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Exact solution for the Poisson field in a semi-infinite strip

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The Poisson equation is associated with many physical processes. Yet exact analytic solutions for the two-dimensional Poisson field are scarce. Here we derive an analytic solution for the Poisson equation with constant forcing in a semi-infinite strip. We provide a method that can be used to solve the field in other intricate geometries. We show that the Poisson flux reveals an inverse square-root singularity at a tip of a slit, and identify a characteristic length scale in which a small perturbation, in a form of a new slit, is screened by the field. We suggest that this length scale expresses itself as a characteristic spacing between tips in real Poisson networks that grow in response to fluxes at tips.
1. Introduction

The Poisson equation appears in many physical phenomena with wide utility in various fields including fluid mechanics [4], electrostatics [19] and mechanical engineering [15]. This equation usually delineates a steady-state diffusive field that can describe a broad array of physical quantities such as chemical concentrations [6], temperature [12], fluid velocity [3,13], groundwater height [16,18] and stress potential [15]. However, in spite of its broad relevance, only a few exact solutions of the Poisson equation exist, in part due to the complexity of the geometry and boundary conditions [3,8,15]. The aim of this paper is to present further analytic progress that culminates in providing an exact analytic solution for the Poisson field with constant forcing in a semi-infinite strip geometry.

The most general form of the two-dimensional Poisson equation is written

\[ \Delta U(x,y) = f(x,y) \]  

(1.1)

where \( f \) is the source term, and \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplacian operator in Cartesian coordinates. As the source term vanishes (\( f = 0 \)) the Poisson equation reduces to the Laplace equation, and the field becomes harmonic. In these cases, the complex geometry can be conformally mapped to a simple geometry and the field can be found using the many theorems of analytic functions in the complex plane. That is not the case when the source term is not zero, and the field becomes non-harmonic.

Recent work shows how stream networks can be considered as absorbing slits in a diffusive groundwater field [7,10]. Motivated by this problem, we refer to these slits as channels and find an exact solution for the Poisson field around a rudimentary channel network. We then derive the corresponding flux \(-\nabla U\) and study the singularity of the field in the vicinity of a channel tip. We next show how the flux to the tip evolves with the length of the channel. Finally, we provide an estimation of the length scale over which a perturbation is screened by the Poisson field, and show how this length scale may manifest itself in the growth of stream networks.

2. Setting Up the Problem

We consider the geometry of an incipient channel (C-D) of length \( L \) that bifurcates from a main channel (B-B') of length 2a, Fig 1. We define a channel as an infinitesimally thin finger, or a slit. The field is confined between two parallel semi-infinite channels (A-B and A'-B') also connected to the main vertical channel. This geometry depicts a stream network fed by groundwater [1] or grounded wires amid uniform charge distribution. The field \( U \) represents the groundwater height squared or the electric potential, respectively. It then becomes the solution of the Poisson equation

\[ \Delta U = -1, \]  

(2.1)

with absorbing boundary condition along the channels, \( U = 0 \).

\( U(x, y) \) is not an analytic function, and thus cannot be solved directly using conformal mapping. However, we can express the field with an analytic function \( \varphi \) as

\[ U(x, y) = \frac{1}{2} \Re\{1 - y^2 + \varphi(x, y)\}, \]  

(2.2)

where \( \Re \) denotes the real part of its argument. Introducing the complex notation \( z = x + iy \) and its complex conjugate \( \bar{z} = x - iy \), the field becomes

\[ U(z) = \frac{1}{2} \Re\{1 + \frac{1}{4}(z - \bar{z})^2 + \varphi(z)\}, \]  

(2.3)

where \( \varphi(z) \) is an analytic function in the whole domain such that

\[ \Delta \varphi = 0 \]  

(2.4)
except on the boundary. The boundary conditions for \( \varphi \) are shown in Fig. 2. This transformation changes the Poisson problem with simple boundary conditions to a problem of finding an analytic function with complicated boundary conditions.

![Figure 2](image-url)

Figure 2: The geometry and boundary conditions for the real part of the analytic function \( \varphi \) (in blue). For simplicity we choose \( a = 1 \), and \( l \) effectively becomes the ratio between the length of the incipient channel and the half-width of the domain.

