Large Primordial Trispectra in General Single Field Inflation

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Large Primordial Trispectra in General Single Field Inflation

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Abstract

We compute the large scalar four-point correlation functions in general single field inflation models, where the inflaton Lagrangian is an arbitrary function of the inflaton and its first derivative. We find that the leading order trispectra have four different shapes determined by three parameters. We study features in these shapes that can be used to distinguish among themselves, and between them and the trispectra of the local form. For the purpose of data analyses, we give two simple representative forms for these “equilateral trispectra”. We also study the effects on the trispectra if the initial state of inflation deviates from the standard Bunch-Davies vacuum.
1 Introduction

Primordial non-Gaussianity is potentially one of the most promising probes of the inflationary universe [1]. Like the role colliders play in particle physics, measurements of primordial non-Gaussian features provide microscopic information on the interactions of the inflatons and/or the curvatons. Constraining and detecting primordial non-Gaussianities has become one of the major efforts in modern cosmology. Theoretical predictions of non-Gaussianities, especially their explicit forms, play an important role in this program. On the one hand, they are needed as inputs of data analyses [2–5] which eventually constrain the parameters defining the non-Gaussian features; on the other hand, different forms of non-Gaussianities are associated with different inflaton or curvaton interactions, and so if detected can help us understand the nature of inflation.

A variety of potentially detectable forms of non-Gaussian features from inflation models have been proposed and classified, in terms of their shapes and running. The scalar three-point functions, i.e. the scalar bispectra, are by far the most well-studied. For single field inflation, a brief summary of the status is as follows. Minimal slow-roll inflation gives undetectable amount of primordial non-Gaussianities [6–8]; non-canonical kinetic terms can generate large bispectra of the equilateral shapes [9, 10]; non-Bunch-Davies vacuum can boost the folded shape [9,11,12]; and features in the Lagrangian (sharp or periodic) can give rise to large bispectra with oscillatory running [13, 14]. Multifield inflation models provide many other possibilities due to various kinds of isocurvature modes, such as curvatons [15], turning [16–21] or bifurcating [22, 23] trajectories, thermal effects [24, 25] and etc. These models give many additional forms of large bispectra, notably ones with a large local shape.

We will be getting much more data in the near future from new generations of experiments, ranging from cosmic microwave background, large scale structure and possibly even 21-cm hydrogen line. Compared with the current WMAP, these experiments will be measuring signals from shorter scales and/or in three dimensions. Therefore a significant larger number of modes will become available. This makes the study of four- or higher point functions interesting, as they provide information on new interaction terms and refined distinctions among models. In this paper we extend the work of Ref. [9] and classify the forms of large scalar trispectra (i.e. the scalar four-point function) in general single field inflation models. There have been some preliminary works in this direction [26, 27], calculating contributions from the contact interaction diagram (Fig. 1 (A)). For models with a large trispectrum, there is yet another set of diagrams involving the exchange of a scalar (Fig. 1 (B)) that contributes at the same order of magnitude. In this paper, we complete this

\footnote{Note that for slow-roll inflation, the contribution from the scalar-exchange diagram Fig. 1 (B) is sub-leading, while the graviton-exchange contribution belongs to the leading order [28].}
program and classify all possible shapes arising in this framework.

For the bispectra in general single field inflation, the leading large non-Gaussianities have two different shapes controlled by two parameters [9]. As we will see here, for trispectra, we have four different shapes controlled by three parameters. Some of them have complicated momentum-dependence. For the purpose of data analyses, we give simple representative shapes that can capture the main features of these functions. We point out the features in the shapes that can be used to distinguish among themselves, as well as to distinguish them from the trispectra of the local form. We also study the effects of a non-Bunch-Davies initial state of inflation on these trispectra.

This paper is organized as follows. In Section 2, we review the basic formalisms and main results for the power spectrum and bispectra in general single field inflation. In Section 3, we calculate the leading order trispectra, and summarize the final results. At leading order, the trispectra can be classified into four shapes, controlled by three parameters. In Section 4, we investigate the shapes of the trispectra, including consistency relations, figures in various limits, and also give two simple representative forms of these equilateral trispectra to facilitate future data analyses. In Section 5, we discuss DBI and K-inflation as two examples to illustrate our results. In Section 6, we study the trispectra when the initial state of inflation is in a non-Bunch-Davies vacuum. We conclude in Section 7.

2 Formalism and review

In this section, we review the formalisms and main results of Ref. [9]. As in Ref. [29], we consider the following general Lagrangian for the inflaton field $\phi$,

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ M_{Pl}^2 R + 2P(X, \phi) \right] ,$$

(2.1)

where $X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ and the signature of the metric is $(-1,1,1,1)$. 

Figure 1: Two diagrams that contribute to the large trispectra.
Irrespective of the specific mechanism that is responsible for the inflation, once it is achieved we require the following set of slow-variation parameters to be small,

\[ \epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = \frac{\dot{\epsilon}}{\epsilon H}, \quad s = \frac{\dot{c}_s}{c_s H}, \quad (2.2) \]

where \( H \) is Hubble parameter and

\[ c_s^2 = \frac{P_{,X} P_{,XX}}{P_{,X} + 2X P_{,XX}}, \quad (2.3) \]

is the sound speed. The slow-variation parameters can be large temporarily or quickly oscillating [13, 14, 30], but we do not consider such cases here.

The power spectrum \( P_\zeta \) is defined from the two-point function of the curvature perturbation \( \zeta \),

\[ \langle \zeta(k_1) \zeta(k_2) \rangle = (2\pi)^5 \delta^3(k_1 + k_2) \frac{1}{2k_1^3} P_\zeta. \quad (2.4) \]

For the class of inflation models that we consider,

\[ P_\zeta = \frac{1}{8\pi^2 M_{Pl}^2} \frac{H^2}{c_s \epsilon}. \quad (2.5) \]

In order to parametrize the three-point function, we need to define a parameter \( \lambda/\Sigma \) related to the third derivative of the inflaton Lagrangian \( P \) with respect to \( X \),

\[ \lambda = X^2 P_{,XX} + \frac{2}{3}X^3 P_{,XXX}, \quad (2.6) \]

\[ \Sigma = XP_{,X} + 2X^2 P_{,XX} = \frac{H^2 \epsilon}{c_s^2}. \quad (2.7) \]

The bispectrum form factor \( A(k_1, k_2, k_3) \) is defined as

\[ \langle \zeta(k_1) \zeta(k_2) \zeta(k_3) \rangle = (2\pi)^7 \delta^3(k_1 + k_2 + k_3) P_\zeta^2 \prod_{i=1}^{3} \frac{1}{k_i^3} A. \quad (2.8) \]

Up to \( \mathcal{O}(\epsilon) \), this bispectrum is determined by five parameters, \( c_s, \lambda/\Sigma, \epsilon, \eta \) and \( s \). For the most interesting cases \( c_s \ll 1 \) or \( \lambda/\Sigma \gg 1 \) where the non-Gaussianities are large, the leading bispectrum is given by

\[ A = \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right) \frac{3k_i^2 k_j^2 k_k^2}{2K^3} \]

\[ + \left( \frac{1}{c_s^2} - 1 \right) \left( -\frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_{i} k_i^3 \right). \quad (2.9) \]
So we have two different forms determined by two parameters. In such cases, the effect of the non-canonical kinetic terms of the inflaton has to become large enough so that the inflationary mechanism is no longer slow-roll (in slow-roll the canonical kinetic term dominates over the non-canonical terms). Since inflation gives approximately scale-invariant spectrum, ignoring the mild running of the non-Gaussianity [31], the bispectrum is approximately a function of two variables in terms of the momentum ratios, $k_2/k_1$ and $k_3/k_1$ [32]. The two forms in (2.9) have very similar shapes and they are usually referred to as the equilateral shapes. Because the two shapes do have a small difference, for fine-tuned model parameters $c_s$ and $\lambda/\Sigma$, they can cancel each other to a large extent and leave an approximately orthogonal component. One can use this component and the one of the originals to form a new bases of the shapes.\footnote{We would like to thank Eiichiro Komatsu for discussions on this point [33].}

3 Large trispectra

As in the bispectrum case, we are most interested in cases where the trispectra are large. In general single field inflationary models, this is achieved by the non-canonical kinetic terms. The origin of large non-Gaussianities come from terms with derivatives of the inflaton Lagrangian $P$ with respect to $X$. The contribution from the gravity sector is negligibly small. The derivative of $P$ with respective to $\phi$ is also small due to the approximate shift symmetry associated with the inflaton. Another equivalent way to see this is to work in the comoving gauge [6] where the scalar perturbation $\zeta$ only appears in the metric. So $P,\phi$ explicitly does not appear in the expansion. Using the leading order relation

$$\zeta \approx -\frac{H}{\dot{\phi}}\alpha$$

(3.1)

to convert $\zeta$ into $\alpha \equiv \delta \phi$, again we see that $P,\phi$ does not appear.

