Non-Gray Radiation Exchange: The Internal Fractional Function Reconsidered

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<td>As Published</td>
<td><a href="https://doi.org/10.1115/IMECE2018-86386">https://doi.org/10.1115/IMECE2018-86386</a></td>
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<tr>
<td>Publisher</td>
<td>American Society of Mechanical Engineers</td>
</tr>
<tr>
<td>Version</td>
<td>Final published version</td>
</tr>
<tr>
<td>Accessed</td>
<td>Mon Apr 01 06:41:43 EDT 2019</td>
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<tr>
<td>Citable Link</td>
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ABSTRACT

The radiation fractional function is the fraction of black body radiation below a given value of $\Delta T$. Edwards and others have distinguished between the traditional, or “external”, radiation fractional function and an “internal” radiation fractional function. The latter is used for simplified calculation of net radiation from a non-gray surface when the temperature of an effectively black source is not far from the surface’s temperature, without calculating a separate total absorptivity. This paper examines the analytical approximation involved in the internal fractional function, with results given in terms of the incomplete zeta function. A rigorous upper bound on the difference between the external and internal emissivity is obtained. Calculations using the internal emissivity are compared to exact calculations for several models and materials. A new approach to calculating the internal emissivity is developed, yielding vastly improved accuracy over a wide range of temperature differences. The internal fractional function can be useful for certain simplified calculations.

NOMENCLATURE

- $g_{\text{net}}$: Net radiant heat flux from surface 1 to 2 [W m$^{-2}$]
- $R_{\text{rad}}$: Radiation thermal resistance, Eqn. (47) [K W$^{-1}$]
- $r_e$: Electrical resistivity [Ω cm]
- $T$: Temperature [K]
- $T_1$: Temperature of surface [K]
- $T_2$: Temperature of environment [K]
- $T_m$: Mean temperature, $(T_2 + T_1)/2$ [K]
- $X$: $c_2/\Delta T$
- $X_c$: Finite value of $X$ for which $dF/dX = 0$, 3.92069
- $Y$: Function in Appendix C, Eqn. (60)
- $\alpha(T_1, T_2)$: Total hemispherical absorptivity
- $\alpha(\lambda, T)$: Spectral hemispherical absorptivity
- $\alpha_{\text{lw}}, \alpha_{\text{sw}}$: See Eqn. (39)
- $\Delta T$: $T_1 - T_2$ [K]
- $\delta T$: $T_2 - T_1$ [K]
- $\Delta \sigma$: $\alpha_{\text{lw}} - \alpha_{\text{sw}}$
- $\varepsilon(T)$: Total hemispherical emissivity (external)
- $\varepsilon_n(T)$: Total normal emissivity (external)
- $\varepsilon(\lambda, T)$: Spectral hemispherical emissivity
- $\varepsilon'(\theta, \lambda, T)$: Spectral directional emissivity
- $\varepsilon_n(\lambda, T)$: Spectral normal emissivity
- $\varepsilon'(T)$: Internal total hemispherical emissivity
- $\varepsilon_n'(T)$: Internal total normal emissivity
- $\zeta(s)$: Riemann zeta function
- $\zeta(X, s)$: Incomplete zeta function
- $\lambda$: Wavelength [m]
- $\sigma$: Stefan-Boltzmann constant [W m$^{-2}$ K$^{-4}$]
INTRODUCTION

The use of the radiation fractional function is well known in computing total emissivity from spectral emissivity. Less well known is the use of the internal radiation fractional function for approximating net heat flux from one surface to another at a similar temperature. Edwards was a strong proponent of the latter approach, beginning in the 1960’s [1], following work published a few years earlier by Czerny and Walther [2]. Edwards’ work on radiative property measurements had well acquainted him with the failure of gray-body exchange approximations at even small temperature differences [3], leading him to seek a better approximation that retained the computational convenience of using only a single radiative property.

Edwards distinguished the traditional fractional function by calling it the external fractional function. The terms external and internal derive from Edwards’ work in the aerospace industry, differentiating between radiation exchange on the exterior of a spacecraft, where temperature differences are large, and on the interior of a spacecraft, where surface temperatures are not far apart [4]. The internal radiation fractional function appears in several textbooks by Edwards and his coworkers [5–7] spanning a period of decades. Most other textbooks have not taken it up. In part, the lack of adoption may reflect the limited range of temperature difference over which the internal emissivity provides an accurate estimate of the net heat flux.

