SO(3) family symmetry and axions

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Motivated by the idea of comprehensive unification, we study a gauged SO(3) flavor extension of the extended Standard Model, including right-handed neutrinos and a Peccei-Quinn symmetry with simple charge assignments. The model accommodates the observed fermion masses and mixings and yields a characteristic, successful relation among them. The Peccei-Quinn symmetry is an essential ingredient.

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I. INTRODUCTION AND MOTIVATION

For all its success, the Standard Model has many loose ends and shortcomings. It leaves unexplained the threefold family replication, the observed pattern of quark and lepton masses and mixings, and the lack of CP violation in the strong interaction [1–3]. It does not account for the cosmological dark matter [4], and in its minimal form it leaves neutrinos massless [5].

In addressing those questions, it is natural to consider extending the ideas of gauge symmetry and its spontaneous breaking beyond their established, central role in the Standard Model. SU(3) and SO(3) suggest themselves as candidate symmetries for family unification, since they support irreducible triplet representations. (Discrete symmetries can also be gauged, and quantum gravity might require that they are [6,7].) SO(3) is particularly attractive, since it arises naturally in the context of comprehensive unification, which brings together forces and flavor [8–12].

In this paper we explore an SO(3) family symmetry model inspired by comprehensive unification. Within a reasonably economical model, several appealing features emerge:

(i) a Peccei-Quinn symmetry, leading to axions, which is both natural and helpful to ensure correct mass relations,
(ii) extreme fine-tuning is not required,
(iii) a characteristic “golden” formula relating quark and lepton masses, given in Eq. (10) [13–15],
(iv) a successful explanatory framework for the Cabibbo-Kobayashi-Maskawa (CKM) matrix, with two predictions, Eqs. (12) and (13), and
(v) a conventional seesaw mechanism [8] [16–21] for neutrino mass generation at the Peccei-Quinn (PQ) scale, supplemented by a connection between lepton number and PQ breaking, which relates the axion and neutrino mass scales, Eq. (18).

II. MODEL CONSTRUCTION

A. SO(3) as family symmetry

Discrete [13–15] and continuous [22,23] horizontal flavor symmetries have been used extensively in model building. Many options have been considered. It is interesting to consider, as a source of guidance, their possible deeper origin. In [12] we revived the idea of comprehensive unification, merging gauge and family symmetry. The striking fact that one can accommodate the observed fermions into a single irreducible spinor multiplet of large orthogonal groups encourages such ideas, which have a long history (see, e.g., [9]). On the face of it, however, they contain too many families, and also an equal number of wrong-chirality “antifamilies” (but no other exotics). We suggested that extraneous families are confined at a high scale ≳10 TeV, while the antifamilies were removed through an orbifold construction.
TABLE I. Particle content and transformation properties under the SU(3)$_c$ $\otimes$ SU(2)$_L$ $\otimes$ U(1)$_Y$ and flavor SO(3) gauge groups. The vacuum expectation values (VEVs) of SU(3)$_c$ $\otimes$ SU(2)$_L$ $\otimes$ U(1)$_Y$ singlets $\sigma$ and $\rho$ break U(1)$_{PQ}$ and lepton number, generating Majorana neutrino masses.

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More specifically, the breaking scheme SO(18) $\rightarrow$ SO(10) $\times$ SO(5) $\times$ SO(3) (see [8]) allows for the standard, attractive, SO(10) gauge unification, together with a hypercolor SO(5) which confines 5 families (leaving 3) and an SO(3) family symmetry group. This motivates consideration of SO(3) as a family unification group.

More generally, SO(3) family symmetry is more easily compatible with gauge unification than is SU(3) family symmetry. In the usual SU(5) and SO(10) theories one embeds the Standard Model particle content in the anomaly free sets of representations: $3 \times (\bar{5} + 10)$ for SU(5) and $3 \times 16$ for SO(10). Assigning these representations as SU(3) triplets generally leads to anomalies. For example, in the SO(10) $\times$ SU(3) theory the standard (16, 3) combination has an [SU(3)$_F$]$^3$ anomaly.