Therefore for our purpose, it is convenient to choose the inflaton gauge where the scalar perturbation only appears in the inflaton [6],

$$\phi = \phi_0(t) + \alpha(t,x) ;$$

(3.2)

and when we expand the inflaton Lagrangian $P$, we only concentrate on terms that have derivatives with respect to $X$. Such a method has also been used in Ref. [34, 35].
3.1 Scalar-exchange diagram

In this subsection, we compute the scalar-exchange diagram, Fig. (B). Using the inflaton gauge, we get the cubic terms of the Lagrangian in the small $c_s$ or large $\lambda/\Sigma$ limit,

$$\mathcal{L}_3 = \left(\frac{1}{2} P_{XX} \phi + \frac{1}{6} P_{XXX} \phi^3\right) a^3 \dot{\alpha}^3 - \frac{1}{2} P_{XX} \phi a \dot{\alpha} (\nabla \alpha)^2 .$$  (3.3)

Written in terms of $\zeta$ using (3.1), we get

$$\mathcal{L}_3 = -2a^3 \frac{\lambda}{H^3} \dot{\zeta}^3 + a \frac{\Sigma}{H^3} (1 - c_s^2) \dot{\zeta} (\partial \zeta)^2 .$$  (3.4)

Despite of its different appearance from the three leading cubic terms in [9], one can show, using the linear equation of motion and integration by part, that the difference is a total derivative.

In terms of the interaction Hamiltonian, $\mathcal{H}_3^I = -\mathcal{L}_3$, we denote the two terms in (3.4) as

$$H_3^I = -\int d^3x \mathcal{L}_3 = H_a + H_b ,$$  (3.5)

where

$$H_a(t) = 2a^3 \frac{\lambda}{H^3} \int \prod_{i=1}^3 \frac{d^3p_i}{(2\pi)^3} \dot{\zeta}_f(p_1, t) \zeta_t(p_2, t) \zeta_t(p_3, t) (2\pi)^3 \delta^3(\sum_{i=1}^3 p_i) ,$$  (3.6)

$$H_b(t) = a \frac{\Sigma}{H^3} (1 - c_s^2) \int \prod_{i=1}^3 \frac{d^3p_i}{(2\pi)^3} (p_2 \cdot p_3) \dot{\zeta}_f(p_1, t) \zeta_t(p_2, t) \zeta_t(p_3, t) (2\pi)^3 \delta^3(\sum_{i=1}^3 p_i) .$$  (3.7)

The $\zeta_f$ is in the interaction picture and satisfies the equation of motion followed from the kinematic Hamiltonian.

The scalar trispectrum is the expectation value of the curvature perturbation $\zeta_t^4$ in the interaction vacuum. According to the in-in formalism [36], there are three terms contributing to the diagram Fig. (B)\footnote{The in-in formalism is often used in the literature in terms of a commutator form which is equivalent to the form presented here. However, for a subset of terms, the algebra in the commutator form is more complicated than the one we use here. We discuss this equivalence in Appendix A.}

$$\langle \zeta_t^4 \rangle = \langle 0| \left[ Te^{-i \int_0^t dt H_I(t')} \right] \zeta_f(k_1, t) \zeta_f(k_2, t) \zeta_I(k_3, t) \zeta_I(k_4, t) \left[ T e^{-i \int_0^t dt H_I(t')} \right] |0 \rangle$$  (3.8)

$$\supset \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \langle 0| H_I(t') \zeta_t^4 H_I(t''') |0 \rangle$$

$$- \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \langle 0| H_I(t'') H_I(t') \zeta_t^4 |0 \rangle$$

$$- \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \langle 0| \zeta_t^4 H_I(t') H_I(t'') |0 \rangle .$$  (3.9)
Here \( t \) is a time several efolds after the modes exit the horizon and \( t_0 \) is a time when modes are all well within the horizon. In terms of the conformal time \( \tau \), \( dt = a(\tau)d\tau \), we take \( \tau = 0 \) and \( \tau_0 = -\infty \).

We evaluate (3.9) using the standard technique of normal ordering. We decompose (omitting the subscript “I” for \( \zeta \) in the following)

\[
\zeta(k, \tau) = \zeta^+ + \zeta^- = u(k, \tau)a_k + u^*(-k, \tau)a^+_k ,
\]

where

\[
u(k, \tau) = \frac{H}{\sqrt{4\epsilon c_s k^3}}(1 + ikc_s \tau)e^{-ikc_s \tau},
\]

and

\[
[a_k, a^+_p] = (2\pi)^3 \delta^3(k - p).
\]

After normal ordering, the only terms that are non-vanishing are those with all terms contracted. A contraction between the two terms, \( \zeta(k, \tau') \) (on the left) and \( \zeta(p, \tau'') \) (on the right), gives

\[
[\zeta^+(k, \tau'), \zeta^-(p, \tau'')] = u(k, \tau')u^*(-p, \tau'')(2\pi)^3 \delta^3(k + p).
\]

We sum over all possible contractions that represent the Feynman diagram Fig. 1 (B), where the four external legs are connected to \( \zeta(k, t) \)’s.

To give an example, we look at the 1st term of (3.9) with the component (3.6). One example of such contractions is

\[
\zeta^+\zeta^+\zeta^-\zeta^-=u(k, \tau'){u^*(-p, \tau'')}\delta^3(k + p).
\]

There are three ways of picking two of the three \( p_i \)’s \( (q_i \)’s), so we have a symmetry factor 9. Also, there are 24 permutations of the \( k_i \)’s. The overall contribution to the correlation
\[
9 \cdot 4 \frac{\lambda^2}{H^6} u_{k_1}^*(t)u_{k_2}^*(t)u_{k_3}(t)u_{k_4}(t) \times \left[ \int_{t_0}^{t_0} dt' \int_{t_0}^{t} dt'' a^3(t')a^3(t'')\dot{u}_{k_1}(t')\dot{u}_{k_2}(t')\dot{u}_{k_3}^*(t'')\dot{u}_{k_4}^*(t'')\dot{u}_{k_{12}}(t')\dot{u}_{k_{12}}^*(t'') \right] \times (2\pi)^3 \delta^3(\sum_{i=1}^4 k_i) + 23 \text{ perm.}
\]

\[
= \frac{9}{8} \left( \frac{\lambda}{\Sigma} \right)^2 \frac{k_{12}}{k_1k_2k_3k_4 (k_1 + k_2 + k_{12})^3 (k_3 + k_4 + k_{12})^3} \times (2\pi)^9 P_\zeta^3 \delta^3(\sum_{i=1}^4 k_i) + 23 \text{ perm.} ,
\]

where

\[
k_{12} = k_1 + k_2 .
\]

The 2nd and 3rd term in (3.9) has a time-ordered double integration, and so is more complicated. Their integrands are complex conjugate to each other, and we get

\[
\begin{align*}
&2 \cdot \frac{9}{8} \left( \frac{\lambda}{\Sigma} \right)^2 \frac{k_{12}}{k_1k_2k_3k_4 (k_1 + k_2 + k_{12})^3 (k_3 + k_4 + k_{12})^3} \times \left[ 6 \frac{(k_1 + k_2 + k_{12})^2}{K^5} + 3 \frac{k_1 + k_2 + k_{12}}{K^4} + 1 \right] \\
&\times (2\pi)^9 P_\zeta^3 \delta^3(\sum_{i=1}^4 k_i) + 23 \text{ perm.} ,
\end{align*}
\]

where

\[
K = k_1 + k_2 + k_3 + k_4 .
\]

The other terms are similarly computed. We leave the details to Appendix B.

### 3.2 Contact-interaction diagram

In this subsection, we compute the contact-interaction diagram, Fig. I (A). We define

\[
\mu \equiv \frac{1}{2} X^2 P_{XX} + 2X^3 P_{XXX} + \frac{2}{3} X^4 P_{XXXX} .
\]

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\[4\] The integrations are conveniently done in terms of the conformal time \( \tau \). Integrals such as \( \int_{-\infty}^{0} dx \ x^2 e^{\pm i x} = \pm 2i \) are constantly used in the evaluation in this paper. As in [6], the convergence at \( x \to -\infty \) is achieved by \( x \to x(1 \mp i \epsilon) \).
The fourth order expansion is [26]
\[ \mathcal{L}_4 = a^3 \frac{\mu}{H^4} \dot{\zeta}^4 - \frac{a}{H} \left( 3\lambda - \Sigma(1-c_s^2) \right) (\partial \zeta)^2 \dot{\zeta}^2 + \frac{1}{4aH^4} \Sigma(1-c_s^2)(\partial \zeta)^4. \] (3.20)

Generally speaking, the Lagrangian of the form
\[ \mathcal{L}_2 = f_0 \dot{\zeta}^2 + j_2, \] (3.21)
\[ \mathcal{L}_3 = g_0 \dot{\zeta}^3 + g_1 \dot{\zeta} + j_3, \] (3.22)
\[ \mathcal{L}_4 = h_0 \dot{\zeta}^4 + h_1 \dot{\zeta}^3 + h_2 \dot{\zeta}^2 + h_3 \dot{\zeta} + j_4 \] (3.23)
gives the following interaction Hamiltonian at the fourth order in \( \dot{\zeta} \) [26],
\[ \mathcal{H}_4 = \left( \frac{9g_0^2}{4f_0} - h_0 \right) \dot{\zeta}_4^4 + \left( \frac{3g_0g_1}{f_0} - h_1 \right) \dot{\zeta}_4^3 + \left( \frac{3g_0g_2}{2f_0} + \frac{g_1^2}{f_0} - h_2 \right) \dot{\zeta}_4^2 + \left( \frac{g_1g_2}{f_0} - h_3 \right) \dot{\zeta}_4 + \frac{g_2^2}{4f_0} - j_4, \] (3.24)
where \( f, g, h \) and \( j \)'s are functions of \( \zeta, \partial_i \zeta \) and \( t \), and the subscripts denote the orders of \( \zeta \). So for (3.20) we have
\[ \mathcal{H}_4 = \frac{a^3}{H^4} (\mu - 9 \frac{\lambda^2}{\Sigma}) \dot{\zeta}_4^4 + \frac{a}{H} \left( 3\lambda c_s^2 - \Sigma(1-c_s^2) \right) (\partial \zeta)^2 \dot{\zeta}_4^2 + \frac{1}{4aH^4} \Sigma(-c_s^2 + c_s^4)(\partial \zeta_4)^4. \] (3.25)

Note that in the second term the order \( \lambda \) term cancelled, in the third term the order \( \Sigma \) term cancelled.