The present paper reconsiders the approximation leading to the internal radiation fractional function, with the objective of understanding its accuracy and how that may be improved. Results are given in terms of the incomplete zeta function, and a rigorous upper bound on the difference between the external and internal emissivity is obtained. Calculations based upon the internal radiation fractional function are compared to exact calculations, and a procedure that sharply increases accuracy of the internal emissivity is established. This procedure is useful, e.g., for computing radiation thermal resistance in multi-mode heat transfer.

BACKGROUND

The spectral hemispherical absorptivity, $\alpha(\lambda, T_1)$, of a surface at $T_1$, is related to the total hemispherical$^2$ emissivity, $\varepsilon(T_1)$, by integration over the Planck distribution for blackbody monochromatic emissive power, $e_{\lambda,b}(T_1)$,

$$\varepsilon(T_1) = \frac{1}{\sigma T_1^4} \int_0^\infty \alpha(\lambda, T_1) e_{\lambda,b}(T_1) \, d\lambda$$  \hspace{1cm} (1)

$$e_{\lambda,b}(T_1) = \frac{\pi c_1}{\lambda^5 \left( \exp(c_2/\lambda T_1) - 1 \right)}$$  \hspace{1cm} (2)

where $c_1$ and $c_2$ are the first and second radiation constants. If this surface is irradiated by an effectively black environment at $T_2$ (for example, if the surroundings are much larger than the surface), the total hemispherical absorptivity is

$$\alpha(T_1, T_2) = \frac{1}{\sigma T_2^4} \int_0^\infty \alpha(\lambda, T_1) e_{\lambda,b}(T_2) \, d\lambda$$  \hspace{1cm} (3)

The net radiation leaving this surface is

$$q_{\text{net}} = \sigma \varepsilon(T_1) T_1^4 - \sigma \alpha(T_1, T_2) T_2^4$$  \hspace{1cm} (4)

NON-GRAY ERRORS

For a gray surface, $\alpha(\lambda, T)$ is independent of wavelength and so $\varepsilon(T_1) = \alpha(T_1, T_2)$. However, some older literature went farther, applying this idea to non-gray surfaces for which $T_1$ is near $T_2$. Edwards notes that the fact that $\lim_{T_2 \rightarrow T_1} \alpha(T_1, T_2) = \varepsilon(T_1)$ is meaningless in calculating heat exchange; and further, working with Nelson, he documented associated “non-gray errors” in radiative property measurements that exceeded 60% [3].

The reason for such errors is not that the limit is incorrect, but that the heat flux is determined by the slope of $q_{\text{net}}$ with temperature difference, which changes if the surface is not truly gray. To see why in a simple way, the last term in Eqn. (4) may be linearized:

$$\alpha(T_1, T_2) T_2^4 \approx \alpha(T_1, T_1) T_1^4 + \frac{d}{dT_2} \left( \alpha(T_1, T_2) T_2^4 \right) \bigg|_{T_1} (T_2 - T_1)$$  \hspace{1cm} (5)

$$= \varepsilon(T_1) T_1^4 + 4T_1^3 \left[ \varepsilon(T_1) + T_1 \frac{d\alpha}{dT_2} \bigg|_{T_1} \right] (T_2 - T_1)$$  \hspace{1cm} (6)

Thus,

$$q_{\text{net}} \approx 4\sigma T_1^3 \left[ \varepsilon(T_1) + T_1 \frac{d\alpha}{dT_2} \bigg|_{T_1} \right] (T_1 - T_2)$$  \hspace{1cm} (7)

For a gray (or black) surface, $d\alpha/dT_2 = 0$, and we recover a common linearization: $q_{\text{net}} \approx 4\sigma \varepsilon T_1^4 \Delta T$ [11]. Thus, the purpose of introducing the internal fractional function is to account for the difference in slope when $d\alpha/dT_2 \neq 0$.

\[\text{Reference:}\]

1 The internal radiation fractional function also arises in models for radiation extinction in stellar interiors [8, §37] and other optically thick, non-gray gases [9], but those applications are unrelated to the present discussion.

2 The spectral hemispherical absorptivity and emissivity are equal if the irradiation is diffuse, $\alpha(\lambda, T_1) = \varepsilon(\lambda, T_1)$ [5, 10]. The main points of the present work also apply to spectral directional values and are so used in certain parts of this study.
EXTERNAL AND INTERNAL RADIATION FRACTIONS

The fraction of blackbody radiation between wavelengths of 0 and $\lambda$ is

$$f(\lambda T) = \frac{1}{\sigma T^4} \int_0^{\lambda} e_{\lambda,b} \, d\lambda \quad (8)$$

$$= 1 - \frac{15}{\pi^4} \int_0^{\lambda c_2/\lambda T} \frac{t^3}{e^t - 1} \, dt \quad (9)$$

$$= 1 - \frac{90}{\pi^4} \zeta(c_2/\lambda T, 4) \quad (10)$$

$\zeta(X, s)$ is the incomplete zeta function (see derivation in Appendix A). Edwards [1, 4, 5] refers to $f$ as the external radiation fraction. The distinct internal radiation fractional function, $f_i(\lambda T)$, arises from a linearization that allows $\varepsilon(T_s)$ and $\alpha(T, T_c)$ to be combined, as described next.

