## Intrinsic and emergent anomalies at deconfined critical points

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Intrinsic and emergent anomalies at deconfined critical points

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It is well known that theorems of Lieb-Schultz-Mattis type prohibit the existence of a trivial symmetric gapped ground state in certain systems possessing a combination of internal and lattice symmetries. In the continuum description of such systems, the Lieb-Schultz-Mattis theorem is manifested in the form of a quantum anomaly afflicting the symmetry. We demonstrate this phenomenon in the context of the deconfined critical point between a Neel state and a valence bond solid in an \( S = 1/2 \) square lattice antiferromagnet and compare it to the case of \( S = 1/2 \) honeycomb lattice where no anomaly is present. We also point out that new anomalies, unrelated to the microscopic Lieb-Schultz-Mattis theorem, can emerge, prohibiting the existence of a trivial gapped state in the immediate vicinity of critical points or phases. For instance, no translationally invariant weak perturbation of the \( S = 1/2 \) gapless spin chain can open up a trivial gap even if the spin-rotation symmetry is explicitly broken.

The same result holds for the \( S = 1/2 \) deconfined critical point on a square lattice.

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I. INTRODUCTION

The Lieb-Schultz-Mattis (LSM) theorem [1] and its generalization to higher dimensions [2,3] state that an insulator with half-odd-integer spin per unit cell cannot have a trivial gapped ground state: In \( 1+1D \), the ground state must either break the translational symmetry or be gapless, while in higher dimensions the system may also spontaneously break the \( SO(3) \), spin-rotation symmetry or support topological order. In recent years, this result has been generalized to a variety of cases where one relies on lattice symmetries other than translation—e.g., rotation, reflection, or glide—in combination with \( SO(3) \), or replaces \( SO(3) \) by time-reversal symmetry, to rule out a trivial gap [4–9]. Furthermore, it was noted that the impossibility of a trivial gap is very reminiscent of the situation occurring on the boundary of a topological insulator, or a more general symmetry-protected topological (SPT) phase. In fact, one may view a system with \( S = 1/2 \) per unit cell as a boundary of a crystalline SPT phase protected by a combination of translational symmetry and \( SO(3) \) [10]. Such a crystalline SPT can be constructed as an array of \( 1+1D \) Haldane chains; then the boundary is an array of “dangling” spin-1/2’s. As we will see, the higher dimensional bulk is a useful conceptual tool, even in cases when it is physically absent.

For SPT phases protected just by internal symmetry, the relationship between the bulk topological invariant and the nontriviality of the surface is very well understood—the boundary realizes the symmetry in a nononsite manner. If one attempts to gauge the symmetry in the boundary theory, one runs into an inconsistency—an anomaly. This anomaly is, however, cured by the bulk of the system. This means that every surface phase, no matter whether it is symmetry broken, gapless, or topologically ordered, must realize the same anomaly which matches the bulk—a property that must be implemented by the low-energy continuum theory describing each surface phase. What about the the bulk-boundary relationship for a crystalline SPT protected by a combination of lattice and internal symmetries or, equivalently, how do LSM constraints enter in the low-energy continuum theory? Here, we discuss two examples: (i) the gapless \( S = 1/2 \) spin chain in \( 1+1D \) and (ii) the deconfined quantum critical point (QCP) in \( 2+1D \) between an \( S = 1/2 \) Neel state and a valence bond solid (VBS) on square and honeycomb lattices [11,12]. In these examples, we focus on the following symmetries: \( SO(3) \), translations, and (in \( 2+1D \)) lattice rotations. We find that the LSM-like anomaly may be determined by treating the lattice symmetries in the low-energy theory as internal symmetries. In the case of rotations, this is done by combining the microscopic rotation symmetry with the emergent Lorentz symmetry of the continuum field theory. In particular, we find that for the \( S = 1/2 \) square lattice the combination of \( SO(3) \), and translations is anomalous and also the combination of \( SO(3) \), and 180-deg rotations is anomalous. This is in complete agreement with LSM-like theorems [5]. On the other hand, on the honeycomb lattice, we find no anomalies for the symmetries listed above. Again, this is consistent since a trivial symmetric gapped state on the honeycomb lattice has been recently constructed [13,14]. The treatment of lattice symmetries as internal symmetries for the purpose of anomaly computation is consistent with Ref. [15], which argues that the classification of crystalline SPTs with a symmetry group \( G \) comprising both lattice and internal symmetries is identical to the classification of SPTs with a purely internal symmetry group \( G \) (see also Ref. [16]). It is also consistent with the results of Ref. [10] obtained in the context of topologically ordered \( 2+1D \) phases with crystalline symmetries.

In addition to the anomalies mandated by LSM-like theorems, we find that new anomalies can emerge in the neighborhood of critical points and phases. This occurs when the microscopic symmetry group \( G \) does not act on the gapless degrees of freedom in the critical theory in a faithful manner: \( G \) may act as \( G/H \), where \( H \) is a normal subgroup. There
are cases when $G/H$ has an anomaly even though $G$ itself does not. Then no $G$-symmetric infinitesimal perturbation of the critical theory can open up a trivial gap. Physically, there are not enough degrees of freedom in the critical theory to drive the system into a trivial phase. However, if we perturb the system strongly, states transforming nontrivially under $H$ may eventually come down in energy and a trivial gapped ground state may be achieved. An example of this is provided by the 1+1D $S = 1/2$ chain. Here the gapless excitations sit at points $k = 0$ and $k = \pi$ in the Brillouin zone. Therefore, the translational symmetry $Z$ acts as $Z_2$ in the continuum theory. It has long been known that this $Z_2$ symmetry is anomalous [18,19]. What this means, however, is that no weak perturbation can gap out the deconfined critical point in an $S = 1/2$ chain where the microscopic SO(3) symmetry is anomalous. Thus, again, no weak perturbation can drive the system into a trivial phase. However, if we perturb the system into an Ising antiferromagnet, $(S^z_i)^2 \sim (-1)^i$, which spontaneously breaks the translation symmetry (the $S^z$ spin-rotation symmetry and time reversal can be further broken with a small uniform Zeeman field). Other nearby gapped states, such as the VBS, also break translations. Of course, if one applies a sufficiently strong Zeeman field, one completely polarizes the chain, consistent with the fact that there is no intrinsic LSM-like anomaly for translational symmetry alone. This, however, requires a critical strength of the Zeeman field and does not occur in the immediate vicinity of the gapless state.

We find that similar new anomalies emerge at the deconfined critical point in an $S = 1/2$ square lattice magnet. Here, the translational symmetry $Z_2 \times Z_2$ acts in a $Z_2 \times Z_2$ manner on the gapless degrees of freedom at the QCP. Furthermore, we find that this $Z_2 \times Z_2$ symmetry is anomalous. Thus, again, no weak perturbation can drive the system into a translationally invariant gapped phase (even if it breaks the SO(3) symmetry). Another emergent anomaly is present for the combination of diagonal translations $T_x, T_y$, and SO(3)$_r$. While one might have naively thought that by staggering the bond strengths as in Fig. 1 one can immediately trivially gap out the deconfined critical point, this is not the case; a finite strength of such staggering is needed for a trivial gap to open. In contrast, we find no emergent anomalies for the combination of translations, rotations, and SO(3)$_r$ for an $S = 1/2$ honeycomb lattice (and as already mentioned, no intrinsic LSM anomalies). This, in principle, opens the possibility that in the CP$^4$ field theory perturbed by triple monopoles governing the deconfined QCP on the honeycomb lattice an intermediate trivial symmetric phase may exist between the Neel state and the VBS state. However, current studies of lattice models on the honeycomb lattice suggest either a continuous direct transition or a weakly first-order transition [20–23]. References [20,23] also argue based on the anisotropy of VBS histograms that the triple monopole operator is nearly marginal at the transition; it may be that the system sizes probed in Refs. [20–23] were not large enough to study the true IR effects of this operator. If this operator is slightly relevant, it is possible that it eventually drives the system to a trivial gapped state, opening up a narrow region of intermediate gapped phase near the putative QCP. Of course, a less exciting scenario where this operator drives a first-order transition or leads to coexistence of the Neel and VBS phases is also possible. In any case, these findings motivate further numerical study of the Neel-VBS transition on the honeycomb lattice.

We would like to point out that the situation of emergent anomalies described above should not be confused with the case when the microscopic symmetry $G$ is dynamically enlarged in the critical state to a larger group $G'$, i.e., when perturbations breaking $G'$ to $G$ are irrelevant in the renormalization group (RG) sense. In such cases, the enlarged symmetry may also be anomalous. An example is provided by the 1+1D $S = 1/2$ chain where the microscopic SO(3)$_r \times Z_2^a$ symmetry is dynamically enlarged to SO(4). Similarly there is evidence that the SO(3)$_r \times [Z_4^{\text{rot}} \times Z_2^a]$ symmetry of the $S = 1/2$ square lattice deconfined QCP is dynamically enlarged to an SO(5) symmetry (here $Z_4^{\text{rot}}$ stands for 90-deg rotations). The anomaly associated with this SO(5) symmetry has been determined in Ref. [24] and may be used as a starting point to derive the intrinsic and/or emergent anomalies associated with the physical symmetries studied here [25]. However, it is not necessary to assume this emergent SO(5) symmetry or to study its consequences.