The following are the contributions to the form factor \( T \) defined as
\[ \langle \zeta^4 \rangle = (2\pi)^9 P^{3} \delta^3(\sum_{i=1}^{4} k_i) \prod_{i=1}^{4} \frac{1}{k_i^3} \mathcal{T}. \] (3.26)

The contribution from the first term in (3.25) is
\[ 36 \left( \frac{\mu}{\Sigma} - 9 \frac{\lambda^2}{\Sigma^2} \right) \frac{\prod_{i=1}^{4} k_i^2}{K^5}. \] (3.27)

from the second term,
\[ -\frac{1}{8} \left( 3\lambda - 1 \right) \sum_{i<j} k_i k_j (k_3 \cdot k_4) \left[ 1 + \frac{3(k_3 + k_4)}{K} + \frac{12k_3k_4}{K^2} \right] + 23 \text{ perm.}; \] (3.28)

from the third term,
\[ \frac{1}{32} \left( \frac{1}{c_s^2} - 1 \right) \frac{(k_1 \cdot k_2)(k_3 \cdot k_4)}{K} \left[ 1 + \sum_{i<j} k_i k_j K^2 + \frac{3k_1 k_2 k_3 k_4}{K^3} \sum_{i=1}^{4} \frac{1}{k_i} + 12 \frac{k_1 k_2 k_3 k_4}{K^4} \right] + 23 \text{ perm.}. \] (3.29)


3.3 Summary of final results

Here we summarize the final results from Sec. 3.1.3.2 and Appendix B. For the general single field inflation \( \mathcal{L}(\phi, X) \), we define

\[
    c_s^2 \equiv \frac{P_{X \phi}}{P_{X \phi} + 2X P_{XX}}, \\
    \Sigma \equiv XP_{X \phi} + 2X^2 P_{XX}, \\
    \lambda \equiv X^2 P_{XX} + \frac{2}{3} X^3 P_{XXX}, \\
    \mu \equiv \frac{1}{2} X^2 P_{XX} + 2X^3 P_{XXX} + \frac{2}{3} X^4 P_{XXXX}.
\]

(3.30)

If any of \( \mu/\Sigma, \lambda^2/\Sigma^2, 1/c_s^4 \gtrsim 1 \), the single field inflation generates a large primordial trispectrum, whose leading terms are given by

\[
    \langle \zeta^4 \rangle = (2\pi)^9 P_\zeta^3 \delta^3 \left( \sum_{i=1}^4 \frac{1}{k_i} \right) \prod_{i=1}^4 T(k_1, k_2, k_3, k_4, k_{12}, k_{14}) ,
\]

(3.31)

where \( T \) has the following six components:

\[
    T = \left( \frac{\lambda}{\Sigma} \right)^2 T_{s1} + \frac{\lambda}{\Sigma} \left( \frac{1}{c_s^2} - 1 \right) T_{s2} + \left( \frac{1}{c_s^2} - 1 \right)^2 T_{s3} + \left( \frac{\mu}{\Sigma} - \frac{9\lambda^2}{\Sigma^2} \right) T_{c1} \\
    + \left( \frac{3\lambda}{\Sigma} - \frac{1}{c_s^2} + 1 \right) T_{c2} + \left( \frac{1}{c_s^2} - 1 \right) T_{c3}.
\]

(3.32)

The \( T_{s1,s2,s3} \) are contributions from the scalar-exchange diagrams and are given in Appendix B. \( T_{c1,c2,c3} \) are contributions from the contact-interaction diagram and are given by (3.27)-(3.29). For the most interesting cases, where any of \( \mu/\Sigma, \lambda^2/\Sigma^2, 1/c_s^4 \gg 1 \), the first four terms in (3.32) are the leading contributions. So we have four shapes determined by three parameters, \( \lambda/\Sigma, 1/c_s^2 \) and \( \mu/\Sigma \). A large bispectrum necessarily implies a large trispectrum, because either \( 1/c_s^4 \) or \( (\lambda/\Sigma)^2 \) is large. But the reverse is not necessarily true. One can in principle have a large \( \mu/\Sigma \) but small \( 1/c_s^4 \) and \( (\lambda/\Sigma)^2 \).

To quantify the size (i.e., magnitude) of the non-Gaussianity for each shape, we define the following estimator \( t_{NL} \) for each shape component,

\[
    \langle \zeta^4 \rangle_{\text{component}} \to \text{RT} \frac{RT \lim}{(2\pi)^9 P_\zeta^3 \delta^3 \left( \sum_{i} \frac{1}{k_i} \right) \prod_{i} \frac{1}{k_i^9} t_{NL}} ,
\]

(3.33)

where the RT limit stands for the regular tetrahedron limit \( (k_1 = k_2 = k_3 = k_4 = k_{12} = k_{14} = k) \). The parameter \( t_{NL} \) is analogous to the \( f_{NL} \) parameter for bispectra. This definition

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3More generally, we have six shapes controlled by three parameters. However, the second line of (3.32) are negligible unless \( \mu/\Sigma, \lambda^2/\Sigma^2, 1/c_s^4 \) are all \( \sim 1 \), in which case the trispectra is only marginally large, \( \sim \mathcal{O}(1) \). Also note that, for \( \mu/\Sigma, \lambda^2/\Sigma^2, 1/c_s^4 \gg 1 \), the second line of (3.32) does not capture all the subleading contributions.
applies to both the cases of interest here, and the non-Gaussianities of the local form that we will discuss shortly. Unlike the convention in the bispectrum case where the normalization of $f_{NL}$ is chosen according to the local form non-Gaussianity, here we conveniently choose the normalization of $t_{NL}$ according to \( (3.33) \). This is because, for the trispectra, even the local form has two different shapes.

The size of non-Gaussianity for each shape in \( (3.32) \) is then given by

\[
\begin{align*}
t_{s1}^{NL} &= 0.250 \left( \frac{\lambda}{\Sigma} \right)^2, \\
t_{s2}^{NL} &= 0.420 \frac{\lambda}{\Sigma} \left( \frac{1}{c_s^2} - 1 \right), \\
t_{s3}^{NL} &= 0.305 \left( \frac{1}{c_s^2} - 1 \right)^2, \\
t_{c1}^{NL} &= 0.0352 \left( \frac{\mu}{\Sigma} - \frac{9\lambda^2}{\Sigma^2} \right), \\
t_{c2}^{NL} &= 0.0508 \left( \frac{3\lambda}{\Sigma} - \frac{1}{c_s^2} + 1 \right), \\
t_{c3}^{NL} &= 0.0503 \left( \frac{1}{c_s^2} - 1 \right)
\end{align*}
\] (3.34)

For comparison, let us also look at the trispectrum of the local form. This is obtained from the ansatz in real space \([39, 40]\),

\[
\zeta(x) = \zeta_g + \frac{3}{5} f_{NL} (\zeta_g^2 - \langle \zeta_g^2 \rangle) + \frac{9}{25} g_{NL} (\zeta_g^3 - 3\langle \zeta_g^2 \rangle\zeta_g),
\] (3.35)

where $\zeta_g$ is Gaussian and the shifts in the 2nd and 3rd terms are introduced to cancel the disconnected diagrams. Such a form constantly arises in multi-field models, where the large non-Gaussianities are converted from isocurvature modes at super-horizon scales. The resulting trispectrum is

\[
T = f_{NL}^2 T_{loc1} + g_{NL} T_{loc2}.
\] (3.36)

The two shapes are

\[
T_{loc1} = \frac{9}{50} \left( \frac{k_1^3 k_2^3}{k_{13}^3} + 11 \text{ perm.} \right),
\] (3.37)

\[
T_{loc2} = \frac{27}{100} \sum_{i=1}^4 k_i^3,
\] (3.38)

where the 11 permutations includes $k_{13} \rightarrow k_{14}$ and 6 choices of picking two momenta such as $k_1$ and $k_2$. The size of the trispectrum for each shape is

\[
t_{loc1}^{NL} = 2.16 f_{NL}^2, \\
t_{loc2}^{NL} = 1.08 g_{NL}.
\] (3.39)

So again a large bispectrum implies a large trispectrum, but not reversely.