B. Field content

We now develop a consistent flavor extension of the Standard Model in which the gauge symmetry is enlarged by adding the local SO(3)$_F$ family symmetry [22]. In addition to Standard Model particles, the model has an enlarged scalar sector and right-handed neutrinos. This minimal extension is enough to accommodate fermion masses and mixings without fine-tuning of parameters and the other features mentioned earlier.

The field content of our model is displayed in Table I. Especially noteworthy are the Peccei-Quinn charge assignments.$^1$ They arise from a transformation that commutes with SO(10): all the fermion fields which occur in the SO(10) spinor have the same PQ charge. It also commutes with SO(3)$_F$.

C. Symmetry breaking

In our model symmetry breaking proceeds through the following set of scalar fields:

$^1$Recently, an alternative framework with flavor-dependent Peccei-Quinn charges has been proposed in [24,25]. Our U(1)$_{PQ}$ symmetry is related to flavor in a rather different way, through the SO(3) family symmetry.

There are two SU(3)$_c$ $\otimes$ SU(2)$_L$ $\otimes$ U(1)$_Y$ singlet scalars, $\sigma \approx (1, 1, 0, 5)$ and $\rho \approx (1, 1, 0, 1)$. All of these fields will acquire nontrivial vacuum expectation values.

Both SO(3) singlets as well as the quintuplet carry nontrivial PQ charges. Therefore, the spontaneous breaking of the Peccei-Quinn symmetry is triggered by their large VEVs. On the other hand, the SO(3) family symmetry breaking is associated to the VEV of the SU(3)$_c$ $\otimes$ SU(2)$_L$ $\otimes$ U(1)$_Y$ singlet scalar $\sigma$:

$$\langle \sigma \rangle = v_\sigma \text{diag}(0, 1, -1) \quad \text{and} \quad \langle \rho \rangle = v_\rho \text{diag}(1, 1, 1).$$

As we will see later, both VEVs play a key role in breaking lepton number, generating Majorana neutrino mass, and accounting for the large neutrino mixing angles observed in neutrino oscillations.

In order to break the electroweak symmetry we assume VEVs for the SU(2)$_L$ scalar doublets, i.e., $\Phi^u$ and $\Phi^d$, transforming as SO(3) quintuplets, as well as $\Psi^u$ and $\Psi^d$, transforming as SO(3) triplets. We assume the following pattern for the VEVs:

$$\langle \Phi^{u,d} \rangle = \begin{pmatrix} 0 \\ -k_{u,d} \\ e_1 \end{pmatrix},$$

$$\langle \Psi^{u,d} \rangle = \begin{pmatrix} 0 \\ e_{1}^{u,d} \\ e_{2}^{u,d} \end{pmatrix},$$

where the small parameters $e_i$ denote a perturbation with respect to the simplest alignments diag(0, 1, 1) and (1, 1, 0). This symmetry breaking pattern minimizes the Higgs potential [22], and provides a good description of the observed fermion mass hierarchy; see below.

An important feature of the model is the existence of a spontaneously broken U(1)$_Y$ global PQ symmetry. For definiteness, we fix the PQ quantum numbers as given in Table I. The VEVs of SU(3)$_c$ $\otimes$ SU(2)$_L$ $\otimes$ U(1)$_Y$ singlets $\sigma$ and $\rho$ break U(1)$_{PQ}$ as well as lepton number. The alignment of the associated Nambu-Goldstone boson $G$ is

$$G \approx \frac{1}{(v_\sigma^2 + v_\rho^2)^{1/2}} (v_\sigma \sigma^d + v_\rho \rho^d + \ldots)$$
where $\rho^I$ etc. denote the imaginary parts of scalars and $\ldots$ denotes components along the isodoublet scalars $\Psi^{ud}$, $\Psi^{dl}$, $\Phi^{ud}$, $\Phi^{dl}$, weighted by their VEVs and times their PQ charges. Notice that, through these projections, $G$ will couple directly to quarks and leptons at the tree level. These couplings are suppressed linearly by the PQ-breaking couple directly to quarks and leptons at the tree level.