In addition to the above anomaly analysis, we discuss the dynamics of the Neel-VBS transition of an $S = 1/2$ rectangular lattice and $S = 1$ square lattice. Some time ago, it was suggested that the Neel-VBS transition of an $S = 1/2$ rectangular lattice may be continuous and may possess an emergent O(4) symmetry [26]. However, numerical simulations of Ref. [22] have found a first-order transition on a
rectangular lattice, so this proposal was abandoned. Here, we would like to revisit this proposal in view of recent theoretical [24,27,28] and numerical progress [29,30]. We suggest that this continuous transition may be accessed by starting with the $S = 1/2$ square lattice Neel-VBS transition and introducing a weak rectangular anisotropy (even weaker than considered in Ref. [22]). We also suggest that the same $O(4)$ symmetric CFT governs the Neel-VBS transition of the $S = 1$ square lattice.

We would like to note that some of our results have been recently independently obtained by other groups. Reference [31] discusses LSM-like anomalies at deconfined critical points using less formal methods. Reference [32] discusses LSM-like anomalies in a number of gapless systems, including the

\[ S \]

This continuous transition may be accessed by starting with the Ref. [22]). We also suggest that the same $O(4)$ symmetric CFT governed the Neel-VBS transition of the $S = 2$ square lattice. Reference [31] provides a field-theoretic derivation of these anomalies here and provides a physical interpretation. While this paper was being completed, Ref. [33] appeared, which also discusses the implication of anomalies of 2+1D CP$^1$ model for lattice antiferromagnets.

This paper is organized as follows. In Sec. II, we discuss the anomalies of the 1+1D $S = 1/2$ chain: In Sec. II A, we use the Abelian bosonization description of the chain, and in Sec. II B, we use the CP$^1$ description. The latter allows for a more complete formal analysis where the SO(3), and translational symmetries are gauged. Section III is devoted to the Neel-VBS deconfined critical point in 2+1D: The case of the $S = 1/2$ square lattice is discussed in Sec. III A, and the case of the $S = 1/2$ honeycomb lattice in is discussed in Sec. III B. A physical picture of the mixed anomaly involving the lattice rotational symmetry and SO(3), is given in Sec. III C. Here we clarify the old arguments of Ref. [34] regarding $S = 1/2$ moment in the VBS vortex core. Section III D discusses some issues involving the breaking of continuous Lorenz (rotation) symmetry of the low-energy field theory description. Section III D also discusses anomalies of the $S = 1$ deconfined critical point on the square lattice. Section IV has a slightly different focus: It is devoted to the possibility that $S = 1/2$ rectangular lattice and $S = 1$ square lattice Neel-VBS transitions might be continuous. Concluding remarks are presented in Sec. V. We also point out Appendixes A and B, which give a careful definition of the CP$^1$ model in 1+1D and 2+1D as a boundary of a higher dimensional SPT phase. Finally, Appendix C discusses VBS vortices in the context of the nearest neighbor dimer model, supplementing the discussion in Sec. III C.

II. $S = 1/2$ SPIN CHAIN IN 1+1D

We begin with the example of the $S = 1/2$ antiferromagnetic chain in 1+1D. While anomalies in this example have been studied at length before [19,32], our interpretation of the “emergent anomaly” and its consequences is somewhat different from that in the literature.

A. Bosonized description

We begin with the bosonized description of the chain (we work in real time here),

\[ L = \frac{1}{2\pi} \partial_\theta \partial_\varphi - \frac{1}{4\pi} [ (\partial_\varphi \varphi)^2 + (\partial_\theta \varphi)^2 ] \tag{1} \]

The microscopic operators are expressed as $S^i_j \sim A(-1)^je^{i\theta}$, $S^c \sim A(-1)^j \sin \varphi + \frac{\partial_\varphi}{\partial_\varphi} \varphi$, and $V \sim \cos \varphi$, where $V_j \sim (-1)^j S^c_j \cdot S^c_{j+1}$ is the VBS order parameter. Here, we use Abelian bosonization, so only the SO(2), subgroup of SO(3), symmetry, corresponding to spin rotations around the $\vec{z}$ axis, is manifest. (Below, we will also discuss the CP$^1$ formulation where the full SO(3), symmetry is manifest.) The SO(2), symmetry acts as

\[ SO(2)_\theta : \theta \rightarrow \theta + \alpha, \quad \varphi \rightarrow \varphi. \tag{2} \]

The translational symmetry acts as

\[ T_z : \theta \rightarrow \theta + \pi, \quad \varphi \rightarrow \varphi + \pi. \tag{3} \]

Note that the microscopic Z translation symmetry acts in a $Z_2$ manner in the low-energy theory, so we will sometimes refer to $T_z$ as $Z_2^z$.

Let us first discuss the manifestation of the LSM anomaly, which involves the combination of SO(2)$_\theta$ and translation symmetry $T_z$. First, consider a closed chain with an odd number of sites. Increasing the number of sites in the chain by one is tantamount to inserting a flux of the $T_z$ symmetry through the cycle of the chain. Using the action of $T_z$ (3), a chain with an odd number of sites corresponds to twisted boundary conditions, $\theta(x + L) = \theta(x) + 2\pi n (n + 1)/2$, $\varphi(x + L) = \varphi(x) + 2\pi (m + 1)/2$. Now, the total SO(2)$_\theta$ charge of the chain is $S^c = \oint dx \partial_\varphi \varphi$. So we see that the chain with an odd number of sites carries $S^c$, which is half-odd integer. Of course, this is precisely the correct physics for an $S = 1/2$ chain. However, if the microscopic symmetry was really SO(3)$_\theta$ [and its subgroup SO(2)$_\theta$], then only integer values of $S^c$ would be allowed, so our theory is anomalous.

Another (more standard) identification of the LSM anomaly proceed via threading flux of SO(2)$_\theta$ through the chain. When flux $\alpha$ of SO(2)$_\theta$ is threaded through the chain, the fields satisfy twisted boundary conditions, $\theta(x + L) = \theta(x) + 2\pi n + \alpha$, $\varphi(x + L) = \varphi(x) + 2\pi m$. Thus, as we insert flux $2\pi n$ of SO(2)$_\theta$, the winding number of $\theta$ increases by $2\pi n$, while the winding number of $\varphi$ remains unchanged. Now, from the action of translational symmetry (3), we can identify the physical momentum

\[ P = \frac{1}{2} \int_0^L dx (\partial_\varphi \varphi - \partial_\theta \varphi). \tag{4} \]

So, after threading flux $2\pi$, the momentum $P$ changes by $\pi$. Of course, this is the result that we expect microscopically from the $S = 1/2$ chain [1]. However, if we treated SO(2), and translation as onsite internal symmetries, then the momentum $P$ cannot change after flux threading. So this again is a signature of the intrinsic LSM anomaly.

Next, we proceed to the emergent anomaly, which is associated with the translation symmetry and does not require spin rotations or time reversal. We observe that the action of translational symmetry (3) coincides precisely with the action of $Z_2$ symmetry on the edge of a 1+1D $Z_2$-protected SPT [35–37]. The edge of a $Z_2$-protected SPT cannot be gapped out without breaking $Z_2$. Now, any translationally invariant weak perturbation that we add to the theory must respect $Z^z_2$, so such perturbations cannot open a symmetric gap.
It is instructive to understand how the argument above breaks down when the perturbation added is not weak. Indeed, we know that, for instance, a sufficiently large uniform Zeeman field can fully polarize the spin chain. A weak Zeeman field corresponds to a perturbation,
\[ \delta L = \frac{\delta}{2\pi} \partial_t \phi \]
with \( \delta \sim B_z \). This perturbation can be eliminated by redefining \( \phi(x) = \phi(x) - \delta x \). Under translations by a lattice spacing \( a \), \( T_a : \phi(x) \to \phi(x + a) + \pi + \delta a \). Thus, translations no longer act on \( \phi \) in a \( Z_2 \) manner. As we keep increasing \( B_z \), eventually we reach a point where \( T_a : \phi(x) \to \phi(x + a) + 2\pi \), so a perturbation
\[ \delta L \sim \cos \phi \]
becomes allowed and can open a gap; this corresponds to a fully polarized chain. Physically, the momenta at which gapless degrees of freedom are present evolve as \( B_z \) is tuned until momentum-preserving backscattering terms are allowed. If we express (6) in terms of the original field \( \phi \),
\[ \delta S \sim \int dxdt \cos(\phi + \pi x / a) \]
Clearly, close to the starting theory \( \delta = 0 \), this term vanishes since the momenta carried by the continuum field \( e^{i\phi} \) are assumed to be small (much smaller than \( \pi \)).

The example considered here is quite general. Any continuum field theory where gapless degrees of freedom sit at isolated points in momentum space will have an emergent continuum translational symmetry [in our example, \( \phi(x) \to \phi(x + \epsilon) \), \( \theta(x) \to \theta(x + \epsilon) \)]. For “kinematic” reasons outlined above, these continuum translations with microscopic translations, we get a purely internal symmetry. If the underlying gapless excitations sit at commensurate points in momentum space, this internal symmetry will act as a finite group \( G \) in the field theory \( (Z_2 \) in our example). \( G \) might be an anomalous symmetry of the theory, in which case weak translation-preserving perturbations cannot open a gap.

The example with the Zeeman field also illustrates how to immediately determine whether an anomaly is intrinsic (of LSM type), i.e., whether it is stable to large perturbations away from points in momentum space, this internal symmetry will act as a finite group \( G \) in the field theory \( (Z_2 \) in our example). \( G \) might be an anomalous symmetry of the theory, in which case weak translation-preserving perturbations cannot open a gap.

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We leave the discussion of bulk-boundary correspondence for the anomalies described above to the next section.