### 4 Shapes of trispectra

In this section, we investigate the shape of the trispectra. We take various limits of the shape functions $T_{s1}$, $T_{s2}$, $T_{s3}$ and $T_{c1}$, and then compare among themselves, and with the local shapes $T_{loc1}$ and $T_{loc2}$. We will summarize the main results at the end of this section.
Before the discussion of the shape functions, we note that the arguments of the shape functions are six momenta $k_1, k_2, k_3, k_4, k_{12}, k_{14}$. In order for these momenta to form a tetrahedron (as in Fig. 2), the following two conditions are required:

Firstly, we define three angles at one vertex:

$$
\cos(\alpha) = \frac{k_1^2 + k_{14}^2 - k_{12}^2}{2k_1 k_{14}}, \\
\cos(\beta) = \frac{k_2^2 + k_{14}^2 - k_{12}^2}{2k_2 k_{14}}, \\
\cos(\gamma) = \frac{k_2^2 + k_2^2 - k_{12}^2}{2k_2 k_2}.
$$

These three angles should satisfy $\cos(\alpha - \beta) \geq \cos(\gamma) \geq \cos(\alpha + \beta)$. This inequality is equivalent to

$$
1 - \cos^2(\alpha) - \cos^2(\beta) - \cos^2(\gamma) + 2 \cos(\alpha) \cos(\beta) \cos(\gamma) \geq 0.
$$

Secondly, the four momenta should satisfy all the triangle inequalities. We need

$$
k_1 + k_4 > k_{14}, \quad k_1 + k_2 > k_{12}, \quad k_2 + k_3 > k_{14},
\k_1 + k_{14} > k_4, \quad k_1 + k_{12} > k_2, \quad k_2 + k_{14} > k_3,
\k_4 + k_{14} > k_1, \quad k_2 + k_{12} > k_1, \quad k_3 + k_{14} > k_2.
$$

The last triangle inequality involving $(k_3, k_4, k_{12})$ is always satisfied given Eq. (4.2) and Eqs. (4.3).
We also would like to mention a symmetry in our trispectrum. As \(k_1, k_2, k_3, k_4\) are symmetric in our model, we have

\[
\mathcal{T}(k_1, k_2, k_3, k_4, k_{12}, k_{14}) = \mathcal{T}(k_1, k_2, k_4, k_3, k_{12}, k_{13}) = \mathcal{T}(k_1, k_3, k_2, k_4, k_{13}, k_{14}) ,
\]

and etc, where

\[
k_{13} \equiv |k_1 + k_3| = \sqrt{k_1^2 + k_2^2 + k_3^2 + k_4^2 - k_{12}^2 - k_{14}^2} .
\]

The first set of limits we would like to take is those involved in the consistency relations. There are two known consistency relations for the trispectra to satisfy.

Firstly, we discuss the consistency relation in the squeezed limit. When one external momentum, say, \(k_4\) goes to zero, this mode can be treated as a classical background for the other modes, and the trispectrum should reduce to the product of a power spectrum and a running of bispectrum \([26, 37, 38]\):

\[
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle \sim -\mathcal{P}(k_4) \frac{d}{d \ln a} \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle ,
\]

where \(\mathcal{P}(k) \equiv \frac{2\pi^2}{k^3} P_\zeta(k)\), and \(P_\zeta(k)\) is the dimensionless power spectrum.

In our case, the leading order contribution to the trispectra scales as \(c_s^{-4}\), or \(c_s^{-2} \lambda/\Sigma\), or \((\lambda/\Sigma)^2\), or \(\mu/\Sigma\). However, RHS scales as \(c_s^{-3}\), or \(c_s^{-1} \lambda/\Sigma\). In order that Eq. (4.6) holds, \(k_4^3 \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle\) must vanish at the leading order in the \(k_4 \to 0\) limit. One can check that our result indeed vanish in this limit, \(T_{s1,2,3, T_{c1}} \to \mathcal{O}(k_4^2)\).

Secondly, we check the folded limit, say \(k_{12} \to 0\). For the \(s\)-channel (in which the exchanged scalar carries the momentum \(k_{12}\)), the four-point function can be regarded as a pair of two-point functions modulated by the same classical background generated by the long wave mode \(k_{12}\), and we have \([28]\)

\[
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle \sim (n_s - 1)^2 \mathcal{P}(k_1) \mathcal{P}(k_3) \langle \zeta_{k_{12}} \zeta_{k_{12}} \rangle .
\]

Note that the RHS takes the same shape as \((3.37)\). Again, RHS scales as \(c_s^{-3}\) in our case. So in the \(k_{12} \to 0\) limit, we expect \(k_{12}^3 \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle\) to vanish for the \(s\)-channel. For the \(t\)-, \(u\)- and the contact interaction channels, there are neither propagators that give rise to the pole behavior \(1/k_{12}^3\) nor inverse Laplacians in our Lagrangian. Therefore in our case \(k_{12}^3 \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle \to 0\) trivially for these channels.

One can check that our results indeed satisfy the condition. In fact, we have \(\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle \to \mathcal{O}(k_{12})\) for the \(s\)-channel, so the pole behavior at \(k_{12} = 0\) is cancelled more than enough to satisfy the condition. (Note that summing over all channels gives \(\langle \zeta^4 \rangle \to \text{constant}\).)

After checking the consistency relations, now we shall plot the shape functions. To do so, we shall take various limits to reduce the number of variables. We set the shape function to zero when the momenta do not form a tetrahedron. We consider the following cases:
1. Equilateral limit: \( k_1 = k_2 = k_3 = k_4 \). In Fig. 3 we plot \( T_{s1}, T_{s2}, T_{s3}, T_{c1}, T_{loc1} \) and \( T_{loc2} \) as functions of \( k_{12}/k_1 \) and \( k_{14}/k_1 \). (We would like to remind the reader that unlike the first four shape functions, \( T_{loc1} \) and \( T_{loc2} \) are not obtained in our model. We plot them for the purpose of comparison.) One observes that \( T_{loc1} \) blows up at all boundaries. This feature can distinguish our shape functions from the local shape \( T_{loc1} \) originated from the local \( f_{NL} \).

2. Folded limit: \( k_{12} = 0 \). (This limit is also related to the parallelogram limit, \( k_1 = k_3 \), by the symmetry \( (4.4) \).) In this limit, \( k_1 = k_2 \) and \( k_3 = k_4 \). We plot \( T_{s1}, T_{s2}, T_{s3}, T_{c1} \) and \( T_{loc2} \) as functions of \( k_4/k_1 \) and \( k_{14}/k_1 \) in Fig. 4 (Note that \( T_{loc1} \) blows up in this limit). We assumed \( k_4 < k_1 \) without losing generality. Note that \( T_{loc2} \) does not vanish in the \( k_4 \to 0 \) limit. This can be used to distinguish our shape functions from the local shape originated from \( g_{NL} \).

3. Specialized planar limit: We take \( k_1 = k_3 = k_{14} \), and additionally the tetrahedron to be a planar quadrangle. In this limit, one can solve for \( k_{12} \) from (4.2):

\[
k_{12} = \left[ k_1^2 + \frac{k_2 k_4}{2k_1^2} \left( k_2 k_4 \pm \sqrt{(4k_1^2 - k_2^2)(4k_1^2 - k_4^2)} \right) \right]^{1/2}.
\]  

(4.8)

The minus sign solution can be related to another plus sign solution in the \( k_1 = k_2 = k_{14} \) limit through a symmetry discussed in Appendix C. We will only consider the plus sign solution in our following discussion. We plot the shape functions as functions of \( k_2/k_1 \) and \( k_4/k_1 \) in Fig. 5. These figures illustrate two important distinctions between our shape functions and the local form shape functions. At the \( k_2 \to k_4 \) limit, we have \( k_{13} \to 0 \), so \( T_{loc1} \) blows up, while the others are all finite. At the \( k_2 \to 0 \) and \( k_4 \to 0 \) boundaries, our shapes functions vanish as \( O(k_2^2) \) and \( O(k_4^2) \) respectively, while \( T_{loc1} \) and \( T_{loc2} \) are non-vanishing.

4. Near the double-squeezed limit: we consider the case where \( k_3 = k_4 = k_{12} \) and the tetrahedron is a planar quadrangle. We are interested in the behavior of the shape functions as \( k_3 = k_4 = k_{12} \to 0 \), i.e. as the planar quadrangle is doubly squeezed. In this case, Eq. (4.2) takes the equal sign. One can solve for \( k_2 \) from (4.2). The solution is presented in Eq. (C.1). We plot \( T_{s1}/(\prod_{i=1}^4 k_i), T_{s2}/(\prod k_i), T_{s3}/(\prod k_i), T_{c1}/(\prod k_i), T_{loc1}/(\prod k_i) \) and \( T_{loc2}/(\prod k_i) \) as functions of \( k_{12}/k_1 \) and \( k_{14}/k_1 \) in Fig. 6. To reduce the range of the plot, we only show the figures partially with \( k_4 < k_1 \). Note that in this figure, we divided the shape functions by \( \prod k_i \) in order to have better distinction between contact-interaction and scalar-exchange contributions. Fig. 6 shows simultaneously the three differences among the four shapes \( T_{s1} (\sim T_{s2,3}), T_{c1}, T_{loc1} \) and \( T_{loc2} \).  