### 3. Properties of the Analytic Function \( \varphi(z) \)

The field \( U \) is symmetric with respect to the \( x \)-axis, thus

\[
U(z) = U(\bar{z}).
\]  

(3.1)

Since \( (z - \bar{z})^2 = (\bar{z} - z)^2 \), it requires that

\[
\Re\{\varphi(z)\} = \Re\{\varphi(\bar{z})\}.
\]  

(3.2)

The left and the right terms give

\[
\varphi(z) + \overline{\varphi(z)} = \varphi(\bar{z}) + \overline{\varphi(\bar{z})}
\]  

(3.3)

and by rearranging we get

\[
\varphi(z) - \varphi(\bar{z}) = -[\varphi(z) - \varphi(\bar{z})].
\]  

(3.4)

Since \( \varphi(z) \) is an analytic function in the whole domain, except maybe on the boundary, the function \( \varphi(z) \) is also an analytic function. Thus, the left member of Eq. (3.4) is an analytic function
in the same domain. Because it equals the negative of its complex conjugated part, it must be an imaginary function. By the Cauchy-Riemann equations, a pure imaginary analytic function is a constant. Hence,
\[ \varphi(z) = \bar{\varphi}(z) + iC. \] (3.5)
We can add any imaginary constant to \( \varphi \) without changing the Poisson field, \( U \). Replacing \( \varphi(z) \rightarrow \varphi(z) + iC \) in Eq. (3.5) gives \( C = 0 \), thus,
\[ \varphi(z) = \bar{\varphi}(z). \] (3.6)

4. Conformal Mapping and Boundary Conditions

To find a solution we use conformal mapping and study the boundary condition of the analytic function. The upper part of the domain (marked in gray in Fig. 2) is mapped into the upper half plane of \( \zeta = \xi + i\eta \) using the following mapping function [20] (cf. Fig. 3),
\[ \zeta = \cosh \left( \frac{\pi l}{a} \right). \] (4.1)
We note that the geometry can be rescaled by the width \( a \) and therefore we set \( a = 1 \) and \( l = L/a \).

![Diagram](image)

Figure 3: The mapping of the upper half of the domain to the upper half plane. The length of the incipient channel is rescaled by \( a \) and located between \( 1 < \xi < \cosh(\pi l) \). The gray area indicates the upper part of the strip in Fig. 1.

The analytic function can be written as a function of the inverse map \( z = \Omega(\zeta) = \frac{1}{\pi} \cosh^{-1}(\zeta) \):
\[ \varphi(z) = \varphi[\Omega(\zeta)] = F(\zeta). \] (4.2)

\( F \) is an analytic function. By using Eq. (3.6) we obtain
\[ F(\zeta) = \bar{F}(\zeta). \] (4.3)

The boundary conditions of \( F \) in the mathematical plane, in Fig 3, are derived from the boundary conditions of \( \varphi \) in the physical plane, as shown in Fig. 2. We now discuss the boundary conditions:

(A-B): In the upper boundary, \( \Re\{\varphi\} = 0 \), which is translated to \( \Re\{F(\xi)\} = 0 \). Applying Eq. (4.3), we obtain
\[ F(\xi + 0i) + F(\xi - 0i) = F(\xi + 0i) + \bar{F}(\xi - 0i) \]
\[ = F(\xi + 0i) + F(\xi - 0i). \] (4.4)
Thus, the boundary condition becomes,
\[ F_+(\xi) + F_-(\xi) = 0 \] (4.5)
We used the subscripts + and −  for the limiting values of the function reaching from the upper and the lower half of the plane, respectively.

(B-C): On the left vertical channel, \( \Re(\varphi) = y^2 - 1 \). Since \( x = 0 \) along this line, we can write the boundary condition as \( \Re(\varphi) = -(x^2 + 1) \). Using Eq. (4.4) and the inverse of the map given in Eq. (4.1), we find

\[
F_+(\xi) + F_-(\xi) = -2 \left[ \frac{1}{\pi} \cosh^{-1} \xi \right]^2 + 1.
\]  

(C-D): \( \Re(\varphi) = -1 \) gives

\[
F_+(\xi) + F_-(\xi) = -2 \tag{4.7}
\]

(D-E): Here, the field is symmetric with respect to the \( x \)-axis. Thus the normal derivative vanishes, and

\[
\frac{\partial U}{\partial y} = -y + \frac{\partial}{\partial y} \Re(\varphi) = 0. \tag{4.8}
\]

We note that \( y = 0 \) along this line. The Cauchy-Reimann equations give \( \frac{\partial}{\partial y} \Re(\varphi) = -\frac{\partial}{\partial \xi} \Im(\varphi) = 0 \), where \( \Im \) stands for the imaginary part of the function. Integrating this equation with respect to \( x \), we find \( \Im(\varphi) \) is constant along the boundary (In general, it is a function of \( y \), but \( y = 0 \) along the boundary). Using Eq. 3.6, and the fact that \( \varphi \) is single-valued along this line, this constant must be zero. Hence,

\[
\Im\{\varphi(x + 0i)\} = \frac{1}{2i}(\varphi(x + 0i) - \bar{\varphi}(x - 0i)) = 0. \tag{4.9}
\]