### B. CP\(^1\) description

We saw in the previous section that the 1+1D \( S = 1/2 \) spin chain possesses anomalies associated with the SO(3), spin-rotation symmetry and translational symmetry. In this section, we discuss an interpretation of these anomalies when the chain is viewed as a surface of a 2+1D (crystalline) SPT phase. Here we describe the gapless phase of the chain using the CP\(^1\) model with a \( \theta \) term at \( \theta = \pi \),
\[ L = \left| (\partial_\mu - ia_\mu) \right|^2 + i \theta \frac{f}{2\pi}, \quad \theta = \pi. \]

Here and below, we work in Euclidean time. \( a_\mu \) is a \( u(1) \) gauge field and \( f = \epsilon_{\mu\nu}\partial_\mu a_\nu \) is the associated field strength. \( z_\alpha, \alpha = 1, 2 \), is a complex scalar transforming in the projective \( S = 1/2 \) representation of spin-rotation group SO(3). The Neel order parameter is identified with \( \eta \sim z^* \bar{z} \), and the VBS order parameter is identified with \( V \sim f \). Under translations by one lattice spacing
\[ T_a : z \to ia^r z^*, \quad a \to -a, \]
so that both \( \eta \) and \( V \) are odd under \( T_a \), as necessary. Note that \( T^2 \) is a rotation by \( \pi \) in the \( u(1) \) gauge group (i.e., \( T^2 \) acts trivially on all physical observables). This means that \( T_a \) acts as a \( Z_2 \) symmetry in the field theory (7). In a recent work [25], it was shown that this \( Z_2 \) symmetry is anomalous. Moreover, the combination of \( Z_2 \times \text{SO}(3) \) is also anomalous [38]. In fact, as found in Ref. [25], one can think of (7) as living on the boundary of a 2+1D SPT with \( Z_2 \times \text{SO}(3) \), symmetry and bulk action,
\[ S_{\text{bulk}} = \pi i \int_{X_3} (x w_2 + x^3), \]
where \( X_3 \) is the bulk three-manifold, \( x \in \mathcal{H}^4(X_3, Z_2) \) is the background gauge field corresponding to \( Z_2 \) symmetry, \( w_2 \in \mathcal{H}^2(X_3, Z_2) \) is the second Stiefel-Whitney class of the background SO(3) bundle, and the product of cohomology classes is the cup product. We give a precise definition and a derivation of the \( \text{bulk + boundary theory} \) corresponding to (7) and (9) in Appendix A. Note that our definition and derivation differ somewhat from the discussion in Ref. [25].

We proceed to discuss the physical interpretation of the bulk action (9). The first term in this action,
\[ S_{1, \text{bulk}} = \pi i \int_{X_3} x w_2, \]
is precisely the intrinsic LSM anomaly for the combined SO(3), and translational symmetry. The second term,
\[ S_{2, \text{bulk}} = \pi i \int_{X_3} x^3, \]
is the emergent anomaly for the translational symmetry alone. Let us begin with the emergent anomaly: We recognize that \( S_{2, \text{bulk}} \) is precisely the bulk action of a \( Z_2 \)-protected 2+1D SPT in the presence of a background \( Z_2 \) gauge field \( x \) [36,39]. It is also immediately clear that this anomaly is not intrinsic if one remembers that the microscopic translation symmetry group is \( Z \) rather than \( Z_2 \). The difference between a \( Z \) gauge field and a \( Z_2 \) gauge field is that for a \( Z \) gauge field \( x \) (without vison defects) \( dx = 0 \), while for a \( Z_2 \) gauge field \( dx = 0 \) (mod 2) the condition for a \( Z_2 \) gauge field is more restrictive.\(^3\) Now, for a \( Z_2 \) gauge field \( x^2 = \frac{4\pi}{\pi} = 0 \) (mod 2) and \( S_{2, \text{bulk}} \) vanishes—no

\(^3\)Here and below, \( d \) denotes the coboundary operation on cochains.
anomaly for translational symmetry alone is present. On the other hand, if we take x to be a Z_2 gauge field, then S_{1, \text{bulk}} is generally nonvanishing. As discussed in Sec. II A, no translationally invariant weak perturbation of the critical chain breaks the internal Z_2^4 symmetry; therefore, to analyze the stability of the chain to weak perturbation one is allowed to couple it to a Z_2 gauge field, whereby one discovers an anomaly. To analyze stability to strong perturbations, one must, however, treat x as a Z gauge field; then no anomaly is found and a gapped phase exists.

Next, we proceed to show that S_{1, \text{bulk}} is the LSM anomaly. Here we may think of the bulk physically as a crystalline SPT obtained as a stack of Haldane chains; the surface is then precisely an S = 1/2 chain. S_{1, \text{bulk}} is the “response theory” of such a crystalline SPT. Let each Haldane chain stretch along the y direction and the chains be stacked along x. Let the length along x be L_x and the length along y be L_y. For a moment, let both x and y be periodic, so that the space-time manifold is S^1_y \times S^1_y \times S^1_x. As noted in Sec. II A, increasing L_x \rightarrow \infty corresponds precisely to threading flux of the T_c gauge field along the x cycle. When \int_{S^1_x} x = 1 (and x vanishes along the other cycles), S_{1, \text{bulk}} = \pi i \int_{S^1_y \times S^1_y} w_2 \pi which is precisely the response of the Haldane phase. Thus, as we increase L_x by one, the system compactified along the x direction goes from being a trivial SO(3)_{\text{SPT}} to the Haldane SO(3)_{\text{SPT}}. But that is precisely a property of a stack of Haldane chains!

Another important manifestation of the LSM anomaly is obtained by thinking about the magnetic flux of SO(3) in the 2+1D bulk. Let us compactify the bulk on Y_2 \times S^1_y, where we think of Y_2 as a spatial manifold. Place flux of SO(3) through Y_2 (for instance, one can take 2\pi flux of the SO(2)_c subgroup). The SO(3)_c flux is defined only mod 2 and is measured precisely by \int_{Y_2} w_2. Therefore, in this geometry, S_{1, \text{bulk}} = \pi i \int_{S^1_y \times S^1_y} x. This means that an SO(3)_c flux carries momentum \pi under x translations. This is precisely right. Indeed, consider the bulk with a boundary. We may take the spatial bulk manifold to be a disk, so that the spatial boundary is a circle. Imagine moving the SO(3) flux—e.g., 2\pi flux in the SO(2)_c subgroup—from the trivial vacuum outside to inside the bulk. Outside the flux carries no momentum, but inside it carries momentum \pi. Therefore, in the process, there must be momentum \pi left on the boundary. That is precisely right. Indeed, from the boundary viewpoint, this process corresponds to threading SO(2)_c flux 2\pi through the chain. We know microscopically that this changes the momentum by \pi.

Note that the gauge fields x that we considered in our discussion of S_{1, \text{bulk}} satisfied dx = 0; i.e., the anomaly is already present when translations are treated as a Z group. Again, this is what we expect for an intrinsic LSM anomaly.

III. DECONFINED CRITICALITY IN 2+1D

In this section, we discuss the Neel to VBS transition in 2+1D on square and honeycomb lattices. The underlying field theory believed to control this transition is the 2+1D CP^1 model,

\[ L = |(\partial_{\mu} - ia_{\mu})z_{\alpha}|^2. \]

where we use the same notation as in 1+1D; see Sec. II B. As written, the action (12) contains no monopole operators. Depending on the lattice and the value of spin S one is considering, the action (12) admits various perturbations (particularly monopole operators) that we will discuss below. The continuum theory (12) has three internal global symmetries: (i) SO(3)_c rotations under which z_{\alpha} transforms in the spinor representation. (ii) U(1)_\phi flux symmetry under which the 2\pi flux monopole of a that we denote by an operator V transforms as

\[ U(1)_\phi : V \rightarrow e^{i\alpha} V. \]

We denote the operator implementing a U(1)_\phi rotation by an angle \alpha as U^\alpha. (iii) A unitary “charge conjugation” symmetry:

\[ C : z \rightarrow i\sigma^y z^*, \quad a \rightarrow -a, \quad V \rightarrow V^\dagger. \]

Note that C^2 = 1 on gauge-invariant degrees of freedom; i.e., C acts as a Z_2 symmetry. Combining C with U(1)_\phi, we get a group O(2)_\phi; therefore, the full internal symmetry group of (12) is O(2)_\phi \times SO(3)_c. As we will discuss in the case of each lattice, the microscopic symmetries are implemented in the continuum theory as a subgroup of this symmetry group (in the case of rotations, combined with continuum symmetries).

Before we specialize in particular physical symmetries, it is useful to compute the anomaly associated with the full continuum symmetry O(2)_\phi \times SO(3)_c. This was done in Ref. [25] (we give a slightly different derivation in Appendix B). It was found that (12) is the boundary of a 3+1D O(2)_\phi \times SO(3)_c protected SPT with the following bulk response:

\[ S_{\text{bulk}} = \pi i \int_{X_4} w_2[\xi_\phi] \cup (w_2[\xi_1] + w_1[\xi_\phi]). \]

Here, X_4 is the bulk four-manifold, \xi_1 is the SO(3)_c bundle, \xi_\phi is the O(2)_\phi bundle, and as before w_{1/2} denote the first and second Stiefel-Whitney classes. In particular, w_{1/2}[\xi_\phi] \in H^1(X_4, Z_2) is just the Z_2 gauge field corresponding to the charge-conjugation symmetry.

While this is not important for the anomaly analysis, let us say a few words about the order of transition in the continuum “noncompact” theory (12). Numerical simulations suggest that it is either continuous or very weakly first order. Further, if the latter situation is the case, the weakly first-order behavior is quasiuniversal; the same critical exponents (and small drifts of these exponents with system size) are seen in microscopically different models. Reference [40] proposed that this quasiuniversal behavior may be controlled by a nearby nonunitary critical point (or equivalently a unitary critical point appears if the parameters such as spatial dimension and/or number of flavors are varied slightly). Our discussion below can also be adapted to the quasiuniversal first-order scenario: In this case, when we talk of relevancy or irrelevancy of a certain operator in (12), we define it with respect to this nonunitary critical point and/or nearby unitary critical point.