1) In the double-squeezed limit, \( k_3 = k_4 \to 0 \), the scalar-exchange contributions \( T_{s1}/(\prod k_i), \)
$T_{s2}/(\prod k_i)$, $T_{s3}/(\prod k_i)$ are nonzero and finite, and the contact-interaction $T_{c1}/(\prod k_i)$ vanishes. As a comparison, the local form terms $T_{loc1}/(\prod k_i)$ and $T_{loc2}/(\prod k_i)$ blow up.  2) In the folded limits, at the $(k_4/k_1 = 1, k_{14}/k_1 = 0)$ corner where $k_{14} \to 0$, and close to the $(k_4/k_1 = 1, k_{14}/k_1 = 2)$ area where $k_{13} \to 0$, $T_{loc1}/(\prod k_i)$ blows up.  3) In the squeezed limit, at $(k_4/k_1 = 1, k_{14}/k_1 = 1)$ where $k_2 \to 0$, the $T_{loc1}/(\prod k_i)$ and $T_{loc2}/(\prod k_i)$ blow up. The last two behaviors have also appeared in the previous figures.

In the second, third and fourth limits, the tetrahedron reduce to a planar quadrangle. We collectively denote this group of limits as the planar limit. This planar limit is of special importance, because one of the most important ways to probe trispectrum is the small (angular) scale CMB experiments. These experiments directly measure signals contributed mainly from the planar quadrangles. The more general plot for the planar limit is presented in Appendix C. We can see that while very different from the two local shapes, the three shapes $T_{s1}$, $T_{s2}$ and $T_{s3}$ are overall similar. Of course like in the bispectrum case, we can tune the parameters to subtract out the similarities and form new bases for the shapes.

We end this section by emphasizing a couple of important points:

- **The equilateral trispectra forms**: The scalar-exchange contributions $T_{s1,2,3}$ and the contact-interaction contribution $T_{c1}$ are similar at most regions, but can be distinguished in the double-squeezed limit (e.g. $k_3 = k_4 \to 0$), where the two kinds of forms approach zero at different speeds, $T_{s1,2,3} \to \mathcal{O}(k_3^2)$, $T_{c1} \to \mathcal{O}(k_3^4)$. Within the scalar-exchange contributions, the three shapes $T_{s1}$, $T_{s2}$, $T_{s3}$ are very similar overall, having only small differences.  

For the purpose of data analyses, one can then use the following two representative forms for the “equilateral trispectra”. One is $T_{c1}$, given in (3.27). This ansatz can be used to represent all four leading shapes at most regions. For a refined data analysis, for example to distinguish shapes in the double-squeezed limit, one can add another form $T_{s1}$, given in [B.3] and [B.4]. This ansatz represents very well the three scalar-exchange contributions $T_{s1,2,3}$. The first ansatz is factorizable (in terms of the six variables $k_{1,2,3,4, k_{12}, k_{14}}$) by introducing an integral $1/K^n = (1/\Gamma(n)) \int_0^\infty t^{n-1} e^{-Kt}$ [4]; while the second ansatz cannot be easily factorized due to the presence of $k_{13}$ given by [4.31].

- **Distinguishing between the equilateral and local forms**: In the following limits, the equilateral and local forms behave very differently. At the folded limit (e.g. $k_{12} \to 0$

---

6For example, in Fig. 3 or 4 if we look at the double folded limit, $k_{12} \to 0$ and $k_{14} \to 0$, $T_{s2}$ and $T_{s3}$ go from positive to negative ($T_{s2} \to -0.066$ and $T_{s3} \to -0.030$), while $T_{s1}$ remains positive ($T_{s1} \to 0.092$).
Figure 3: In this group of figures, we consider the equilateral limit $k_1 = k_2 = k_3 = k_4$, and plot $T_{s1}$, $T_{s2}$, $T_{s3}$, $T_{c1}$, $T_{loc1}$ and $T_{loc2}$, respectively, as functions of $k_{12}/k_1$ and $k_{14}/k_1$. Note that $T_{loc1}$ blows up when $k_{12} \ll k_1$ and $k_{14} \ll k_1$. $T_{loc1}$ also blows up in the other boundary, because this boundary corresponds to $k_{13} \ll k_1$. So $T_{loc1}$ is distinguishable from all other shapes in this limit. We also note that $T_{c1}$ and $T_{loc2}$ are both independent of $k_{12}$ and $k_{14}$. 
Figure 4: In this group of figures, we consider the folded limit $k_{12} = 0$, and plot $T_s_1$, $T_s_2$, $T_s_3$, $T_c_1$ and $T_{loc_2}$, respectively, as functions of $k_{14}/k_1$ and $k_4/k_1$. $T_{loc_1}$ blows up in this limit. Note that when $k_4 \to 0$, all shape functions except $T_{loc_1}$ and $T_{loc_2}$ vanish.
Figure 5: In this group of figures, we consider the specialized planar limit with $k_1 = k_3 = k_{14}$, and plot $T_{s1}, T_{s2}, T_{s3}, T_{c1}, T_{loc1}$ and $T_{loc2}$, respectively, as functions of $k_2/k_1$ and $k_4/k_1$. Again, in the $k_2 \to 0$ or $k_4 \to 0$ limit, our shape functions vanish as $O(k_2^2)$ and $O(k_4^2)$ respectively. This is different from that of the local shape. $T_{loc1}$ blows up when $k_2 \to k_4$. This is because in this limit, $k_{13} \to 0$. 
Figure 6: In this group of figures, we look at the shapes near the double squeezed limit: we consider the case where $k_3 = k_4 = k_{12}$ and the tetrahedron is a planar quadrangle. We plot $T_{s1}/(\prod_{i=1}^{4} k_i), T_{s2}/(\prod k_i), T_{s3}/(\prod k_i), T_{c1}/(\prod k_i), T_{loc1}/(\prod k_i)$ and $T_{loc2}/(\prod k_i)$, respectively, as functions of $k_{12}/k_1$ and $k_{14}/k_1$. Note that, taking the double-squeezed limit $k_4 \to 0$, the scalar-exchange contributions $T_{s1}/(\prod k_i), T_{s2}/(\prod k_i), T_{s3}/(\prod k_i)$ are nonzero and finite, and the contact-interaction $T_{c1}/(\prod k_i)$ vanishes. As a comparison, the local form terms $T_{loc1}/(\prod k_i)$ and $T_{loc2}/(\prod k_i)$ blow up. The different behaviors in the folded and squeezed limit can also be seen from this figure (see the main text for details).
0), $T_{\text{loc}1}$ generically blows up, while the four equilateral shapes and $T_{\text{loc}2}$ approach constants. At the squeezed limit (e.g. $k_4 \to 0$), the four equilateral shapes all vanish as $O(k_4^2)$, while the local forms $T_{\text{loc}1}$ and $T_{\text{loc}2}$ do not.

5 Examples

For DBI inflation [41–46], $P = -f(\phi)^{-1} \sqrt{1 - 2X f(\phi)} + f(\phi)^{-1} - V(\phi)$,

$$c_s \ll 1, \quad \frac{\lambda}{\Sigma} = \frac{1}{2} \left( \frac{1}{c_s^2} - 1 \right), \quad \frac{\mu}{\Sigma} = \frac{1}{4} \left( \frac{5}{c_s^2} - 4 \right) \left( \frac{1}{c_s^2} - 1 \right). \quad (5.1)$$

The dominant contribution come from the scalar-exchange terms $T_{s1,s2,s3}$ and one contact-interaction term $T_{c1}$, which are of order $1/c_s^4$. The $T_{c2}$ and $T_{c3}$ are of order $1/c_s^2$, so belong to subleading contributions. Therefore the shape function defined in (3.32) takes the form

$$T^{\text{DBI}} \approx \left( \frac{T_{s1}}{4} + \frac{T_{s2}}{2} + T_{s3} - T_{c1} \right) \frac{1}{c_s^4}. \quad (5.2)$$

According to the definition (3.33), $t_{\text{NL}}^{\text{DBI}} \approx 0.542/c_s^4$.

For k-inflation [47–49], we look at the example $P \sim (-X + X^2)/\phi^2$,

$$c_s \ll 1, \quad \frac{\lambda}{\Sigma} = \frac{2X}{-1 + 6X} = \frac{1 - c_s^2}{2}, \quad \frac{\mu}{\Sigma} = \frac{X}{-1 + 6X} = \frac{1 - c_s^2}{4}. \quad (5.3)$$

The only dominant term is one of the scalar-exchanging terms $T_{s3}$, the others all belong to subleading contributions. So

$$T^{\text{K}} \approx \frac{T_{s3}}{c_s^4}, \quad (5.4)$$

and $t_{\text{NL}}^{\text{K}} \approx 0.305/c_s^4$.

As mentioned in the previous section, there are some differences among the shapes $T_{s1}, T_{s2}$ and $T_{s3}$, and especially between them and $T_{c1}$. These differences may be used to distinguish some special models within this class. But unfortunately, for the above two examples, after summing over all contributions for DBI inflation, the trispectrum difference between the DBI inflation and this specific k-inflation example becomes smaller, and we do not find any features that can very sharply distinguish them. This is because the four leading shapes $T_{si}$ and $T_{c1}$ are similar in most regions and in the discriminating double-squeezed limit, the trispectra in both examples take the form $T_{si}$ (which are similar among themselves) since $T_{c1}$ vanishes faster.
6 Non-Bunch-Davies vacuum

We now study the shape of trispectrum if the initial state of inflation deviates from the standard Bunch-Davies vacuum of de Sitter space. This is an interesting question because short distance physics may give rise to such deviation \[52, 53\], and one might argue whether its effects on the power spectrum of the CMB are observable \[54\]. The effects of the non-Bunch-Davies vacuum on the bispectrum have been studied in \[9, 11, 12\], where it was found that the non-Gaussianities are boosted in the folded triangle limit (e.g. \( k_1 + k_2 - k_3 \sim 0 \)).