Edwards [1] defines the internal hemispherical emissivity such that

$$e^i(T_1) = \lim_{T_2 \to T_1} \frac{\sigma \varepsilon(T_1) T_1^4 - \sigma \alpha(T_1, T_2) T_2^4}{\sigma T_1^4 - \sigma T_2^4} \quad (11)$$

Because the numerator and denominator both go to zero in the limit, L'Hôpital's rule (or Taylor expansion) is required:

$$e^i(T_1) = \lim_{T_2 \to T_1} \frac{\sigma \varepsilon(T_1) \frac{\partial}{\partial T_1} e_{\lambda,b}(T_2) \, d\lambda}{4\sigma T_2^3} \quad (12)$$

$$= \int_0^\infty \alpha(\lambda, T_1) \left[ \frac{1}{4\sigma T_1^3} \frac{\partial}{\partial T_2} e_{\lambda,b}(T_2) \right]_{T_1} \, d\lambda \quad (13)$$

Thus, when $T_2$ is not too much different from $T_1$

$$q_{net} \approx e^i(T_1) \frac{4\sigma T_1^4}{4\sigma T_1^4 (T_1 - T_2)} \quad (14)$$

with

$$e^i(T) = \frac{1}{4\sigma T^3} \int_0^\infty \alpha(\lambda, T) \frac{\partial e_{\lambda,b}}{\partial T} \, d\lambda = \int_0^1 \alpha(\lambda, T) \, df(\lambda T) \quad (15)$$

Equation (15) stands in contrast to the external total hemispherical emissivity, which from Eqns. (1) and (8) maybe written

$$e(T) = \frac{1}{\sigma T^4} \int_0^\infty \alpha(\lambda, T) e_{\lambda,b}(T) \, d\lambda = \int_0^1 \alpha(\lambda, T) \, df(\lambda T) \quad (17)$$

From these relationships, one can show that

$$f_i(\lambda T) - f(\lambda T) = F(X) = \frac{15}{4\pi^4} \frac{X^4}{e^X - 1} \quad (18)$$

where $X = c_2/\lambda T$. The functions $f$, $f_i$, and $F$ are plotted as a function of $\lambda T$ in Fig. 1.

The numerical implementation is described in Appendix B.

DIFFERENCE BETWEEN EXTERNAL AND INTERNAL EMISSIVITIES

We now derive an upper bound on the difference of these two emissivities.

$$e - e^i = \int_0^1 \alpha(\lambda, T) \, df(\lambda T) - \int_0^1 \alpha(\lambda, T) \, df_i(\lambda T) \quad (19)$$

$$= \int_0^\infty \alpha(\lambda, T) \frac{dF}{dX} \, dX \quad (20)$$

$$= \int_0^X \alpha(\lambda, T) \frac{dF}{dX} \, dX + \int_X^{\infty} \alpha(\lambda, T) \frac{dF}{dX} \, dX \quad (21)$$

where $X = X_c$ is the finite zero point of $dF/dX$, $X_c = 3.92069$ (see Appendix C). The function $dF/dX$ is positive for $X < X_c$.
and negative for $X > X_z$, and $0 < \alpha(\lambda, T) \leq 1$. Thus we can form two bounds on the difference:

$$
\varepsilon - \varepsilon' \leq \int_0^{X_z} \frac{dF}{dX} dX = F(X_z) \quad \text{if} \quad \varepsilon - \varepsilon' > 0, \quad \text{and} \quad (22)
$$

$$
\varepsilon' - \varepsilon \leq \int_{X_z}^\infty \frac{dF}{dX} dX = F(X_z) \quad \text{if} \quad \varepsilon' - \varepsilon > 0
$$

(23)

A simple calculation shows that

$$
F(X_z) = \frac{15}{\pi^4} X_z^3 e^{-X_z} = 0.18400
$$

(24)

In other words,

$$
|\varepsilon - \varepsilon'| \leq 0.18400
$$

(25)

The value of $X_z$ corresponds to

$$
(\Delta T)_z = \frac{14387.77}{3.92069} = 3699.70 \mu m \cdot K
$$

(26)

At 300 K, the wavelength $\lambda_z = 12.2323 \mu m$.