We now specialize in the particular lattices.
A. $S = 1/2$ square lattice

The lattice symmetries we focus on are translations $T_x$, $T_y$, and $Z_4$ rotations about a site. These act in the following way. The $Z_4$ rotation $R_{\pi/2}$ is just a $\pi/2$ rotation in the $U(1)_{\Phi}$ group $U(1)/\Phi_1$ (together with a $\pi/2$ emergent continuum rotation), i.e.,

$$R_{\pi/2} : V(x) \rightarrow i V(R_{-1/2}^{-1}x).$$

(16)

The translations act $T_x = C$, $T_y = U(1)_{\Phi}C$, i.e. (apart from action on $z, a$),

$$T_x : V \rightarrow V^\dagger, \quad T_y : V \rightarrow -V^\dagger.$$  

(17)

From these transformations, we can identify $V = V_x + i V_y$ with the VBS order parameter (see Fig. 2). Further, we see that $T_x$, $T_y$, and $Z_4^{rot}$ act in the field theory as a $D_4$ subgroup of $O(2)_{\Phi}$, and the anomaly can be obtained by replacing the $O(2)_{\Phi}$ bundle $\xi_{\Phi}$ in (15) by the $D_4$ bundle. Let us focus on two subgroups of this $D_4$.

(1) Imagine restricting the lattice symmetry to $Z_4$ rotations. Then we are interested in the $Z_4$ subgroup of $U(1)_{\Phi}$, so there are no $C$ gauge fields and $w_1[\xi_{\Phi}] = 0$. Further, for a $U(1)_{\Phi}$ gauge field, $w_2[\xi_{\Phi}] = \frac{F}{2\pi}$ (mod 2), where $F \in H^2(X, \mathbb{Z})$ is the Chern class of the $U(1)_{\Phi}$ bundle. In our case, if we denote the $Z_4$ gauge field by $\gamma \in H^1(X_4, \mathbb{Z}_4)$, $\frac{F}{2\pi} \in \mathbb{Z}$. The anomaly (15) then becomes

$$S_{\text{bulk}} = \pi i \int \frac{dy}{4} \cup w_z^i.$$  

(18)

This is a mixed anomaly involving $Z_4$ rotations and $SO(3)_s$ symmetry. It is generally nonvanishing. Indeed, even if we restrict discussion to only 180-deg rotations, i.e., take $\gamma = 2\tilde{\gamma}$ with $\tilde{\gamma} \in H^1(X_4, \mathbb{Z}_2)$, the action (18) is still nontrivial:

$$S_{\text{bulk}} = \pi i \int \frac{d\tilde{\gamma}}{2} \cup w_z^i = \pi i \int \tilde{\gamma}^2 w_z^i.$$  

(19)

The presence of the anomalies (18) and (19) is in exact accord with an LSM-like theorem, stating that a trivial gap is impossible in a system with spin $S = 1/2$ located at a 180-deg rotation center [5]. Thus, these anomalies are intrinsic anomalies.

(2) Imagine restricting the lattice symmetry to translations $T_x$, $T_y$. In the field theory, these act as a $Z_2^x \times Z_2^y$ subgroup of the $O(2)_{\Phi}$ group, corresponding to $O(2)$ transformations $\text{diag}(1, -1)$ and $\text{diag}(-1, 1)$. By denoting the $Z_2^x$ and $Z_2^y$
gauge fields as $x$ and $y$, we have $\xi_\Phi = \xi_x \oplus \xi_y$; i.e., $\xi_\Phi$ is a direct sum of line bundles $\xi_x$ and $\xi_y$. Using the Whitney formula, $w^i_1[\xi_\Phi] = w^i_1[\xi_x] + w^i_1[\xi_y] = x + y$, $w^2_3[\xi_\Phi] = w^2_1[\xi_x] w^1_1[\xi_y] = xy$. So the anomaly reduces to

$$S_{\text{bulk}} = \pi i \int_{x_1} (xy w^1_1 + x^3 y + xy^3).$$

The first term $xy w^1_1$ corresponds to the mixed LSM anomaly for the SO(3)$_x$ symmetry and translations. The last two terms comprise an emergent anomaly for the translation symmetry.

Let us first discuss the LSM anomaly. Again, we can think of the $S = 1/2$ square lattice as a boundary of a stack of Haldane chains. We let the chains run along the $z$ direction and be stacked in a square lattice along $x$ and $y$. Let the $x$, $y$, and $z$ directions be periodic. Increasing $L_y$ by 1 corresponds to threading $T_y$ flux along the $y$ cycle. Then, with such a $y$ flux, the bulk action becomes $S_{\text{bulk}} = \pi i \int_{S^1} x y w^1_1$ is nontrivial even if $x$ and $y$ are Z gauge fields rather than $Z_2$ gauge fields, as befits an intrinsic LSM anomaly.

We next discuss the emergent anomaly $S_{\text{bulk}} = \pi i \int_{S^1\times S^1\times S^1} x y w^1_1$. Again, if $x$ and $y$ are Z gauge fields, then $x^2 = y^2 = 0$ (mod 2) so $S_{\text{bulk}}$ vanishes. However, if $x$ and $y$ are $Z_2$ gauge fields, then $S_{\text{bulk}}$ is nontrivial; in fact, it is precisely the response of a $Z_2 \times Z_2$ protected SPT in 3+1D. Since translations act in a 1D array of Haldane chains. This is the correct physics: Fixing $L_y$ to be large by finite, we can view our bulk as a 2+1D phase protected by $T_x$ and SO(3)$_y$. At each $x$ "site," there is a Haldane phase if $L_y$ is odd and an SO(3)$_y$ trivial phase if $L_y$ is even. Again, we note that the action $S_{\text{bulk}} = \pi i \int_{x_1} x y w^1_1$ is nontrivial even if $x$ and $y$ are Z gauge fields, rather than $Z_2$ gauge fields, as befits an intrinsic LSM anomaly.

Another emergent anomaly is present for the combination of diagonal translations $T_x T_y$ and SO(3)$_x$. In the continuum theory, $T_x T_y$ acts in the same way as 180-deg rotations, so we indeed expect such a mixed anomaly. If only the $Z_2$ symmetry corresponding to $T_x T_y$ is gauged, we have $x = y$. Then

$$S_{\text{bulk}} = \pi i \int_{x_1} x^2 w^2_2,$$

which is again generally nontrivial for $x$, a $Z_2$ gauge field, but vanishes for $x$, a $Z$ gauge field. From a lattice perspective,
the honeycomb lattice (besides the perturbation $|z_a|^2$ that tunes between the two phases). If this perturbation is irrelevant, the transition is described by the “noncompact” theory (12) with an emergent $O(2)_b$ symmetry, whose anomaly prohibits a trivial gap. On the other hand, if the perturbation is relevant, then the symmetry of the low-energy theory is really only $D_3$. Since this symmetry is not anomalous, it is possible that a region of trivial gapped symmetric phase opens up between the Neel and VBS phases on the honeycomb lattice.\footnote{Strictly speaking, we also need to demonstrate that when the reflection and time-reversal symmetries are added to symmetries considered above, no emergent anomalies are present. We leave this to future work.}

Numerically, the Neel-VBS transition on the honeycomb lattice appears continuous or very weakly first order [20–23]. Further, on finite but large systems the same critical exponents are observed as on the square lattice. This suggests that the same “noncompact” theory (12) governs the transition on the honeycomb lattice as on the square lattice. However, while on the square lattice nearly $U(1)$ symmetric histograms of the VBS order parameter are seen, which has been interpreted as evidence for the irrelevancy of the quadruple monopole operator $V^4$, on the honeycomb lattice a clear $Z_3$ anisotropy of the histogram is observed. Thus, it may be the case that the $V^3$ operator is close to marginality. If it is slightly relevant, it may be that system sizes where its effects start to play a role have not been reached in Refs. [20–23]. In this light, it would be interesting to numerically study the Neel-VBS transition for the $S = 1/2$ honeycomb lattice in more detail. As already pointed out in Ref. [23], it would be particularly interesting to look for new microscopic models realizing this transition with the hope that some of them have larger values of $V^3$ perturbation than those studied previously.

C. Vortices and domain walls

In this section, we give a more physical picture of the mixed anomaly between lattice rotational symmetry and SO(3), clarifying the previous discussion in Ref. [34].

It has long been appreciated that the essential feature of the Neel-VBS transition on the square lattice is that VBS vortices carry spin $S = 1/2$ [34]. At the field-theory level, this is seen as follows [24]. Imagine first that no monopoles of $a$ are present in the action, so that the $Z^4_{rot}$ symmetry is dynamically enlarged to $U(1)_V = U(1)_a$. To nucleate a vortex of $U(1)_V$, one couples the system to a background $U(1)_a$ gauge field $A$,

$$L = |(\partial_\mu - ia_\mu z_a)|^2 + \frac{i}{2\pi} A \wedge da,$$

(24)

and considers a configuration with flux $2\pi$ of $A$. In order for this configuration to carry no $a$ charge (i.e., be gauge invariant), we must additionally nucleate a $z_a$ particle, so the vortex carries $S = 1/2$. This matches the bulk anomaly (15). Indeed, if we compactify the bulk theory (15) on $S^2 \times Y_2$ with flux $2\pi$ of $U(1)_b$ through $S^2$, then (15) reduces to $S = \pi i \int_{Y_2} w_2^3$, the partition function of a Haldane chain. Considering $Y_2$ to be open, we see that a monopole of $U(1)_b$ is just like an end of a Haldane chain; i.e., it carries $S = 1/2$. When a monopole of $U(1)_b$ sits in the 3+1D bulk, there is flux $2\pi$ of $A$ emanating through the 2+1D surface, so a VBS vortex is present on the surface and carries spin $1/2$.

