \xrightarrow{\gamma} \text{Mot}_{\text{loc}}^\mathbb{A}^1(k),$$

where $\text{Mot}_{\text{add}}(k)$ is a certain compactly generated triangulated category of noncommutative mixed motives, $U_{\text{add}}$ is a certain functor sending sequential (homotopy) colimits to sequential homotopy colimits, and $\gamma$ is a certain homotopy colimit preserving functor; consult [14, §8.4.2] for details. The triangulated category $\text{Mot}_{\text{add}}(k)$ is moreover enriched over spectra; we write $\text{Hom}_{\text{Spt}}(-, -)$ for this enrichment. Let $\text{NM}$ be a compact object of $\text{Mot}_{\text{add}}(k)$. In order to prove our claim, it is then enough to show that the (transfinite) composition obtained by applying the functor $\text{Hom}_{\text{Spt}}(\text{NM}, -)$ to (6.11) (with $E = U_{\text{add}}$) is an isomorphism. Since the spectrum $\text{Hom}_{\text{Spt}}(\text{NM}, U_{\text{add}}(C))$ is the sequential homotopy colimit of $\text{Hom}_{\text{Spt}}(\text{NM}, U_{\text{add}}(A))$, with respect to the transition morphism(s) $\text{Hom}_{\text{Spt}}(\text{NM}, U_{\text{add}}(\phi))$, the proof follows now automatically from the general result [4, Lem. 3.3] concerning spectra. This finishes the proof of Theorem 3.1.
7. Proof of the Generalization of Theorem 3.1

The triangulated category Mot_{loc}(k) carries a symmetric monoidal structure making the functor \( U_{loc} \) symmetric monoidal; see [14, §8.3.1]. Therefore, the distinguished triangle (6.9) (with \( E = U_{loc} \)) gives rise to the distinguished triangle:

\[
U_{loc}(A \otimes C[\epsilon]) \xrightarrow{U_{loc}(\text{id} \otimes \delta B)} U_{loc}(A \otimes C/\delta N) \longrightarrow U_{loc}(A \otimes C/\delta Z) \xrightarrow{\partial} \Sigma U_{loc}(A \otimes C[\epsilon]).
\]

Since the functor \( \otimes - \) preserves (sequential) homotopy colimits, the combination of the preceding triangle with the commutative diagram (6.8) and with Proposition 6.6 leads then to the following sequential homotopy colimit diagram

\[
\text{hocolim} U_{loc}(idA \otimes B) \longrightarrow \text{hocolim} U_{loc}(idA \otimes B) \longrightarrow \cdots \longrightarrow U_{loc}(A \otimes A[t_+, t_-; \phi]),
\]

where the transition morphism(s) is induced by the corner isomorphism \( \phi \). The proof of the first claim follows now automatically from the fact that \( U_{loc} \) is the initial localizing invariant; see [14, Thm. 8.5].

The triangulated category Mot_{loc}^A(k) carries a symmetric monoidal structure making the functor \( U_{loc}^A \) symmetric monoidal; see [14, §8.5.2]. Therefore, the distinguished triangle (3.3) (with \( E = U_{loc}^A \)) gives rise to the distinguished triangle:

\[
U_{loc}^A(A \otimes A) \xrightarrow{id - U_{loc}^A(idA \otimes A)} U_{loc}^A(A \otimes A) \longrightarrow U_{loc}^A(A \otimes A[t_+, t_-; \phi]) \xrightarrow{\partial} \Sigma U_{loc}^A(A \otimes A).
\]

The proof of the second claim follows now automatically from the fact that \( U_{loc}^A \) is the initial \( A^1 \)-homotopy invariant; see [14, Thm. 8.25].

8. Proof of Theorem 3.7

Similarly to the arguments used in §7, it suffices to prove Theorem 3.7 in the particular case where \( E = U_{loc}^A \) and \( A = k \). As mentioned in Example 1.4, the Leavitt path algebra \( L := L_Q \) is a corner skew Laurent polynomial algebra. Let \( L_0 \) be the homogeneous component of degree 0 and \( \phi: L_0 \to eL_0e \) the “corner” isomorphism. Thanks to Theorem 3.1 (with \( E = U_{loc}^A \)), we have a triangle

\[
U_{loc}^A(L_0) \xrightarrow{id - U_{loc}^A(idA \otimes \phi A)} U_{loc}^A(L_0) \longrightarrow U_{loc}^A(L_0) \xrightarrow{\partial} \Sigma U_{loc}^A(L_0)
\]