The following two extreme examples are designed to illustrate these bounds. First, suppose that we have a 300 K surface that is black for $\lambda \leq 12.2323 \mu m$, but perfectly reflective on other wavelengths. Numerical integration of Eqns. (15) and (17) yields:

$$
\varepsilon = 0.4177, \quad \varepsilon' = 0.6017, \quad \text{and} \quad \varepsilon' - \varepsilon = 0.1840
$$

(27)

Equation (18) gives $\varepsilon'$, and the results for the long wavelength example may be obtained by subtracting these values from unity.

LINEARIZATION ERRORS

The linearization itself produces error in calculation of the heat flux, apart from non-gray error. To quantify this error, we
may compare the linearized and exact values of $q_{net}$ for a black surface. Their ratio is

$$
\frac{4T_1^3 \Delta T}{T_1^4 - T_2^4} = \frac{4T_1^3}{(T_1^2 + T_2^2)(T_1 + T_2)} = \frac{4}{1 + (T_2/T_1)^2(1 + T_2/T_1)}
$$

(Eqn. (32))

Eqn. (32) is plotted in Fig. 3. For $T_2 = 320$ K and $T_1 = 300$ K, the linearized flux is 90.5% of the exact flux. The ratio is asymptotic to $1 - 3\Delta T/2T_1 + 5(\Delta T)^2/4T_1^2$ as $\Delta T \to 0$.

The linearization around $T_1$ is considerably less accurate than the traditional approach (i.e., for radiation heat transfer coefficients) of linearizing around $T_m = (T_1 + T_2)/2$. For that case, the ratio is

$$
\frac{4T_m^3 \Delta T}{T_1^4 - T_2^4} = \frac{4T_m^3}{(2T_m^2 + \Delta T^2/2)(2T_m)} = \frac{1}{1 + \Delta T^2/4T_m^2}
$$

(Eqn. (33))

and has errors on the order of $(\Delta T/T_m)^2/4$ [11]. At seen in Fig. 3, this linearization is far more accurate. The reason for the greater accuracy is that linearization about $T_m$ is equivalent to a second-order Runge-Kutta method, with third-order truncation error in heat flux, while linearization about $T_1$ is simply a forward Euler method (see Appendix D).

From these considerations, we see that $\varepsilon'$ will also have a limited range of accuracy if evaluated at $T_1$. Instead, $\varepsilon'$ should be computed at $T_m$, following the analysis in Appendix D,

$$
\varepsilon'(T_m) = \frac{1}{4\sigma T_m^4} \int_0^\infty \alpha(\lambda, T_1) \frac{\partial \varepsilon_{\lambda, h}}{\partial T} \bigg|_{T_m} d\lambda = \int_0^1 \alpha(\lambda, T_1) df_{\lambda}(\lambda T_m)
$$

(Eqn. (34))

with the heat flux evaluated, to an accuracy of $O(\Delta T^3)$, as

$$
q_{net} \approx 4\varepsilon'(T_m) \cdot \sigma T_m^3 \Delta T
$$

(Eqn. (35))

The results for heat flux are compared in Fig. 4 and 5 for the same two example surfaces at $T_1 = 300$ K. The charts show:

1. The gray surface approximation, $\varepsilon = \alpha$, has the wrong slope as $T_2 \to T_1$, consistent with Eqn. 7. This approximation has poor accuracy for even the smallest temperature differences. The ratio of gray slope to exact slope is $\varepsilon/\varepsilon'$. Using the values from Eqn. (27) or (28), that ratio is equal to 0.69 and 1.46 for Figs. 4 and 5, respectively.

2. The approximation using $\varepsilon'(T_1)$ has the correct slope as $T_2 \to T_1$, but is accurate for only a limited range of temperatures. For the surface black on short wavelengths, with $T_2 = 320$ K the magnitude of the linearized flux from Eqn. (14) is 13% less than the exact value, Eqn. (4). This case corresponds to the seemingly small temperature difference $\Delta T/T_1 = 0.067$.

3. The approximation using $\varepsilon'(T_m)$ remains accurate over a very broad range of temperature variation, to the degree that the curves for this equation and the exact equation are difficult to distinguish. The charts in Fig. 4 and 5 include values as high as $\Delta T/T_1 = 0.33$.

Although Figs. 4 and 5 use 300 K as a reference point, Eqns. (29)–(31) show that, for a surface at $T_1 = T_2$ that switches from black to reflective behavior at $\lambda_z$, both $\varepsilon(T_z)$ and $\varepsilon'(T_z)$ are fixed. The values of $\varepsilon'(T_m) = \varepsilon'(T_z - \Delta T/2)$ and $\alpha(T_z, T_z - \Delta T)$ depend additionally only upon $\Delta T/T_z$. Hence, the trends seen will be effectively the same at any other temperature level.