Now, what happens when the $U(1)_b$ symmetry is broken to $Z^4_{rot}$? If we work in the VBS phase, a VBS vortex will break up into a junction of four domain walls of $Z^4_{rot}$; see Fig. 2. This vortex still traps $S = 1/2$ as is clear from Fig. 2, top left. This is in agreement with the anomaly surviving when $U(1)_b \rightarrow Z^4_{rot}$. A crucial point is that one must consider a vortex, which is invariant under $Z^4_{rot}$ (for an alternative viewpoint appropriate for the nearest neighbor dimer model, see Appendix C). For instance, the configuration in Fig. 2, top right, has the same four domains as in Fig. 2, top left. However, it is not $Z^4_{rot}$ symmetric; one of the domain walls differs from the other three. We can think of this configuration as obtained from Fig. 2, top left, by dressing one of the domain walls with a Haldane chain. The Haldane chain contributes an extra $S = 1/2$ to the vortex, so that the total spin is an integer. If we, instead, dress all four domain walls with Haldane chains, so that the configuration is again $Z^4_{rot}$ symmetric, Fig. 2, bottom, we again have a half-odd-integer spin trapped in the vortex core.

What about the $S = 1/2$ honeycomb lattice? Here, the rotational symmetry of interest is $Z^3_{rot}$, corresponding to $2\pi/3$ rotations about a site.\footnote{This is a composition of $R^{a}_{2/3} \cdot 2\pi/3$ rotation about a plaquette center and a translation by one lattice spacing $T_1$.} In the present case, there exist $Z^3_{rot}$ symmetric $Z^3_{rot}$ vortices with both half-odd integer and integer spins; see Fig. 3. Schematically, one goes from the $S = 1/2$ vortex to an integer spin vortex by dressing each of the $Z_3$ domain walls with a Haldane chain. Indeed, in Fig. 3, bottom, there are two $S = 0$ states that the four “dangling” $S = 1/2$’s can be locked into. These two states carry $Z^4_{rot}$ quantum numbers of $e^{\pm 2\pi i/3}$. This is not a projective representation of $Z^3_{rot}$ (in fact, there are no projective representations of $Z_3$); it may be screened by local degrees of freedom to give a completely trivial vortex. This is consistent with the absence of an anomaly on a honeycomb lattice.

D. $S = 1$ square lattice and breaking of continuous rotation symmetry

So far, when discussing the anomalies we have treated the translational symmetry and rotational symmetry as internal symmetries of the theory. More formally, the low-energy theory (12) has a full emergent Poincare symmetry and we have combined elements of this Poincare symmetry with microscopic lattice symmetries to obtain purely internal symmetries. The anomalies associated with these internal symmetries allow us to place constraints on the dynamics when the Poincare symmetry is present. But what if it is broken? By comparing our anomaly computations so far with the microscopic LSM theorem, we guess that the anomaly found for the internal symmetry at the Lorentz invariant point is, in fact, the correct anomaly.

For instance, consider the case of $S = 1/2$ square lattice. One allowed perturbation in this case is the quadruple monopole operator,

$$\delta L \sim V^4 + (V^1)^4.$$

(25)
Throughout our discussion above, when we wrote $V^q$ we understood this to be a Lorentz scalar, which creates a flux $2\pi q$. Such perturbations do not break the Lorentz symmetry, although they do break $U(1)_\phi \to Z_4$. However, there also exist operators which carry quantum numbers under both the Lorentz symmetry and $U(1)_\phi$; let us denote these by $V^q_{\ell}$, where $q$ is still the $U(1)_\phi$ charge and $\ell$ is the angular momentum, such that under continuum spatial rotations $SO(2)_L$,

$$SO(2)_L: V^q_{\ell}(x) \to e^{i\ell\alpha} V^q_{\ell}(R^{-1}_\ell x)$$

(26)

(here, the subscript $L$ stands for Lorentz). Consider, for instance, the perturbation

$$\delta L \sim V^q_{\ell=1} + V^q_{\ell=-1}.$$  

(27)

While this perturbation breaks $U(1)_\phi$ and $SO(2)_L$ individually, it preserves their combination; i.e., the microscopic lattice rotation. The microscopic LSM theorem for $Z^2 \times SO(3)$, symmetry [5] tells us that such a perturbation (even if relevant) cannot open a trivial gap. Note that the perturbation (27) breaks the lattice translational symmetry. A perturbation consistent with all the symmetries of the square lattice is

$$\delta L \sim V^q_{\ell=2} + V^q_{\ell=-2} + V^{q=2}_{\ell=-2} + V^{q=-2}_{\ell=2}.$$  

(28)

Again, LSM theorem guarantees that this perturbation cannot open a trivial gap. In fact, this perturbation is very likely irrelevant: Unitarity implies that the scaling dimension of an operator with angular momentum $\ell \neq 0$ satisfies $\Delta_\ell \geq l + D - 2$, where $D$ is the space-time dimension [43], so in our case, $\Delta_{\ell=2} \geq 3$. It is unlikely that an operator other than the energy-momentum tensor exactly saturates the unitarity bound (if it does, it gives rise to a conserved $\ell = 2$ current). The numerically observed emergent $U(1)_{\text{VBS}}$ symmetry of the deconfined critical point [29,44] is also consistent with the irrelevance of (28).

With the above remarks in mind, we proceed to the case of $S = 1$ Neel-VBS transition on the square lattice (see Fig. 4).

Here the symmetries are implemented in the following way:

$$T_x = C, \quad T_y = C, \quad R_{\pi/2} = U^{\phi}_{\pi/2}, \quad \text{i.e.,}$$

$$T_x: V \to V^\dagger, \quad T_y: V \to V^\dagger,$$

$$R_{\pi/2}: V(x) \to -V(R^{-1}_{\pi/2} x).$$

(29)

Note that when combined with the spacial rotation in the Lorentz group, the 90-deg rotation symmetry acts in a $Z_2$ manner. So when treated as internal symmetries, $T_x, T_y, R_{\pi/2}$ act as a $Z_2 \times Z_2$ subgroup of $O(2)_\phi$. Since $T_x$ and $T_y$ act in the same way, let us focus on just one of them, say, $T_x$. Denote the $Z_2$ gauge field corresponding to $T_x$ as $x$, and the $Z_2$ gauge field corresponding to $R_{\pi/2}$ as $\alpha$. The $O(2)_\phi$ bundle $\xi_\phi$ is then a direct sum of two $Z_2$ bundles: $x$ and $\alpha$, so $w_1[\xi_\phi] = x$, $w_2[\xi_\phi] = \alpha(x + \alpha)$. The bulk action (15) then is

$$S_{\text{bulk}} = \pi i \int_{X_2} ((ax + \alpha^2)w_2 + ax^3),$$

(30)

which is generally nonvanishing for arbitrary $Z_2$ gauge fields $x, \alpha$. This anomaly implies that as long as we allow only Lorentz invariant (more specifically rotationally invariant) perturbations to the action (12), no trivial gap is possible. However, we know from an explicit construction that a trivial gapped state does exist for an $S = 1$ square lattice [14]. So there must be no intrinsic anomaly present. To see this, we note that the microscopic symmetry group generated by $T_x, T_y$, and $R_{\pi/2}$ is actually $(Z^2 \times Z^2) \times Z^2$. As shown in Appendix B 1, for an $O(2)_\phi$ bundle corresponding to this group, $w_2[\xi_\phi] = 0$, so $S_{\text{bulk}} = 0$ in accord with microscopics.

This leaves the question: If we allow for weak Lorentz breaking perturbations to the CP$^1$ model consistent with $S = 1$ square lattice symmetry, can a trivial gap be opened? For instance, we can envision a perturbation

$$\delta L \sim V^q_{\ell=2} + V^q_{\ell=-2} + V^{q=2}_{\ell=-2} + V^{q=-2}_{\ell=2},$$

(31)

10This question is quite formal since in a microscopic lattice model there is no way to control the strength of Lorentz breaking perturbations.

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*We thank Adam Nahum for pointing out this fact.*
which preserves both translation and rotation symmetry. Again, unitarity implies that the scaling dimension of this operator is greater or equal to 3, so it is likely irrelevant. Suppose we did not know this fact, or wish to consider the combined effect of this perturbation and other relevant perturbations. It turns out that just from anomaly considerations, we can argue that (31) cannot open a trivial gap. Indeed, $T_x$ and $R_{\pi/2}$ act in the continuum theory as a $Z_2 \times Z_4^{rot}$ symmetry. Note that since the action no longer possesses Lorentz symmetry, rotations must be treated as a $Z_4$ group rather than $Z_2$ group. On the other hand, for weak perturbations, we may still continue to treat $T_x$ as a $Z_2$ symmetry. Then $\alpha^2 w_x^2 = \frac{d\alpha}{\pi} w_x^2 = 0$ (mod 2), since $d\alpha = 0$ (mod 4). However, the other two terms in (30) are generally nontrivial,

$$S_{\text{bulk}} \to \pi i \int x_i \left( \alpha x w_x^2 + \alpha x^3 \right),$$

so a trivial gap cannot be opened.