A general vacuum state for the fluctuation of the inflaton during inflation can be written as

\[
u_k = u(k, \tau) = \frac{H}{\sqrt{4\epsilon c_s k^3}} (C_+(1 + i k c_s \tau) e^{-ikc_s \tau} + C_-(1 - i k c_s \tau) e^{ikc_s \tau} ). \tag{6.1}
\]

Here a small and non-zero \( C_- \) parametrizes a deviation from the standard Bunch-Davies vacuum which has \( C_+ = 1, C_- = 0 \). We consider the corrections of a non-zero \( C_- \) to the leading shapes assuming \( C_- \) is small, we keep only terms up to linear order in \( C_- \). Similar to the case of the bispectrum \[9\], the first sub-leading corrections come from replacing one of the \( u(\tau, k) \)’s with their \( C_- \) components, and since the correction from \( u(0, k) \) only has the same shape as that of the Bunch-Davies vacuum, we only need to consider the case that \( u(\tau, k) \) comes from the interacting Hamiltonian, where \( \tau \) is not zero. In the following we discuss the contact-interaction diagram and the scalar-exchange diagram respectively.

For the contact-interaction diagram, the corrections consist of four terms from replacing \( k_i \) with \(-k_i \) in the shape for Bunch-Davies vacuum. For the leading shape \( T_{c1} \) we denote the correction as \( \tilde{T}_{c1} \) and we find

\[
\tilde{T}_{c1} = 36 \text{ Re}(C_-) \prod_{i=1}^{4} k_i^2 \left[ \frac{1}{(k_1 + k_2 + k_3 - k_4)^5} + \frac{1}{(k_1 + k_2 - k_3 + k_4)^5} \right. \\
\left. + \frac{1}{(k_1 - k_2 + k_3 + k_4)^5} + \frac{1}{(-k_1 + k_2 + k_3 + k_4)^5} \right]. \tag{6.2}
\]

Then let us consider scalar-exchange diagram. For illustration we only consider the \( T_{s1} \) term. The calculations for \( T_{s2} \) and \( T_{s3} \) are similar but more complicated. Now we have six \( u_k \) modes from the interaction Hamiltonian. Replacing each mode with its \( C_- \) component gives rise to six terms in the corrections. They correspond to replacing \( k_i \) with \(-k_i \), or one of the two \( k_{12} \)’s with \(-k_{12} \) in the calculations for the Bunch-Davies vacuum. The corrections

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\footnote{The choice of initial state is often discussed in the context of trans-Planckian effects \[55\] though the issue has more general applicability. See e.g. \[56\] for a review and references, and \[57\] for a discussion of how to capture the initial state effects in terms of a boundary effective field theory.}
to Eq. (B.3) are

\[
9 \text{Re}(C_-) \frac{k_1^2 k_2^2 k_3^2 k_4^2 k_{12}}{8} \left\{ \sum_{k_i \rightarrow -k_i} \frac{1}{(k_1 + k_2 + k_4 + k_{12})^3 (k_1 + k_2 + k_{12})^3} \right\} + \frac{1}{(k_1 + k_2 - k_{12})^3} \left( \frac{6(k_3 + k_4 - k_{12})^2}{(K - 2k_{12})^3} + \frac{3(k_3 + k_4 - k_{12})}{(K - 2k_1^2) + 1} \right) \left( \frac{6M^2}{K^5} + \frac{3M}{K^4} + \frac{1}{K^3} \right) \right. \\
\left. + \frac{1}{M^3} \left( \frac{6M^2}{(K + 2k_{12})^5} + \frac{3M}{(K + 2k_{12})^4} + \frac{1}{(K + 2k_{12})^3} \right) \right\} + 23 \text{ perm. (6.3)}
\]

In Eq. (B.4), the two $k_{12}$ cancelled in the final expression for the Bunch-Davies vacuum, so we have to recover them in the calculations. Denoting $M = k_3 + k_4 + k_{12}$ and $K = k_1 + k_2 + k_3 + k_4$, we find the corrections

\[
9 \text{Re}(C_-) \frac{k_1^2 k_2^2 k_3^2 k_4^2 k_{12}}{4} \left\{ \sum_{k_i \rightarrow -k_i} \frac{1}{M^3} \left( \frac{6M^2}{K^5} + \frac{3M}{K^4} + \frac{1}{K^3} \right) \right. \\
\left. + \frac{1}{M^3} \left( \frac{6M^2}{(K + 2k_{12})^5} + \frac{3M}{(K + 2k_{12})^4} + \frac{1}{(K + 2k_{12})^3} \right) \right\} + 23 \text{ perm. (6.4)}
\]

To summarize, the correction $\tilde{T}_{s1}$ to $T_{s1}$ is the summation of Eqs. (6.3) and (6.4).

We can look for the analogue of the folded triangle limit discovered in the study of bispectrum where the corrections due to deviation from the Bunch-Davies vacuum diverge. Here we see when any one triangle, e.g., $(k_1, k_2, k_{12})$, in the momentum tetrahedron becomes folded, the corrections to $T_{s1}$ (see (6.3) and (6.4)) become divergent. Furthermore, when $k_1 + k_2 + k_3 - k_4 = 0$, the two triangles $(k_1, k_2, k_{12})$ and $(k_3, k_4, k_{12})$ become folded simultaneously, and the correction (6.2) to $T_{c1}$ also diverges. We will refer to such configurations as the folded sub-triangle configurations.

As discussed in [9], these divergences are artificial, and do not correspond to real infinities in observables. Rather, the divergences appear because it is not realistic to assume a non-standard vacuum to exist in the infinite past. A cutoff on momenta should be imposed at the same time when a non-Bunch-Davies vacuum is considered.

We would like to point out two interesting aspects of the effects of the non-Bunch-Davies vacuum on trispectra, and more generally on higher point functions.

Let us first look at the regions away from the folded sub-triangle configurations. In the regular tetrahedron limit, in terms of (3.33), the corresponding $t_{NL}$ for the non-Bunch-Davies contribution are

\[
\tilde{t}_{NL}^{c1} = 4.50 \text{Re}(C_-) \left( \frac{\mu}{\Sigma} - \frac{9\lambda^2}{\Sigma^2} \right), \quad \tilde{t}_{NL}^{s1} = 401 \text{Re}(C_-) \left( \frac{\lambda}{\Sigma} \right)^2 . \quad (6.5)
\]

\[8\]Due to permutations, the three momenta do not have to be next to each other in terms of Fig. 2, for example, $k_1, k_3, k_{13}$.  

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Figure 7: In this group of figures, we plot the $\tilde{T}_{c1}/\text{Re}(C_-)$ (the left column) and $\tilde{T}_{s1}/\text{Re}(C_-)$ (the right column) in the equilateral limit, specialized planar limit, and near double squeezed limit (we plot $\tilde{T}_{c1}/[\text{Re}(C_-)\Pi_i k_i]$ and $\tilde{T}_{s1}/[\text{Re}(C_-)\Pi_i k_i]$ in near double squeezed limit) respectively. Note that, in order to show clearly the locations of the divergence, in some figures we have taken the cutoffs of the z-axes to be extremely large.
Note that $\tilde{t}_{NL}^{cl}/\text{Re}(C_-)$ is 128 times larger than $t_{NL}^{cl}$; $\tilde{t}_{NL}^{s1}/\text{Re}(C_-)$ is about 1600 times larger than $t_{NL}^{s1}$. These large numbers arise because some plus signs become minus signs in the denominators, and there are also more terms to consider in the non-Bunch-Davies case. So in the context of general single field inflation, even if $\text{Re}(C_-)$ is as small as one part in one thousand, it becomes important phenomenologically. Generalize this to higher point functions, we see that, no matter how small the $C_-$ is, there always exists a high point function beyond which the contribution from the $C_-$ component becomes comparable to that from the $C_+$ component. For such functions, we should use the whole wavefunction \cite{6.1} to get the correct shapes instead of treating the $C_-$ component as a correction, even away from the folded sub-triangle limit. Therefore generally speaking, higher point function is a more sensitive probe to the non-Bunch-Davies component. However, on the other hand, higher point functions contribute less to the total non-Gaussianities and will be more difficult to measure experimentally.

We next look at the region near the folded sub-triangle limits. Because the denominators here have larger powers than those in 3pt, the corrections grow faster as we approach the folded sub-triangle limit. This also indicates that the trispectrum, and more generally higher point functions, is a nice probe of the non-Bunch-Davies vacuum. But, on the other hand, because the higher point function has a larger momentum phase space (three more dimensions here in trispectra) than the 3pt, the phase space for the folded limits becomes relatively smaller.

It will be interesting to see how the two factors in each of the above two aspects play out in the data analyses.

In Fig. 7 we plot $\tilde{T}_{cl}/\text{Re}(C_-)$ and $\tilde{T}_{s1}/\text{Re}(C_-)$ in the equilateral limit, specialized planar limit, and near double squeezed limit ($\tilde{T}_{cl}/[\text{Re}(C_-)\Pi_i k_i]$ and $\tilde{T}_{s1}/[\text{Re}(C_-)\Pi_i k_i]$ in this case) respectively.