Edwards [1, 4, 5] and, later, Mills [7] were both very specific in recommending to evaluate $\varepsilon'$ at $T_1$. Edwards et al. [6] did
not specify the temperature, but without comment included an example worked using $T_m$. Likewise, Czerny and Walther [2] provide an example worked using $T_m$, again without comment. The present results show clearly that only $T_m$ should be used when evaluating $e'$. 

**APPLICATION TO REPRESENTATIVE MATERIALS**

The previous example surfaces were contrived to maximize the differences between $e$ and $e'$. In this section, we examine the differences for representative non-gray materials.

**Alumina**

Experimental data for polycrystalline alumina (99.5% Al$_2$O$_3$, 6 mm thick, 1 μm roughness) have been reported by Teodorescu and Jones [12] for $T = 823$ K and $2 \leq \lambda \leq 25$ μm. The experimental uncertainty is 3.5%. This data is available in tabular form on 1 μm intervals and as a chart (Fig. 6). These results are generally consistent with other reports on polycrystalline alumina.

In this case, we first integrate the data to obtain the spectral normal emissivity, $e_n$. One additional data point is extrapolated, $e(1 \mu m, 823 K) \approx 0.128$, because $f_i$ at 2 μm still has the finite value 0.059; the extrapolation extends the range to $f_i = 0.00012$. At the other end of this spectrum, $f = 0.987$ at 25 μm, but no extrapolation was done. Thus, the value for total normal emissivity $e_n$ is thus likely to be low by ~1%. Integration of the discrete data by a simple trapezoidal rule yields the following values, both at 823 K:

$$e_n = 0.506, \quad e'_n = 0.404, \quad \text{and} \quad e'_n - e_n = 0.102 \quad (36)$$

**FIGURE 6. SPECTRAL NORMAL EMISSIVITY OF ALUMINA AT 823 K FROM TEODORESCU AND JONES [12]. REPRINTED BY PERMISSION FROM SPRINGER, J. MATER. SCI. © 2008**

Our interest, ultimately, lies with estimating heat exchange, so we require the spectral hemispherical emissivity. Teodorescu and Jones [12] also measured the spectral directional emissivity. We have integrated those data, as described in Appendix E, to obtain the spectral hemispherical values from 1 to 25 μm.

We consider exchange with an effectively black environment at temperatures down to 625 K. At 625 K, $f = 0.972$ at 25 μm, so that the absorptivity calculation may be low by 2% or so. The results are shown in Fig. 7. Once again, Eqn. (35) using $e'(T_m)$ is in excellent agreement with the exact result, Eqn. (4). 

**Drude/Hagens-Rubens Metal**

The Drude/Hagen-Rubens model for emissivity applies approximately to some metals in the infrared regime [5, 10, 13]. Baehr and Stephan [13] provide the following equation for the spectral, hemispherical emissivity under this model:

$$e(\lambda, T) = 48.70 \sqrt{\frac{r_e}{A}} \left[ 1 + \left( 31.62 + 6.849 \ln \left( \frac{r_e}{A} \right) \right) \sqrt{\frac{r_e}{A}} - 166.78 \frac{r_e}{A} + \cdots \right] \quad (37)$$

for $r_e/\lambda < 5 \times 10^{-4} \Omega \cdot \text{cm}/\mu \text{m}$, where $r_e(T)$ the electrical resistivity in $\Omega \cdot \text{cm}$.

We consider a surface at $T_1 = 373$ K with $r_e = 13.1 \times 10^{-6} \Omega \cdot \text{cm}$, a value representative of platinum. At this temperature, $0.01 < f < 0.99$ for $3.88 \leq \lambda \leq 61.3$ μm and $f_i = 0.01$ at $\lambda = 3.36$ μm. Over this range of wavelengths, $e(\lambda)$ varies from 0.086 down to 0.022 according to Eqn. (37). We may consider heat exchange with a surface at $T_2$ down to 275 K; at 275 K, $f > 0.99$ for $\lambda > 83.1$ μm. The integrations are therefore done from 3 to 100 μm. Results are shown in Fig. 8.
Once again, Eqn. (35) using \( \epsilon'(T_m) \) closely tracks the exact solution over a wide range (down to at least 300 K or \( \Delta T / \Delta T_1 = 20\% \)). In this instance, non-gray effects are present but less pronounced; the gray approximation, while not the best alternative for even small \( \Delta T \), shows less divergence than in previous examples. In addition, at 373 K

\[
\begin{align*}
\epsilon &= 0.0497, \quad \epsilon' = 0.0553, \quad \text{and} \quad \epsilon' - \epsilon &= 0.0056 \quad (38)
\end{align*}
\]

The internal and external emissivities are closer, as would be expected when non-gray effects are smaller.