IV. $S = 1/2$ RECTANGULAR LATTICE AND $S = 1$ SQUARE LATTICE: DYNAMICS

The present section has a slightly different emphasis from the rest of the paper. Here, we discuss a possibility that the easy-plane transition is first order. Recall that numerics suggests that the easy-plane Neel-VBS transition match with those of a different model with an explicit SO(4) symmetry. Previously, it was thought that the easy-plane transition is first order. However, recent simulations [30] suggest that when the easy-plane anisotropy $\lambda_3$ is small, the transition is actually continuous and described by an O(4) invariant CFT where the O(4) vector is $\vec{Y} = (n^x, n^y, V^x, V^y)$. The transition on the rectangular lattice is then described by the same O(4) invariant CFT with the O(4) vector $\vec{Z} = (n^x, n^y, n^z, V^z)$. (As already noted, this possibility was first raised some time ago in Ref. [26].) If we form the SO(4) traceless symmetric tensor, $Z_{ab}$, then the perturbation driving the Neel-VBS transition on the rectangular lattice is

$$\delta L \sim Z_{44},$$

which breaks the emergent O(4) symmetry to SO(3)$_v \times Z_2^{rot} \times Z_2^{s}$. This should be compared to the perturbation driving the easy-plane square lattice transition

$$\delta L \sim Y_{33} + Y_{44}.$$  

The perturbations driving the transitions in the two cases are different (albeit in the same multiplet), so the phases are also different (e.g., the Neel phase in the easy-plane case has only one Goldstone, while it has two Goldstones in the SO(3)$_v$ case). As for other perturbations on the rectangular lattice besides (38), we have, e.g., the component of a four-index traceless symmetric tensor $Z_{4444}$. This should be compared to a perturbation of the easy-plane theory $\sum_{a=1}^{4} \sum_{i=1}^{4} Y_{iaab}$, which is in the same multiplet. This perturbation must be irrelevant for the easy-plane transition to be continuous and to possess SO(4) symmetry (as numerics suggest).

So far, we have only discussed Lorentz invariant perturbations on the rectangular lattice. There are also symmetry-allowed Lorentz breaking perturbations. The most simple of these is $|D_x z|^2 - |D_z z|^2$, which however, can be eliminated by a coordinate rescaling. We assume that other Lorentz breaking perturbations are irrelevant.

For the case of $S = 1$ magnet on a square lattice, the double monopole perturbation (33) is again allowed, so we again conjecture a transition described by the same O(4) invariant CFT. Note that a set of Lorentz breaking perturbations distinct from those of a rectangular lattice are allowed here, e.g., Eq. (31). We again assume that these perturbations are irrelevant.

\[\text{References}\]

\[\text{Footnotes}\]
V. FUTURE DIRECTIONS

In this paper, we have focused on the anomalies of lattice systems associated with the combination of spin-rotation symmetry and lattice translations and rotations. It will be interesting to extend this analysis to include time-reversal and reflection symmetries. In particular, it will be interesting to see if there are any emergent anomalies associated with these symmetries in the vicinity of the deconfined QCP on the honeycomb lattice (we expect that there is no intrinsic anomaly, since a trivial symmetric gapped state on the honeycomb lattice exists). If no emergent anomaly is found, then an intermediate trivial phase whose appearance is driven by the $V^3$ operator might, indeed, be possible.

The entire anomaly analysis carried out in this paper has been performed by tuning the critical theory to a Lorentz invariant point and treating lattice symmetries as internal symmetries. While our results agree with LSM-like theorems, this procedure is still very much a conjecture. A stronger argument in favor of this conjecture (perhaps, utilizing the technology of Ref. [15]) is left to future work.

Finally, in this work we have not considered LSM-like theorems relying on (usually fractional) $U(1)$-number filling per unit cell. Additional subtleties arise in the formal treatment of this situation, so we leave it to future investigation.

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APPENDIX A: CP$^1$ MODEL IN 1 + 1D

In this appendix, we deduce the bulk action (9), which matches the anomalies of the 1+1D CP$^1$ model at $\theta = \pi$,

$$L = |(\partial_{\mu} - ia_{\mu}z_{a}|^2 + i\theta \frac{f}{2\pi}, \quad \theta = \pi.$$ (A1)

Let us begin by considering just the $Z^2_3$ symmetry and ignore $SO(3)$. Let us attempt to gauge the global $Z^2_3$ symmetry of (A1). Then the scalar $z$ sees a combination of transition functions in the $u(1)_g$ gauge group and in the $Z^2_3$ group. Since $T^2_3 = u^3_z$, overall $z$ sees transition functions in $pin(2)_g$.

Now, the immediate difficulty that one is faced with when trying to gauge $Z^2_3$ symmetry is how to define the $\theta$ term in (A1). Indeed, locally $f \rightarrow -f$ under $Z^2_3$, so as written, the $\theta$ term is not well defined. Instead, when $Z^2_3$ is gauged, we will define the theory in the following way. We think of the theory as living on the surface of a 2+1D SPT for the $Z^3_2$ symmetry. We call the bulk three-manifold $X_3$ and the surface $\partial X_3$. There is a $Z^2_3$ gauge field $x \in H^1(X_3, Z_2)$ living in the bulk and on the surface. On the surface, $x$ is replaced with the $u(1)_g$ gauge field $a$ a form a $pin(2)_g$ gauge field (note $a$ lives only on the boundary $M$, not in the bulk $X_3$). Let us call the $pin(2)_g$ gauge bundle $\xi_g$. We find a three-manifold $Y_3$ such that $\partial Y_3 = M$ and $\xi_g$ extends to $Y_3$ as a $pin(2)_g$ bundle (therefore, also extends to $Y_3$). We define the action of our theory as

$$S_{\text{bulk+bound}} = S_{\text{bound}}[M] + \pi i \int_{X_3 \cup Y_3} x^3.$$ (A2)

with

$$S_{\text{bound}}[M] = \int_M d^2x \sqrt{g} \left( \partial_a + ia_{\mu} z_{a} \right)^2 (\partial^\mu - ia^\mu) z.$$ (A3)

Note $Y_3$ is not the “physical” bulk manifold $X_3$ but rather an auxiliary manifold used to define the action. Further, observe that the “boundary” action (A3) is purely real and contains no topological terms. All the topological terms have been shifted to the second term on the right-hand side (RHS) of (A2). While it is not immediately obvious, we will shortly show that when the $Z^2_3$ gauge field on the physical space $X_3$ is absent, (A2) reduces to our original theory (A1).

In order for (A2) to be a well-defined action on a “physical” bulk $X_3$ with a boundary $M$, we have to make sure that it is independent of the manifold $Y_3$ and the particular extension of the boundary $pin(2)_g$ bundle to $Y_3$ that we have chosen. To see this, it suffices to show that for a $pin(2)_g$ gauge field on a closed manifold $Y_3$, $f_{\xi_g} x^3 = 0$ (mod 2). Indeed, if we project our $pin(2)_g$ bundle $\xi_g$ to an $O(2)$ bundle $\xi_0$, $f_{\xi_0}$, so if and only if $w_2(\xi_0) = 0$ (45). Thus, $w_2(\xi_0) = x^3 = \frac{d^3 x}{\pi^2}$. Furthermore, $w_3 = w_1 w_2 = \frac{d^3 x}{\pi^2}$, so $w_3 = 0$, i.e., $x^3 = 0$ and $f_{\xi_0} x^3 = 0$ for $Y_3$. Indeed, if $x^3 = 0$ and prior relations hold only in the sense of $Z_2$ cocycles, so it is important for $Y_3$ to be closed. In particular, we cannot just drop the $Y_3$ part of (A7); in fact, the resulting expression will not be gauge invariant.

We note that while (A2) does not depend on $Y_3$, it clearly depends on the gauge field $x$ on the “physical” three-dimensional manifold $X_3$. Crucially, the boundary $pin(2)_g$ bundle need not extend to the “physical” bulk $X_3$, so in general $f_{X_3} x^3 \neq 0$. Indeed, when $X_3$ has no boundary, (A2) reduces to (9), which is the topological response of a $Z^2_3$ protected SPT. This tells us that the surface theory has a $Z^2_3$ anomaly.

It remains to show that (A2) coincides with (A1) when the $Z^2_3$ symmetry on $X_3$ is not gauged, i.e., when $x = 0$ on $X_3$. The boundary $M$ of $X_3$ is an oriented surface with a $u(1)$ gauge field $a$. When the flux $m \equiv \int_{\partial X_3} x^3$ is not zero, we cannot extend a from $M$ to some $Y_3$ as a $u(1)$ gauge field. However, as we will now show, we can extend a from $M$ to $Y_3$ as a $pin(2)_g$ gauge field. First, it suffices to consider the case when $M$ is a two-sphere $S^2$ with flux $2 \pi$. Indeed, $M$ is always bordant to $m$ such surfaces. So specializing to $M$, a two-sphere $S^2$ with flux $2 \pi$, we must show that $f_{X_3} x^3 = 1$, so that the topological part of the action is given by $\pi i$, in accord with (A1). We take $Y_3$ to be $RP^3 \times D_3$, where $D_3$ is a three-dimensional ball. It is convenient to think of $RP^3$ as a three-dimensional ball of radius $R$ with antipodal points on the boundary identified. We obtain $Y_3$ by cutting out a ball of radius 1 centered at the origin from this realization of $RP^3$ (we take $R > 1$). The boundary $M$ of $Y_3$ is a sphere $S^2$ of radius 1. We place flux $2 \pi$ on this sphere. In polar coordinates, we choose

$$a_\varphi (r, \theta, \varphi) = \frac{1}{2} (1 - \cos \theta), \quad a_\theta = 0, \quad a_r = 0.$$ (A4)