The boundary condition for \( F \) becomes

\[
F_+(\xi) - F_-(\xi) = 0. \tag{4.10}
\]

To summarize, we define an analytic function \( F(\xi) \) in the upper half of the mathematical plane with the following boundary conditions along the \( \xi \)-axis:

\[
\begin{align*}
F_+(\xi) + F_-(\xi) &= 0 \quad &\xi < -1 \\
F_+(\xi) + F_-(\xi) &= -2 \left[ \frac{1}{\pi} \cosh^{-1} \xi \right]^2 + 1 \quad &-1 < \xi < 1 \\
F_+(\xi) + F_-(\xi) &= -2 \quad &1 < \xi < \cosh(\pi l) \\
F_+(\xi) - F_-(\xi) &= 0 \quad &\cosh(\pi l) < \xi
\end{align*}
\]  

(5.11)

5. The Field

Equations (4.11) constitute a Hilbert problem [14,15]. The solution of this problem can be achieved by bringing these equations to the same form, by introducing the auxiliary function

\[
G(\zeta) = (\zeta - \cosh(\pi l))^{1/2} \tag{5.1}
\]

with a branch cut along \( \{\xi < \cosh(\pi l), \eta = 0\} \) and a choice of a branch that \( G(\zeta) \to \zeta \) as \( \zeta \to \infty \). Now,

\[
\frac{G_-(\xi)}{G_+(\xi)} = \begin{cases} 
0 & : \xi < \cosh(\pi l) \\
1 & : \xi > \cosh(\pi l)
\end{cases} \tag{5.2}
\]

and Eq. (4.11) can be written as the single equation

\[
F_+(\xi) - \frac{G_-(\xi)}{G_+(\xi)} F_-(\xi) = s_0(\xi). \tag{5.3}
\]

Here \( s_0(\xi) \) signifies the boundary values of the field:

\[
s_0(\xi) = -2\left[ \frac{1}{\pi^2} \left[ \cosh^{-1}(\xi) \right]^2 \right] \left[ H(\xi + 1) - H(\xi - 1) \right] + \left[ H(\xi - 1) - H(\xi - \cosh(\pi l)) \right], \tag{5.4}
\]

where \( H(\star) \) is Heaviside step function.
Multiplication of both sides of Eq. (5.4) with the function \( G_+ (\xi) \) results in

\[
G_+ (\xi) F_+ (\xi) - G_- (\xi) F_- (\xi) = G_+ (\xi) s_0 (\xi).
\] (5.5)

The left-hand side gives the difference between the values of the analytic function \( G(\zeta) F(\zeta) \) on the upper and lower side of the \( \xi \)-axis. Using Plemelj’s formulae [17] (for more details see chapter 10 in [14] and appendix A4 in [5]), we can determine the function by solving the following integral:

\[
G(\zeta) F(\zeta) = \frac{1}{2\pi i G(\zeta)} \int_{-\infty}^{\infty} \frac{G_+ (\xi)s_0 (\xi)}{\xi - \zeta} d\xi.
\] (5.6)

However, the solution is not complete; any analytic function in the whole plane can be added, since it vanishes in the left member of equation (5.5). Hence,

\[
F(\zeta) = \frac{1}{2\pi i G(\zeta)} \int_{-\infty}^{\infty} \frac{G_+ (\xi)s_0 (\xi)}{\xi - \zeta} d\xi + \frac{S(\zeta)}{G(\zeta)},
\] (5.7)

where \( S(\zeta) \) is an analytic function in the whole plane. According to Liouville’s theorem, an analytic function can be written as a polynomial of finite degree. Since the field is finite as \( |\zeta| \to \infty \), it must be constant. This constant can be determined by applying the condition that the field must be continuous everywhere. In particular, the first term of the function \( F \) in Eq. (5.7) has a singularity as \( \zeta \to \cosh (\pi l)^+ \). The function \( S \) is defined to annul this singularity, i.e.

\[
\lim_{\zeta \to \cosh (\pi l)^+} F(\zeta) = 0,
\] (5.8)

where the superscript \( + \) indicates the direction in which the limit is taken, in this case decreasing from larger values of \( \xi \).