**Spectrally Selective Materials**

We may also consider an idealized spectrally selective surface that makes a step transition at \( \lambda_c \) from a short-wavelength absorptivity, \( \alpha_{sw} \), to a long-wavelength absorptivity, \( \alpha_{lw} \):

\[
\alpha(\lambda) = \begin{cases} 
\alpha_{sw} & \text{for } \lambda \leq \lambda_c \\
\alpha_{lw} & \text{for } \lambda > \lambda_c 
\end{cases} \quad (39)
\]

From Eqns. (10) and (17),

\[
\begin{align*}
\epsilon(T) &= \alpha_{sw} f(\lambda_c, T) + \alpha_{lw} [1 - f(\lambda_c, T)] \\
&= \alpha_{sw} + \frac{90}{\pi^4} \Delta \alpha \zeta(X_c, 4) \quad (40)
\end{align*}
\]

\[
\alpha(T_1, T_2) = \alpha_{lw} + \frac{90}{\pi^4} \Delta \alpha \zeta(X_{c,2}, 4) \quad (46)
\]

with \( X_{c,2} = c_2 / \lambda_c T_2 \). The previous results show that the impact of selectivity will be greatest when \( X_c \) and \( X_e \) are close.

As an example, we may consider the following crude approximation to the soft-anodized aluminum described in [5]: \( \alpha_{sw} = 0.1, \alpha_{lw} = 0.85, \) and \( \lambda_c = 7 \mu m \). If this selective solar reflector is at \( T_1 = 360 \) K and exchanges radiation with sky at \( T_2 = 290 \) K, we obtain the results in Table 1. At these temperatures, most of the radiant energy is on wavelengths above 7 \( \mu m \), and non-grayiness is not pronounced. Nonetheless, as in previous examples, the agreement between Eqn. (4), 371.8 W/m\(^2\), and Eqn. (35), 371.0 W/m\(^2\), is excellent, whereas Eqn. (14), 462.1 W/m\(^2\), performs quite poorly.
ACCURACY VERSUS EFFORT

One rationale for introducing \( e^i \) is simply to avoid computing the total hemispherical emissivity and absorptivity separately when the spectral absorptivity has a significant wavelength dependence. The same labor-saving rationale was used in some older literature for setting \( \varepsilon(T_1) \approx \alpha(T_1, T_2) \) when the temperatures are close. At the time the internal fractional function was developed, personal computers (and pocket calculators, and spreadsheets, and symbolic integration, etc.) did not exist. Routine analyses required either tedious hand calculations or cumbersome trips to the mainframe computer center (to type up Hollerith cards and then wait for the SysOp to run the job!). So one attraction of the method was clearly to reduce labor.

Numerical integration is very easy today, but if a simple approximate formula or chart for the spectral hemispherical data is available, \( e^i(T_m) \) can still save some time in doing basic calculations. The greater challenge may be to obtain complete spectral, hemispherical data spanning the necessary range of wavelengths. If electromagnetic theory is used, working from the complex refractive index and spectral directional emissivity computations, much coding has to be undertaken, and the single integration saved by \( e^i \) will not be at all significant.

However, a second very important consideration is that a linearized expression for radiation heat transfer allows the use of a radiation thermal resistance, perhaps in parallel to a natural convection resistance, in network analyses of heat transfer. For an area \( A \), such a resistance has the form

\[
R_{\text{rad}} = \frac{1}{4\varepsilon\sigma T_m^4 A} \quad (47)
\]

\( R_{\text{rad}} \) is routinely evaluated using a gray body approximation for \( \varepsilon \). Clearly, using the internal emissivity, \( e^i(T_1) \), in Eqn. (47) provides a more appropriate value than a gray body approximation. The present work has shown \( e^i(T_m) \) to be the most robustly accurate among the three, with excellent performance for \( \Delta T / T_1 \) up to 30% or so. Despite modern computing power, the need for an accurate linearized radiation resistance remains as important today as it was 60 years ago. Thus, we conclude that the internal fractional function has an ongoing value when surfaces do not politely behave as “gray bodies.”