### 7 Conclusion

To conclude, we have calculated the leading order trispectra for general single field inflation. As in the case of bispectra, the trispectra turns out to be of “equilateral shape” in general single field inflation. Compared with the local shape trispectra, the equilateral shape trispectra has not been extensively investigated in the literature. It is clear that there are a lot of work worthy to be done on this topic in the future. Directions for future work include:

- It is useful to extend our calculation to more general cases. We have focused here on the leading order contribution and single field inflation. It is interesting to generalize this calculation to next-to-leading order which may be also potentially observable, or
multifield inflation [58]. It is also useful to perform a unified analysis for general single field inflation and slow roll inflation.

- The shape of the trispectra is much more complicated compared with that of the bispectra. In our paper, we have obtained a lot of features of the shape functions by taking various limits. However, it is still a challenge to find improved representations to understand the shape functions. For example, one can find new bases for the shapes by tuning parameters to subtract out the similarities. Also, we focus on the planar limit in plotting the figures (except for Fig. 3), because the planar limit is of special importance for the CMB data analysis. It is interesting to investigate the non-planar parameter region in more details for the large scale structure and 21-cm line surveys.

- In the discussion of non-Bunch-Davies vacuum, we have calculated contributions from the two representative shapes. However, as we have seen in Section 6, trispectra is a powerful probe for non-Bunch-Davies vacuum. Even one part in $10^3$ deviation from the Bunch-Davies vacuum could lead to an order one correction to the trispectra. So it is valuable to perform the full calculation for the non-Bunch-Davies vacuum, and to study the effects of the cutoff.

- Most importantly, one would like to apply these shape functions to data analyses and see how they are constrained.

Note added: On the day this work appeared on the arXiv, the paper [59] was also submitted to the arXiv, which overlaps with our Sec.3 and 5.

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A Commutator form of the in-in formalism

In Sec. 3.1, we have used the original definition of the in-in formalism (3.8) and (3.9) to calculate the correlation function due to the scalar-exchange diagram. Another equivalent and commonly-used form is written in terms of the nested commutators. For the diagrams that we considered in Sec. 3.1, it takes the following form,

\[ \langle \zeta^4 \rangle \supset - \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \langle 0 | [\zeta^4_I(t), H_I(t')] , H_I(t'') | 0 \rangle . \]  

(A.1)

The main difference is that now the first term in (3.9) is separated into two integrals, each has a time- or anti-time-ordered double integration,

\[ \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \langle 0 | H_I(t'') \zeta^4_I(t) H_I(t') | 0 \rangle + \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \langle 0 | H_I(t') \zeta^4_I(t) H_I(t'') | 0 \rangle . \]  

(A.2)

If one uses this form, besides the fact that the algebra becomes much more complicated due to the time-ordered double integrations, one also encounters spurious divergences at special momentum configurations, such as \( k_1 + k_2 - k_3 - k_4 = 0 \), for each of the two terms in (A.2). These divergences can be seen, after some complicated algebra, to cancel each other once the two terms are summed up [28, 50, 51]. Therefore using the form of the first term in (3.9) is both algebraically simpler and free of spurious divergences. This conclusion can be generalized to the more nested terms.

For example, using formula (A.1) we can get

\[ \langle \zeta^4 \rangle_{aa} \propto \frac{9 \lambda^2}{8 \Sigma^2} \frac{8k_1^2k_2^2k_3^2k_4^2k_{12}}{K_{(-)}(2k_{12} - K_{(-)} + K_{(+))} \{3 (K_{(-)}^7 + K_{(+)}^7) + (12k_{12}^2 - 9K_{(-)}K_{(+))} \}
\times (K_{(-)}^5 + K_{(+))}^5) + 8K_{(-)}^2K_{(+)}^2 (K_{(-)}^3 + K_{(+)}^3) - 6k_{12} \left[ 2 (K_{(-)}^6 - K_{(+))}^6) \right.
\left. - 3K_{(-)}K_{(+)} (K_{(-)}^4 - K_{(+))}^4) \right] } + 23 \text{ perm.} , \]  

(A.3)

with \( K_{(-)} \equiv k_1 + k_2 - k_3 - k_4 \) and \( K_{(+))} \equiv k_1 + k_2 + k_3 + k_4 \). Although the terms before 23 perm. are not equivalent to the one in (B.3) plus (B.4), however, after including the permutation terms and performing lots of lengthy but straightforward calculations, one can find that the two expressions are the same.
B Details on the scalar-exchange diagram

In this Appendix, we give the details of the scalar-exchange diagram. We denote
\[ k_{12} = k_1 + k_2, \quad M = k_3 + k_4 + k_{12}, \quad K = k_1 + k_2 + k_3 + k_4. \]  
(B.1)

The following are the various contributions to the trispectrum form factor \( T \) defined as
\[ \langle \zeta^4 \rangle \equiv (2\pi)^9 P^3\delta^3(\sum_{i=1}^4 k_i) \prod_{i=1}^4 \frac{1}{k_i^3} T(k_1, k_2, k_3, k_4, k_{12}, k_{14}). \]  
(B.2)

The interaction Hamiltonian has two components, (3.6) and (3.7). So there are four different combinations.

B.1 The component \( \langle \zeta^4 \rangle_{aa} \)

This component is given in (3.15) and (3.17). The contribution from the first term of Eq. (3.9):
\[ -\frac{3}{32} \lambda \left( \frac{1}{c_s^2} - 1 \right) \left( k_3 \cdot k_4 \right) k_{12} k_2^2 k_4^2 \frac{1}{(k_1 + k_2 + k_{12})^3} F(k_3, k_4, M) + 23 \text{ perm.} \]  
(B.3)

The contribution from the second and third terms of Eq. (3.9):
\[ -\frac{3}{16} \lambda \left( \frac{1}{c_s^2} - 1 \right) \left( k_{12} \cdot k_4 \right) \frac{k_2^2 k_4^2 k_{12}}{k_{12}} \frac{1}{(k_1 + k_2 + k_{12})^3} F(k_{12}, k_4, M) + 23 \text{ perm.} \]  
(B.4)

B.2 The component \( \langle \zeta^4 \rangle_{ab} \)

There are two sub-diagrams contributing to this case. In the first sub-diagram, the exchanged scalar propagator is due to the contraction between two \( \dot{\zeta} \)'s; in the second, between \( \dot{\zeta} \) and \( \zeta \). The result of these two sub-diagrams for the first term of Eq. (3.9), and the second and third term of Eq. (3.9) are as follows.

The contribution from the first term:
- The first sub-diagram:
\[ -\frac{3}{32} \lambda \left( \frac{1}{c_s^2} - 1 \right) \left( k_3 \cdot k_4 \right) k_{12} k_2^2 k_4^2 \frac{1}{(k_1 + k_2 + k_{12})^3} F(k_3, k_4, M) + 23 \text{ perm.} \]  
(B.5)
- The second sub-diagram:
\[ -\frac{3}{16} \lambda \left( \frac{1}{c_s^2} - 1 \right) \left( k_{12} \cdot k_4 \right) \frac{k_2^2 k_4^2 k_{12}}{k_{12}} \frac{1}{(k_1 + k_2 + k_{12})^3} F(k_{12}, k_4, M) + 23 \text{ perm.} \]  
(B.6)
The contribution from the second and third terms:

- The first sub-diagram:
  \[- \frac{3}{16} \lambda \left( \frac{1}{c_s^2} - 1 \right) (k_3 \cdot k_4)k_1^2 k_2^2 k_{12} G_{ab}(k_3, k_4) + 23 \text{ perm.} \quad (B.7)\]

- The second sub-diagram:
  \[- \frac{3}{8} \lambda \left( \frac{1}{c_s^2} - 1 \right) (k_{12} \cdot k_4)k_1^2 k_2^2 G_{ab}(k_{12}, k_4) + 23 \text{ perm.} \quad (B.8)\]

**B.3 The component \(\langle \zeta^4 \rangle_{ba}\)**

Similar to the \(\langle \zeta^4 \rangle_{ab}\) case, here we also have two terms, each term includes two sub-diagrams:

The contribution from the first term:

Same as \(B.5\) and \(B.6\).

The contribution from the second and third terms:

- The first sub-diagram:
  \[- \frac{3}{16} \lambda \left( \frac{1}{c_s^2} - 1 \right) (k_1 \cdot k_2)k_3^2 k_4^2 k_{12} G_{ba}(k_1, k_2) + 23 \text{ perm.} \quad (B.9)\]

- The second sub-diagram:
  \[\frac{3}{8} \lambda \left( \frac{1}{c_s^2} - 1 \right) (k_2 \cdot k_{12})k_1^2 k_2^2 k_{12} G_{ba}(-k_{12}, k_2) + 23 \text{ perm.} \quad (B.10)\]

**B.4 The component \(\langle \zeta^4 \rangle_{bb}\)**

In this case, we have four sub-diagrams for each term. In the first sub-diagram, the exchanged scalar propagator is due to the contraction between two \(\dot{\zeta}\)'s; in the second, between \(\dot{\zeta}\) and \(\zeta\); in the third, between \(\zeta\) and \(\dot{\zeta}\); in the fourth, between two \(\zeta\)'s. The result of these four sub-diagrams for the first term of Eq. (3.9), and the second and third term of Eq. (3.9) are as follows.