These couplings are given as \[22\] simple VEV alignment, the eigenvalues of the matrices

$$m_{u,d,e} = 0,$$

$$m_{e,s,\mu} = |y_{2,4,6}u^{ud} - y_{1,3,5}v^{ud}|,$$

$$m_{t,b,r} = |y_{2,4,6}u^{ud} + y_{1,3,5}v^{ud}|.$$

When one takes into account the small perturbations, $\epsilon_i$, one finds that the golden formula

$$\frac{m_t}{\sqrt{m_c m_{\mu}}} \approx \frac{m_b}{\sqrt{m_s m_s}}.$$

This successful formula nicely relates down-type quark and charged lepton masses. On the other hand, the doubled Higgs structure forced by PQ symmetry allows us to avoid the unwanted top quark mass prediction $\frac{m_{\mu}}{\sqrt{m_c m_{\mu}}} \approx \frac{m_t}{\sqrt{m_s m_s}}$ present in [22]. Let us note that the golden formula relating quark and lepton masses in Eq. (10) also emerges in other flavor symmetry schemes, such as the ones proposed in [13–15], but without connection to an underlying Peccei-Quinn symmetry.

### III. “GOLDEN FORMULA” FOR QUARKS AND LEPTON MASSES

Given the $SO(3)$ multiplication rules, $3 \times 3 = 1 + 3 + 5$, one can use the vector (triplet) and the two-index symmetric traceless tensor (quintuplet) representations to build the following invariant Yukawa Lagrangian:

$$\mathcal{L} = \bar{q}_L(y_1\Psi^u + y_2\Phi^u)_{LR} + \bar{q}_L(y_3\Psi^d + y_4\Phi^d)d_R$$

$$+ \bar{l}_L(y_5\Psi^d + y_6\Phi^d)e_R + \text{H.c.}$$

Note that the “duplicated” scalar sector, with two scalar doublets selectively coupled to up-type/down-type fermions, does not imply a nonminimal low-energy Higgs sector, as we shall discuss further below.

After electroweak breaking, Eq. (5) leads to the quark mass matrices

$$M^u = \begin{pmatrix} 0 & y_1 e^u_2 & 0 \\ -y_1 e^u_2 & -y_2 k^u & y_1 v^u + y_2 e^u_1 \\ 0 & -y_1 v^u + y_2 e^u_1 & y_2 k^u \end{pmatrix},$$

$$M^d = \begin{pmatrix} 0 & y_3 e^d_2 & 0 \\ -y_3 e^d_2 & -y_4 k^d & y_3 v^d + y_4 e^d_1 \\ 0 & -y_3 v^d + y_4 e^d_1 & y_4 k^d \end{pmatrix},$$

and for the charged leptons

$$M^e = \begin{pmatrix} 0 & y_5 e^e_2 & 0 \\ -y_5 e^e_2 & -y_6 k^e & y_5 v^e + y_6 e^e_1 \\ 0 & -y_5 v^e + y_6 e^e_1 & y_6 k^e \end{pmatrix},$$

where we take into account the VEV alignment patterns of the $SO(3)$ triplet and quintuplet scalars, respectively.

These matrices allow a good description of the charged fermion masses. Indeed, neglecting the $\epsilon_i$ parameters, assumed small, which describe the departure from the simplest VEV alignment, the eigenvalues of the matrices are given as [22]

$$m_{u,d,e} = 0,$$

$$m_{e,s,\mu} = |y_{2,4,6}u^{ud} - y_{1,3,5}v^{ud}|,$$

$$m_{t,b,r} = |y_{2,4,6}u^{ud} + y_{1,3,5}v^{ud}|.$$

for up- and down-type quarks, with eigenvalues given by Eq. (9). This means that the CKM matrix, defined as $V_{\text{CKM}} = V^T_L V_L^d$, is naturally “aligned” to be just the identity matrix.

The perturbations of the eigenvectors of $M \cdot M^T$ which result from turning on the perturbations around the minima get translated into a small shift of the matrices in Eq. (11), which no longer coincide. Their mismatch is the CKM matrix. After turning on these perturbations, the electron and the up and down quarks, all get nonzero masses, while small quark mixing angles emerge naturally.