SUMMARY

The internal radiation fractional function has been in literature since the late 1950’s, although not widely adopted in textbooks covering thermal radiation. The most visible recommendations have been from Edwards and co-workers, generally advising the use of \( e^i(T_1) \) to calculate non-gray exchange for enclosures with modest temperature differences. In this study, the following new findings are reported:

1. The maximum absolute difference between \( e(T_1) \) and \( e^i(T_1) \) is 0.18400, Eqn. (25).
2. The internal emissivity should be evaluated at the mean temperature, \( T_m \), not \( T_1 \) as has often been suggested. The differences in accuracy are shown to be significant because the evaluation at \( T_m \) using Eqn. (35) is effectively a second-order accurate numerical method, with truncation error in the heat flux of \( O(\Delta T^3) \). In contrast, evaluation at \( T_1 \) is a first-order approximation.
3. Theory and examples for several non-gray materials show that the gray-body approximation gives the wrong slope for heat flux as \( T_2 \rightarrow T_1 \).
4. Calculations involving both the internal and external fractional functions can be conveniently implemented using the incomplete zeta function.
5. \( e^i(T_m) \) should be used in calculating radiation thermal resistances for non-gray surfaces.

ACKNOWLEDGEMENT

Don Edwards was on the faculty of UCLA’s CNTE Department when I studied there, although he was not my instructor. Nevertheless, I felt his positive influence through the curriculum and the other faculty in the heat transfer program.

REFERENCES

APPENDIX A: INCOMPLETE ZETA FUNCTION

The radiation fractional function may be written in terms of the incomplete zeta function for convenience:

\[ f(\lambda T) = \frac{1}{\sigma T^4} \int_0^4 \frac{2\pi \hbar c_o^2}{\lambda^3 \left[ \exp(\hbar c_o / k_B T \lambda) - 1 \right]} \, d\lambda \]

\[ = \frac{1}{\sigma T^4} \frac{2 \pi k_B T^4}{\hbar^3 c_o^3} \int_{c_o / \lambda T}^{\infty} \frac{t^3}{e^t - 1} \, dt \]

(48)

(49)

When \( \lambda T \to \infty \), \( f = 1 \) and the last equation yields the well-known result

\[ \sigma T^4 = \frac{2 \pi k_B T^4}{\hbar^3 c_o^3} \int_0^\infty \frac{t^3}{e^t - 1} \, dt = \zeta(4) \Gamma(4) \]

(50)

where the Gamma function \( \Gamma(4) = 3! \) and the Riemann zeta function, \( \zeta(4) \), has the indicated integral representation \([14, \S 13.12]\).

A classical result due to Euler \([15]\) gives \( \zeta(4) = \pi^4 / 90 \) (see also \([16, \S 167]\)), from which we recover the usual definition of the Stefan-Boltzmann constant, \( \sigma \). Returning to Eqn. (49) with this
information, we have

\[ f(\lambda T) = \frac{15}{\pi^4} \int_0^\infty \frac{t^3}{e^t - 1} \, dt - \frac{15}{\pi^4} \int_0^{c_2/\lambda T} \frac{t^3}{e^t - 1} \, dt \]

\[ = 1 - \frac{15}{\pi^4} \int_0^X \frac{t^3}{e^t - 1} \, dt \]

\[ = 1 - \frac{15}{\pi^4} \Gamma(4) \zeta(X, 4) \]

where \( X = c_2/\lambda T \) and we have identified the incomplete zeta function, \( \zeta(X, s) \) [17, §8.22]. Hence,

\[ f(\lambda T) = 1 - \frac{90}{\pi^4} \zeta(X, 4) \]

**APPENDIX B: NUMERICAL IMPLEMENTATION**

The integrals were computed using the GNU Scientific Library [18] with FFI bindings [19] to Lua code [20] under LuaLaTeX [21] using TeXShop over TeX Live [22]. The C code was compiled using XCode under Mac OS X. Integration was performed using GSL’s QAG adaptive integration with 61 point Gauss-Kronrod rules. Convergence was checked, and the numerical values given in the text are believed to be accurate to the usual process:

\[ \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} \]

This equation may be solved using the Lambert \( W \) function

\[ X_z = 4 - W(4e^{-4}) = 3.92069 \cdots \]

We may show by contradiction that \( X_z \) is irrational. Assuming \( X_z \) is rational, \( X_z = a/b \) for nonzero integers \( a \) and \( b \). Then the lefthand side of Eqn. (55) is \( 4 - 4e^{-a/b} \); however, \( e^{a/b} \) is itself irrational (see, e.g., [28, §4.7]). Hence, \( 4 - 4e^{-a/b} \) is an irrational number and cannot equal the assumed righthand side, \( a/b \).