in the category Mot_{loc}^A(k). Following Ara-Brustenga-Cortiñas [4, §5], the \( k \)-algebra \( L_0 \) admits a “length” filtration \( L_0 = \bigcup_{n=0}^{\infty} L_{0,n} \). Concretely, \( L_{0,n} \) is the \( k \)-linear span of the elements of the form \( \sigma \zeta^* \), where \( \sigma \) and \( \zeta \) are paths such that \( r(\sigma) = r(\zeta) \) and \( \text{deg}(\sigma) = \text{deg}(\zeta) = n \). It turns out that the \( k \)-algebra \( L_{0,n} \) is isomorphic to the product of \( (n+1)v' + (v-v') \) matrix algebras with \( k \)-coefficients. Making use of the (derived) Morita equivalences between a matrix algebra with \( k \)-coefficients and \( k \), we conclude that \( U_{loc}^A(L_{0,n}) \) is isomorphic to the direct sum of \( (n+1)v' + (v-v') \) copies of \( U_{loc}^A(k) \). Recall from [14, Thm. 8.28] that we have an isomorphism \( \text{Hom}_{Mot_{loc}^A(k)}(U_{loc}^A(k), U_{loc}^A(k)) \simeq K_0(k) \simeq \mathbb{Z} \). Under this identification, the inclusion \( L_{0,n} \subset L_{0,n+1} \) corresponds to the matrix morphism (see [4, §5]):

\[
\begin{pmatrix}
\text{id} & 0 \\
0 & I_{Q}
\end{pmatrix} : \bigoplus_{i=1}^{(n+1)v' + (v-v')} U_{loc}^A(k) \longrightarrow \bigoplus_{i=1}^{(n+1)v' + v} U_{loc}^A(k).
\]
In the same vein, the homomorphism $\phi: L_{0,n} \to L_{0,n+1}$, which increases the degree of the filtration by 1, corresponds to the matrix morphism

$$
(8.3) \begin{pmatrix} 0 & & & & \cr & & & & \cr & & & & \cr & & & & \cr & & & & \cr \vdots & & & & \cr \end{pmatrix} : \bigoplus_{i=1}^{n+1} U_{\text{loc}}^k(L) \to \bigoplus_{i=1}^{n+1} U_{\text{loc}}^k(L).
$$

Since the functor $U_{\text{loc}}^k$ sends sequential (homotopy) colimits to sequential homotopy colimits, we hence obtain the following sequential homotopy colimit diagram

$$
\begin{array}{c}
U_{\text{loc}}^k(L_{0,0}) \xrightarrow{(8.2)} U_{\text{loc}}^k(L_{0,1}) \xrightarrow{(8.2)} \cdots \xrightarrow{\text{id} - U_{\text{loc}}^k(\partial L_0)} U_{\text{loc}}^k(L_0) \\
\downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
U_{\text{loc}}^k(L_{0,1}) \xrightarrow{(8.2)} U_{\text{loc}}^k(L_{0,2}) \xrightarrow{(8.2)} \cdots \xrightarrow{\text{id} - U_{\text{loc}}^k(\partial L_0)} U_{\text{loc}}^k(L_0).
\end{array}
$$

Simple matrix manipulations show that the homotopy cofibers of the vertical morphisms of the diagram (8.4) are all equal to the homotopy cofiber of the morphism

$$(\partial_{id}) - I_{\mathbb{Q}} : \bigoplus_{i=1}^{\nu-n'} U_{\text{loc}}^k(L_i) \to \bigoplus_{i=1}^{\nu-n'} U_{\text{loc}}^k(L_i).$$

This allows us then to conclude that distinguished triangle (8.1) yields the following distinguished triangle

$$\bigoplus_{i=1}^{\nu-n'} U_{\text{loc}}^k(L_i) \xrightarrow{(\partial_{id}) - I_{\mathbb{Q}}} \bigoplus_{i=1}^{\nu-n'} U_{\text{loc}}^k(L_i) \to U_{\text{loc}}^k(L_{0}) \xrightarrow{\partial} \bigoplus_{i=1}^{\nu-n'} \sum U_{\text{loc}}^k(L_i).$$

Consequently, the proof is finished.

9. Proof of Proposition 3.11

By construction, the triangulated category $\text{Mot}_{\text{loc}}^k(k)$ comes equipped with an action of the homotopy category of spectra (see [14, §A.3]):

$$\text{Spt} \times \text{Mot}_{\text{loc}}^k(k) \to \text{Mot}_{\text{loc}}^k(k) \quad (S, NM) \mapsto S \otimes NM.$$

Consider the distinguished triangle of spectra $S \xrightarrow{nu} S \to S/n \to \Sigma S$, where $S$ stands for the sphere spectrum. Since $S/n \otimes U_{\text{loc}}^k(A)$ identifies with the mod-$n$ Moore object of $U_{\text{loc}}^k(A)$ and we have an isomorphism $S/n \simeq S/1^r \oplus \cdots \oplus S/1^r$ in $\text{Spt}$, we then conclude from Example 3.9 (with $E = U_{\text{loc}}^k$) that

$$U_{\text{loc}}^k(A \otimes L_n) \simeq U_{\text{loc}}^k(A \otimes L_n') \oplus \cdots \oplus U_{\text{loc}}^k(A \otimes L_n').$$

The proof follows now automatically from the fact that $U_{\text{loc}}^k$ is the initial $A^\wedge$-homotopy invariant; see [14, Thm. 8.25].

Acknowledgments: The author is very grateful to Guillermo Cortiñas for sharing his views on Leavitt path algebras and their algebraic $K$-theory, for useful comments on a previous version of this note, and for mentioning the reference [1].

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