Thanks to the structured breaking of the $SO(3)$ family symmetry, one can predict mixing angles in terms of quark masses. We have the well-known Gatto-Sartori-Tonin [27] relation for the Cabibbo angle

$$\theta_C \approx \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}}.$$  

while for $|V_{ub}|$ we get

$$|V_{ub}| \approx \sqrt{\frac{m_d m_s}{m_b}} - \sqrt{\frac{m_u m_c}{m_t}}.$$
which extends a relation found in Ref. [22]. Finally, the doubling of scalar quintuplets \( \langle \Phi^a \rangle \) plays a crucial role in generating \( |V_{cb}| \), given as

\[
|V_{cb}| = \frac{e_1^{|d|}}{2k^{|d|}} - \frac{e_1^{|d|}}{2k^{|d|}}. \tag{14}
\]

In contrast to \( \theta_C \) and \( |V_{ub}| \), the \( |V_{cb}| \) matrix element can only emerge from the duplicated set of quintuplets, i.e., from the fact that \( \Phi^a \) and \( \Phi^d \) are different fields. Otherwise, the \( b \) quark would decay predominantly to up quarks through the weak charged current. Thus, in the present framework, mass hierarchies and mixing angles arise as perturbations around the symmetry breaking minima of the scalar potential, rather than hierarchies in the Yukawa couplings. \( CP \) violation can be accommodated through nontrivial phases in the Yukawa couplings, but no useful prediction emerges.

In short, the predictions we found confirm the expectation from a simple parameter counting. Including Yukawa couplings and VEVs we have in total 10 relevant parameters to describe 13 observables, namely the 9 charged fermion masses plus the 4 CKM parameters.\(^2\) This leads to three successful relations. Two of these are the predictions for the charged fermion masses, Eq. (10), and for the Cabibbo angle, Eq. (12). The third relation is given in Eq. (13). Notice that, thanks to the PQ symmetry, the golden relation in Eq. (10) is a successful one, as it involves only the down-type fermions, in contrast to Ref. [22]. Likewise, we have that \( |V_{cb}|, |V_{ub}| \neq 0 \), as required. Note also the important role played by the scalar potential dynamics, namely, the need for VEV misalignment in Eq. (3).

VI. HIGGS SCALAR SPECTRUM

Our explicit implementation of \( SO(3) \) flavor symmetry requires several scalar multiplets. In the context of renormalizable quantum field theory, without further constraints, there are many scalar coupling terms, and—given that most of the spectrum is lifted to a high mass scale—few observational handles on them. Thus a complete analysis is both impractical and pointless, but we do need to ensure that an acceptable low-energy sector can emerge.

Generically, all the fields other than the axion will acquire mass terms of order the flavor and PQ breaking scale, barring cancellations between bare and induced mass terms. For purposes of \( SU(2) \times U(1) \) breaking, we require at least one much lighter doublet. Notoriously, that requires a conspiracy or fine-tuning among parameters. This is an aspect of the hierarchy problem, which we do not address here. The only slight good news is that the existence of more than one doublet would require additional fine-tuning, so that the minimal one doublet structure, which so far is supported by experimental observations, is minimally unnatural.

To illustrate the mechanism whereby induced mass terms arise, consider the quartic operator \( \Psi_u^\dagger \Psi_u \sigma^\dagger \rho \). Its contraction is unique and can be easily visualized in matrix form. If \( \langle \sigma \rangle \) is aligned in the diagonal (recall it is symmetric and traceless) and \( \rho \) is an \( SO(3) \) singlet we get, after \( SO(3) \) breaking takes place,

\[
\langle \rho \rangle \Psi_u^\dagger (\sigma^\dagger) \Psi_u
\]

\[
= v_\rho \begin{pmatrix} \Psi_{1u}^\dagger & \Psi_{2u}^\dagger & \Psi_{3u}^\dagger \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -v_\sigma & 0 \\ 0 & 0 & v_\sigma \end{pmatrix} \begin{pmatrix} \Psi_{1u} \\ \Psi_{2u} \\ \Psi_{3u} \end{pmatrix}
\]

\[
= -v_\sigma v_\rho |\Psi_{2u}|^2 + v_\sigma v_\rho |\Psi_{3u}|^2. \tag{16}
\]