The contribution from the first term:

- The first sub-diagram:
  \[\frac{1}{2^7} \left( \frac{1}{c_s^2} - 1 \right)^2 (k_1 \cdot k_2)(k_3 \cdot k_4)k_{12} F(k_1, k_2, k_1 + k_2 + k_{12})F(k_3, k_4, M) + 23 \text{ perm.} \quad (B.11)\]

- The second and third sub-diagrams:
  \[\frac{1}{2^5} \left( \frac{1}{c_s^2} - 1 \right)^2 (k_1 \cdot k_2)(k_{12} \cdot k_4)k_{12}^2 F(k_1, k_2, k_1 + k_2 + k_{12})F(k_{12}, k_4, M) + 23 \text{ perm.} \quad (B.12)\]
• The fourth sub-diagram:

\[- \frac{1}{2^5} \left( \frac{1}{c_s^2} - 1 \right)^2 (k_{12} \cdot k_2)(k_{12} \cdot k_4) \frac{k_3^2}{k_{12}^3} F(k_{12}, k_2, k_1 + k_2 + k_{12}) F(k_{12}, k_4, M) + 23 \text{ perm}(B.13)\]

The contribution from the second and third terms:

• The first sub-diagram:

\[ \frac{1}{2^6} \left( \frac{1}{c_s^2} - 1 \right)^2 (k_1 \cdot k_2)(k_3 \cdot k_4) k_{12} G_{bb}(k_1, k_2, k_3, k_4) + 23 \text{ perm. } \tag{B.14} \]

• The second sub-diagram:

\[ \frac{1}{2^6} \left( \frac{1}{c_s^2} - 1 \right)^2 (k_1 \cdot k_2)(k_{12} \cdot k_4) \frac{k_3^2}{k_{12}^3} G_{bb}(k_1, k_2, k_{12}, k_4) + 23 \text{ perm. } \tag{B.15} \]

• The third sub-diagram:

\[- \frac{1}{2^5} \left( \frac{1}{c_s^2} - 1 \right)^2 (k_{12} \cdot k_2)(k_3 \cdot k_4) \frac{k_1^2}{k_{12}^3} G_{bb}(-k_{12}, k_2, k_3, k_4) + 23 \text{ perm. } \tag{B.16} \]

• The fourth sub-diagram:

\[- \frac{1}{2^4} \left( \frac{1}{c_s^2} - 1 \right)^2 (k_{12} \cdot k_2)(k_{12} \cdot k_4) \frac{k_3^2}{k_{12}^3} G_{bb}(-k_{12}, k_2, k_{12}, k_4) + 23 \text{ perm. } \tag{B.17} \]

The function \( F, G_{ab}, G_{ba}, G_{bb} \) are defined as follows:

\[ F(\alpha_1, \alpha_2, m) \equiv \frac{1}{m^3} \left[ 2\alpha_1 \alpha_2 + (\alpha_1 + \alpha_2)m + m^2 \right], \tag{B.18} \]

\[ G_{ab}(\alpha_1, \alpha_2) \equiv \frac{1}{M^3 K^3} \left[ 2\alpha_1 \alpha_2 + (\alpha_1 + \alpha_2)M + M^2 \right] + \frac{1}{M^2 K^3} [2\alpha_1 \alpha_2 + (\alpha_1 + \alpha_2)M] + \frac{12}{MK^5} \alpha_1 \alpha_2, \tag{B.19} \]

\[ G_{ba}(\alpha_1, \alpha_2) \equiv \frac{1}{M^3 K^3} + \frac{1}{M^3 K^2} (\alpha_1 + \alpha_2 + M) + \frac{1}{M^3 K^3} [2\alpha_1 \alpha_2 + 2(\alpha_1 + \alpha_2)M + M^2] \]
\[ + \frac{1}{M^2 K^4} [2\alpha_1 \alpha_2 + (\alpha_1 + \alpha_2)M] + \frac{12}{MK^5} \alpha_1 \alpha_2, \tag{B.20} \]

\[ G_{bb}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \]
\[
\equiv \frac{1}{M^3 K} \left[ 2\alpha_3 \alpha_4 + (\alpha_3 + \alpha_4)M + M^2 \right] \\
+ \frac{1}{M^3 K^2} \left[ 2\alpha_3 \alpha_4 (\alpha_1 + \alpha_2) + (2\alpha_3 \alpha_4 + (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4))M + \sum_{i=1}^{4} \alpha_i M^2 \right] \\
+ \frac{2}{M^3 K^3} \left[ 2 \prod_{i=1}^{4} \alpha_i + (2\alpha_3 \alpha_4 (\alpha_1 + \alpha_2) + \alpha_1 \alpha_2 (\alpha_3 + \alpha_4))M + \sum_{i<j} \alpha_i \alpha_j M^2 \right] \\
+ \frac{6}{M^2 K^4} \left( \prod_{i=1}^{4} \alpha_i \right) \left( 2 + M \sum_{i=1}^{4} \frac{1}{\alpha_i} \right) + \frac{24}{MK^6} \prod_{i=1}^{4} \alpha_i .
\]  

(B.21)

Note that in \( G_{ab}, G_{ba} \) and \( G_{bb} \), the \( K \) and \( M \) are defined as \( K = k_1 + k_2 + k_3 + k_4 \) and \( M = k_3 + k_4 + k_{12} \), but not in terms of \( \alpha_i \)'s.

To summarize we denote the overall contribution from the scalar-exchange diagrams as

\[
T_s = \left( \frac{\lambda}{\Sigma} \right)^2 T_{s1} + \frac{\lambda}{\Sigma} \left( \frac{1}{c_s^2} - 1 \right) T_{s2} + \left( \frac{1}{c_s^2} - 1 \right)^2 T_{s3} ,
\]

where \( T_{s1} \) is given by (B.3) and (B.4), \( T_{s2} \) is given by (B.5)-(B.10), \( T_{s3} \) is given by (B.11)-(B.17).

C The planar limit of the trispectra

In Sec. \ref{sec:trispectra} we discussed various properties of the shape functions. As we have stated, the planar limit has special importance for CMB experiments. So in this appendix, we investigate the trispectra in more detail in the planar limit and perform a survey of parameters for the shape functions.

In the planar limit, Eq. (4.2) takes the equal sign. One can solve \( k_2 \) from Eq. (B.2),

\[
k_2 = \sqrt{k_1^2 \left( -k_{12}^2 + k_3^2 + k_4^2 \right) \pm k_{s1}^2 k_{s2}^2 + k_{12}^2 k_{14}^2 + k_{12}^2 k_{14}^2 + k_{14}^2 k_{14}^2 - k_{11}^2 k_{14}^2 - k_{14}^2 - k_3^2 k_4^2}} \sqrt{2k_4} ,
\]

(C.1)

where \( k_{s1} \) and \( k_{s2} \) are defined as

\[
k_{s1}^2 = 2 \sqrt{(k_{14}^2 + k_1 \cdot k_4)(k_{14}^2 - k_1 \cdot k_4)} ,
\]

\[
k_{s2}^2 = 2 \sqrt{(k_{34}^2 + k_3 \cdot k_4)(k_{34}^2 - k_3 \cdot k_4)} .
\]

We first take a close look at the \( \pm \) sign in Eq. (C.1). The \( - \) sign and the \( + \) sign correspond to two different quadrangles. In Fig. \ref{fig:quadrangles} the \( - \) and \( + \) solutions correspond to the black (with edge \( k_i \)) and blue (with edge \( q_i \)) quadrangles respectively. The former has all internal angles \( \leq \pi \), while the latter has one \( > \pi \). The blue quadrangle can be transformed into another quadrangle belonging to the same class as the black one by the symmetry discussed in Eq.
Figure 8: The quadrangles in black and blue represents the solution (C.1) with minus and plus sign respectively. The term with plus sign corresponds to the blue quadrangle, and can be transformed into another quadrangle corresponding to the minus solution by a symmetry discussed in Eq. (4.4).

So without losing generality, we will only consider the $-$ solution in the following discussion.

To scan the parameter space, we plot $T_{s1}$, $T_{s2}$, $T_{s3}$, $T_{c1}$, $T_{loc1}$ and $T_{loc2}$ as functions of $k_{12}/k_1$ and $k_{14}/k_1$ for different values of $k_3/k_1$ and $k_4/k_1$ in Fig. 9. The momenta $(k_3/k_1, k_4/k_1)$ take the following values in the 4 figures in each group.

$$(k_3/k_1, k_4/k_1) = \{(0.6, 0.6), (0.6, 1.0), (1.0, 0.6), (1.0, 1.0)\}. \quad (C.3)$$

We assume $k_1 > k_2, k_3, k_4$ in the plot without losing generality. Note that when $k_3 = k_4 = k_{12} = k_{14} = k_1$ (the center of the last figure in each group), $T_{s1}$, $T_{s2}$, $T_{s3}$ and $T_{c1}$ vanishes because in this case $k_2 = 0$. We can see from these graphs that the shapes of $T_{s1}$, $T_{s2}$ and $T_{s3}$ are overall very similar.
Figure 9: In the six rows, we plot $T_{s1}$, $T_{s2}$, $T_{s3}$, $T_{c1}$, $T_{loc1}$ and $T_{loc2}$ respectively as functions of $k_{12}/k_1$ and $k_{14}/k_1$ in the planar limit. Within each row, the momenta configuration is $(k_3/k_1, k_4/k_1) = \{(0.6, 0.6), (0.6, 1.0), (1.0, 0.6), (1.0, 1.0)\}$ respectively in the four columns, as given in (C.3).
References


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