Diophantine approximations to \( X_z \) may be constructed by calculating a continued fraction representation of \( X_z \) through the usual process:

\[ X_z \approx \left\{ \frac{4}{12}, \frac{149}{38}, \frac{247}{63}, \frac{1137}{290}, \ldots \right\} \]

The second of these is within 0.1% of the exact value; the last agrees to six digits.

**APPENDIX C: THE CONSTANT \( X_z \)**

\( X_z \) is the finite zero point of \( dF/dX \), specifically the nonzero root of

\[ 4 \left( 1 - e^{-X_z} \right) = X_z \]

The equation may be solved by iteration to find \( X_z = 3.92069 \cdots \). Alternatively, by defining \( y = 4 - X_z \), Eqn. (55) becomes

\[ ye^y = 4e^4 \]

Successive convergents (finite truncations) of the fraction give an infinite number of rational approximations to \( X_z \):

\[ X_z \approx \left\{ \frac{4}{12}, \frac{149}{38}, \frac{247}{63}, \frac{1137}{290}, \ldots \right\} \]

**APPENDIX D: LINEARIZATIONS AS FINITE DIFFERENCE APPROXIMATIONS**

We may consider \( q_{\text{net}} \) to be a function of \( T \) that we wish to determine approximately at \( T = T_2 \) by evaluating at other temperatures, i.e., \( T = T_1 \) and \( T = T_m \). With

\[ Y(T) = \sigma e(T_1)T_1^4 - \sigma \alpha(T_1, T)T^4 \]

we seek to estimate

\[ q_{\text{net}} = Y(T_2) = \sigma e(T_1)T_1^4 - \sigma \alpha(T_1, T_2)T_2^4 \]

A first-order Euler method starting at \( T = T_1 \) would approximate \( Y(T_2) \) based only on conditions at \( T_1 \) as follows:

\[ Y(T_2) \approx Y(T_1) + Y'(T_1) \cdot \delta T = Y'(T_1) \cdot \delta T \]
for $\delta T = T_2 - T_1 = -\Delta T$, which gives Eqn. (7) or (14). The local truncation error of this single-step approximation is $O(T''\Delta T^2)$ [29].

A second-order Runge-Kutta method works from $T_m$ with expansions toward both $T_1$ and $T_2$, subtracting the former from the latter:

$$Y(T_2) = Y(T_m) + Y'(T_m) \frac{\delta T}{2} + Y''(T_m) \frac{\delta T^2}{8} + O(\delta T^3) \tag{63}$$

$$Y(T_1) = Y(T_m) - Y'(T_m) \frac{\delta T}{2} + Y''(T_m) \frac{\delta T^2}{8} - O(\delta T^3) \tag{64}$$

$$Y(T_2) = Y(T_1) + Y'(T_m) \cdot \delta T + O(\delta T^3) \tag{65}$$

$$Y(T_2) \approx Y'(T_m) \cdot \delta T \tag{66}$$

This single-step Runge-Kutta approximation has local truncation error $O(T''\Delta T^2)$ [29]. In these calculations, $\alpha(\lambda, T_1)$ is evaluated at the surface temperature, $T_1$. With Eqns. (3) and (13),

$$Y''(T_m) = - \left. \frac{d}{dT} \left( \sigma T^4 \alpha(T_1, T) \right) \right|_{T_m} \tag{67}$$

$$= - \int_0^\infty \alpha(\lambda, T_1) \frac{\partial \varepsilon \lambda, \alpha}{\partial T} \bigg|_{T_m} \, d\lambda \tag{68}$$

$$= - 4\sigma T_m^3 \cdot \varepsilon'(T_m) \tag{69}$$

which results in Eqn. (35). Given lower truncation error of the Runge-Kutta approach, we should expect significantly better accuracy using Eqn. (35) than when using Eqn. (7) or (14); and indeed the numerical results confirm this conclusion.

**APPENDIX E: INTEGRATION OF DIRECTIONAL EMISIVITY FOR ALUMINA**

The spectral hemispherical emissivity was obtained by integration of the spectral directional emissivity data from [12] as:

$$\varepsilon(\lambda, T) = \int_0^{\pi/2} \varepsilon'(\theta, \lambda, T) \sin(2\theta) \, d\theta \tag{70}$$

The spectral directional emissivity data are in 12° increments of polar angle $\theta$ from 0° to 72°. In all cases, the data are essentially constant from 0 to 36°, and this range was integrated analytically. From 36° to 84° a five-point trapezoidal rule was used, and the integral from 84° to 90° was approximated as a trapezoid. The value at 90° was set to zero, in line with theory. This procedure was found to have a numerical truncation error of 1.0% for a gray surface.

3 Standard theory shows that for multi-step integration, the global error of Euler’s method is $O(\Delta T)$ [30, §9.3].