\(^2\)The neutrino sector is discussed separately, see below.
One sees that the vacuum expectation value of the above operator generates a splitting of order the flavor/PQ-breaking scale among the electroweak doublet components of \( \Psi \), so that two of them can be made heavy, i.e., at the large symmetry breaking scale, leaving the other massless. This argument may be escalated to the full scalar potential, which contains many relevant quartics, viz.

\[
\begin{align*}
\Phi_u^i \Phi_u^j \sigma^i \sigma^j, & \quad \Phi_u^i \Phi_d^j \sigma^i \rho^j, \\
\Phi_d^i \Phi_d^j \sigma^i \sigma^j, & \quad \Phi_d^i \Phi_d^j \sigma^i \rho^j, \\
\Phi_u^i \Psi_u^j \sigma^i \sigma^j, & \quad \Phi_u^i \Psi_d^j \sigma^i \rho^j, \\
\Phi_d^i \Psi_u^j \sigma^i \sigma^j, & \quad \Phi_d^i \Psi_d^j \sigma^i \rho^j, \\
\Phi_u^i \Phi_u^j \rho^i \rho^j, & \quad \Phi_u^i \Phi_d^j \rho^i \rho^j, \\
\Phi_d^i \Phi_u^j \rho^i \rho^j, & \quad \Phi_d^i \Phi_d^j \rho^i \rho^j, \\
\Phi_u^i \Psi_u^j \rho^i \rho^j, & \quad \Phi_u^i \Psi_d^j \rho^i \rho^j, \\
\Phi_d^i \Psi_u^j \rho^i \rho^j, & \quad \Phi_d^i \Psi_d^j \rho^i \rho^j.
\end{align*}
\]

These can be obtained in a systematic way; see [29.]) One finds that, after breaking, the scalar mass squared matrix typically contains suitable off-diagonal terms, ensuring that the light doublet is a linear combination of \( \Psi_u, \Psi_d, \Phi_u, \Phi_d \) wherein each appears with a nonzero coefficient.

VII. DISCUSSION

Before closing, we comment briefly on three issues which deserve mention.

1. The PQ symmetry \( U(1)_{\text{PQ}} \) is conserved at the classical level, and to all orders in perturbation theory, but violated nonperturbatively. One can visualize the breaking using QCD instantons, and infer its character by analyzing anomalies. In this way, one may discover that a nontrivial \( Z_N \) subgroup of \( U(1)_{\text{PQ}} \) is valid even nonperturbatively. Our model, as it stands, has \( N = 12 \), with doubly charged scalar fields. If scalar fields which are not \( Z_N \) singlets acquire VEVs, the possibility of domain walls arises. Such domain walls are very dangerous for early universe cosmology [30]. The most straightforward way to avoid this difficulty is to assume that the \( Z_N \) breaking is followed by a period of cosmic inflation, so that potential domain walls get pushed beyond the horizon. Another possibility is to arrange that \( N = 1 \). (We could also allow \( N = 2 \), since the PQ breaking VEVs have PQ charge 2). This does not occur in our model as it stands, but it can be achieved by adding suitable colored fermions. In the absence of other motivations, however, that construction seems contrived.

2. The vacuum expectation values of the \( \rho \) and \( \sigma \) scalars are responsible both for Peccei-Quinn and lepton number symmetry breaking. This entails an interesting conceptual relation between the axion and neutrino mass scales of the form

\[
m_a \sim (\Lambda_{\text{QCD}} m_\pi / v^2) m_\nu.
\]

VIII. SUMMARY

Motivated by ideas arising in comprehensive unification based on spinors, we have considered possible consequences of supplementing the Standard Model gauge symmetry with commuting \( SO(3) \) flavor and PQ symmetries in a way consistent with \( SO(10) \) embedding. Proceeding in a bottom-up way, we analyzed a minimal \( SO(3)_F \times U(1)_{\text{PQ}} \) extension of the Standard Model unifying the three families of matter. Fairly simple choices of multiplet structure and symmetry breaking pattern allowed us to accommodate the known phenomenology of quark and lepton masses and mixings and to make several nontrivial connections among them. The PQ symmetry
was important to this success, and of course it continues to serve its familiar roles in ensuring accurate strong $T$ symmetry and in providing, in axions, a good dark matter candidate.

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