Finally, using this condition on Eq. (5.7), we obtain

\[
F(\zeta) = \frac{1}{2\pi i G(\zeta)} \left( \int_{-\infty}^{\infty} \frac{G_+ (\xi)s_0 (\xi)}{\xi - \zeta} d\xi - \int_{-\infty}^{\infty} \frac{G_+ (\xi)s_0 (\xi)}{\xi - \cosh (\pi l)} d\xi \right).
\] (5.9)

Note that the second integral is the constant \( S \).

We next substitute Eq. (5.4), (5.2), and the conformal map Eq. (4.1) into Eq. (5.9). After several mathematical steps, we find that the solution of the analytic function in the physical plane is given by

\[
\phi(z) = \frac{1}{\pi^3} I(z) \sqrt{\cosh (\pi z) - \cosh (\pi l)} - \frac{2}{\pi} \tan^{-1} \left( \sqrt{\frac{1 + \cosh (\pi l)}{\cosh (\pi z) - \cosh (\pi l)}} \right),
\] (5.10)

where

\[
I(z) = \int_{-1}^{1} \frac{\cosh^{-1}(\xi)^2}{(\xi - \cosh (\pi z))\sqrt{\xi - \cosh (\pi l)}} d\xi
\] (5.11)

is obtained from Cauchy’s principle value.

Using the solution of the analytic function \( \phi(z) \), Eq. (5.10), in Eq. (2.2) and we obtain the Poisson field \( U(z, \bar{z}) \) for the semi-infinite strip:

\[
U(z) = \frac{1}{2} \Re \left\{ 1 - y^2 + \frac{i}{\pi} I(z) \sqrt{\cosh (\pi z) - \cosh (\pi l)} - \frac{2}{\pi} \tan^{-1} \left( \sqrt{\frac{1 + \cosh (\pi l)}{\cosh (\pi z) - \cosh (\pi l)}} \right) \right\}.
\] (5.12)

Examples of the solution of the Poisson field are shown in Fig. 4.

6. The Poisson Flux: Intensity and Singularity

The growth of a channel in a diffusive field is usually correlated with the flow lines that enter its tip. In many physical problems including combustion [21] and stream networks [1], these flow lines are characterized by the gradient of the field, \(-\nabla U\). In this section, we analyse the flux and its singularity in the vicinity of the channel head.
In complex notation, the gradient operator can be written as

$$\nabla = \partial_x + i\partial_y = 2\partial_z.$$  

(6.1)

Thus, the gradient of the Poisson field as defined in Eq. (2.3) becomes

$$\nabla U = \partial_z \Re\{1 + \frac{1}{4}(z - \bar{z})^2 + \varphi(z)\} = \frac{1}{2}[(\bar{z} - z) + \partial_z \varphi(z)].$$  

(6.2)

We calculate the derivative of the analytic function \(\varphi(z)\), Eq. (5.10), and obtain the flux in the whole domain:

$$\nabla U = \frac{1}{2}(z - \bar{z}) + \frac{1}{\sqrt{2} \pi} \sqrt{\cosh(\pi z) - \cosh(\pi l)} \cdot \left( \frac{i}{4\pi^2} \sinh(\pi z) I(z) + \frac{1}{\sqrt{2}} \cosh\left(\frac{\pi l}{2}\right) \tanh\left(\frac{\pi z}{2}\right) \right)$$

$$+ \frac{i}{2\pi} \sqrt{\cosh(\pi z) - \cosh(\pi l)} I'(z).$$  

(6.3)

For simplicity, we present the complex conjugate of the flux \(\nabla U = \partial_x U - i\partial_y U\). \(I'(z)\) is the first derivative of the Cauchy integral, Eq. (5.11), with respect to \(z\).

In Fig. 5, we show the flux direction and density for several channel lengths. The void in the vicinity of the channel head indicates a divergence of the flux. Next, we discuss the singularity of the flux at this point.

In a Laplacian field, the flux in the vicinity of the tip of a channel or a slit is determined by a dominant term with an inverse square-root singularity [2]. This result is universal and depends neither on the geometry nor on the boundary conditions of the domain. Recent studies [7,9] show that the general expansion near the tip also has a similar form in a Poisson field. Here we study the nature of the singularity at the channel head \((x = l; y = 0)\) of the analytic solution of the Poisson flux (6.3).