Now, we must glue the fields at $r = R$. Clearly, we need to use the $Z^2_3$ symmetry to do so. We impose at $r = R$,

$$z(x) = e^{i(\alpha(x))} z^*(a(x)), \quad a(x) = -(\alpha^*(x)) + da(x),$$ (A5)
Indeed, the left- and right-hand sides are precisely the obstruction functions for \( \pi(2)_- \) and \( \text{SU}(2)_- \). In fact, the transition functions for \( z_a \) live in \([\pi(2)_- \times \text{SU}(2)_-]/Z_2\). Thus, the \( \pi(2)_- \) transition functions and the \( \text{SU}(2)_- \) transition functions may not individually satisfy the cocycle condition, but the combination does. If we project our \( \pi(2)_- \) bundle \( \xi_z \) to an \( \text{O}(2)_\theta \) bundle \( \xi_Y \), and \( \text{SU}(2)_\theta \) to an \( \text{SO}(3)_\theta \) bundle \( \xi_\phi \), then the resulting bundles satisfy

\[
\omega(\xi_Y) + w_2^1(\xi_\phi) = \omega(\xi_\phi).
\]

Indeed, the left- and right-hand sides are precisely the obstructions to lifting \( \xi_Y \) and \( \xi_\phi \) to \( \pi(2)_- \) and \( \text{SU}(2)_- \) bundles, respectively. Now, we extend the \([\pi(2)_- \times \text{SU}(2)_-]/Z_2\) bundle from the surface \( M \) to some \( Y \); the condition \((A6)\) continues to be satisfied on \( Y \). This also automatically extends the \( Z_2 \) gauge field \( x = w_2(\xi_\phi) \) to \( Y \). Now, we want to check if \((A2)\) is still independent of the extension to \( Y \). It suffices to compute \( \int_Y x^3 \) for \( Y \) closed. We have \( x^3 = x(\omega(\xi_Y) + \omega(\xi_\phi)) = \frac{2\omega(\xi_Y)}{2} + xw_2(\xi_\phi) \), so \( \int_Y x^3 = \int_Y xw_2(\xi_\phi) \), which generally does not vanish. However, there is an easy fix: We modify the action to

\[
S_{\text{bulk+bound}} = S_{\text{bound}}(M) + \pi i \int_{\partial Y} (\alpha^3 + xw_2(\xi_\phi)), \tag{A7}
\]

which now does not depend on the extension to \( Y \) chosen. For \( X \) closed, we recover \((9)\). The first term is a pure \( Z_2 \) anomaly, while the second term is a mixed \( Z_2 \times \text{SO}(3)_\theta \) anomaly.

**APPENDIX B: \text{CP}^1 \text{ MODEL IN 2+1D}**

In this appendix, we deduce the bulk action \((15)\), which matches the anomalies of \(2+1\)D \text{CP}^1 model,

\[
L = |D_\alpha z_a|^2 + \frac{i}{2\pi} A \wedge da + \frac{A}{2\pi} da
\]

with \( a \) being the dynamical gauge field and \( A \) being a gauge field coupling to the flux current \( \frac{1}{2\pi} db \). The symmetries of the \text{CP}^1 model we consider are \( O(2)_\theta = U(1)_\theta \times C \) and \( \text{SO}(3)_\theta \) (see Sec. III).

We denote the associated bundles with \( \xi_\theta \) and \( \xi_\phi \), \( \text{SO}(3)_\theta \), and \( C \) act on the spinons \( z \) in a projective manner (the \( U(1)_\theta \) group does not act on the spinons). Indeed, \( C^2 : z \rightarrow -z \). So, \( C \) combines with the gauge group \( u(1)_\theta \) to a group \( \pi(2)_- \). The overall transition functions seen by \( z \) live in \([\text{SU}(2)_\theta \times \pi(2)_-]/Z_2\). The transition functions of \( \text{SU}(2)_\theta \) generally will satisfy the cocycle condition only up to a factor of \(-1\), and so will the transition functions of \( \pi(2)_- \). Let us project \( \pi(2)_- \) down to an \( O(2)_\theta \) group that we call \( O(2)_\xi \), and let the associated bundle be labeled by \( \xi_{\xi} \). Then the obstruction to lifting \( O(2)_\xi \) to \( \pi(2)_- \) must be exactly equal to \( w_2(\xi_{\xi}) \). But the obstruction to lifting an \( O(n) \) bundle to a \( \pi(n)_- \) bundle is \( w_2 + w_3 \) \([45]\). So, we must have \( w_2(\xi_{\xi}) + w_3(\xi_{\xi}) = w_2(\xi_{\xi}) \). We now extend the full \( O(2)_\phi \times [\text{SU}(2)_\theta \times \pi(2)_-]/Z_2 \) bundle from our original three-manifold \( M \) to a four-manifold \( Y \), such that \( dY = M \), and define

\[
\int \frac{i}{2\pi} \int_M A \wedge da = \int_Y \frac{dA \wedge da}{2\pi} = \pi i w_2(\xi_{\xi} \otimes \xi_{\phi}), Y \tag{B2}
\]

where \( dA \) is the field strength of the \( O(2)_\phi \) bundle and \( da \) is the field strength of the \( \pi(2)_- \) bundle. Equivalently, \( 2da \) is the field strength of the \( O(2)_\phi \) bundle. We want to see if \( (B2) \) is independent of the extension to \( Y \); i.e., we want to find what values it takes for \( Y \) closed. Since \( w_2(\xi_{\phi}) = w_1(\xi_{\phi}) \), we may combine the \( O(2)_\phi \) and \( O(2)_\phi \) bundles into an \( \text{SO}(4)_\phi \) bundle \( \xi_{\phi} \otimes \xi_{\phi} \). We claim, for closed \( Y \),

\[
2\pi i \int \frac{dA}{2\pi} \wedge \frac{da}{2\pi} = \pi i w_2(\xi_{\phi} \otimes \xi_{\phi}), Y \tag{B3}
\]

Indeed, let us project \( \text{SO}(4)/Z_2 = \text{SO}(3)_\theta \times \text{SU}(2)_\theta \) \( \text{SO}(2)_\theta \) rotations by angles \( \alpha \), \( \beta \) in \( \text{O}(2)_\theta \), \( \text{O}(2)_\theta \) becomes rotations by \( \alpha - \beta \) and \( \alpha + \beta \) around (say) the \( z \) axis in \( \text{SO}(3)_\theta \) and \( \text{SO}(3)_\theta \) respectively. The reflection diag \((-1, -1)\) performed simultaneously in \( \text{O}(2)_\theta \) and \( \text{O}(2)_\theta \) becomes a simultaneous \( \pi \) rotation around \( y \) axis in \( \text{SO}(3)_\theta \) and \( \text{SO}(3)_\theta \). Therefore, the \( \text{SO}(3)_\theta \) and \( \text{SO}(3)_\theta \) connections are (locally)

\[
A_L = (A - 2a)(\begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) \quad \text{and} \quad A_R = (A + 2a)(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix})
\]

Now, for an \( \text{SO}(4)_\phi \) bundle,

\[
w_4 = \frac{1}{4}(p_1[\text{SO}(4)_\theta] - p_1[\text{SO}(3)_\phi]) \mod 2 \tag{B4}
\]

[see Ref. [24], Eq. (141)]. Here, \( p_1 \) is the Pontryagin number of an \( \text{SO}(n) \) bundle, which has an integral formula:

\[
p_1[\text{SO}(n)] = \frac{1}{2(2\pi)^2} \int_{\text{SO}(n)} F \wedge F \tag{B5}
\]

So,

\[
w_4(\xi_{\phi} \otimes \xi_{\phi}) = \frac{1}{4(2\pi)^2} \int_Y [(dA - 2da) \wedge (dA - 2da)] - (dA + 2da) \wedge (dA + 2da)] - \frac{2}{(2\pi)^2} \int_a dA \wedge da
\]

which proves \((B3)\). Next, let us use the Whitney sum formula,

\[
w_4(\xi_{\phi} \otimes \xi_{\phi}) = w_2(\xi_{\phi}) \cup w_2(\xi_{\phi}) \tag{B6}
\]

all the other terms vanish, since \( \xi_{\phi} \) and \( \xi_\phi \) are \( O(2) \) bundles. Recalling \( w_2(\xi_{\phi}) + w_3(\xi_{\phi}) = w_2(\xi_{\phi}) \) and \( w_1(\xi_{\phi}) = w_1(\xi_{\phi}) \), we have

\[
w_4(\xi_{\phi} \otimes \xi_{\phi}) = w_2(\xi_{\phi}) \cup (w_2(\xi_{\phi}) + w_3(\xi_{\phi})) \tag{B7}
\]

Notice that any dependence on the gauge bundle \( \xi_{\phi} \) has disappeared—the above expression only depends on the background gauge bundles of the global symmetries \( O(2)_\phi \) and
SO(3). This means that although (B3) is dependent on the extension to $Y_4$, this dependence can be canceled by thinking of the theory as living on the surface of a $3 + 1$D SPT. The bulk partition function of this SPT on a closed manifold $X_4$ is just

$$S_{\text{bulk}} = \pi i \int_{X_4} w_2[\xi\phi] \cup (w_2[\xi] + w_1^2[\xi\phi]).$$

If $X_4$ has a boundary $M$, then we define

$$S_{\text{bulk-k-bound}} = \int_M |D_a\gamma|_2^2 + 2\pi i \int_M \frac{dA}{2\pi} \wedge \frac{da}{2\pi} + \pi i \int_{X_4 \setminus Y_4} w_2[\xi\phi] \cup (w_2[\xi] + w_1^2[\xi\phi]).$$

(B9)

Now, any dependence on the extension to $Y_4$ is canceled between the second and third terms above. However, the action does depend on the values of the background $O(2) \times SO(3)_L$ gauge fields on the “physical” four-manifold $X_4$.