The flux near the tip \((z = l + \varepsilon; \varepsilon \ll l)\) becomes

$$\nabla U(\varepsilon) = \frac{1}{\sqrt{2\pi\varepsilon}} \left( \frac{i}{\sqrt{8\pi^2}} \sqrt{\sinh(\pi l) I(l)} + \frac{1}{2} \sqrt{\tanh\left(\frac{\pi l}{2}\right)} \right) + O(\varepsilon).$$  

(6.4)

We find that similar to the Laplacian field, the flux diverges as we asymptotically reach the channel head as \(\varepsilon \to 0^+\). Therefore, it is not well-determined at the tip. The inverse square-root term \(\varepsilon^{-1/2}\) is the only term that is singular at \(\varepsilon = 0\). The next dominant term in the expansion is \(1/\varepsilon\) and it vanishes at the tip. In fracture mechanics, the singularity of the field near the crack tip is characterized by the stress intensity factor, which expresses the strength of the stress field near the tip [11]. Using this analogy, we define the flux intensity factor as

$$K = \lim_{\varepsilon \to 0} \sqrt{2\pi\varepsilon} \nabla U(\varepsilon).$$  

(6.5)
Applying equation (6.4), we find

$$K = \frac{i}{\sqrt{8\pi^2}} \sinh \frac{\pi l I}{2} + \sqrt{\frac{1}{2}} \tanh \left( \frac{\pi l}{2} \right).$$

(6.6)

Note that $K$ is a real number for $l > 0$. Using the flux intensity factor we can express the cartesian components of the flux in terms of $\varepsilon = r e^{i\theta}$ up to order of $O(\sqrt{\varepsilon})$ as

$$j_x = -\frac{\partial U}{\partial x} = -\frac{K}{\sqrt{2\pi r}} \cos(\theta/2)$$

(6.7)

and

$$j_y = -\frac{\partial U}{\partial y} = -\frac{K}{\sqrt{2\pi r}} \sin(\theta/2),$$

(6.8)

where $r$ is the distance from the channel head and $\theta = 0$ points in the positive $x$-direction. These expressions are universal for any slit in a Poisson field with constant forcing. The properties of the geometry and the boundary conditions are embedded in the intensity of the field $K$ given by Eq. (6.6).

7. Screening Lengths

The growth of a channel is tightly correlated to the flux entering the tip, which can be characterized solely by the intensity factor $K$. For a semi-infinite channel with no vertical boundaries amid two parallel infinite channels (see supplementary materials), the flux intensity factor is given by a constant value $K = \frac{1}{\sqrt{2}}$. For finite length, however, the flux intensity depends also on the length of the channel. Fig. 6 shows that the flux monotonically increases with the channel length and converges asymptotically as $l \to \infty$ to the solution of a semi-infinite channel. The inset shows that the solution converges to a constant $K \sim \frac{1}{\sqrt{2}} (1 - e^{-\pi l/a})$. The non-normalized length scale characterizing the convergence is therefore $a = \frac{\pi}{2}$, where $a$ is the half-width of the channel.
Lastly, we study how the field ahead of the channel converges to the solution at infinity. The solution of the field between two infinite absorbing walls is \( U = \frac{1}{2} (1 - y^2) \). Thus \( \varphi(z) \to 0 \) as \( x \to \infty \). Fig. 7 shows that the function exponentially decays to zero with a characteristic length of \( 2a/\pi \). This result provides the length scale in which a small perturbation in the geometry, in a form of an incipient channel, affects the field and other channels in its vicinity. It also indicates the typical distance for interaction between channels—a screening length—that may be related to the regular spacing of channels in ramified networks [1]. Fig. 8 suggests how this length scale manifests itself in a network incised by groundwater seepage located near Bristol, Florida [1,10]. In this network, the field \( U \) represents the groundwater height squared. Previous work [9] showed numerically that the ratio between two characteristic lengths in such networks gives rise to the appearance of the golden ratio \( \phi = 1.618... \). Here, Our analysis predicts that a related interaction length is given by a quantitatively similar number, \( \pi/2 = 1.57 .... \).
8. Discussion and Summary

This paper presents an analytic solution for the Poisson field in a semi-infinite strip. We show that the Poisson flux develops an inverse square-root singularity in the vicinity of the tip of a channel. Since the flux diverges at the tip, we suggest a definition for the flux intensity, and show that it monotonically increases with the channel length. The flux intensity converges exponentially to a constant with a length scale of $a/\pi$. We also find that a perturbation of the Poisson field due to the small central channel decays exponentially as we moving away from the channel’s tip, with a length scale of $2a/\pi$. This length scale may exhibit itself as a characteristic spacing in networks growing in such fields.

Data accessibility

The data supporting this article have been uploaded as part of the electronic supplementary material.

Competing interests

The authors declare no conflicts of interest.

Authors’ contributions

Y.C. derived the exact solution. Y.C. and D.H.R. contributed equally to its physical interpretation. Both authors wrote the paper.

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