Note that we may also rewrite $w_2[\xi\phi] + w_1^2[\xi\phi] = w_2[\xi\phi]$, where $\xi = \xi_\mu \otimes \det(\xi\phi)$ is an $(O,3)$ bundle derived from the original $SO(3)_L$ bundle $\xi$, by multiplying the transition functions by $-1$ whenever the rotation in $O(2)_L$ is improper.

Further note that as shown in Ref. [25], we obtain the same anomaly by working with a different proposed formulation of the deconfined critical point—the $N_f = 2$ QCD$_3$ theory [24]. Recall that the QCD$_3$ formulation has an anomalous global $SO(5)$ symmetry, and the anomaly is given by $S_{\text{bulk}} = \pi i w_4[\xi, X_4]$, where $\xi^5$ is an $SO(5)$ bundle. For the symmetries explicit in the CP$^1$ model, we $\xi^5 = \xi_\mu \otimes \hat{\xi}_\mu$ is a direct sum of $(O,2)_4$ bundle and $(O,3)$. Using the Whitney formula,

$$w_4[\xi^5] = w_1[\xi_\phi]w_3[\hat{\xi}_\mu] + w_2[\xi_\phi]w_2[\hat{\xi}_\mu].$$

(B10)

But $w_1[\xi_\phi] = w_1[\hat{\xi}_\mu]$ and $w_1w_3 = w_2^2$, so the first term is a total derivative and does not contribute to the bulk action. We then recover $w_4[\xi^5] = w_1[\xi_\phi]w_2[\hat{\xi}_\mu]$, in agreement with the computation in the CP$^1$ model.

1. Vanishing of anomaly

We now show that the anomaly (B8) vanishes for the symmetry appropriate to the honeycomb lattice and for the intrinsic symmetry appropriate to the $S = 1$ square lattice.

We begin with the honeycomb lattice. Here, the relevant subgroup of $O(2)_L$ is $D_3$. We want to show that $w_2[\xi\phi] = 0$. Recall that $w_2$ is the obstruction to lifting an $(O,n)$ bundle to a $pin(n)_+$ bundle [45]. Let $\pi : pin(2)_+ \to O(2)$ be the projection map. Now, $pin(2)_+ = O(2)$. Write $O(2) = U(1) \times Z_2$ with $Z_2$ generated by $C$. Then $\pi(u_\alpha) = u_\alpha C$ and $\pi(C) = C$, where $u_\alpha$ is a rotation by $\alpha$ in $U(1)$. Furthermore, if we restrict $O(2)$ to a $D_3$ subgroup, $\pi : D_3 \to D_1$ is an isomorphism. In fact, $\pi^2 = 1$. Thus, for any $D_3$ bundle we obtain a lift to $pin(2)_+$ simply by applying $\pi$ to the transition functions. Therefore, $w_2[D_3] = 0$.

Next, we proceed to the $S = 1$ square lattice. Here, we want to show that the intrinsic anomaly vanishes. For this, we have to consider bundles associated with the microscopic symmetry group $(Z^2 \times Z^2) \times Z_4^\text{rot}$. Let $x$ be the generator of $Z^2$, $y$ be the generator of $Z^\text{rot}$, and $r$ be the generator of $Z_4^\text{rot}$. The associated $O(2)_\phi$ bundle is a $Z_2 \times Z_2$ bundle obtained via the projection $p : (Z^2 \times Z^2) \times Z_4^\text{rot} \to Z_2 \times Z_2$, with $p(x) = C$, $p(y) = C$, $p(r) = u_\pi$. We can also form a $D_4$ representation $s : (Z^2 \times Z^2) \times Z_4^\text{rot} \to D_4$, with $s(x) = C$, $s(y) = u_\pi C$, and $s(r) = u_\pi/2$. We then have the sequence $(Z^2 \times Z^2) \times Z_4^\text{rot} \to D_4 \to Z_2 \times Z_2$, with $\pi : pin(2)_+ \to O(2)$ as before. Further, $\pi \circ s = p$. So to obtain a lift of $Z_2 \times Z_2$ to $D_4$, we simply apply $s$ to the parent $(Z^2 \times Z^2) \times Z_4^\text{rot}$. Therefore, $w_2[\xi\phi] = 0$.

APPENDIX C: ASYMMETRIC VORTICES

In Sec. III C, we revisited the well-known fact that $Z_4^\text{rot}$ VBS vortices on the square lattice carry $S = 1/2$ in their core. We emphasized that in general one needs to consider $Z_4^\text{rot}$ symmetric VBS vortices in order to reach this conclusion. In our analysis, we defined a vortex as having four macroscopic VBS domains in a clock configuration. The details of the domain walls separating the domains did not affect the counting of the vortex winding. In this appendix, we show that for the nearest neighbor dimer model there is an alternative way to define the vorticity by a closed line integral around a contour enclosing the vortex core, so that the vorticity does depend on the microscopic details of the domain walls. Further, with this definition, the vorticity is always equal to $N_A - N_B$, where $N_A/B$ is the number of “dangling spins” on $A/B$ sites in the vortex core. This holds even when the vortex is not rotationally symmetric. Further, we use this definition of vorticity to make contact with the anomaly formula (18): $S = \pi i \int_{X_4 \setminus Y_4} w_1 + w_2$.

For a dimer configuration on the square lattice, we want to compute the “vortex charge” $Q(U)$ of a region $U$. We assume that if any “dangling” spins are present, they are away from the boundary $\partial U$. We define $Q(U) = \frac{1}{4} \int_{\partial U} \gamma$, where $\gamma$ is a 1-cochain living on the links of the square lattice. This cochain is defined by counting VBS domain walls crossing the (oriented) contour $\partial U$ in the following way. First, we assign numbers $1, i, -1, -i$ to the links of the square lattice using a $2 \times 2$ unit cell as shown in Fig. 5 (1 is represented by a right arrow, $i$ by an up arrow, $-1$ by a left arrow, and $-i$ by a down arrow). For each site $j$, we define the VBS order parameter $V_j$ by the number on the dimer covering $j$; this is the standard definition of the columnar dimer order parameter. Now, to define $\gamma$ on a link $j \mu$, $\mu = \hat{x}, \hat{y}$, we consider $\frac{V_j}{V_j^{\text{vir}}}$. If $\frac{V_j^{\text{vir}}}{V_j} = 1$, we set $\gamma_{j\mu} = 0$. If $\frac{V_j^{\text{vir}}}{V_j} \neq 1$, the link crosses a VBS

FIG. 5. A unit cell for the branching structure of the usual VBS convention. Edges occupied with a dimer are considered part of a domain associated with the direction labeling that edge.
domain wall. For $V_{ijx}/V_{ijy} = \pm i$, this is a “single” domain wall, and we assign $\gamma_{jx} = \pm 1$. For $V_{ijx} = -1$, we have a double domain wall and assign $\gamma_{jx} = \pm 2$. The sign can be determined by breaking up the double domain wall into two single domain walls, as demonstrated in Fig. 6. Another example of this procedure is shown in Fig. 7. Using this procedure, we obtain the following general expression for the sign of $\gamma_{jx}$. Let $\lambda_{jx} = (-1)^{V_{ijx}}$, $\lambda_{jy} = i (-1)^{V_{ijy}}$ (so that $\lambda_{jx}$ coincides with the number we assigned to the corresponding link in Fig. 5). If $V_{ijx} = -1$, $-\frac{\pi}{2} = \pm i$ and we define $\gamma_{jx} = \pm 2$.

A direct computation shows that away from “dangling” spins $d\gamma = 0$. Therefore, $Q(U)$ is invariant under deforming the boundary of $U$ (as long as we do not push the boundary through sites with dangling spins). One can also show that in terms of the two sublattices $A$ (those vertices with all arrows incoming or all arrows outgoing) and $B$ (those vertices with two incoming arrows and two outgoing arrows), $Q(U)$ with a counterclockwise contour counts the number of unoccupied $A$ sites minus the number of unoccupied $B$ sites in $U$. Modulo 2, this just counts the number of unoccupied sites. Note that this identification works independent of the details of domain walls. For instance, in Fig. 2, top left $Q = 1$, top right $Q = 0$, and bottom $Q = -3$, in agreement with $N_A - N_B$.

(we take the unoccupied site in Fig. 2, top left, to be an $A$ site). Note, however, that there is no obvious way to extend this formula to more general dimer configurations (not just nearest neighbor). In particular, the integer nature of the invariant $Q$ is an artifact of only bipartite configurations being considered. Nevertheless, the formula for $Q$ is very reminiscent of the anomaly formula (18): $S = \pi i \int_{\Sigma} \frac{d\gamma}{\Sigma} \cup w_2^i$. Indeed, this formula indicates that in a spatial boundary slice $\Sigma$, $\int_{\Sigma} \frac{d\gamma}{\Sigma}$ (mod 2), tells us whether we have a projective SO(3) representation or not. Identifying the cochain $\gamma$ extracted from the domain walls with the background $Z_2$ gauge field $\gamma$ in the spirit of [15], we see a geometric confirmation of the anomaly formula.

We can also extend the definition of the vortex charge $Q$ to the honeycomb lattice. Here, we have three different Kekulé VBS domains with $V = 1, e^{2\pi i/3}, e^{4\pi i/3}$. For a given link $(ij)$, we compute $V_{ij}^{-1}$. If $V_{ij} = 1$, we assign $\gamma_{ij} = 0$ to the link. If $V_{ij} = e^{2\pi i/3}$, we assign $\gamma_{ij} = \pm 1$. Note that in this case there are no double domain walls. It is again true that $Q = N_A - N_B$. For instance, the vortex in Fig. 3, left, has $Q = 1$ and the vortex in Fig. 3, right, has $Q = -2$, as required.